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On the action of the mapping class group for Riemann surfaces of infinite type

Dedicated to Professor Hiroki Sato on his 60th birthday

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Abstract. We consider Riemann surfaces of infinite type and their reduced Teichmüller spaces. The reduced Teichmüller space admits the action of the reduced mapping class group. Generally, the action is not discrete while it is faithful. We give sufficient conditions for the discreteness of the action in terms of the geometry of Riemann surfaces.

1. Introduction.

The mapping class group (or the Teichmüller modular group) $\mathrm{Mod}(R)$ for a Riemann surface R is the set of equivalence classes of quasiconformal automorphisms of R (see [**10**]). Two quasiconformal automorphisms h_1 and h_2 of R are equivalent if $h_2^{-1} \circ h_1$ is homotopic to the identity by a homotopy that keeps every point of ideal boundary ∂R fixed throughout. In the theory of Teichmüller spaces of Riemann surfaces of analytically finite type, the mapping class group plays an important role in various fields. This is a group of the biholomorphic automorphisms of the Teichmüller space and it acts faithfully and properly discontinuously. On the other hand, it seems that there are few studies on $\mathrm{Mod}(R)$ for a Riemann surface R of infinite type. Recently, Earle-Gardiner-Lakic showed in [**3**] that it acts faithfully on the Teichmüller space $T(R)$. In this paper, we consider the discreteness of the action of the mapping class group. We say that a subgroup G of $\mathrm{Mod}(R)$ is discrete if the orbit of any point of $T(R)$ under the G action is discrete.

For a Riemann surface of analytically finite type, $\mathrm{Mod}(R)$ is discrete, while in the case of infinite type, $\mathrm{Mod}(R)$ is not necessarily discrete. In particular, if R has a boundary curve (border), $\mathrm{Mod}(R)$ is not discrete since a slight change of the boundary value of a quasiconformal map produces a different mapping class in $\mathrm{Mod}(R)$. Thus, it is natural that we consider another group, the reduced mapping class group. The reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the set of homotopy classes of quasiconformal automorphisms of R whose homotopy maps do not necessarily keep points of ∂R fixed. The reduced mapping class group is also important because it naturally acts on the reduced Teichmüller space.

We explore the problem of discreteness of the reduced mapping class group for Riemann surfaces of infinite type. Actually, if R is a Riemann surface of topologically

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finite type, then $\mathrm{Mod}^{\#}(R)$ is discrete. However, $\mathrm{Mod}^{\#}(R)$ is not discrete in general. For example, if R has a sequence of disjoint simple closed geodesics which are not freely homotopic to a boundary component and whose lengths tend to 0, then we see that $\mathrm{Mod}^{\#}(R)$ is not discrete (See §3 and §6). The purpose of this paper is to give a sufficient condition for discreteness.

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2. The mapping class group for the reduced Teichmüller space.

Throughout this paper, we assume that a Riemann surface R is *hyperbolic*, that is, it is represented by $\boldsymbol{H}/\Gamma$ for some Fuchsian group Γ acting on the upper half-plane $\boldsymbol{H}$ with the hyperbolic metric $|dz|/y$ $(z = x + y\sqrt{-1})$. We also assume that the Fuchsian group Γ is always non-elementary. In other words, we assume that the group Γ is non-abelian. A Riemann surface is called of *analytically finite type* if the hyperbolic area is finite, and is called of *analytically infinite type* if the area is not finite.

For an open Riemann surface R, a relatively non-compact connected component of the complement of a compact subset of R is called an *end*. An end V of R is called a *hole* if it is doubly connected and the hyperbolic area of V is infinite. A doubly connected end of R is called a *cusp* if the hyperbolic area of V is finite. A cusp V with smooth relative boundary is conformally equivalent to the punctured disk $\{0 < |z| < 1\}$. An ideal boundary of R corresponding to the origin $z = 0$ is called a *puncture*.

Notation. The hyperbolic distance on $\boldsymbol{H}$ and on a Riemann surface R is denoted by $d_{\boldsymbol{H}}(\cdot,\cdot)$ and $d_R(\cdot,\cdot)$ respectively. Further the hyperbolic length of a curve c in $\boldsymbol{H}$ or in R is denoted by $\ell(c)$.

We review the theory of Teichmüller spaces and mapping class groups. See [**4**], [**6**] and [**10**] for the details.

Definition 1. Fix a Riemann surface R. For pairs (S_j, f_j) of Riemann surfaces S_j and quasiconformal maps f_j of R onto S_j $(j = 1, 2)$, we say that (S_1, f_1) and (S_2, f_2) are *RT* (*reduced Teichmüller*) *equivalent* if there exists a conformal map h of S_1 onto S_2 such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R. The *reduced Teichmüller space* $T^{\#}(R)$ with the base Riemann surface R is the set of all the RT equivalence classes $[S, f]$ of such pairs (S, f) as above.

Definition 2. We say that two quasiconformal automorphisms h_1 and h_2 of R are *RT equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity on R. The *reduced mapping class group* $\mathrm{Mod}^{\#}(R)$ is the set of all the RT equivalence classes $[h]$ of quasiconformal automorphisms h of R. Furthermore, for a simple closed geodesic c on R, we set

$$\mathrm{Mod}_c^{\#}(R) = \{[h] \in \mathrm{Mod}^{\#}(R) \mid h(c) \text{ is freely homotopic to } c\}.$$

Every quasiconformal map of $R = \boldsymbol{H}/\Gamma$ induces an isomorphism of Γ into $PSL(2, \boldsymbol{R})$. We see that if two automorphisms h_1 and h_2 are RT equivalent then they induce the same isomorphism modulo $PSL(2, \boldsymbol{R})$ conjugacy.

If R is a compact Riemann surface, then the reduced Teichmüller space $T^{\#}(R)$ is nothing but the ordinary Teichmüller space $T(R)$ of R and the reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the ordinary mapping class group $\mathrm{Mod}(R)$.

Similar to the case of $T(R)$, the reduced Teichmüller space $T^{\#}(R)$ is equipped with the reduced Teichmüller distance $d_T(\cdot,\cdot)$ defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \frac{1}{2} \inf_{g_1, g_2} \log K(g_1 \circ g_2^{-1}),$$

where $K(\cdot)$ is the maximal dilatation of a quasiconformal map and the infimum is taken over all quasiconformal maps g_1 and g_2 determining $[S_1, f_1]$ and $[S_2, f_2]$, respectively. It is known that $T^{\#}(R)$ is a complete metric space with respect to this d_T. An element $\omega = [h] \in \mathrm{Mod}^{\#}(R)$ induces an automorphism of $T^{\#}(R)$ by

$$[S, f] \mapsto [S, f \circ h^{-1}].$$

This is an isometric automorphism with respect to d_T and denoted by ω_*. Namely, we have a homomorphism $\mathrm{Mod}^{\#}(R) \to \mathrm{Aut}(T^{\#}(R))$.

Remark 1. In [**3**], it is proved that for any Riemann surface R of analytically infinite type (and if $2g + n > 4$ when R is of finite (g, n)-type), the homomorphism $\mathrm{Mod}^{\#}(R) \to \mathrm{Aut}(T^{\#}(R))$ as above is faithful. Therefore we can identify ω_* with ω and omit the asterisk hereafter.

Definition 3. We say that a subgroup G of $\mathrm{Mod}^{\#}(R)$ is *discrete* if every sequence $\{\omega_n\} \subset G$ satisfying $\lim_{n\to\infty} \omega_n(p) = q$ for some pair of points p, q in $T^{\#}(R)$ is eventually a constant sequence, that is, there exists an $N \in \boldsymbol{N}$ such that $\omega_n = \omega_N$ for every $n \geq N$.

3. Examples.

As we noted in the introduction, if R is a compact Riemann surface, then the action of $\mathrm{Mod}(R)$ on $T(R)$ is discrete. Contrary to this case, there are various kinds of examples which show non-discreteness of $\mathrm{Mod}^{\#}(R)$ for a Riemann surface R of infinite type.

