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Variable instability of a constant blow-up solution in a nonlinear heat equation

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Abstract. This paper is concerned with positive solutions of the semilinear diffusion equation $u_t = \triangle u + u^p$ in Ω under the Neumann boundary condition, where $p > 1$ is a constant and Ω is a bounded domain in $\boldsymbol{R}^N$ with C^2 boundary. This equation has the constant solution $(p-1)^{-1/(p-1)}(T_0 - t)^{-1/(p-1)}$ $(0 \le t < T_0)$ with the blow-up time $T_0 > 0$. It is shown that for any $\varepsilon > 0$ and open cone Γ in $\{f \in C(\bar{\Omega}) \mid f(x) > 0\}$, there exists a positive function $u_0(x)$ in $\bar{\Omega}$ with $\partial u_0/\partial \nu = 0$ on $\partial\Omega$ and $\|u_0(x) - (p-1)^{-1/(p-1)} T_0^{-1/(p-1)}\|_{C^2(\bar{\Omega})} < \varepsilon$ such that the blow-up time of the solution $u(x,t)$ with initial data $u(x,0) = u_0(x)$ is larger than T_0 and the function $u(x, T_0)$ belongs to the cone Γ. A theorem on the blow-up profile is also given.

1. Introduction.

This paper is concerned with solutions of the nonlinear diffusion equation

$$\begin{cases} u_t = \triangle u + u^p & \text{in } \Omega \times (t_0, T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (t_0, T), \\ u(x, t_0) = u_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with C^2 boundary, ν is the unit outward normal vector on $\partial\Omega$, $p > 1$ is a constant and $u_0 \in C(\bar{\Omega})$ is a positive function. The *blow-up time* $T > t_0$ of the solution $u(x,t)$ of (1.1) is defined by

$$T = \sup\{\tau > t_0 \mid u(x,t) \text{ is bounded in } \bar{\Omega} \times (t_0, \tau)\}.$$

For the problem (1.1), T is finite. Hence, $\overline{\lim}_{t \to T} \|u(x,t)\|_{C(\bar{\Omega})} = +\infty$ holds.

The equation (1.1) has the constant solution $(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)}$ with the blow-up time T. Our main theorem states that *for any* $\varepsilon > 0$ *and open cone* Γ *in* $\{f \in C(\bar{\Omega}) \mid f(x) > 0\}$, *there exists a positive function* $u_0 \in C^2(\bar{\Omega})$ *with* $\partial u_0/\partial \nu = 0$ *on* $\partial\Omega$ *and* $\|u_0(x) - (p-1)^{-1/(p-1)} T^{-1/(p-1)}\|_{C^2(\bar{\Omega})} < \varepsilon$ *such that the blow-up time of the solution* $u(x,t)$ *of* (1.1) *with* $t_0 = 0$ *is larger than* T *and the function* $u(x,T)$ *belongs to the cone* Γ. Precisely, we show the following theorem in this paper.

THEOREM 1. *Let* $f \in C(\bar{\Omega})$ *be a positive function, and let* δ *and* T_0 *be positive constants. Then, there exist* C *and* $\varepsilon_0 > 0$ *satisfying the following: For any* $\varepsilon \in (0, \varepsilon_0]$, *there exists* $u_0^\varepsilon \in C^2(\bar{\Omega})$ *satisfying* $\partial u_0^\varepsilon/\partial \nu = 0$ *on* $\partial\Omega$ *and*

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$$\|u_0^\varepsilon(x) - (p-1)^{-1/(p-1)}T_0^{-1/(p-1)}\|_{C^2(\bar{\Omega})} \le C\varepsilon^{p-1}$$

such that the blow-up time of the solution $u^\varepsilon(x,t)$ *of* (1.1) *with initial data* $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ *is larger than* T_0 *and the inequality*

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \le \delta$$

holds.

Also, the *blow-up set* of the solution $u(x,t)$ is defined as the set

$$\{x \in \bar{\Omega} \mid \text{there is a sequence } (x_n, t_n) \text{ in } \bar{\Omega} \times (t_0, T) \text{ such that } (x_n, t_n) \to (x, T) \text{ and } u(x_n, t_n) \to +\infty \text{ as } n \to \infty\}.$$

This set is a nonempty closed set in $\bar{\Omega}$. From standard parabolic estimates, we can obtain the *blow-up profile*, which is a continuous function defined by

$$u_*(x) = \lim_{t \to T} u(x,t)$$

outside the blow-up set.

