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Some Diophantine approximation inequalities and products of hyperbolic spaces

Dedicated to Professor Yukio Matsumoto on his sixtieth birthday

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Abstract. We find new generalizations of Dirichlet's theorem to real quadratic fields and quartic fields with two complex places, which are closely related to geometry of products of hyperbolic spaces.

1. Introduction.

For any irrational number θ there are infinitely many rational approximations p/q to θ satisfying

$$|\theta - p/q| < C/q^2, \tag{1.1}$$

where $C = 1/\sqrt{5}$. This Dirichlet's theorem was generalized to approximations by imaginary quadratic numbers as follows. Let $\boldsymbol{Q}(\sqrt{-d})$ be an imaginary quadratic field, where d is a positive square-free rational integer, and let $\mathscr{O}_{-d}$ be its ring of integers. Then there exists a positive number C such that for any complex number $\alpha \notin \boldsymbol{Q}(\sqrt{-d})$ there are infinitely many solutions u/v of the inequality

$$|\alpha - u/v| < C/|v|^2, \tag{1.2}$$

with $u, v \in \mathscr{O}_{-d}$. The infimum of such C is called the Hurwitz constant for the field and has been studied by various authors (see for example [**19**] and references therein).

In this paper we find new generalizations of Dirichlet's theorem to some other number fields, which are closely related to geometry of products of hyperbolic spaces. Let $\boldsymbol{k}$ be a real quadratic field, $\mathscr{O}$ its ring of integers and $\triangle$ the discriminant of $\boldsymbol{k}$. For any $\xi \in \boldsymbol{k}$ we denote by $\overline{\xi}$ its conjugate and $H_{\boldsymbol{k}}(\xi)$ the field height of ξ (with respect to $\boldsymbol{k}$) (cf. Section 2 of [**16**, VIII]). We define an embedding $\sigma : \boldsymbol{k} \longrightarrow \boldsymbol{R}^2$ by $\sigma(\xi) = (\xi, \overline{\xi})$. Let $\boldsymbol{k}'$ be a number field of degree 4 over $\boldsymbol{Q}$ with exactly two complex places, $\mathscr{O}'$ its ring of integers and $\triangle'$ the discriminant of $\boldsymbol{k}'$. We choose one of the two field monomorphisms $\boldsymbol{k}' \longrightarrow \boldsymbol{C}$ which are neither the identity embedding nor its complex conjugate. For each $\xi \in \boldsymbol{k}'$ we denote by $\widetilde{\xi}$ the image of ξ under this monomorphism and $H_{\boldsymbol{k}'}(\xi)$ the field height of ξ

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(with respect to $\boldsymbol{k}'$). We also denote by $\widetilde{\boldsymbol{k}'}$ the image of $\boldsymbol{k}'$ under this monomorphism. Let $\sigma' : \boldsymbol{k}' \longrightarrow \boldsymbol{C}^2$ be the embedding defined by $\sigma'(\xi) = (\xi, \widetilde{\xi})$. Our main results are as follows.

Theorem 1. *There exists a positive number $C \leqslant 2\sqrt{\triangle}$ depending only on $\boldsymbol{k}$ such that the following holds. Let α, β be real numbers not in $\boldsymbol{k}$. Then there are infinitely many solutions $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$ of the inequality*

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| < \frac{C}{H_{\boldsymbol{k}}(q)}. \tag{1.3}$$

The exponent in the denominator of the right-hand side is best possible.

Theorem 2. *There exists a positive number $C \leqslant 2\sqrt[4]{\triangle'}$ depending only on $\boldsymbol{k}'$ such that the following holds. Let α be a complex number not in $\boldsymbol{k}'$, β a complex number not in $\widetilde{\boldsymbol{k}'}$. Then there are infinitely many solutions $p/q \in \boldsymbol{k}'$ with $p, q \in \mathscr{O}'$ of the inequality*

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\widetilde{p}}{\widetilde{q}}\right| < \frac{C}{\sqrt{H_{\boldsymbol{k}'}(q)}}. \tag{1.4}$$

The exponent in the denominator of the right-hand side is best possible.

By the classical results of L. R. Ford ([**8**], [**9**]), it is known that the inequalities (1.1) and (1.2) are closely related to geometry of the upper half-plane $\boldsymbol{H}$ and the 3-dimensional upper half-space $\mathscr{H}$ equipped with the Poincaré metrics, respectively. We recall the case of $\boldsymbol{H}$. Let

$$B = \left\{x + y\sqrt{-1} \in \boldsymbol{H} \mid y > 1/(2C)\right\}$$

be a horoball. The group $SL(2, \boldsymbol{Z})$ acts on $\boldsymbol{H}$ as a group of linear fractional transformations:

$$g \cdot z = \frac{pz + r}{qz + s} \qquad \text{for } g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \boldsymbol{Z}) \text{ and } z \in \boldsymbol{H}.$$

If $q \neq 0$, the image of B under g is the interior of the circle tangent to the real axis at p/q with radius C/q^2. Let $\gamma : [0, \infty) \longrightarrow \boldsymbol{H}$ be the geodesic defined by $\gamma(t) = \theta + e^{-t}\sqrt{-1}$. Then p/q is the solution of (1.1) if and only if γ meets $g \cdot B$.

There exist analogous links between the inequality (1.3) and geometry of the product space $\boldsymbol{H} \times \boldsymbol{H}$, (1.4) and geometry of $\mathscr{H} \times \mathscr{H}$, respectively. For any positive number C, we put

$$\begin{aligned} HB(C) &= \left\{(x_1 + y_1\sqrt{-1}, x_2 + y_2\sqrt{-1}) \in \boldsymbol{H} \times \boldsymbol{H} \mid y_1 y_2 > 1/C^2\right\}, \\ HB'(C) &= \left\{((z_1, \lambda_1), (z_2, \lambda_2)) \in \mathscr{H} \times \mathscr{H} \mid \lambda_1 \lambda_2 > 1/C^2\right\}, \end{aligned}$$

where $\mathscr{H}$ is regarded as $\{(z, \lambda) \in \boldsymbol{C} \times \boldsymbol{R} \mid \lambda > 0\}$. These subsets are also called horoballs (see Section 4). For any real numbers α, β, we define a geodesic $\tau = \tau(\alpha,\beta)$ in $\boldsymbol{H} \times \boldsymbol{H}$ by

$$\tau(t) = \left(\alpha + \sqrt{-1}e^{-t/\sqrt{2}},\, \beta + \sqrt{-1}e^{-t/\sqrt{2}}\right). \tag{1.5}$$