Example 1. Suppose that R has a sequence $\{c_n\}$ of distinct simple closed geodesics that are not freely homotopic to a boundary component and that these hyperbolic lengths tend to 0. Then the Dehn twist along each c_n gives an element ω_n of $\mathrm{Mod}^{\#}(R)$ such that the sequence $\{\omega_n(p_0)\}$ converges to p_0 as $n \to \infty$, where $p_0 = [R, \mathrm{id}]$ is the base point of $T^{\#}(R)$. Hence $\mathrm{Mod}^{\#}(R)$ is not discrete.

There exists a Riemann surface R such that it has no short geodesics but that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Example 2. We construct a Riemann surface R such that it has no short geodesics and but contains a point with arbitrarily large injectivity radius with respect to the hyperbolic metric (in fact, R does not satisfy the second condition in Theorem 1, which is stated in Section 4), and that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Set

$$R = \boldsymbol{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \boldsymbol{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\}.$$

To show that $\mathrm{Mod}^{\#}(R)$ is not discrete, set

$$f_n(z) = \begin{cases} x - (y - 2n - 2)/n + y\sqrt{-1} & (2n+1 \le y < 2n+2) \\ x + (y - 2n)/n + y\sqrt{-1} & (2n \le y < 2n+1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

Then f_n are quasiconformal automorphisms of R and the maximal dilatations of $\{f_n\}$ tend to 1. Thus $\mathrm{Mod}^{\#}(R)$ is not discrete.

Now, we see that R does not satisfy the second condition in Theorem 1. We put $A_n = R \cap \{z \mid \mathrm{Im}\, z = 2n+1\}$ and $a_{m,n} = m/n + (2n+1)\sqrt{-1}$ $(n \in \boldsymbol{N}, m \in \boldsymbol{Z})$. Then we shall prove that

$$d_R(A_n, A_{n+1}) \to \infty \quad (n \to \infty). \tag{1}$$

To prove this, we show that the injectivity radii at $b_n = 2n\sqrt{-1}$ tend to ∞ as $n \to \infty$. The length of any non-trivial closed curve passing through b_n is greater than $d_n = \inf_m d_R(b_n, I_{m,n} \cup I_{m,n+1})$, where $I_{m,n}$ is the segment connecting $a_{m,n}$ and $a_{m+1,n}$. Set

$$\varphi_{m,n}(z) = n(z - a_{m,n}).$$

Then, $\varphi_{m,n}$ is a conformal mapping from $\boldsymbol{C} - \{a_{m,n}, a_{m+1,n}\}$ onto the Riemann surface $S = \boldsymbol{C} \backslash \{0, 1\}$. From the decreasing property of the hyperbolic distance, we have

$$\begin{aligned} d_R(b_n, I_{m,n}) &\ge d_S(\varphi_{m,n}(b_n), \varphi_{m,n}(I_{m,n})) \\ &= d_S(-m - n\sqrt{-1}, (0,1)). \end{aligned}$$

Obviously, $d_S(-m - n\sqrt{-1}, (0,1)) \to \infty$ as $|m| + |n| \to \infty$. Therefore, $\lim_{n \to \infty} d_n = \infty$ and the injectivity radii at b_n tend to ∞ as $n \to \infty$. Hence we see that, for any $M > 0$, there exists $n \in \boldsymbol{Z}$ such that A_n and A_{n+1} belong to distinct components of R_M each other. This implies that R does not satisfy the second condition in Theorem 1.

Next, we show that R has no short geodesics. Suppose that there exists a sequence $\{c_k\}$ of simple closed geodesics on R such that $\lim_{k \to \infty} \ell(c_k) = 0$. From (1) we may assume that c_k contains two distinct points $a_{m,n}$ and $a_{m',n}$. By translation, we may also assume that c_k contains $a_{0,n}, a_{m,n}$ but does not contain $a_{-m,n}$. From the decreasing property of the hyperbolic metric as above, we see that the hyperbolic length of c_k in R is greater than the length in R', where $R' = \hat{\boldsymbol{C}} \backslash \{\infty, a_{-m,n}, a_{0,n}, a_{m,n}\}$ which is conformally equivalent to $R_0 = \hat{\boldsymbol{C}} \backslash \{\infty, -1, 0, 1\}$. It is well known that the length of any closed geodesic in R_0 is greater than some positive constant L. Thus, we have $\ell(c_k) > L > 0$ and it is a contradiction.

We exhibit an example of a planar Riemann surface R without cusps but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that $\mathrm{Mod}^{\#}(R)$ is not discrete.

EXAMPLE 3. We construct a Riemann surface R without cusps such that it satisfies the first condition in Theorem 1 but does not satisfy the second condition and that $\mathrm{Mod}^{\#}(R)$ is not discrete.

For each $n \geq 2$, we set

$$I_n = [-1, 1] \cup \bigcup_{k=1}^{\infty} I_{n,k},$$

where

$$I_{n,k} = \{x + (1 - 1/n)^k \sqrt{-1} \mid -1 \leq x \leq 1\}$$
$$\cup \{x + (1 + (k-1)/n)\sqrt{-1} \mid -1 \leq x \leq 1\}.$$

We take infinitely many copies $\{R_n\}$ of $\boldsymbol{C} - \{y\sqrt{-1} \mid y \leq -1\}$ and set $R_n' = R_n - I_n$ for each $n \geq 2$. We make a Riemann surface R by gluing the right hand side of $\{y\sqrt{-1} \mid y < -1\}$ on R_n' with the left hand side of $\{y\sqrt{-1} \mid y < -1\}$ on R_{n-1}' $(n = 3, 4, \dots)$ along the imaginary axis. By using the same argument as that of Example 2, we can show that the Riemann surface R satisfies the first condition in Theorem 1 but does not satisfy the second condition in Theorem 1.

Consider a quasiconformal map f_n of R_n' defined by

$$f_n(z) = \begin{cases} x + (1 - 1/n)y\sqrt{-1} & (0 < y \leq 1) \\ x + (y - 1/n)\sqrt{-1} & (y > 1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

It is easily seen that the maximal dilatations of f_n converge to 1 as $n \to \infty$. Obviously, f_n is extended to a quasiconformal automorphism of R by setting it the identity on $R - R_n'$ and we will write it by the same letter f_n. Thus the quasiconformal map f_n gives an element $[f_n]$ of $\mathrm{Mod}^{\#}(R)$ such that $\{[f_n](p_0)\}$ converges to p_0 as $n \to \infty$, where $p_0 = [R, \mathrm{id}]$ is the base point of $T^{\#}(R)$. Hence we conclude that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Even if a Riemann surface R has no short geodesics and no points with arbitrarily large injectivity radius, $\mathrm{Mod}^{\#}(R)$ may not be discrete.

EXAMPLE 4. We construct a Riemann surface R such that it has no short geodesics and no points with arbitrarily large injectivity radius but that $\mathrm{Mod}^{\#}(R)$ is not discrete. Consider a torus S with two geodesic borders with the same length one another. We take infinity many copies $\{S_n\}_{n=-\infty}^{\infty}$ of S. We denote the two geodesic borders of S_n by $\ell_{n,1}$ and $\ell_{n,2}$. Construct a Riemann surface R by gluing the $\ell_{n-1,2}$ with $\ell_{n,1}$ and gluing $\ell_{n,2}$ with $\ell_{n+1,1}$ for each n. Let f be a conformal automorphism of R which sends S_n to S_{n+1}, and we set $f_n := f^n$. Then we see that $[f_n] \neq \mathrm{id}$ as an element of $\mathrm{Mod}^{\#}(R)$. However, $[f_n](p_0) = p_0$ for all n, where $p_0 = [R, \mathrm{id}] \in T^{\#}(R)$ because $f_n : R \to R$ is a conformal mapping. Hence, $\mathrm{Mod}^{\#}(R)$ is not discrete.

4. Main Results.

As Example 1 shows, for the discreteness of the mapping class group, it is necessary that there exist no sequences of geodesics on the Riemann surface whose lengths converge to zero. Examples 2, 3 show that some conditions for the injectivity radius are required for the discreteness.