There are a number of results for the nature of the blow-up set. For the Cauchy problem with $(N-2)p < N+2$, Velázquez [**31**] showed that the $(N-1)$-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of $\boldsymbol{R}^N$ whenever the solution is not the constant blow-up one $(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)}$. For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with $(N-2)p < N+2$, Merle and Zaag [**22**] showed that for any finite set $D \subset \Omega$, there exists u_0 such that the blow-up set is D (See also [**1**] and [**3**]). For the Cauchy problem with $N=1$, Herrero and Velázquez [**16**] showed that for any point $\bar{x}$ in the blow-up set of a solution $\bar{u}$ and $\varepsilon > 0$, there exists u_0 with $\|u_0 - \bar{u}_0\|_C \le \varepsilon$ such that the blow-up set of u consists of a single point x with $|x - \bar{x}| \le \varepsilon$. For the Cauchy-Dirichlet problem in an ellipsoid centered at the origin with $(N-2)p < N$, Filippas and Merle [**10**] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with $(N-2)p < N+2$, Mizoguchi [**24**] showed the following. For any continuous function $\phi \ge 0$ and $\delta > 0$, if $\varepsilon > 0$ is small, then any point x in the blow-up set satisfies $\phi(x) \ge \max_y \phi(y) - \delta$ for $u_0 = \varepsilon^{-1}\phi$. See also [**32**]. Recently, for the Cauchy-Neumann problem with $(N-2)p < N+2$, Ishige and Mizoguchi [**18**] obtained the following. Let P be the orthogonal projection in $L^2(\Omega)$ onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function $\phi \in L^\infty(\Omega)$ and $\delta > 0$, if $\varepsilon > 0$ is small, then any point x in the blow-up set satisfies $(P\phi)(x) \ge \max_y (P\phi)(y) - \delta$ for $u_0 = \varepsilon\phi$.

On the other hand, from the results by Baras and Cohen [**2**] and Lacey and Tzanetis [**20**], we might see that it was natural to interpret the value of the solution after the blow-up time as infinity at all points in the domain (See also [**4**] and [**29**]). See, e.g., the references in this paper for related results or other studies on blow-up formation in $u_t = \triangle u + u^p$.

The following theorem shows that *for any positive constant* c *and positive function* $f \in C^2(\bar{\Omega})$ *satisfying* $\partial f / \partial \nu = 0$ *on* $\partial\Omega$, *there exists a sequence* $\{u_0^\varepsilon\}_{\varepsilon \in (0,1]} \subset C^2(\bar{\Omega})$ *with*

$\partial u_0^\varepsilon/\partial\nu = 0$ *on* $\partial\Omega$ *and* $\lim_{\varepsilon\to 0}\|u_0^\varepsilon - c\|_{C^2(\bar\Omega)} = 0$ *such that* $\varepsilon u_*^\varepsilon(x)$ *approaches* $(f(x)^{-(p-1)} - (\max_{y\in\bar\Omega} f(y))^{-(p-1)})^{-1/(p-1)}$ *uniformly on compact sets of* $\{x\in\bar\Omega \mid f(x) < \max_{y\in\bar\Omega} f(y)\}$ *as* $\varepsilon\to 0$, *where the function* u_*^ε *is the blow-up profile of the solution* u^ε *of* (1.1) *with initial data* u_0^ε. This means that the constant blow-up solution has very variable instability.