We also define a geodesic $\tau' = \tau'(\alpha,\beta)$ in $\mathscr{H} \times \mathscr{H}$ for any complex numbers α, β by

$$\tau'(t) = \left(\left(\alpha,\, e^{-t/\sqrt{2}}\right),\, \left(\beta,\, e^{-t/\sqrt{2}}\right)\right). \tag{1.6}$$

Let h be the class number of $\boldsymbol{k}$. For any subset S of $\mathscr{O}$, we denote by $\langle S\rangle$ the ideal of $\mathscr{O}$ generated by S. We choose in the h ideal classes, fixed integral ideals $\mathfrak{a}_1 = \langle a_1, b_1\rangle, \dots, \mathfrak{a}_h = \langle a_h, b_h\rangle$ with $a_i, b_i \in \mathscr{O}$, so that each $\mathfrak{a}_i$ is of minimum norm among all the integral ideals of its class. Let c_i, d_i be elements of $(\mathfrak{a}_i)^{-1}$ with $a_i d_i - b_i c_i = 1$ and $g_i = \begin{pmatrix} a_i & c_i \\ b_i & d_i \end{pmatrix}$ for each $i = 1, \dots, h$. Let h' be the class number of $\boldsymbol{k}'$. We choose integral ideals $\mathfrak{a}'_i = \langle a'_i, b'_i\rangle$ and $g'_i = \begin{pmatrix} a'_i & c'_i \\ b'_i & d'_i \end{pmatrix}$ for $i = 1, \dots, h'$, similarly. The special linear group $SL(2,\boldsymbol{k})$ acts on the product $\boldsymbol{H} \times \boldsymbol{H}$ isometrically, and $SL(2,\boldsymbol{k}')$ acts on $\mathscr{H} \times \mathscr{H}$ isometrically (see Section 4). Then we have the following.

THEOREM 3. *Let α, β be real numbers with $(\alpha,\beta) \notin \sigma(\boldsymbol{k})$. Then the following two conditions are equivalent.*

(1) *There are infinitely many solutions $p/q \in \boldsymbol{k}$ of* (1.3) *with $p, q \in \mathscr{O}$.*

(2) *The geodesic $\tau(\alpha,\beta)$ intersects infinitely many translates of $HB(C)$ by elements of $\bigcup_{i=1}^{h} SL(2,\mathscr{O}) \cdot g_i$.*

THEOREM 4. *Let α, β be complex numbers with $(\alpha,\beta) \notin \sigma'(\boldsymbol{k}')$. Then the following two conditions are equivalent.*

(1) *There are infinitely many solutions $p/q \in \boldsymbol{k}'$ of* (1.4) *with $p, q \in \mathscr{O}'$.*

(2) *The geodesic $\tau'(\alpha,\beta)$ intersects infinitely many translates of $HB'(C)$ by elements of $\bigcup_{i=1}^{h'} SL(2,\mathscr{O}') \cdot g'_i$.*

By combining algebraic and geometric arguments based on this link, we prove Theorems 1, 2.

Let $C(\boldsymbol{k})$ be the infimum of the constant C in the right-hand side of (1.3) such that the condition (1) of Theorem 3 holds for all $(\alpha,\beta) \in (\boldsymbol{R} - \boldsymbol{k})^2$. Let $C(\boldsymbol{k}')$ be the infimum of the constant C in the right-hand side of (1.4) such that the condition (1) of Theorem 4 holds for all $(\alpha,\beta) \in (\boldsymbol{C} - \boldsymbol{k}') \times (\boldsymbol{C} - \widetilde{\boldsymbol{k}'})$. Then $C(\boldsymbol{k})$ and $C(\boldsymbol{k}')$ are positive (see Proposition 8.5). The numbers $C(\boldsymbol{k})$ might correspond to the Hurwitz constants for real quadratic fields.

REMARKS.

(1) It is possible to show the existence of infinitely many solutions of the inequality (1.3) (resp. (1.4)) from the inequality in the main theorem of [**14**] (see (9.5)) and Proposition 2.1 in the next section. In the case, however, the constant C in the right-hand side of the inequality becomes larger, and one cannot show that the exponent in

the right-hand side is best possible. As is seen from the inequalities in Section 9, our inequalities are of slightly different type. We discuss generalization of Theorems 1, 2 to other number fields, together with inequalities by R. Quême ([**14**]) and E. Burger ([**5**]), in the last section.

(2) After L. R. Ford, the relationship between the inequality (1.1) (resp. (1.2)) and geometry of hyperbolic space $\boldsymbol{H}$ (resp. $\mathscr{H}$) was studied in more detail by many authors. For this and generalization of the geometric problems, which occured in this way, to negatively curved manifolds including complex hyperbolic spaces, see [**12**] and the references therein.

2. Outline of the proofs.

Let us consider the following inequalities, where $N(q)$ is the norm of q in $\boldsymbol{k}$, $\boldsymbol{k}'$, respectively:

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| < \frac{C}{|N(q)|}, \tag{2.1}$$

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\widetilde{p}}{\widetilde{q}}\right| < \frac{C}{\sqrt{N(q)}}. \tag{2.2}$$

Although the inequality (2.1) itself is weaker than (1.3), and (2.2) is weaker than (1.4), we have the following proposition.

Proposition 2.1. *For $(\alpha, \beta) \in \boldsymbol{R}^2 - \sigma(\boldsymbol{k})$, there are infinitely many solutions of* (1.3) *if and only if there are infinitely many solutions of* (2.1). *For $(\alpha, \beta) \in \boldsymbol{C}^2 - \sigma'(\boldsymbol{k}')$, there are infinitely many solutions of* (1.4) *if and only if there are infinitely many solutions of* (2.2).

Hence we may replace (1.3) with (2.1), (1.4) with (2.2). Then, the existence of infinitely many solutions with estimates for C in Theorems 1, 2 follow from the linear forms theorem of Minkowski. We can also show that the condition (2) in Theorem 1.3 (resp. Theorem 1.4) is satisfied for some $C > 0$ by studying the behavior of the geodesic $\tau(\alpha, \beta)$ in $\boldsymbol{H} \times \boldsymbol{H}$ (resp. $\tau'(\alpha, \beta)$ in $\mathscr{H} \times \mathscr{H}$). As a result, the inequality (1.3) (resp. (1.4)) has infinitely many solutions for some C. In some cases where the shape of the fundamental domain for $SL(2, \boldsymbol{k})$ or $SL(2, \boldsymbol{k}')$ is known, sharper estimates for C follow from this geometric argument.