DEFINITION 4. For a given $M > 0$, we define R_M to be the subset of points $p \in R$ such that there exists a non-trivial simple closed curve passing through p whose hyperbolic length is less than M. The set R_ε is called the *ε-thin part* of R if $\varepsilon > 0$ is smaller than the Margulis constant. Further, a connected component of the ε-thin part that corresponds to a puncture is called the *cusp neighborhood*.

Now, we exhibit our main results.

THEOREM 1. *Let R be a Riemann surface with the non-abelian fundamental group. Suppose that R satisfies the following two conditions*:

(1) *There exists a constant $\varepsilon > 0$ such that the ε-thin part of R consists only of cusp neighborhoods.*

(2) *There exist a constant $M > 0$ and a connected component R_M^* of R_M such that the homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ which is induced by the inclusion map of R_M^* to R is surjective.*

Then $\mathrm{Mod}_c^{\#}(R)$ is discrete for any simple closed geodesic c on R.

REMARK 2. Example 1 shows that the first condition in Theorem 1 is necessary for the discreteness. On the other hand, the Riemann surfaces in Examples 2 and 3 satisfy the first condition but do not satisfy the second condition. Example 4 shows that there exists a Riemann surface such that it satisfies both the conditions but $\mathrm{Mod}^{\#}(R)$ is not discrete.

REMARK 3. If R satisfies the second condition in Theorem 1 for a constant M, then it satisfies the condition for all $M' \geq M$.

REMARK 4. The region R_M is not necessarily connected for large M even if the homomorphism: $\pi_1(R_M) \to \pi_1(R)$ is surjective. Moreover, in Example 7 of §6, we give a Riemann surface R and divergent sequences $\{M_n\}, \{M_n'\}$ such that

- $M_n < M_n' < M_{n+1} < M_{n+1}'$ $(n = 1, 2, \ldots)$;
- R_{M_n} is connected for all n and the homomorphism: $\pi_1(R_{M_n}) \to \pi_1(R)$ is surjective;
- $R_{M_n'}$ is not connected for all n but the homomorphism: $\pi_1(R_{M_n'}^*) \to \pi_1(R)$ is surjective for some component $R_{M_n'}^*$ of $R_{M_n'}$.

For a hyperbolic Riemann surface $R = \boldsymbol{H}/\Gamma$, we consider the *convex core* $C(\Gamma)$ of the limit set of Γ, that is, the hyperbolic convex envelope of $\Lambda(\Gamma) \subset \boldsymbol{R} \cup \{\infty\}$ in $\boldsymbol{H}$. Since the convex core $C(\Gamma)$ is Γ-invariant, it determines a region $C(R)$ in R and we call the region the *convex core* of R.

DEFINITION 5. We say that a Riemann surface R has *ε-uniform geometry* if the following two conditions are satisfied for some $\varepsilon > 0$:

(1) The ε-thin part of R consists of cusp neighborhoods.

(2) The injectivity radius on the convex core $C(R)$ of R is less than ε^{-1}.

Since $C(R)$ is connected and it contains any closed geodesic on R, from Theorem 1 we have the following immediately.

Corollary 1. *Let R be a Riemann surface with ε-uniform geometry for some $\varepsilon > 0$. Then $\mathrm{Mod}_c^{\#}(R)$ is discrete for any simple closed geodesic c on R.*

Remark 5. The conditions in Theorem 1 do not imply the uniform geometry. For example, set $R = \boldsymbol{C} - \boldsymbol{Z}$. Then R has a Fuchsian model of the first kind, and hence the convex core $C(R)$ coincides with R. By considering a sequence $\{z_n\}$ in R with $|\mathrm{Im}\, z_n| \to \infty$ as $n \to \infty$, we see that R has points with arbitrarily large injectivity radius. Hence, R does not have ε-uniform geometry for any $\varepsilon > 0$. On the other hand, it is easily seen that R satisfies the conditions in Theorem 1.

Remark 6. The conditions having uniform geometry were first stated as no short geodesics and no large disk condition. Nakanishi and Yamamoto [**11**] shows that under these conditions the outradius of the Teichmüller space is strictly less than 6. Ohtake [**12**] uses these conditions to show that the norm of the Poincaré series is strictly less than one which generalizes a result in McMullen [**9**].

It is important to give conditions for the mapping class group to be discrete. By using the above results, we have the following.

Theorem 2. *Let R be a Riemann surface satisfying the conditions in Theorem* 1 *or Corollary* 1. *Suppose that either the genus, the number of cusps or the number of holes of R is positive finite. Then $\mathrm{Mod}^{\#}(R)$ is discrete.*

5. Proofs of main results.

First of all, we note the geometry of a component of R_M.

Proposition 1. *For $M > 0$, let R_M^* be a connected component of R_M defined in Definition* 4 *and R_ε the ε-thin part of R for some small $\varepsilon < M$. We assume that $R_M^* - R_\varepsilon$ is not of type $(0,3)$. Then there exists a constant $M_1 > 0$ depending only on M and ε such that for any point $p \in R_M^* - R_\varepsilon$ there exists a simple closed curve c_p passing through p with $\ell(c_p) < M_1$ which does not surround a puncture of R.*

Proof. Let Γ be a Fuchsian group representing R. Take an arbitrary point p in $R_M^* - R_\varepsilon$. From the definition, we may find a simple closed curve $c_p \ni p$ whose length is less than M. If c_p is not homotopic to a simple closed curve which surrounds a puncture of R, then there is nothing to prove.

Thus, we suppose that c_p surrounds a puncture of R. Then, a parabolic transformation $\gamma \in \Gamma$ represents c_p. We may assume that $\gamma(z) = z + 1$. For $r > 0$, we take $\delta(r)$ so that

$$d_{\boldsymbol{H}}(\delta(r)\sqrt{-1}, \delta(r)\sqrt{-1} + 1) = r,$$

It is easily seen that $\delta(r) = (2\sinh(r/2))^{-1}$ for $r > 0$. We put

$$S(M,\varepsilon) = \{z \in \boldsymbol{H} \,|\, \delta(M) \le \mathrm{Im}\, z \le \delta(\varepsilon),\ 0 \le \mathrm{Re}\, z \le 1\}.$$

Since $\ell(c_p) < M$ and $p \notin R_\varepsilon$, a lift C_p of c_p contains a point in $S(M,\varepsilon)$.

Let L_z $(z \in \boldsymbol{H})$ denote the geodesic arc from z to $z+1$. Suppose that there exists a point $z \in S(M,\varepsilon)$ such that the projection l_z in R of L_z via the canonical projection

$\pi : \boldsymbol{H} \to R = \boldsymbol{H}/\Gamma$ is not simple. Then l_z contains a non-trivial simple closed curve c_z' with $\ell(c_z') < \ell(l_z) < M$.

If c_z' does not surround a puncture of R, then connect p and c_z' by a simple arc on R. Then, we see that there exists a simple closed curve passing through p with length less than $M_1 = 2(M + d_{\boldsymbol{H}}(\delta(\varepsilon)\sqrt{-1}, \delta(M)\sqrt{-1}))$ which does not surround a puncture of R.

Next, suppose that c_z' surrounds a puncture of R. Noting that l_z is the projection of the geodesic arc L_z, we verify that c_z' is not homotopic to c_p. In other words, the curve c_z' surrounds another puncture of R. Connecting c_z' and c_p, we have a simple closed curve passing through p with length less than M_1. Since $R_M^* - R_\varepsilon$ is not of type $(0,3)$, the curve does not surround a puncture of R.