Theorem 2. *Let* $f\in C^2(\bar\Omega)$ *be a positive function satisfying* $\partial f/\partial\nu = 0$ *on* $\partial\Omega$, *and let* δ *and* c *be positive constants. Then, there exist* C *and* $\varepsilon_0 > 0$ *satisfying the following: For any* $\varepsilon\in(0,\varepsilon_0]$, *there exists* $u_0^\varepsilon\in C^2(\bar\Omega)$ *with* $\partial u_0^\varepsilon/\partial\nu = 0$ *on* $\partial\Omega$ *and* $\|u_0^\varepsilon - c\|_{C^2(\bar\Omega)} \le C\varepsilon^{p-1}$ *such that the blow-up set of the solution* u^ε *of* (1.1) *with initial data* u_0^ε *is contained in the set* $S := \{x\in\bar\Omega \mid f(x)\ge \max_{y\in\bar\Omega} f(y) - \delta\}$ *and the blow-up profile* u_*^ε *satisfies the inequality*

$$\left\|\varepsilon u_*^\varepsilon(x) - \left(f(x)^{-(p-1)} - \left(\max_{y\in\bar\Omega} f(y)\right)^{-(p-1)}\right)^{-1/(p-1)}\right\|_{C(\bar\Omega\setminus S)} \le \delta.$$

2. Proof of Theorems 1 and 2.

The following lemma shows that the actual solution of (1.1) for the early stage is well approximated by the solution of the linearized equation of (1.1), when the initial data is well done by a constant.

Lemma 3. *There exists* $C > 0$ *such that for any* $D > 0$, *there exists* $\tau_0 < 0$ *satisfying the following: Suppose that* $\tau\in[\tau_0, 0)$ *and* $t_0\le\tau$ *are constants, and that* $\phi\in C(\bar\Omega)$ *satisfies* $\|\phi\|_{C(\bar\Omega)}\le D$. *Then, the blow-up time* T *of the solution* $u(x,t)$ *of* (1.1) *with initial data*

$$u(x,t_0) = (p-1)^{-1/(p-1)}(-t_0)^{-1/(p-1)}\left(1 + \frac{\tau^2}{t_0}\phi(x)\right)$$

is larger than τ *and the inequality*

$$\left\|(p-1)^{1/(p-1)}(-t)^{1/(p-1)}u(x,t) - \left(1 + \frac{\tau^2}{t}(e^{\triangle(t-t_0)}\phi)(x)\right)\right\|_{C(\bar\Omega)}$$

$$\le C\frac{\tau^4}{t^2}\|\phi\|_{C(\bar\Omega)}^2$$

holds for all $t\in[t_0,\tau]$.

Proof. We first note

$$\sup_{t\ge 0}\|e^{\triangle t}w\|_{C(\bar\Omega)} \le \|w\|_{C(\bar\Omega)}, \tag{2.1}$$

in virtue of the maximum principle. Let C be a positive constant satisfying

$$|(1+w)^p - 1 - pw| \le \frac{p-1}{4}C|w|^2 \tag{2.2}$$

for all $w\in[-1/2, 1/2]$.

Let $D > 0$. Throughout this proof, we choose $-\tau_0 > 0$ smaller if necessary. By choosing $-\tau_0 > 0$ sufficiently small, we have

$$u(x, t_0) \geq (p-1)^{-1/(p-1)}(-t_0)^{-1/(p-1)}(1 + \tau D) > 0$$

and

$$u(x, t_0) \leq (p-1)^{-1/(p-1)}(-t_0)^{-1/(p-1)}\left(1 - \frac{\tau^2}{t_0} D\right)$$
$$< (p-1)^{-1/(p-1)}(-t_0 - (p-1)\tau^2 D)^{-1/(p-1)}.$$

Hence, the blow-up time T of u satisfies $T > -(p-1)\tau^2 D > \tau$.

Take functions v and U on $\Omega \times [t_0, \tau]$ such that

$$u(x, t) = (p-1)^{-1/(p-1)}(-t)^{-1/(p-1)}(1 + v(x, t))$$

and

$$v(x, t) = \frac{\tau^2}{t}(e^{\triangle(t-t_0)}\phi)(x) + U(x, t)$$

hold. Then, we can see $U(t_0) = 0$ and

$$U_t - \left(\triangle + \frac{1}{-t}\right) U = \frac{1}{(p-1)(-t)}((1+v)^p - 1 - pv).$$