On the other hand, in order to deal with the exponents we have to use the links between the inequalities (1.3), (1.4) and geometry of products of hyperbolic spaces. We show the best possibility of the exponent in (1.3) (resp. (1.4)) by finding a geodesic in $\boldsymbol{H} \times \boldsymbol{H}$ (resp. $\mathscr{H} \times \mathscr{H}$) of the form (1.5) (resp. (1.6)) which does not intersect any translates of $HB(C)$ (resp. $HB'(C)$) for sufficiently small C. Let Π (resp. Π') be the natural projection from $\boldsymbol{H} \times \boldsymbol{H}$ (resp. $\mathscr{H} \times \mathscr{H}$) to its quotient space V (resp. V') by $SL(2, \mathscr{O})$ (resp. $SL(2, \mathscr{O}')$). From the structure of the ends of the quotient space, this is also equivalent to find a geodesic τ (resp. τ') such that $\Pi \circ \tau$ (resp. $\Pi' \circ \tau'$) is contained in some compact subset of V (resp. V').

In Section 3 we collect some basic facts on the geometric boundaries of nonpositively curved manifolds, which we need to deal with the product spaces. We consider horoballs and their images under the action of $SL(2,\boldsymbol{k})$, or $SL(2,\boldsymbol{k}')$, in Section 4. Section 5 is for proofs of Theorems 3, 4. We show that the condition (2) in Theorem 3, or Theorem 4, is satisfied for some $C>0$ in Section 6. In Section 7 we replace this argument with one based on the linear forms theorem. We complete the proofs of Theorems 1, 2 in Section 8.

We have encountered the inequality (2.1) while studying the limit sets of non-uniform lattices of higher rank symmetric spaces in [**11**] by using the description in [**10**] of the asymptotic cones of locally symmetric spaces of finite volume. In fact the condition (2) of Theorem 3 is a sufficient condition for the point at infinity of $\tau(\alpha,\beta)$ to be a conical limit point of $SL(2,\mathscr{O})$. Further, if $\Pi\circ\tau(\alpha,\beta)$ is contained in some compact set, the point at infinity of $\tau(\alpha,\beta)$ might be called a bounded conical limit point as in the case of Fuchsian groups. From this interest, we consider generalization to other number fields in the last section.

In the rest of this section we prove Proposition 2.1. There are only a finite number of (algebraic) integers q in $\boldsymbol{Q}$ or imaginary quadratic fields such that the norm $N(q)$ is bounded. In the cases of other number fields, however, this is not true due to the existence of infinitely many units. So we first show the following, which we use also in Sections 5 and 8.

PROPOSITION 2.2. *Let δ, D be any positive numbers. If $(\alpha,\beta)\in\boldsymbol{R}^2-\sigma(\boldsymbol{k})$, then there are only finitely many (distinct) solutions $p/q\in\boldsymbol{k}$ of the inequality*

$$\left|\alpha-\frac{p}{q}\right|+\left|\beta-\frac{\overline{p}}{\overline{q}}\right|<\frac{C}{|N(q)|^{\delta}} \tag{2.3}$$

with $p,q\in\mathscr{O}$ such that $|N(q)|\leqslant D$. Similarly, if $(\alpha,\beta)\in\boldsymbol{C}^2-\sigma'(\boldsymbol{k}')$, then there are only finitely many (distinct) solutions $p/q\in\boldsymbol{k}'$ of the inequality

$$\left|\alpha-\frac{p}{q}\right|+\left|\beta-\frac{\widetilde{p}}{\widetilde{q}}\right|<\frac{C}{N(q)^{\delta}} \tag{2.4}$$

with $p,q\in\mathscr{O}'$ such that $N(q)\leqslant D$.

PROOF. If p/q is such a solution of (2.3), then, from the triangle inequality, the following hold:

$$|p/q|<C+|\alpha|,\quad |\overline{p}/\overline{q}|<C+|\beta|.$$

By multiplying these inequalities, we have

$$|N(p)|<(C+|\alpha|)(C+|\beta|)D.$$

Hence there are only a finite number of such pairs (p,q) up to pairs of units in $\boldsymbol{k}$ (cf. 5.2 of [**3**, Chapter 2]).

Suppose that there are infinitely many such solutions. Then there exist $\xi_0\in\boldsymbol{k}-\{0\}$

and a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of distinct units in $\boldsymbol{k}$ such that

$$|\alpha - \xi_0\varepsilon_k| + \left|\beta - \overline{\xi}_0\overline{\varepsilon}_k\right| < C. \tag{2.5}$$

On the other hand, for any positive number D' there are only a finite number of $\nu \in \mathscr{O}$ with $|\nu| < D'$, $|\overline{\nu}| < D'$ (cf. 5.3 of [**3**, Chapter 2]). By taking a subsequence if necessary, we may suppose that $\lim_{k\to\infty} |\varepsilon_k| = \infty$, or $\lim_{k\to\infty} |\overline{\varepsilon}_k| = \infty$, which contradicts the inequality (2.5). This proves the assertion for the inequality (2.3). For (2.4) we can argue in the same way. □

Proof of Proposition 2.1. Since $H_{\boldsymbol{k}}(q) \geqslant |N(q)|$, the inequality (1.3) implies (2.1). Similarly, (1.4) implies (2.2).

Suppose that there are infinitely many solutions of (2.1). It is known (cf. 5.4 of [**3**, Chapter 2]) that for any real numbers ξ, η with $\xi\eta \neq 0$, we can find a unit ε' in $\boldsymbol{k}$ such that

$$|\xi\varepsilon'| \leqslant c_1\sqrt{|\xi\eta|}, \quad |\eta\overline{\varepsilon}'| \leqslant c_1\sqrt{|\xi\eta|},$$

where c_1 is the square root of the fundamental unit ε with $\varepsilon > 1$. From Proposition 2.2, there are infinitely many solutions p/q of (2.1) with $|N(q)| > 4(c_1)^2$. For each such solution p/q, we can find a unit $\varepsilon' \in \boldsymbol{k}$ such that

$$|q\varepsilon'| \leqslant c_1\sqrt{|N(q)|}, \quad |\overline{q\varepsilon}'| \leqslant c_1\sqrt{|N(q)|}.$$