Finally, we suppose that l_z is simple for any $z \in S(M, \varepsilon)$. Let us consider a geodesic L_z for $z \in \widetilde{R_M^*} \cap \{z \in \boldsymbol{H} \mid \operatorname{Im} z = \delta(M)\}$, where $\widetilde{R_M^*}$ is a lift of R_M^* with $\widetilde{R_M^*} \cap S(M, \varepsilon) \neq \emptyset$. From the definition, $\ell(L_z) = M$. Therefore, there exists a simple closed curve c_z in R_M^* passing through $\pi(z)$ with $\ell(c_z) < M = \ell(L_z) = \ell(l_z)$. Obviously, the curve c_z is not homotopic to $l_z = \pi(L_z)$ because l_z is the shortest simple closed curve which passes through $\pi(z)$ and surrounds the puncture. Therefore, by using the same argument as above, we have a non-trivial simple closed curve passing through p with length less than M_1 which does not surround a puncture of R. $\square$

To prove the main results, the following proposition on the hyperbolic geometry is crucial.

Proposition 2. *Let Γ be a Fuchsian model on the upper half-plane $\boldsymbol{H}$ of a Riemann surface R. Assume that Γ is non-elementary. Let M and D be positive constants. Then there is a constant $A > 1$ depending only on M and D that satisfies the following property: for a quasiconformal automorphism f of $\boldsymbol{H}$ such that $f \circ \Gamma \circ f^{-1} = \Gamma$, suppose that there exist distinct hyperbolic elements g_1, g_2 and g_3 in Γ such that*

(1) *translation lengths of g_j $(j = 1, 2, 3)$ are less than M,*
(2) *the projections of the axes ℓ_j of g_j to R are simple closed geodesics,*
(3) *the distances between a point z_1 on ℓ_1 and ℓ_j $(j = 2, 3)$ are less than D, and*
(4) *an isomorphism χ of Γ induced by f satisfies*

$$\chi(g_1) = g_1, \quad \chi(g_2) = g_2, \quad \chi(g_3) \neq g_3.$$

Then, $K(f) \geq A$.

To prove this proposition, we prepare some known results.

Lemma 1 ([7], Theorem 1). *Let f be a quasiconformal automorphism of $\boldsymbol{C}$ fixing* 0 *and* 1, *and suppose that there is a point z_0 in $\boldsymbol{C} - \{0, 1\}$ such that*

$$\log M = d_1(z_0, f(z_0)) > 0.$$

Then $K(f) \geq M^2$, where $d_1(\,,\,)$ is the hyperbolic distance on $\boldsymbol{C} - \{0, 1\}$.

Lemma 2 ([13], Lemma 3.1). *Let f be a quasiconformal mapping of a Riemann surface R onto another Riemann surface S, and c be a simple closed geodesic on R with hyperbolic length L. Then the hyperbolic length of a closed geodesic on S homotopic to $f(c)$ is not greater than $K(f)L$.*

Lemma 3 ([**5**]). *For a given $M > 0$, let g and g' be arbitrary two distinct hyperbolic elements of Γ with translation lengths less than M. Suppose that the projections of the axes of g and g' to R are simple closed geodesics which are coincident or disjoint. Then the axes have a distance greater than $C > 0$ depending only on M.*

We also need a variant of the above lemma.

Lemma 4. *For a given $M > 0$, let g and g' be arbitrary two hyperbolic elements of Γ with translation lengths less than M. Suppose that the projections of the axes of g and g' to R are simple closed geodesics which intersect one another. Then the axes make an angle greater than $C > 0$ depending only on M.*

Proof. Assume that the translation length of g' is not less than that of g. Then, on R, the closed geodesic ℓ induced by g can not round more than once in the collar of the geodesic ℓ' induced by g'. Hence we have a desired lower bound for the intersection angle of ℓ and ℓ'. □

Proof of Proposition 2. We may assume that the fixed points of g_1 are 0 and ∞, and that $z_1 = \sqrt{-1} \in \boldsymbol{H}$, hence $d_{\boldsymbol{H}}(\sqrt{-1}, \ell_j) \le D$ for $j = 2, 3$. We may also assume that the maximal dilatation of f is less than 2. Then at least one of the fixed points of g_j $(j = 2, 3)$ is not in $U = \{x \in \boldsymbol{R} \mid |x| < \delta \text{ or } |x| > 1/\delta\}$ for sufficiently small $\delta > 0$ which depends only on M and D. Indeed, if both fixed points of g_j are in $U_1 = \{x \in \boldsymbol{R} \mid |x| < \delta\}$ for small $\delta > 0$, then it contradicts $d_{\boldsymbol{H}}(\sqrt{-1}, \ell_j) \le D$. The same argument works when both fixed points are in $U_2 = \{x \in \boldsymbol{R} \mid |x| > 1/\delta\}$. If one fixed point of g_j is in U_1 and the other is in U_2, then it contradicts Lemma 3 if $\ell_1 \cap \ell_j = \emptyset$ and it contradicts Lemma 4 if $\ell_1 \cap \ell_j \ne \emptyset$ $(j = 2, 3)$. Therefore, we verify that at least one of the fixed points of g_j $(j = 2, 3)$ is not in U. By using the same argument, we see that there exists a constant $\delta' > 0$ depending only on M and D such that all fixed points of g_2 and g_3 are in $\{x \in \boldsymbol{R} \mid \delta' < |x| < 1/\delta'\}$.

Then, since $d_{\boldsymbol{H}}(\sqrt{-1}, \ell_3) \le D$, the Euclidean diameter $\mathrm{diam}(\ell_3)$ of ℓ_3 is greater than some $r = r(M, D) > 0$ which depends only on M and D. Set $g_4 = f \circ g_3 \circ f^{-1}$. By the assumption we have $g_4 \ne g_3$. Then, we see that there exists a constant $C = C(M, D) > 0$ depending only on M and D such that an inequality

$$|b - f(b)| > C \tag{2}$$

holds for at least one fixed point b of g_3. Indeed, since $K(f) < 2$, the translation length of g_4 is less than $2M$ by Lemma 2. Noting that $\mathrm{diam}(\ell_3) > r$, we see that if $\ell_3 \cap \ell_4 \ne \emptyset$, then we have the assertion from Lemma 4, and that Lemma 3 yields the assertion if $\ell_3 \cap \ell_4 = \emptyset$.

Take a fixed point a of g_2 with

$$\delta < |a| < 1/\delta. \tag{3}$$

Let ϕ be a Möbius transformation with $\phi(0) = 0$, $\phi(a) = 1$ and $\phi(\infty) = \infty$. As we noted, $\delta' < |b| < 1/\delta'$. Hence, (2) and (3) imply that

$$d_a(b, f(b)) > \log L$$

holds for some constant $L > 1$ depending only on M and D, where $d_a(\,,\,)$ is the hyperbolic distance on $\boldsymbol{C} - \{0, a\}$. Considering $\{0, a, b\}$ instead of $\{0, 1, z_0\}$ for $z_0 = \phi(b)$ in Lemma 1, we verify that the assertion follows for $A = L^2$. □

Next, we show a fundamental property of $\mathrm{Mod}_c^{\#}(R)$.

Proposition 3. *Let R be a Riemann surface. For an arbitrary simple closed geodesic c on R, let $\{[f_n]\}$ be a sequence of transformations of $\mathrm{Mod}_c^{\#}(R)$ that satisfies $\lim_{n\to\infty} K(f_n) = 1$. Then there exists a subsequence $\{[f_{n_j}]\}$ of $\{[f_n]\}$ such that $\{f_{n_j}\}$ locally uniformly converges to a conformal automorphism f of R which determines a transformation $[f] \in \mathrm{Mod}_c^{\#}(R)$.*

Proof. First we suppose that c is not homotopic to a boundary component of R. Then there exists a simple closed geodesic c' on R with $c \cap c' \neq \emptyset$. Hence Lemma 5 (with $C = c$ and $K = c'$) below shows the desired result.