Hence,

$$U(t) = \frac{1}{(p-1)(-t)} \int_{t_0}^{t} e^{\triangle(t-s)}((1 + v(s))^p - 1 - pv(s))\, ds$$

holds. Let

$$R = \sup\left\{ r \in (t_0, \tau] \mid \|U(t)\|_{C(\bar{\Omega})} \leq \frac{\tau^2}{-t}\|\phi\|_{C(\bar{\Omega})} \text{ for all } t \in [t_0, r] \right\}.$$

Then, by (2.1), (2.2) and choosing $-\tau_0 > 0$ sufficiently small, we obtain

$$\|U(t)\|_{C(\bar{\Omega})} \leq \frac{C}{-4t} \int_{t_0}^{t} \|v(s)\|^2_{C(\bar{\Omega})}\, ds$$
$$\leq C\frac{\tau^4}{-t}\|\phi\|^2_{C(\bar{\Omega})} \int_{t_0}^{t} \frac{1}{s^2}\, ds \leq C\frac{\tau^4}{t^2}\|\phi\|^2_{C(\bar{\Omega})}$$

for $t \in [t_0, R]$. Also,

$$\|U(t)\|_{C(\bar{\Omega})} \leq \frac{\tau^2}{-2t}\|\phi\|_{C(\bar{\Omega})}$$

holds for all $t \in [t_0, R]$. Hence, we see $R = \tau$. □

Lemmas 4 and 5 show that the actual solution of (1.1) until a little before the blow-up time is well approximated by the solution of the ordinary differential equation $u_t = u^p$, when the initial data is well done by a large constant. Lemma 4 gives a super-solution $\bar{u}(x, t)$ of (1.1).

Lemma 4. *Suppose that a positive function $\varphi \in C^2(\bar{\Omega})$ satisfies $\partial\varphi/\partial\nu = 0$ on $\partial\Omega$. Let $\bar{D}_\varphi$ be a constant defined by*

$$\bar{D}_\varphi := \max_{x \in \bar{\Omega}} \left((p-1)|(\triangle\varphi)(x)| + 2p \frac{\sum_{i=1}^N |\varphi_{x_i}(x)|^2}{|\varphi(x)|} \right).$$

Then, the positive function $\bar{u} \in C^2(\bar{\Omega} \times [0, \varepsilon])$ defined by

$$\bar{u}(x,t) := (p-1)^{-1/(p-1)} (\varepsilon(1 + (p-1)\varphi(x)\varepsilon) - t(1 + \bar{D}_\varphi \varepsilon^2))^{-1/(p-1)}$$

is a super-solution for all $\varepsilon > 0$ satisfying $2\bar{D}_\varphi \varepsilon \le (p-1) \min_{x \in \bar{\Omega}} \varphi(x)$.

Proof. We have

$$\bar{u}_t(x,t) = \bar{u}(x,t)^p (1 + \bar{D}_\varphi \varepsilon^2),$$
$$\bar{u}_{x_i}(x,t) = -\bar{u}(x,t)^p (p-1) \varphi_{x_i}(x) \varepsilon^2$$

and

$$\begin{aligned} &\bar{u}_{x_i x_i}(x,t) \\ &\quad = -\bar{u}(x,t)^p (p-1) \varphi_{x_i x_i}(x) \varepsilon^2 \\ &\qquad + \bar{u}(x,t)^p p(p-1)(\varphi_{x_i}(x))^2 (\varepsilon(1 + (p-1)\varphi(x)\varepsilon) - t(1 + \bar{D}_\varphi \varepsilon^2))^{-1} \varepsilon^4. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\bar{u}_t - (\triangle \bar{u} + \bar{u}^p)}{\bar{u}^p \varepsilon^2} \\ &\quad = \bar{D}_\varphi + (p-1)(\triangle\varphi) \\ &\qquad - p(p-1) \left(\sum_{i=1}^N |\varphi_{x_i}(x)|^2 \right) (\varepsilon(1 + (p-1)\varphi(x)\varepsilon) - t(1 + \bar{D}_\varphi \varepsilon^2))^{-1} \varepsilon^2 \end{aligned}$$

holds. From $t \le \varepsilon$ and $2\bar{D}_\varphi \varepsilon \le (p-1) \min_{x \in \bar{\Omega}} \varphi(x)$, we also have