Then we have

$$H_{\boldsymbol{k}}(q\varepsilon') = \max\left\{1,\ |q\varepsilon' + \overline{q\varepsilon}'|,\ |N(q)|\right\} \leqslant |N(q)|.$$

By replacing p, q with $p\varepsilon', q\varepsilon'$, respectively, we obtain a solution p/q of

$$|\alpha - p/q| + |\beta - \overline{p}/\overline{q}| < \frac{C}{H_{\boldsymbol{k}}(q)}.$$

Suppose that there are infinitely many solutions p/q of (2.2). Let ε be a fundamental unit in $\boldsymbol{k}'$ and c_2 (> 1) the square root of $\max\{|\varepsilon|, |\widetilde{\varepsilon}|\ |\varepsilon^{-1}|, |\widetilde{\varepsilon}^{-1}|\}$. Then, for any solution p/q we can find a unit ε'' in $\boldsymbol{k}'$ such that

$$|q\varepsilon''| \leqslant c_2\sqrt[4]{N(q)}, \quad |\widetilde{q\varepsilon}''| \leqslant c_2\sqrt[4]{N(q)}.$$

If $N(q) > 256(c_2)^{12}$, we have $H_{\boldsymbol{k}'}(q\varepsilon'') \leqslant N(q)$ and, by replacing p, q with $p\varepsilon'', q\varepsilon''$, respectively,

$$|\alpha - p/q| + |\beta - \widetilde{p}/\widetilde{q}| < \frac{C}{\sqrt{H_{\boldsymbol{k}'}(q)}}.$$

Hence there are infinitely many solutions of (1.4) from Proposition 2.2. □

3. Geometric boundaries of product spaces.

The product spaces $\boldsymbol{H} \times \boldsymbol{H}$ and $\mathscr{H} \times \mathscr{H}$ equipped with the product metrics are Hadamard manifolds, that is, complete, simply connected Riemannian manifolds of non-positive sectional curvature. We collect some basic facts on the geometric boundaries of Hadamard manifolds. Main references in this section are [**7**] and [**1**].

Let M be an n-dimensional Hadamard manifold and d the distance function on it. A smooth curve $\gamma : [0, \infty) \longrightarrow M$ is a geodesic ray if and only if this curve realizes the distance between any two points on it. Any geodesic $\gamma : [0, \infty) \longrightarrow M$ of a Hadamard manifold is a geodesic ray. We say that two geodesic rays γ and γ' are asymptotic if the convex function $t \longmapsto d(\gamma(t), \gamma'(t))$ is uniformly bounded on $[0, \infty)$. In this case we also have $d(\gamma(t), \gamma'(t)) \leqslant d(\gamma(0), \gamma'(0))$ for all $t \geqslant 0$. Being asymptotic is an equivalence relation. Let $M(\infty)$ be the set of these equivalence classes of geodesic rays in M. The equivalence class of γ is denoted by $\gamma(\infty)$. The union $M \cup M(\infty)$ equipped with the "cone topology" (see [**7**, Section 1.7]) is homeomorphic to the n-dimensional ball and this provides a natural compactification of M. The boundary $M(\infty)$, which is called the geometric boundary of M, is homeomorphic to the $(n-1)$-dimensional sphere.

In the case of $\boldsymbol{H}$, the geometric boundary $\boldsymbol{H}(\infty)$ can be regarded as the real line $\boldsymbol{R}$ compactified by adding one point ∞. Similarly, in the case of $\mathscr{H}$, $\mathscr{H}(\infty)$ is the sphere S^2 obtained from the complex plane $\boldsymbol{C}$ and the point ∞ at infinity. In order to distinguish the position of points on $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$, we regard this boundary as the join of two circles. Recall that any geodesic ray γ in $\boldsymbol{H} \times \boldsymbol{H}$ can be written as

$$\gamma(t) = (\gamma_1(a_1 t), \gamma_2(a_2 t)),$$

where γ_1 and γ_2 are geodesic rays in $\boldsymbol{H}$, $a_1 \geqslant 0$, $a_2 \geqslant 0$, and $(a_1)^2 + (a_2)^2 = 1$. From the triangle inequality, the numbers a_1, a_2 depend only on the equivalence class of γ. If $a_1 \neq 0$ (resp. $a_2 \neq 0$), then the point $\gamma_1(\infty)$ (resp. $\gamma_2(\infty)$) is uniquely determined by the equivalence class of γ (cf. [**11**, Lemma 6.1]). Let $S^1 * S^1$ be the join obtained from $\boldsymbol{H}(\infty) \times \boldsymbol{H}(\infty) \times [0, \pi/2]$ by collapsing $\{z\} \times \boldsymbol{H}(\infty) \times \{0\}$ (resp. $\boldsymbol{H}(\infty) \times \{w\} \times \{\pi/2\}$) into one point for each $z \in \boldsymbol{H}(\infty)$ (resp. $w \in \boldsymbol{H}(\infty)$). We denote by $[z, w, \phi]$ the equivalence class of (z, w, ϕ) in $S^1 * S^1$. Then we can define a bijective map $F : (\boldsymbol{H} \times \boldsymbol{H})(\infty) \longrightarrow S^1 * S^1$ as follows: if $a_1 a_2 \neq 0$, then

$$F(\gamma(\infty)) = [\gamma_1(\infty), \gamma_2(\infty), \theta],$$

where θ is the angle such that

$$\cos\theta = a_1, \ \sin\theta = a_2, \ 0 \leqslant \theta \leqslant \pi/2.$$

In the case where $a_1 = 0$ we put

$$F(\gamma(\infty)) = [\infty, \gamma_2(\infty), \pi/2] = [\gamma_1(\infty), \gamma_2(\infty), \pi/2],$$

and when $a_2 = 0$ we put

$$F(\gamma(\infty)) = [\gamma_1(\infty), \infty, 0] = [\gamma_1(\infty), \gamma_2(\infty), 0].$$

Thus we can regard $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$ as the join $(\boldsymbol{R} \cup \{\infty\}) * (\boldsymbol{R} \cup \{\infty\})$.