Next suppose that c is homotopic to a boundary component of R. We may assume that the Riemann surface R is not topologically finite. Consider the double $\hat{R}$ of R. Then, $\hat{R}$ is still hyperbolic and the curve c is not homotopic to a boundary component of $\hat{R}$. And it is easily seen that quasiconformal mappings $f_n : R \to R$ $(n = 1, 2, \dots)$ are extended to quasiconformal mappings $\hat{f_n} : \hat{R} \to \hat{R}$ with the same maximal dilatations. Therefore, by the same argument as above, we have the desired result. □

Lemma 5. *Let $\{f_n\}$ be a sequence of quasiconformal automorphisms of a hyperbolic Riemann surface R that satisfies $\lim_{n\to\infty} K(f_n) = 1$. Suppose that there exist compact subsets C and K of R such that $f_n(C) \cap K \neq \emptyset$ for all n. Then there exist a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and a conformal automorphism f of R such that $\{f_{n_j}\}$ converges to f locally uniformly on R.*

Proof. From the assumption, there exists a sequence $\{p_n\}$ on C such that $f_n(p_n) \in K$. Since C and K are compact, there exist $p \in C$ and $q \in K$ such that $p_n \to p$ and $f_n(p_n) \to q$ as $n \to \infty$. Take lifts of p_n, p and q in $\boldsymbol{H}$, say $\tilde{p}_n, \tilde{p}$ and $\tilde{q}$, respectively, so that $\tilde{p}_n \to \tilde{p}$ as $n \to \infty$. We can take lifts $\tilde{f_n} : \boldsymbol{H} \to \boldsymbol{H}$ of f_n satisfying $\tilde{f_n}(\tilde{p}_n) \to \tilde{q}$. Since $\{\tilde{f_n}\}$ is a normal family, a subsequence $\{\tilde{f}_{n_j}\}$ of $\tilde{f_n}$ converges locally uniformly, and the limit function $\tilde{f}$ is either a quasiconformal automorphism of $\boldsymbol{H}$ or a constant in $\boldsymbol{R} \cup \{\infty\}$ (see [**8**], Theorem 5.3). Since $\tilde{f}(\tilde{p}) = \tilde{q}$ is in $\boldsymbol{H}, \tilde{f}$ is not a constant. Thus, it follows from $\lim_{n\to\infty} K(f_n) = 1$ that $\tilde{f}$ is a conformal automorphism of $\boldsymbol{H}$. Hence, $\{f_{n_j}\}$ converges locally uniformly to a conformal automorphism f of R which is the projection of $\tilde{f}$. □

Before proving our main theorems, we shall give a sufficient condition for discreteness of a sequence of $\mathrm{Mod}^{\#}(R)$ under the conditions in Theorem 1.

Proposition 4. *Let R be a Riemann surface satisfying two conditions in Theorem* 1, *and $\{f_n\}$ be a sequence of quasiconformal automorphisms of R satisfying the following conditions*:

- $\{(f_n)_*\}$ *converges to the identity, where $(f_n)_* : \pi_1(R) \to \pi_1(R)$ is an isomorphism induced by f_n.*
- $\lim_{n\to\infty} K(f_n) = 1$.

Then, f_n are homotopic to the identity for sufficiently large n.

PROOF. Let Γ be a Fuchsian model of R, and $\tilde{f}_n$ a lift of f_n for each n. We may take $\tilde{f}_n$ so that the isomorphisms $\chi_n : \Gamma \to \Gamma$ induced by $\tilde{f}_n$ converge to the identity. Suppose that χ_n are not eventually the identity. Then the following lemma gives us three hyperbolic elements $g_{1,n}, g_{2,n}$ and $g_{3,n}$ in Γ for each n which satisfy the conditions in Proposition 2 for some constants M' and D. Hence, we have

$$K(f_n) \geq A = A(M', D) > 1.$$

Since constants M' and D are independent of n, this contradicts $\lim_{n\to\infty} K(f_n) = 1$. Hence we have proved this proposition. □

LEMMA 6. *Let R be a Riemann surface satisfying the two conditions in Theorem* 1, *and χ_n are isomorphisms of the Fuchsian model Γ of R such that $\chi_n \to \mathrm{id}$ and that they are not eventually the identity. Then, for each n, there exist hyperbolic elements $g_{j,n}$ $(j = 1, 2, 3)$ of Γ with axes $\ell_{j,n}$ such that they satisfy the following four conditions*:

(1) *the projections $L_{j,n}$ of $\ell_{j,n}$ to R are simple closed geodesics,*

(2) *there is a constant M' independent of n such that the lengths of $L_{j,n}$ are less than M',*

(3) *there is a constant D independent of n such that the distances between a point on $\ell_{1,n}$ and $\ell_{j,n}$ $(j = 2, 3)$ are less than D, and*

(4) *$\chi_n(g_{j,n}) = g_{j,n}$ for $j = 1, 2$, and $\chi_n(g_{3,n}) \neq g_{3,n}$.*

PROOF. First, we observe a fundamental property of R_M. For an arbitrary point p_0 in $R_M^* - R_\varepsilon$, there exists a non-trivial simple closed curve C_{p_0} passing through p_0 such that it is not homotopic to a puncture and $\ell(C_{p_0}) < M_1$, where $M_1 = M_1(M, \varepsilon)$ is a constant in Proposition 1 depending only on M and ε. Then there exists a simple closed geodesic L_{p_0} which is homotopic to C_{p_0}. The length of L_{p_0} is greater than ε and we have

$$0 < \varepsilon/M_1 \leq \ell(L_{p_0})/\ell(C_{p_0}).$$

Hence there exists a constant $B = B(M, \varepsilon)$ depending only on ε and M such that the hyperbolic distance between p_0 and L_{p_0} on R is less than B (Figure 1). This implies

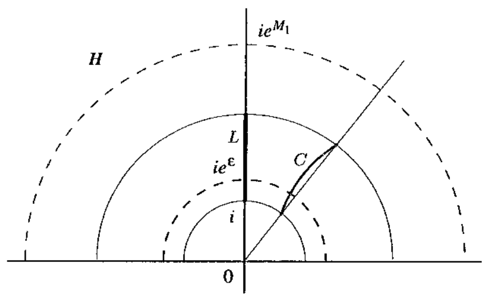

Figure 1. L: a lift of L_{p_0}, C: a lift of C_{p_0}.

that for every point z_0 in a lift of R_M^*, say $\widetilde{R_M^*}$, if it is not projected to R_ε, then there is an axis ℓ_0 of a hyperbolic element of Γ such that $d_H(z_0, \ell_0) \leq B$ and that the projection to R is a simple closed geodesic with length less than M_1.

By Remark 3, the homomorphism $\pi_1(R^*_{M''}) \to \pi_1(R)$ is surjective for all $M'' \geq M$. Thus, we take a constant M sufficiently large so that there exist two disjoint simple closed geodesics L_1^0 and L_2^0 on R^*_M whose lengths are less than M. Let γ_j $(j = 1, 2)$ be hyperbolic elements of Γ which represent L_j^0. Since $\chi_n \to \mathrm{id}$ $(n \to \infty)$, $\chi_n(\gamma_1) = \gamma_1$ and $\chi_n(\gamma_2) = \gamma_2$ for sufficiently large n. Since χ_n is not eventually the identity, we may find a $\gamma_n \in \Gamma$ so that $\chi_n(\gamma_n) \neq \gamma_n$. The following lemma shows more, that is, we may take better one as γ_n.

LEMMA 7. *Let γ_1, γ_2 and χ_n be the same ones as above. For sufficiently large n, there exists a hyperbolic element γ_n of Γ that satisfies the following two conditions*:

(1) $\chi_n(\gamma_n) \neq \gamma_n$,

(2) *the projection of the axis of γ_n on R is a simple closed geodesic with length less than M*,

PROOF. Since $\chi_n \neq \mathrm{id}$, there exists an element α_n of Γ such that $\chi_n(\alpha_n) \neq \alpha_n$. We will show that either $\alpha_n \circ \gamma_1 \circ \alpha_n^{-1}$ or $\alpha_n \circ \gamma_2 \circ \alpha_n^{-1}$ is a desired element. It is obvious that both of them satisfy the second condition of the lemma. Hence, it suffices to show that one of them satisfies the first condition.