$$\begin{aligned} &(\varepsilon(1 + (p-1)\varphi(x)\varepsilon) - t(1 + \bar{D}_\varphi \varepsilon^2))^{-1} \varepsilon^2 \\ &\qquad \le ((p-1)\varphi(x) - \bar{D}_\varphi \varepsilon)^{-1} \le \frac{2}{(p-1)\varphi(x)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\bar{u}_t - (\triangle \bar{u} + \bar{u}^p)}{\bar{u}^p \varepsilon^2} \\ &\qquad \ge \bar{D}_\varphi + (p-1)(\triangle\varphi) - 2p \frac{\sum_{i=1}^N |\varphi_{x_i}(x)|^2}{\varphi(x)} \ge 0 \end{aligned}$$

holds. □

Lemma 5 gives a sub-solution $\underline{u}(x,t)$ of (1.1).

LEMMA 5. *Suppose that a positive function* $\varphi \in C^2(\bar{\Omega})$ *satisfies* $\partial\varphi/\partial\nu = 0$ *on* $\partial\Omega$. *Let* $\underline{D}_\varphi$ *be a constant defined by*

$$\underline{D}_\varphi := (p-1)\max_{x\in\bar{\Omega}} |(\triangle\varphi)(x)|.$$

Then, the positive function $\underline{u} \in C^2(\bar{\Omega}\times[0,\varepsilon])$ *defined by*

$$\underline{u}(x,t) := (p-1)^{-1/(p-1)}(\varepsilon(1+(p-1)\varphi(x)\varepsilon) - t(1-\underline{D}_\varphi\varepsilon^2))^{-1/(p-1)}$$

is a sub-solution for all $\varepsilon > 0$.

PROOF. We have

$$\underline{u}_t(x,t) = \underline{u}(x,t)^p(1-\underline{D}_\varphi\varepsilon^2)$$

and

$$\begin{aligned}&\underline{u}_{x_ix_i}(x,t)\\ &\quad = -\underline{u}(x,t)^p(p-1)\varphi_{x_ix_i}(x)\varepsilon^2\\ &\qquad + \underline{u}(x,t)^p p(p-1)(\varphi_{x_i}(x))^2(\varepsilon(1+(p-1)\varphi(x)\varepsilon) - t(1-\underline{D}_\varphi\varepsilon^2))^{-1}\varepsilon^4.\end{aligned}$$

Hence,

$$\frac{(\triangle\underline{u}+\underline{u}^p)-\underline{u}_t}{\underline{u}^p\varepsilon^2} \ge \underline{D}_\varphi - (p-1)(\triangle\varphi) \ge 0$$

holds. □

The following theorem is the main technical result in this paper. By using Lemmas 3, 4 and 5, we prove this theorem.

THEOREM 6. *For any* $D > 0$ *and* $d > 0$, *there exist* $C_0 > 0$ *and* $\varepsilon_0 > 0$ *satisfying the following*:

Suppose that $\varepsilon \in (0,\varepsilon_0]$ *and* $t_0 < -\varepsilon$ *are constants and that* $\phi \in C(\bar{\Omega})$ *satisfies* $\psi := e^{-\triangle(\varepsilon+t_0)}\phi \ge d$ *in* $\bar{\Omega}$ *and* $\|\phi\|_{C(\bar{\Omega})} + \|\psi\|_{C^2(\bar{\Omega})} \le D$. *Then, the blow-up time* T *of the solution* $u(x,t)$ *of* (1.1) *with initial data*

$$u(x,t_0) = (p-1)^{-1/(p-1)}(-t_0)^{-1/(p-1)}\left(1+\frac{\varepsilon^2}{t_0}\phi(x)\right)$$

is larger than 0 *and the inequality*

$$|(p-1)^{2/(p-1)}\varepsilon^{2/(p-1)}u(x,0) - \psi(x)^{-1/(p-1)}| \le C_0\varepsilon$$

holds in Ω.

PROOF. Let $C > 0$ be the constant given by Lemma 3. Also, let $D > 0$ and $d > 0$. Throughout this proof, we choose $C_0 > 0$ larger and $\varepsilon_0 > 0$ smaller, respectively, if necessary.