For each $\alpha, \beta \in \boldsymbol{R} \cup \{\infty\}$, where α is allowed to be equal to β, we put

$$\mathscr{C}_{\alpha,\beta} = \{[\alpha, \beta, \theta] \mid 0 \leqslant \theta \leqslant \pi/2\}.$$

Since $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$ is decomposed as

$$(\boldsymbol{H} \times \boldsymbol{H})(\infty) = \bigcup_{\alpha,\beta \in \boldsymbol{R} \cup \{\infty\}} \mathscr{C}_{\alpha,\beta},$$

we can regard $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$ as a simplicial complex consisting of 1-dimensional simplices $\mathscr{C}_{\alpha,\beta}$ of length $\pi/2$. This simplicial complex is called a (spherical) Tits building and each $\mathscr{C}_{\alpha,\beta}$ is called a (closed) Weyl chamber at infinity of $\boldsymbol{H} \times \boldsymbol{H}$. On each $\mathscr{C}_{\alpha,\beta}$, we can define the distance between $[\alpha, \beta, \theta]$ and $[\alpha, \beta, \theta']$ to be $|\theta - \theta'|$. This distance is extended to the distance on the whole $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$ in the usual manner by considering the lengths of curves on this simplicial complex. The resultant distance is called the Tits metric on $(\boldsymbol{H} \times \boldsymbol{H})(\infty)$ and is denoted by $Td(\ ,\ )$ (for more details on Tits buildings, see [**1**], [**4**], [**18**]).

Similarly we can regard $(\mathscr{H} \times \mathscr{H})(\infty)$ as the join $(\boldsymbol{C} \cup \{\infty\}) * (\boldsymbol{C} \cup \{\infty\}) = S^2 * S^2$. For $z, w \in \boldsymbol{C} \cup \{\infty\}$, and $\phi \in [0, \pi/2]$ we denote the equivalence class of (z, w, ϕ) in $S^2 * S^2$ by the same notation $[z, w, \phi]$, and put $\mathscr{C}_{\alpha,\beta} = \{[\alpha, \beta, \theta] \mid 0 \leqslant \theta \leqslant \pi/2\}$. The geometric boundary $(\mathscr{H} \times \mathscr{H})(\infty)$ is regarded as a Tits building equipped with the Tits metric $Td(\ ,\ )$ in the same manner.

4. Horoballs in the direct products.

Let M be a Hadamard manifold.

Definition 4.1 ([**7**]). Let $\gamma : [0, \infty) \longrightarrow M$ be a geodesic ray. The Busemann function $b(\gamma) : M \longrightarrow \boldsymbol{R}$ associated with γ is given by

$$b(\gamma)(v) = \lim_{t \to \infty} \{d(v, \gamma(t)) - t\} \quad \text{for } v \in M.$$

Proposition 4.2 (cf. [**1**], [**7**]).

(1) *The function $b(\gamma)$ on M is convex and continuously 2 times differentiable.*

(2) *If two geodesic rays γ, γ' are asymptotic, then $b(\gamma)$ differs from $b(\gamma')$ only by an additive constant.*

(3) *The gradient vector of $b(\gamma)$ at $v \in M$ is the initial velocity vector of the geodesic ray ω with $\omega(0) = v$, $\omega(\infty) = \gamma(\infty)$.*

Definition 4.3 ([**7**]). Let $\gamma : [0, \infty) \longrightarrow M$ be a geodesic ray. For any real number

C, we call the set $b(\gamma)^{-1}((-\infty, C))$ a horoball centered at $\gamma(\infty)$. We call $\gamma(\infty)$ the center of this horoball.

REMARK 4.4. It follows from Proposition 4.2(3) that if two horoballs have different centers, then these horoballs do not coincide.

Let G be the group of isometries of M. Then the action of G on M is extended to the action on the geometric boundary $M(\infty)$ as follows: $g \cdot \gamma(\infty) = (g \cdot \gamma)(\infty)$ for each geodesic ray γ in M and $g \in G$. The Busemann function associated with the translated geodesic ray $g \cdot \gamma$ is given by

$$b(g \cdot \gamma)(v) = b(\gamma)(g^{-1} \cdot v) \qquad \text{for all } v \in M. \tag{4.1}$$

In the rest of this paper we denote by ω the geodesic ray in $\boldsymbol{H} \times \boldsymbol{H}$ defined by

$$\omega(t) = \left(\sqrt{-1}e^{t/\sqrt{2}},\, \sqrt{-1}e^{t/\sqrt{2}}\right) \quad \text{for } t \geqslant 0 \tag{4.2}$$

and ω' the geodesic ray in $\mathscr{H} \times \mathscr{H}$ defined by

$$\omega'(t) = \left(\left(0, e^{t/\sqrt{2}}\right),\, \left(0, e^{t/\sqrt{2}}\right)\right) \quad \text{for } t \geqslant 0. \tag{4.3}$$

We can show, by direct computation, that

$$b(\omega)\big((x_1 + y_1\sqrt{-1},\, x_2 + y_2\sqrt{-1})\big) = -\frac{1}{\sqrt{2}}\log(y_1 y_2)$$

for $x_1, x_2 \in \boldsymbol{R}$, $y_1, y_2 > 0$, and that

$$b(\omega')\big(((z_1, \lambda_1),\, (z_2, \lambda_2))\big) = -\frac{1}{\sqrt{2}}\log(\lambda_1 \lambda_2)$$

for $z_1, z_2 \in \boldsymbol{C}$, $\lambda_1, \lambda_2 > 0$. Hence we have, for any $C > 0$,

$$HB(C) = b(\omega)^{-1}\big((-\infty, \sqrt{2}\log C)\big) \tag{4.4}$$

and

$$HB'(C) = b(\omega')^{-1}\big((-\infty, \sqrt{2}\log C)\big). \tag{4.5}$$

The center of $HB(C)$ (resp. $HB'(C)$) is $[\infty, \infty, \pi/4]$.