Suppose that χ_n fixes $\alpha_n \circ \gamma_j \circ \alpha_n^{-1}$ $(j = 1, 2)$. Then $\beta_n \circ \gamma_j \circ \beta_n^{-1} = \gamma_j$ $(j = 1, 2)$, where $\beta_n = \alpha_n^{-1} \circ \chi_n(\alpha_n)$. Thus, β_n fixes all fixed points of γ_1 and γ_2. Since γ_1 and γ_2 are non-commutative, the Möbius transformation β_n fixes four points and it must be the identity map. This contradicts $\chi_n(\alpha_n) \neq \alpha_n$. □

Let γ_n be an element in Lemma 7. By the proof of Lemma 7, we may assume that $\gamma_n = \alpha_n \circ \gamma_1 \circ \alpha_n^{-1}$ for some $\alpha_n \in \Gamma$. We denote by ℓ_1^0, ℓ_2^0 and ℓ_n the axes of γ_1, γ_2 and γ_n, respectively. The projection of ℓ_n to R is the same as that of ℓ_1^0.

Fix a point z_1 on ℓ_1^0. There exists the nearest point z_n on ℓ_n from z_1. Since z_1 and z_n belong to $\widetilde{R^*_M}$ and since $\widetilde{R^*_M}$ is connected by the second condition in Theorem 1, there exists an oriented smooth curve C_n in $\widetilde{R^*_M}$ from z_n to z_1. Furthermore, we can take the curve C_n so that the projection of C_n is in $R^*_M - R_\varepsilon$.

Now, we shall show the statement for $M' = M_1$ and $D = \max(4(B + M_1 + 1), d_{\boldsymbol{H}}(z_1, \ell_2^0))$; we consider the following two cases for $d_{\boldsymbol{H}}(z_1, \ell_n)$.

1: $d_{\boldsymbol{H}}(z_1, \ell_n) \leq 4(B + M_1 + 1)$.

In this case, we set $g_{1,n} = \gamma_1$, $g_{2,n} = \gamma_2$ and $g_{3,n} = \gamma_n$. Then the third condition of the lemma holds for D. Other three conditions are trivial from the choice of these transformations.

2: $d_{\boldsymbol{H}}(z_1, \ell_n) > 4(B + M_1 + 1)$.

In this case, there are points z_2 and w_2 on C_n such that z_n, z_2 and w_2 are located in this order with respect to the orientation of C_n and they satisfy

$$d_{\boldsymbol{H}}(z_n, z_2) = d_{\boldsymbol{H}}(z_2, w_2) = 2(B + M_1 + 1).$$

Since z_2 and w_2 are points on $\widetilde{R^*_M}$ which are not projected to R_ε, it follows from the above observation that there exists an axis ℓ_2' (resp. ℓ_2'') such that $d_{\boldsymbol{H}}(z_2, \ell_2') \leq B$ (resp. $d_{\boldsymbol{H}}(w_2, \ell_2'') \leq B$) and that the projections of ℓ_2' and ℓ_2'' to R are simple closed geodesics whose lengths are less than M_1. Since $d_{\boldsymbol{H}}(z_2, w_2) > 2(B + M_1)$, we see that ℓ_2' and ℓ_2'' are distinct. Let γ_2' and γ_2'' be hyperbolic elements of Γ whose axes are ℓ_2' and ℓ_2'' respectively. Take a point $\zeta_2 \in \ell_2'$ so that $d_{\boldsymbol{H}}(z_2, \zeta_2) \leq B$ (Figure 2).

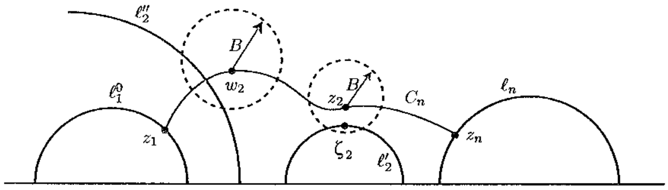

Figure 2.

If $\chi_n(\gamma_2') = \gamma_2'$ and $\chi_n(\gamma_2'') = \gamma_2''$, set $g_{1,n} = \gamma_2'$, $g_{2,n} = \gamma_2''$ and $g_{3,n} = \gamma_n$. Noting that

$$d_{\boldsymbol{H}}(\zeta_2, \ell_2'') \leq 2(B + M_1 + 1) + 2B$$

and

$$d_{\boldsymbol{H}}(\zeta_2, \ell_n) \leq 2(B + M_1 + 1) + B,$$

we see that the third condition of the lemma holds for $D > 4B + 2(M_1 + 1)$. Thus, we obtain desired elements.

We consider the case where $\chi_n(\gamma_2') \neq \gamma_2'$ or $\chi_n(\gamma_2'') \neq \gamma_2''$. We may assume that $\chi_n(\gamma_2') \neq \gamma_2'$ because the argument below works for the case where $\chi_n(\gamma_2'') \neq \gamma_2''$.

If $\chi_n(\gamma_2') \neq \gamma_2'$ and $d_{\boldsymbol{H}}(z_1, \ell_2') \leq 4(B + M_1 + 1)$, then we see that $g_{1,n} = \gamma_1$, $g_{2,n} = \gamma_2$ and $g_{3,n} = \gamma_2'$ are desired ones as in the first case.

If $\chi_n(\gamma_2') \neq \gamma_2'$ and $d_{\boldsymbol{H}}(z_1, \ell_2') > 4(B + M_1 + 1)$, then we use the argument in the second case and we have z_3, w_3 on C_n such that z_2, z_3 and w_3 are located in this order with respect to the orientation of C_n and $d_{\boldsymbol{H}}(z_2, z_3) = d_{\boldsymbol{H}}(z_3, w_3) = 2(B + M_1 + 1)$. Also, we have axes ℓ_3', ℓ_3'' and $\gamma_3', \gamma_3'' \in \Gamma$ as above. Repeating this argument, we get desired elements since $d_{\boldsymbol{H}}(z_1, \ell_k') \leq 4(B + M_1 + 1) \leq D$ for some $k \in \boldsymbol{N}$. □

Proof of Theorem 1. Let $p_0 = [R, \mathrm{id}]$ be the base point of $T^{\#}(R)$. We first suppose that there exists a sequence $\{g_n\}$ of quasiconformal automorphisms of R which determine distinct elements of $\mathrm{Mod}_c^{\#}(R)$ such that $\lim_{n\to\infty} g_n(p_0) = p$ for some p in $T^{\#}(R)$. Consider the sequence $\{f_n' = g_{n+1}^{-1} \circ g_n\}$. Then we see that $f_n'(p_0)$ converges to p_0. Thus there exist quasiconformal mappings $f_n : R \to R$ $(n = 1, 2, \ldots)$ such that f_n is RT-equivalent to f_n' and that $\lim_{n\to\infty} K(f_n) = 1$. From Proposition 3, there exists a conformal automorphism f of R such that $[f_n \circ f] \in \mathrm{Mod}_c^{\#}(R)$ and $f_n \circ f$ converge to the identity on R locally uniformly. Since $\lim_{n\to\infty} K(f_n \circ f) = \lim_{n\to\infty} K(f_n) = 1$, it follows from Proposition 4 that $[f_n \circ f] = [\mathrm{id}]$ for sufficiently large n. Hence $[f_n] = [f^{-1}]$ for sufficiently large n. This contradicts the assumption that all f_n are distinct.