Put $\tau = -\varepsilon$. Then, by Lemma 3, we obtain

$$\|(p-1)^{1/(p-1)}\varepsilon^{1/(p-1)}u(x,-\varepsilon) - (1-\psi(x)\varepsilon)\|_{C(\bar{\Omega})} \le CD^2\varepsilon^2. \tag{2.3}$$

Hence, as we put $\bar{\varphi}_\varepsilon = \psi - CD^2\varepsilon$ for ε,

$$u(x,-\varepsilon) \le (p-1)^{-1/(p-1)}\varepsilon^{-1/(p-1)}(1-\bar{\varphi}_\varepsilon(x)\varepsilon)$$
$$\le (p-1)^{-1/(p-1)}\varepsilon^{-1/(p-1)}(1+(p-1)\bar{\varphi}_\varepsilon(x)\varepsilon)^{-1/(p-1)}$$

holds. Since $\min_{x\in\bar{\Omega}}\bar{\varphi}_\varepsilon(x) \ge d/2$ and $\bar{D}_{\bar{\varphi}_\varepsilon} \le (p-1)ND + 4pN(D^2/d)$ also hold, we have

$$u(x,0) \le (p-1)^{-1/(p-1)}\varepsilon^{-2/(p-1)}((p-1)\bar{\varphi}_\varepsilon(x) - \bar{D}_{\bar{\varphi}_\varepsilon}\varepsilon)^{-1/(p-1)}$$

from Lemma 4. Hence, we get

$$u(x,0) \le (p-1)^{-2/(p-1)}\varepsilon^{-2/(p-1)}\left(\psi(x) - \left(CD^2 + ND + \frac{4pND^2}{(p-1)d}\right)\varepsilon\right)^{-1/(p-1)}$$
$$\le (p-1)^{-2/(p-1)}\varepsilon^{-2/(p-1)}(\psi(x)^{-1/(p-1)} + C_0\varepsilon).$$

We can see

$$(1-(\psi(x)+CD^2\varepsilon)\varepsilon)^{-(p-1)}$$
$$\le 1 + (p-1)(\psi(x)+CD^2\varepsilon)\varepsilon + 2p(p-1)(\psi(x)+CD^2\varepsilon)^2\varepsilon^2$$
$$\le 1 + (p-1)(\psi(x)+(C+8p)D^2\varepsilon)\varepsilon.$$

Hence, from (2.3), we have

$$u(x,-\varepsilon) \ge (p-1)^{-1/(p-1)}\varepsilon^{-1/(p-1)}(1-(\psi(x)+CD^2\varepsilon)\varepsilon)$$
$$\ge (p-1)^{-1/(p-1)}\varepsilon^{-1/(p-1)}(1+(p-1)\underline{\varphi}_\varepsilon(x)\varepsilon)^{-1/(p-1)},$$

as we put $\underline{\varphi}_\varepsilon = \psi + (C+8p)D^2\varepsilon$ for ε. By Lemma 5,

$$u(x,0) \ge (p-1)^{-1/(p-1)}\varepsilon^{-2/(p-1)}((p-1)\underline{\varphi}_\varepsilon(x) + \underline{D}_{\underline{\varphi}_\varepsilon}\varepsilon)^{-1/(p-1)}$$
$$\ge (p-1)^{-2/(p-1)}\varepsilon^{-2/(p-1)}(\psi(x) + ((C+8p)D^2+ND)\varepsilon)^{-1/(p-1)}$$
$$\ge (p-1)^{-2/(p-1)}\varepsilon^{-2/(p-1)}(\psi(x)^{-1/(p-1)} - C_0\varepsilon)$$

holds. □

Now, we prove Theorem 1 by Theorem 6.