The special linear groups $SL(2, \boldsymbol{k})$ and $SL(2, \boldsymbol{k}')$ act isometrically on the products $\boldsymbol{H} \times \boldsymbol{H}$ and $\mathscr{H} \times \mathscr{H}$, respectively, as follows. First, $SL(2, \boldsymbol{R})$ acts on $\boldsymbol{H}$ by fractional linear transformations, which we write as $g \cdot z$ for $g \in SL(2, \boldsymbol{R})$, $z \in \boldsymbol{H}$. The direct product $SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R})$ acts isometrically on $\boldsymbol{H} \times \boldsymbol{H}$ by

$$(g_1, g_2) \cdot (z, w) = (g_1 \cdot z, g_2 \cdot w) \quad \text{for } \; g_1, g_2 \in SL(2, \boldsymbol{R}) \text{ and } (z, w) \in \boldsymbol{H} \times \boldsymbol{H}.$$

For $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \boldsymbol{k})$, we denote by $\overline{g}$ the matrix $\begin{pmatrix} \overline{p} & \overline{r} \\ \overline{q} & \overline{s} \end{pmatrix}$. Let $\iota : SL(2, \boldsymbol{k}) \longrightarrow SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R})$ be the embedding given by $\iota(g) = (g, \overline{g})$. Then the group $SL(2, \boldsymbol{k})$ acts isometrically on $\boldsymbol{H} \times \boldsymbol{H}$ through this embedding:

$$\iota(g) \cdot (z, w) = (g \cdot z, \overline{g} \cdot w) \quad \text{for } g \in SL(2, \boldsymbol{k}) \text{ and } (z, w) \in \boldsymbol{H} \times \boldsymbol{H}.$$

Similarly, $SL(2, \boldsymbol{C})$ acts on $\mathscr{H}$ by the Poincaré extension of the linear fractional transformations on $\boldsymbol{C}$. For $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \boldsymbol{k}')$, we denote by $\widetilde{g}$ the matrix $\begin{pmatrix} \widetilde{p} & \widetilde{r} \\ \widetilde{q} & \widetilde{s} \end{pmatrix}$. Let $\iota' : SL(2, \boldsymbol{k}') \longrightarrow SL(2, \boldsymbol{C}) \times SL(2, \boldsymbol{C})$ be the embedding given by $\iota'(g) = (g, \widetilde{g})$. The group $SL(2, \boldsymbol{k}')$ acts isometrically on $\mathscr{H} \times \mathscr{H}$ through this embedding.

From (4.1) and (4.4), (4.5), we have

$$\iota(g) \cdot HB(C) = b(\iota(g) \cdot \omega)^{-1}\big((-\infty, \sqrt{2} \log C)\big) \qquad \text{for } g \in SL(2, \boldsymbol{k}), \tag{4.6}$$

$$\iota'(g) \cdot HB'(C) = b(\iota'(g) \cdot \omega)^{-1}\big((-\infty, \sqrt{2} \log C)\big) \qquad \text{for } g \in SL(2, \boldsymbol{k}'). \tag{4.7}$$

The center of $\iota(g) \cdot HB(C)$ (resp. $\iota'(g) \cdot HB'(C)$) is $[p/q, \overline{p}/\overline{q}, \pi/4]$ (resp. $[p/q, \widetilde{p}/\widetilde{q}, \pi/4]$) if $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, where p/q means ∞ if $q = 0$.

Let

$$U_{\boldsymbol{k}} = \left\{ \begin{pmatrix} \varepsilon & * \\ 0 & \varepsilon^{-1} \end{pmatrix} \in SL(2, \boldsymbol{k}) \,\middle|\, \varepsilon \text{ is a unit in } \boldsymbol{k}. \right\}$$

and

$$U_{\boldsymbol{k}'} = \left\{ \begin{pmatrix} \varepsilon & * \\ 0 & \varepsilon^{-1} \end{pmatrix} \in SL(2, \boldsymbol{k}') \,\middle|\, \varepsilon \text{ is a unit in } \boldsymbol{k}'. \right\}.$$

Then we can show, by direct computation, that $\iota(U_{\boldsymbol{k}})$ (resp. $\iota'(U_{\boldsymbol{k}'})$) fixes the point $\omega(\infty) = [\infty, \infty, \pi/4]$ (resp. $\omega'(\infty) = [\infty, \infty, \pi/4]$) and that the Busemann function $b(\omega)$ (resp. $b(\omega')$) is $\iota(U_{\boldsymbol{k}})$-invariant (resp. $\iota'(U_{\boldsymbol{k}'})$-invariant). Moreover, if $g \in SL(2, \mathscr{O})$ and $\iota(g)$ fixes the point $\iota(g_i) \cdot \omega(\infty)$, then $g \in g_i U_{\boldsymbol{k}} g_i{}^{-1}$. If $g \in SL(2, \mathscr{O}')$ and $\iota'(g)$ fixes the point $\iota'(g'_i) \cdot \omega'(\infty)$, then $g \in g'_i U_{\boldsymbol{k}'} {g'_i}^{-1}$.

Proposition 4.5. *Let $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, $g' = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$ be any two elements of $\bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$ (resp. $\bigcup_{i=1}^{h'} SL(2, \mathscr{O}') \cdot g'_i$). Then for any $C > 0$, $\iota(g) \cdot HB(C)$ (resp. $\iota'(g) \cdot HB'(C)$) coincides with $\iota(g') \cdot HB(C)$ (resp. $\iota'(g') \cdot HB'(C)$) if and only if $p/q = p'/q'$.*

Proof. We only consider the case of $\boldsymbol{k}$. We can show the case of $\boldsymbol{k}'$ in the same way.

Let $p/q \neq p'/q'$. Then, from Remark 4.4, $\iota(g) \cdot HB(C)$ does not coincide with $\iota(g') \cdot HB(C)$.

Suppose that $p/q = p'/q'$. Let $g = kg_i$, $g' = k'g_j$ with $k = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $k' = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} \in SL(2, \mathscr{O})$. Then we can show that $i = j$ as follows. If $q' = 0$, then $q = 0$ and

$$a_i = dp,\ b_i = -bp,\ a_j = d'p',\ b_j = -b'p'.$$

We have $\mathfrak{a}_i = \langle a_i, b_i \rangle = \langle p \rangle \langle d, -b \rangle = \langle p \rangle \mathscr{O}$ and $\mathfrak{a}_j = \langle a_j, b_j \rangle = \langle p' \rangle \langle d', -b' \rangle = \langle p' \rangle \mathscr{O}$. Since the norm of $\mathscr{O}$ is 1 and $\mathfrak{a}_i$ (resp. $\mathfrak{a}_j$) is of minimum norm among all the integral ideals of its class, $|N(p)| = |N(p')| = 1$ and $\mathfrak{a}_i = \mathfrak{a}_j = \mathscr{O}$. Hence $i = j$. If $q' \neq 0$, then we have ${q'}^{-1} q \begin{pmatrix} a_j \\ b_j \end{pmatrix} = {k'}^{-1} k \begin{pmatrix} a_i \\ b_i \end{pmatrix}$, $\langle a_i, b_i \rangle = \langle a_j, b_j \rangle$, so that $i = j$.