Finally, we see that the same argument as above is valid for an arbitrary point $q = [S, f]$ in $T^{\#}(R)$. To see this, it suffices to show that the conditions of Theorem 1 are invariant under the quasiconformal deformation. Namely, the following lemma concludes the theorem. □

Lemma 8. *Let R and S be Riemann surfaces, and $f : R \to S$ be a K-quasiconformal map. If R satisfies the conditions in Theorem* 1, *then S also satisfies them.*

Proof. Let $\tilde{f} : \boldsymbol{H} \to \boldsymbol{H}$ be a lift of K-quasiconformal map f. The quasiconformal map $\tilde{f}$ can be extended to $\boldsymbol{H} \cup \hat{\boldsymbol{R}}$ with $\tilde{f}(\infty) = \infty$ and the restriction $\tilde{f}|\boldsymbol{R}$ of $\tilde{f}$ to $\boldsymbol{R}$ is a quasisymmetric function. The Douady-Earle extension $\Phi(\tilde{f})$ of $\tilde{f}|\boldsymbol{R}$ to $\boldsymbol{H}$ is a quasiconformal and bilipschitz map, and the bilipschitz constant K' depends only on K (cf. [**2**]). The projection $\phi_f : R \to S$ of $\Phi(\tilde{f})$ satisfies

$$(1/K')\ell(c) \le \ell(\phi_f(c)) \le K'\ell(c)$$

for an arbitrary curve c on R, and $[S, f] = [S, \phi_f]$ in $T^{\#}(R)$. Then for an arbitrary point a in $\phi_f(R_M^*)$, there exists a non-trivial simple closed curve c_0 containing a such that $\ell(c_0) \le K'M$. Thus, $\phi_f(R_M^*) \subset S_{K'M}$. Therefore, we see that the Riemann surface S satisfies the second condition in Theorem 1 for a connected component of $S_{K'M}$ containing $\phi_f(R_M^*)$.

The same argument also shows that S satisfies the first condition. □

PROOF OF THEOREM 2. We may assume that R is a Riemann surface of infinite type. Suppose that R is a Riemann surface of positive finite genus g and satisfies the conditions in Theorem 1. Further suppose that $\mathrm{Mod}^{\#}(R)$ is not discrete. Then there exists a sequence $\{f_n\}$ of quasiconformal automorphisms of R which determine distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} K(f_n) = 1$. Let l be a dividing simple closed curve such that one of components of $R - l$ is a Riemann surface S of genus g with only one boundary component. Take a non-dividing simple closed geodesic c on S. Then $f_n(c) \cap \bar{S} \ne \emptyset$ for all n. Indeed, if $f_n(c) \cap \bar{S} = \emptyset$, then $f_n(c)$ should be a dividing curve. Since c is a non-dividing curve and f_n is a homeomorphism, it can not occur. Then from Lemma 5, there exists a subsequence of $\{f_n\}$ which converges to a conformal automorphism f of R locally uniformly on R. Hence we can apply Proposition 4, and we conclude a contradiction.

Next suppose that R has finite positive number of cusps and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as above. Let V be a cusp neighborhood of a puncture of R. Since R has only finitely many cusps, we may assume that $f_n(V) \cap V \ne \emptyset$ for all n by taking a subsequence of $\{f_n\}$. Let S be a pair of pants in R such that it contains V and the boundary of S consists of the puncture and two dividing simple closed geodesics, say c_1 and c_2. We may assume that two geodesics c_1 and c_2 are not homotopic to a boundary component of R. If $f_n(c_1)$ is homotopic to c_1 for infinity many n, then they determine elements of $\mathrm{Mod}_{c_1}^{\#}(R)$. Hence, they must be discrete from Theorem 1. Assume that $f_n(c_1)$ is not homotopic to c_1 for all n. Since $f_n(V) \cap V \ne \emptyset$ and $f_n(S)$ is still a pair of pants for each n, we see that $f_n(c_1) \cap (\bar{S} \backslash V) \ne \emptyset$ or $f_n(c_2) \cap (\bar{S} \backslash V) \ne \emptyset$. We may assume that $f_n(c_1) \cap (\bar{S} \backslash V) \ne \emptyset$. Then from Lemma 5 and Proposition 4, we conclude a contradiction.

Finally, suppose that R has finite positive number of borders and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as before. Let B be a one of borders of R. Since R has only finite number of borders, we may assume that $f_n(B) = B$ for all n. Let c be a simple closed geodesic which is homotopic to B. Then $f_n(c)$ is homotopic to c. Thus $f_n \in \mathrm{Mod}_c^{\#}(R)$, and $\{f_n\}$ is discrete by Theorem 1. This contradicts $\lim_{n\to\infty} K(f_n) = 1$. Hence $\mathrm{Mod}^{\#}(R)$ is discrete. □

6. Further examples.

In Example 4, we showed that there exists a Riemann surface R that satisfies the two conditions in Theorem 1, but that $\mathrm{Mod}^{\#}(R)$ is not discrete. In this case, there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\omega_n(p_0) = p_0$ for any n, where $p_0 = [R, \mathrm{id}] \in T^{\#}(R)$. By modifying this example, we exhibit another kind of examples of Riemann surfaces R which also show that $\mathrm{Mod}^{\#}(R)$ are not discrete.

Example 5. We construct a Riemann surface R such that there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} d_T(\omega_n(p), p) = 0$ for some $p \in T^{\#}(R)$ and $\omega_n(p) \neq p$ for any n.

First, we consider a torus A_0 with two geodesic borders of the same length. Let B_0 be another torus obtained via the $(1+\varepsilon_0)$ quasiconformal deformation of A_0 for some $\varepsilon_0 > 0$. Attach two copies of B_0 to A_0 along the borders suitably, and we obtain a Riemann surface A_1. Hence, it is a Riemann surface of genus 3 with two geodesic borders.

Next we take a Riemann surface B_1 which is the $(1+\varepsilon_1)$ quasiconformal deformation of A_1 for some $\varepsilon_1 > 0$. Attach two copies of B_1 to A_1 along the borders suitably, and we obtain a Riemann surface A_2 which is a Riemann surface of genus 9 with two geodesic borders. Repeating this process for some sequence $\{\varepsilon_n\}$ of positive numbers, we have a sequence of Riemann surfaces $\{A_n\}$. More precisely, A_{n+1} is a Riemann surface consisting of A_n and two copies of B_n which is $(1+\varepsilon_n)$ quasiconformal deformation of A_n. Thus, A_n is obtained by gluing 3^n surfaces homeomorphic to A_0, say $S_{-\alpha(n)}, S_{-\alpha(n)+1}, \ldots, S_{-1}, S_0, S_1, \ldots, S_{\alpha(n)-1}, S_{\alpha(n)}$ for $\alpha(n) = (3^n - 1)/2$. We construct a Riemann surface R as the inductive limit of these A_n. Namely, R is a Riemann surface obtained by gluing S_k and S_{k+1} $(k = 0, \pm 1, \pm 2, \ldots)$. If the sequence $\{\varepsilon_n\}$ is bounded, then we see that R satisfies the above conditions on the injectivity radius.

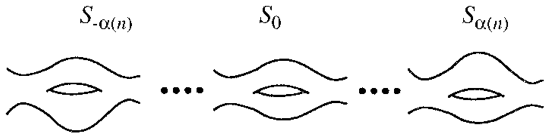

Figure 3.

Let g_n be a quasiconformal automorphism of R which sends a part corresponding to S_k to a part corresponding to S_{k+3^n} $(k = 0, \pm 1, \pm 2, \ldots)$. We shall show that there exists a quasiconformal automorphism f_n homotopic to g_n such that the maximal dilatations of f_n $(n = 1, 2, \ldots)$ converge to one as $n \to \infty$.

We construct such maps inductively. If $0 \le |k| \le \alpha(n)$, then we set $f_n|_{S_k} = h_n$, where $h_n : A_n \to B_n$ is the $(1+\varepsilon_n)$-quasiconformal mapping as above. If $\alpha(n) < |k| \le \alpha(n+1)$, then we may set $f_n|_{S_k} = h_{n+1} \circ h_n \circ h_{n+1}^{-1}$ and the maximal dilatation of $f_n|_{S_k}$ is less than $(1+\varepsilon_n)(1+\varepsilon_{n+1})^2$. Similarly, if $\alpha(m-1) < |k| \le \alpha(m)$ for $m (> n)$ and $h_m^{-1}(S_k) = S_\ell$ for some ℓ with $|\ell| < \alpha(p)$, then we may take $f_n|_{S_k} = h_m \circ h_\ell \circ h_m^{-1}$ on S_k. Therefore, we see that the maximal dilatation of f_n is less than $(1+\varepsilon_n)\prod_{k=n+1}^{\infty}(1+\varepsilon_k)^2$.