Proof of Theorem 1. Let $\delta' = \min\{\delta, \min_{x\in\bar{\Omega}} f(x)\}$. We take $g_1 \in W^{2,2N}(\Omega)$ with $\partial g_1/\partial\nu = 0$ on $\partial\Omega$ and $c_1 > 0$ such that

$$\|f - (e^{\triangle c_1}g_1)^{-1/(p-1)}\|_{C(\bar{\Omega})}$$
$$\le \|f - g_1{}^{-1/(p-1)}\|_{C(\bar{\Omega})} + \|g_1{}^{-1/(p-1)} - (e^{\triangle c_1}g_1)^{-1/(p-1)}\|_{C(\bar{\Omega})} \le \delta'/4$$

holds. We also take $\lambda_k \geq 0$ and $\psi_k \in H^2(\Omega)$ $(k = 1, 2, \ldots, n)$ satisfying $-\triangle\psi_k = \lambda_k\psi_k$ in Ω and $\partial\psi_k/\partial\nu = 0$ on $\partial\Omega$ such that $\|e^{\triangle c_1}g_1 - \sum_{k=1}^n \psi_k\|_{H^{2N}(\Omega)}$ is sufficiently small. Then, we have

$$\left\| f - \left(\sum_{k=1}^n \psi_k\right)^{-1/(p-1)} \right\|_{C(\bar{\Omega})} \leq \|f - (e^{\triangle c_1}g_1)^{-1/(p-1)}\|_{C(\bar{\Omega})} + \left\| (e^{\triangle c_1}g_1)^{-1/(p-1)} - \left(\sum_{k=1}^n \psi_k\right)^{-1/(p-1)} \right\|_{C(\bar{\Omega})} \leq \delta'/2. \tag{2.4}$$

We put $\psi = \sum_{k=1}^n \psi_k$, $D = \sum_{k=1}^n (1 + e^{\lambda_k T_0})\|\psi_k\|_{C^2(\bar{\Omega})}$ and $d = \min_{x\in\bar{\Omega}} \psi(x)$. Then, by Theorem 6 with $t_0 := -T_0$, there exist $C_0 > 0$ and $\varepsilon_0' > 0$ such that for any $\varepsilon' \in (0, \varepsilon_0']$, the blow-up time of the solution $u(x,t)$ of (1.1) with initial data

$$u(x,0) = (p-1)^{-1/(p-1)} T_0^{-1/(p-1)} \left(1 - \frac{\varepsilon'^2}{T_0} \sum_{k=1}^n e^{\lambda_k (T_0 - \varepsilon')} \psi_k(x)\right)$$

is larger than T_0 and the inequality

$$\left\| (p-1)^{2/(p-1)} \varepsilon'^{(2/(p-1))} u(x, T_0) - \left(\sum_{k=1}^n \psi_k(x)\right)^{-1/(p-1)} \right\|_{C(\bar{\Omega})} \leq C_0 \varepsilon'$$

holds. Hence, from (2.4), the conclusion of Theorem 1 follows with $\varepsilon_0 := (p-1)^{2/(p-1)} \min\{\varepsilon_0', \delta'/2C_0\}^{2/(p-1)}$ and $C := (p-1)^{-1/(p-1)-2} T_0^{-1/(p-1)-1} D$. □

We prove Theorem 2 by combining Theorem 1 with Theorem 6 of [**32**].

Proof of Theorem 2. [Step 1] In this step, we show the following.

Let $C_0 = ((p-1)/2)^{-1/(p-1)}$. Then, for any $u_0 \in C^2(\bar{\Omega})$ satisfying $\partial u_0/\partial\nu = 0$ on $\partial\Omega$ and

$$\|u_0(x) - c\|_{C^2(\bar{\Omega})} \leq \min\left\{\frac{c}{2}, \frac{c^p}{2^{p+1}N}\right\},$$

the solution $u(x,t)$ of (1.1) *with the blow-up time T satisfies*

$$u(x,t) \leq C_0 (T-t)^{-1/(p-1)}$$

in $\bar{\Omega} \times [t_0, T)$.

Let $v(x,t)$ denote the function $2\triangle u(x,t) + u(x,t)^p$. Then, we have $v(x,t_0) = 2\triangle u_0(x) + u_0(x)^p \geq -2N\|u_0 - c\|_{C^2(\bar{\Omega})} + (c - \|u_0 - c\|_{C^2(\bar{\Omega})})^p \geq 0$. As well as Proof of Proposition 7 in [**32**], we can also see $\partial v/\partial\nu = 0$ on $\partial\Omega$ and $v_t \geq \triangle v + pu^{p-1}v$. Therefore, we have $2u_t - u^p = v \geq 0$. Because $1/2 \leq u_t/u^p$ holds, we obtain $(T-t)/2 \leq \int_t^T (1/2)\,ds \leq \int_{u(x,t)}^{u(x,T)} (1/u^p)\,du \leq u(x,t)^{-(p-1)}/(p-1)$.