Since

$$\iota(g) \cdot \omega(\infty) = [p/q, \overline{p}/\overline{q}, \pi/4] = [p'/q', \overline{p}'/\overline{q}', \pi/4] = \iota(g') \cdot \omega(\infty),$$

there exists $u \in U_{\boldsymbol{k}}$ with ${k'}^{-1} k = g_i u {g_i}^{-1}$. We have

$$\begin{aligned} b(\iota(g) \cdot \omega)(v) &= b(\iota(kg_i) \cdot \omega)(v) = b\big(\iota(k'g_i)\iota(u) \cdot \omega\big)(v) = b(\omega)\big(\iota(u)^{-1}\iota(k'g_i)^{-1} \cdot v\big) \\ &= b(\omega)\big(\iota(k'g_i)^{-1} \cdot v\big) = b\big(\iota(k'g_i) \cdot \omega\big)(v) = b(\iota(g') \cdot \omega)(v) \end{aligned}$$

for any $v \in \boldsymbol{H} \times \boldsymbol{H}$. Hence $\iota(g) \cdot HB(C) = \iota(g') \cdot HB(C)$ by (4.6). □

5. Proofs of Theorems 3, 4.

We only prove Theorem 3 because we can show Theorem 4 in exactly the same way.

Proof of Theorem 3. From Proposition 2.1, the condition (1) is equivalent to the following condition.

(3) There are infinitely many solutions $p/q \in \boldsymbol{k}$ of (2.1) with $p, q \in \mathscr{O}$.

Hence it suffices to show that the two conditions (3) and (2) are equivalent. Let us consider the following condition.

(4) There are infinitely many $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$ satisfying

$$\big\{|\alpha - p/q|^2 + 1\big\}\big\{|\beta - \overline{p}/\overline{q}|^2 + 1\big\} \geqslant C^2/N(q)^2, \tag{5.1}$$

$$\big(|\alpha - p/q| + |\beta - \overline{p}/\overline{q}|\big)^2 < C^2/N(q)^2, \tag{5.2}$$

$$|\alpha - p/q|^2 + |\beta - \overline{p}/\overline{q}|^2 + 2 > C^2/N(q)^2, \tag{5.3}$$

simultaneously.

We show the following proposition, which completes the proof of Theorem 3. □

Proposition 5.1. *The three conditions* (2), (3), (4) *are equivalent.*

Proof. If p/q does not satisfy (5.1), then we have $1 < C^2/N(q)^2$. Similarly, if p/q does not satisfy (5.3), then we have $2 \leqslant C^2/N(q)^2$. Hence, from Proposition 2.2, for a given $C > 0$, there are only a finite number of solutions of (5.2) which do not satisfy one of (5.1), (5.3). This implies that (3) and (4) are equivalent.

It remains only to show that (4) follows from (2) and (2) follows from (3). Let us consider the following condition.

(5) The geodesic $\tau = \tau(\alpha, \beta)$ intersects $\iota(g) \cdot HB(C)$ for $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$.

This is equivalent to the condition that $\iota(g)^{-1} \cdot \tau$ intersects $HB(C)$, which is equivalent to the existence of a solution of the following inequality in $[0, \infty)$:

$$\frac{e^{-t/\sqrt{2}}}{|-q\alpha + p|^2 + q^2 e^{-\sqrt{2}t}} \cdot \frac{e^{-t/\sqrt{2}}}{|-\overline{q}\beta + \overline{p}|^2 + \overline{q}^2 e^{-\sqrt{2}t}} > 1/C^2. \tag{5.4}$$

We first suppose that $\alpha, \beta \in \boldsymbol{R} - \boldsymbol{k}$. If $q = 0$, then it follows from Proposition 4.5 and its proof that $\iota(g) \cdot HB(C) = HB(C)$, $|N(p)| = 1$, and the inequality (5.4) is equivalent to

$$e^{\sqrt{2}t} < C^2/N(p)^2 = C^2,$$

which has a solution in $[0, \infty)$ if and only if $C > 1$.

Suppose that $q \neq 0$. By substituting $\lambda = e^{-\sqrt{2}t}$, the existence of a solution of (5.4) in $[0, \infty)$ is equivalent to the existence of a solution of the inequality

$$f(\lambda) = \lambda^2 + A\lambda + B < 0 \tag{5.5}$$

in $(0, 1]$, where

$$A = |\alpha - p/q|^2 + |\beta - \overline{p}/\overline{q}|^2 - C^2/N(q)^2, \quad B = |\alpha - p/q|^2 |\beta - \overline{p}/\overline{q}|^2.$$

Since $f(0) = B > 0$, there are only two cases to be considered.

(a) *The case where* $f(1) < 0$. We have

$$\bigl\{|\alpha - p/q|^2 + 1\bigr\}\bigl\{|\beta - \overline{p}/\overline{q}|^2 + 1\bigr\} < C^2/N(q)^2 \tag{5.6}$$

and consequently

$$\bigl(|\alpha - p/q| + |\beta - \overline{p}/\overline{q}|\bigr)^2 < C^2/N(q)^2. \tag{5.2}$$

(b) *The case where* $f(1) \geqslant 0$. We have

$$\bigl\{|\alpha - p/q|^2 + 1\bigr\}\bigl\{|\beta - \overline{p}/\overline{q}|^2 + 1\bigr\} \geqslant C^2/N(q)^2. \tag{5.1}$$

In this case, the discriminant of f, which is equal to $A^2 - 4B$, must be positive and $0 < -A/2 < 1$. The latter condition means that the axis of the graph of f lies in the region $0 < \lambda < 1$. Hence we have $A < -2\sqrt{B}$ and $-2 < A$. The first inequality is equivalent to

$$\left(|\alpha - p/q| + |\beta - \overline{p}/\overline{q}|\right)^2 < C^2/N(q)^2 \tag{5.2}$$

and the second inequality is equivalent to

$$|\alpha - p/q|^2 + |\beta - \overline{p}/\overline{q}|^2 + 2 > C^2/N(q)^2. \tag{5.3}$$

As a result, in the case where $\alpha, \beta \in \boldsymbol{R} - \boldsymbol{k}$, it is necessary and sufficient for (5) that one of the following three conditions is satisfied.

(6) $q = 0$ and $C > 1$.

(7) The inequality (5.6) holds.

(8) Three inequalities (5.1), (5.2), (5.3) hold simultaneously.

From Proposition 2.2, for a given $C > 0$, there are only a finite number of solutions p/q of (5.6), since (5.6) implies (5.2) and $1 < C^2/N(q)^2$. This shows that (4) follows from (2).