If we take a sequence $\{\varepsilon_n\}$ converges to zero rapidly so that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then we verify that the maximal dilatations of f_n converge to 1 as $n \to \infty$. Thus, the quasiconformal automorphism f_n induces an element of $\mathrm{Mod}^{\#}(R)$ whose orbits of $p_0 = [R, \mathrm{id}]$ converge to p_0 in $T^{\#}(R)$.

The following example shows that Theorem 2 does not necessarily hold for a planar Riemann surface.

EXAMPLE 6. Set $R = \boldsymbol{C} - \boldsymbol{Z}$, which is a planar Riemann surface satisfying the conditions in Theorem 1, and set $f_n(z) = z + n$ $(n = 1, 2, 3, \dots)$. Since $f_n(z)$ is a conformal automorphism of R, we see that $[f_n](p_0) = p_0$ for all n, where $p_0 = [R, \mathrm{id}] \in T^{\#}(R)$. Hence $\mathrm{Mod}^{\#}(R)$ is not discrete (cf. Example 4).

Further we see that there exists a point p in $T^{\#}(R)$ such that the set of the orbit of p under the action of $\mathrm{Mod}^{\#}(R)$ is not discrete. To show this, consider a following Riemann surface S: Set

$$z_n = \begin{cases} n + \dfrac{\sqrt{-1}}{j(n)+1} & (n \neq 0) \\ 0 & (n = 0), \end{cases}$$

where $j(n)$ is the power of the factor 2 when we decompose $|n|$ to the product of primes, and set $S = \boldsymbol{C} - \bigcup_{n=-\infty}^{\infty}\{z_n\}$. Since there exists a quasiconformal automorphism h of $\boldsymbol{C}$ such that $h(n) = z_n$ $(n \in \boldsymbol{Z})$, S is a quasiconformal deformation of R.

For every positive m, we take a locally affine quasiconformal automorphism g_m of S such that $\mathrm{Re}\, g_m(z) = \mathrm{Re}\, z + 2^m$ (and hence $g_m(z_n) = z_{(n+2^m)}$). Then, since $j(n + 2^m) = j(n)$ for $j(n) < m$ and $j(n + 2^m) = m$ for $j(n) \geq m$, we may take the locally affine maps g_m so that the maximal dilatations of g_m tend to 1. Hence we see that the set of the orbit of $p = [S, h] \in T^{\#}(R)$ under the action of $\mathrm{Mod}^{\#}(R)$ is not discrete.

We shall construct a Riemann surface R and sequences $\{M_n\}, \{M_n'\}$ having the properties referred in Remark 3 in §4.

EXAMPLE 7. We consider right-angled hexagons H_n $(n = 1, 2, \dots)$ in the hyperbolic plain $\boldsymbol{H}$. The sides of the hexagon H_n are labelled $a_{j,n}$ $(j = 1, 2, \dots, 6)$ counter-clockwise. We construct the hexagon so that $\ell(a_{2,n}) = \ell(a_{6,n})$, $\ell(a_{3,n}) = \ell(a_{5,n}) = 1$ and $\ell(a_{1,n}) = (2n)^{-1}$. Then $\{H_n\}$ converges to a pentagon with one cusp as $n \to \infty$. Thus, we see that

$$d_{\boldsymbol{H}}(P_n, a_{2,n}) = d_{\boldsymbol{H}}(P_n, a_{6,n}) \leq M < \infty \tag{4}$$

holds for some M independent of n, where P_n is the midpoint of $a_{4,n}$. Take the perpendicular line $L_{j,n}$ $(j = 2, 6)$ from P_n to $a_{j,n}$. Since $d_{\boldsymbol{H}}(P_n, a_{1,n}) \to \infty$ as $n \to \infty$, it follows from (4) that $d_{\boldsymbol{H}}(a_{1,n}, L_{2,n}) = d_{\boldsymbol{H}}(a_{1,n}, L_{6,n}) \to \infty$ (Figure 4).

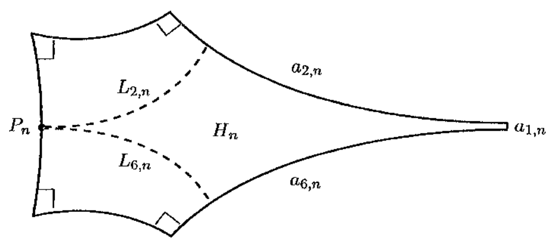

Figure 4. A hexagon close to a pentagon with one cusp.

Now, we take $k(n)$ copies of H_n, say $H_n^1, \dots, H_n^{k(n)}$, so that

$$\frac{1}{3} d_{\boldsymbol{H}}(a_{1,n}, L_{2,n}) \le 2k(n)\ell(a_{1,n}) = \frac{1}{n} k(n) \le \frac{1}{2} d_{\boldsymbol{H}}(a_{1,n}, L_{2,n}). \tag{5}$$

Obviously, $k(n)/n \to \infty$ as $n \to \infty$. Let $a_{j,n}^i$ $(i = 1, 2, \dots k(n); j = 1, 2, \dots, 6)$ denote the sides of H_n^i corresponding to $a_{j,n}$. Glue H_n^i and H_n^{i+1} along $a_{6,n}^i$ and $a_{2,n}^{i+1}$. Then, we have a right-angled $(2k(n)+4)$-gon D_n in $\boldsymbol{H}$. Label the side of D_n formed by $a_{1,n}^1 \cup \cdots \cup a_{1,n}^{k(n)}$ as $b_{1,n}$ and the rest of sides as $b_{2,n}, \dots, b_{2k(n)+4,n}$ counterclockwise.

We take a copy of D_n' of D_n with sides $b_{j,n}'$ $(j = 1, 2, \dots, 2k(n)+4)$ corresponding to $b_{j,n}$ of D_n. We glue D_n and D_n' along $b_{j,n}$ and $b_{2k(n)+6-j,n}'$ for $j = 2, 4, \dots, 2k(n)+2$ and $2k(n)+4$. Then we have a hyperbolic bordered surface S_n of type $(0, k(n)+1)$. The boundary ∂S_n consists of one long curve $c_{1,n}$ and $k(n)$ short curves $c_{2,n}, \dots, c_{k(n),n}$. It follows from the construction that

$$\ell(c_{1,n}) = \frac{k(n)}{n},$$

$$\ell(c_{2,n}) = \ell(c_{k(n),n}) = 2,$$

and

$$\ell(c_{3,n}) = \cdots = \ell(c_{k(n)-1,n}) = 4.$$

From (4), we verify that $(S_n)_{4M}$ is connected and the natural map of $\pi_1((S_n)_{4M})$ to $\pi_1(S_n)$ is surjective. On the other hand, it follows from (5) that $(S_n)_{k(n)/n}$ is not connected while both $(S_n)_{k(n)/2n}$ and $(S_n)_{2k(n)/n+4M}$ are connected.

We take a sequence $\{j_n\}$ so that

$$4M < \frac{k(j_n)}{j_n} < \frac{k(j_{n+1})}{10 j_{n+1}}. \quad (n = 1, 2, \dots)$$

We glue S_{j_n} and $S_{j_{n+1}}$ along $c_{k(j_n),j_n}$ of ∂S_{j_n} and $c_{2,j_{n+1}}$ of $\partial S_{j_{n+1}}$. Then we have a bordered Riemann surface S, and a Riemann surface R whose convex core is S. From the construction we verify that R_{M_n} is connected for $M_n = k(j_n)/2j_n$ but $R_{M_n'}$ is not connected for $M_n' = k(j_n)/j_n$. Since $M_n, M_n' > 4M$, the natural maps of $\pi_1(R_{M_n})$ and $\pi_1(R_{M_n'}^*)$ to $\pi_1(R)$ are surjective, where $R_{M_n'}^*$ is the "core component" of $R_{M_n'}$. Thus, $R, \{M_n\}$ and $\{M_n'\}$ are our desired ones.

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