[Step 2] By Theorem 6 in [**32**], we have the following.

There exists $\varepsilon_0' > 0$ *such that for any positive constant* ε' *and function* $v_0 \in C(\bar{\Omega})$ *with* $\varepsilon' \le \varepsilon_0'$ *and* $\|v_0 - f\|_{C(\bar{\Omega})} \le \varepsilon_0'$, *if the solution* $v(x,\tau)$ *of*

$$\begin{cases} v_\tau = \varepsilon'^2 \triangle v + v^p & \text{in } \Omega \times (0, T'), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T'), \\ v(x,0) = v_0(x) & x \in \bar{\Omega} \end{cases} \tag{2.5}$$

with the blow-up time T' *satisfies* $v(x,\tau) \le C_0(T'-\tau)^{-1/(p-1)}$ *in* $\bar{\Omega} \times [0, T')$, *then the blow-up set is contained in the set* $S := \{x \in \bar{\Omega} \mid f(x) \ge \max_{y \in \bar{\Omega}} f(y) - \delta\}$ *and the blow-up profile* $v_*(x)$ *satisfies*

$$\left\| v_*(x) - \left(f(x)^{-(p-1)} - \left(\max_{y \in \bar{\Omega}} f(y) \right)^{-(p-1)} \right)^{-1/(p-1)} \right\|_{C(\bar{\Omega} \setminus S)} \le \delta.$$

[Step 3] In this step, we prove Theorem 2 by Steps 1, 2 and Theorem 1.

By Theorem 1, the following holds. There exist C and $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ with $\partial u_0^\varepsilon / \partial \nu = 0$ on $\partial\Omega$ and $\|u_0^\varepsilon(x) - c\|_{C^2(\bar{\Omega})} \le C\varepsilon^{p-1}$ such that the blow-up time of the solution $u^\varepsilon(x,t)$ of (1.1) with initial data $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ is larger than $T_0 := (p-1)^{-1} c^{-(p-1)}$ and the inequality

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \le \varepsilon_0'$$

holds.

Now, we take a constant $\varepsilon_0 > 0$ such that $\varepsilon_0 \le \varepsilon_1$, $\varepsilon_0^{p-1} \le \varepsilon_0'^2$ and $C\varepsilon_0^{p-1} \le \min\{c/2, c^p/2^{p+1}N\}$ hold. Let $0 < \varepsilon \le \varepsilon_0$.

Then, by Step 1 and $\|u_0^\varepsilon(x) - c\|_{C^2(\bar{\Omega})} \le C\varepsilon^{p-1}$, the inequality

$$u^\varepsilon(x,t) \le C_0(T-t)^{-1/(p-1)}$$

holds in $\bar{\Omega} \times [0, T)$, where T is the blow-up time of $u^\varepsilon(x,t)$. Hence, as we put $\varepsilon' = \varepsilon^{(p-1)/2}$ and $v(x,\tau) = \varepsilon u^\varepsilon(x, T_0 + \varepsilon'^2 \tau)$,

$$v(x,\tau) \le C_0(T'-\tau)^{-1/(p-1)} \tag{2.6}$$

holds in $\bar{\Omega} \times [0, T')$, where $T' = \varepsilon'^{-2}(T - T_0)$ is the blow-up time of $v(x,\tau)$. The function v also satisfies the equation (2.5) with $v_0(x) = \varepsilon u^\varepsilon(x, T_0)$. Therefore, by Step 2 and (2.6), the blow-up set of $v = \varepsilon u^\varepsilon$ is contained in the set $S := \{x \in \bar{\Omega} \mid f(x) \ge \max_{y \in \bar{\Omega}} f(y) - \delta\}$ and the inequality

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(f(x)^{-(p-1)} - \left(\max_{y \in \bar{\Omega}} f(y) \right)^{-(p-1)} \right)^{-1/(p-1)} \right\|_{C(\bar{\Omega} \setminus S)} \le \delta$$

holds. □

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