Suppose that (3) holds. Then, there are infinitely many solutions $p/q \in \boldsymbol{k}$ of (2.1) with $\langle p, q\rangle = \mathfrak{a}_i$ for some i. To see this, let p/q be a solution of (2.1) and suppose that $\langle p, q\rangle$ is equivalent to $\mathfrak{a}_j$. Then there exists $\theta \in \boldsymbol{k}$ such that $\mathfrak{a}_j = \langle\theta\rangle\langle p, q\rangle$. Since the norm of $\mathfrak{a}_j$ is minimum among the integral ideals of its class, $|N(\theta)| \leqslant 1$ and $|N(q\theta)| \leqslant |N(q)|$. Hence $p\theta/(q\theta)$ is a solution of (2.1) with $p\theta, q\theta \in \mathscr{O}$ and $\langle p\theta, q\theta\rangle = \mathfrak{a}_j$.

For each solution p/q of (2.1) with $\langle p, q\rangle = \mathfrak{a}_i$, we can find $r, s \in (\mathfrak{a}_i)^{-1}$ such that $ps - qr = 1$. Let $g = \begin{pmatrix} p & r \\ q & s\end{pmatrix}$. Since

$$g{g_i}^{-1} = \begin{pmatrix} p & r \\ q & s\end{pmatrix}\begin{pmatrix} d_i & -c_i \\ -b_i & a_i\end{pmatrix} = \begin{pmatrix} pd_i - rb_i & -pc_i + ra_i \\ qd_i - sb_i & -qc_i + sa_i\end{pmatrix} \in SL(2, \mathscr{O}),$$

we have $g \in SL(2, \mathscr{O}) \cdot g_i$. Hence (2) follows from Proposition 4.5.

Finally we suppose that at least one of α, β lies in $\boldsymbol{k}$ and $(\alpha, \beta) \notin \sigma(\boldsymbol{k})$. Consider the condition (5). Then $B = 0$ only if $\alpha = p/q$ or $\beta = \overline{p}/\overline{q}$. From Proposition 4.5, for each of these two cases, the translates $\iota(g) \cdot HB(C)$ provide only one horoball. Hence we can exclude the possibility that $B = 0$, and for the remainder the situation is the same as in the case already considered. This completes the proof of Proposition 5.1. □

Corollary 5.2. *Let* $g = \begin{pmatrix} p & r \\ q & s\end{pmatrix} \in \bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$ *(resp.* $\in \bigcup_{i=1}^{h'} SL(2, \mathscr{O}') \cdot {g'}_i$*) with* $|N(q)| > C > 0$ *and* $p/q \neq \alpha, \beta$. *If* p/q *is a solution of* (2.1) *(resp.* (2.2)*), then after intersecting* $\iota(g) \cdot HB(C)$ *(resp.* $\iota'(g) \cdot HB'(C)$*) in the interval*

$$\left(\frac{-1}{\sqrt{2}} \log\left(\frac{C^2}{N(q)^2} - \left|\alpha - \frac{p}{q}\right|^2 - \left|\beta - \frac{\overline{p}}{\overline{q}}\right|^2\right),\ \sqrt{2}\log\frac{C}{|N(q)|\left|\alpha - \frac{p}{q}\right|\left|\beta - \frac{\overline{p}}{\overline{q}}\right|}\right),$$

the geodesic ray $\tau = \tau(\alpha, \beta)$ *(resp.* $\tau' = \tau'(\alpha, \beta)$*) goes outside this horoball and never intersects it.*

This tells us how to find horoballs of the form $\iota(g) \cdot HB(C)$ (resp. $\iota'(g) \cdot HB'(C)$) which intersect the geodesic ray τ (resp. τ') consecutively.

6. Intersection of geodesics and horoballs.

We denote by X either $\boldsymbol{H}\times\boldsymbol{H}$ or $\mathscr{H}\times\mathscr{H}$. Let γ, $\gamma' : [0,\infty) \longrightarrow X$ be geodesic rays. Suppose that γ' intersects a horoball $b(\gamma)^{-1}((-\infty, C))$ for some C. From the distance $Td(\gamma(\infty), \gamma'(\infty))$ one can know whether γ' will remain in this horoball or γ' will go out from it later.

Lemma 6.1 ([**11**, Lemma 3.4]). *Let $\gamma, \gamma' : [0,\infty) \longrightarrow X$ be two geodesic rays. Then there exists a positive number C_1 depending on γ and γ' such that the following hold.*

(1) *If $Td(\gamma(\infty), \gamma'(\infty)) > \pi/2$, then*

$$b(\gamma)(\gamma'(t)) \geqslant -t\cdot\cos\bigl(Td(\gamma(\infty), \gamma'(\infty))\bigr) - C_1 \quad \text{for all } t \geqslant 0.$$

(2) *If $Td(\gamma(\infty), \gamma'(\infty)) \leqslant \pi/2$, then $b(\gamma)(\gamma'(t))$ is monotone decreasing in t.*

In case (1) of this lemma, even if γ' meets the horoball $b(\gamma)^{-1}((-\infty, C))$, γ' goes outside this horoball later. While in case (2), if once γ' meets the horoball $b(\gamma)^{-1}((-\infty, C))$, then γ' stays within this horoball on and after that time.

Proposition 6.2. *Let $\alpha, \beta \in \boldsymbol{R} - \boldsymbol{k}$ (resp. $\alpha \in \boldsymbol{C} - \boldsymbol{k}', \beta \in \boldsymbol{C} - \widetilde{\boldsymbol{k}'}$). Then the following hold.*

(1) *Suppose that $\tau = \tau(\alpha, \beta)$ (resp. $\tau' = \tau'(\alpha, \beta)$) intersects $\iota(g)\cdot HB(C)$ (resp. $\iota'(g)\cdot HB'(C)$) for some $C > 0$ and $g \in \bigcup_{i=1}^{h} SL(2, \mathscr{O})\cdot g_i$ (resp. $\bigcup_{i=1}^{h'} SL(2, \mathscr{O}')\cdot {g'}_i$). Then τ (resp. τ') goes out this horoball later and never meets it again.*

(2) *The condition (2) of Theorem 3 (resp. Theorem 4) is satisfied for some $C > 0$.*

Proof. We only consider the case of $\boldsymbol{k}$. We can show the case of $\boldsymbol{k}'$ in the same way.

For $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \boldsymbol{k})$, the center of the horoball $\iota(g)\cdot HB(C)$ is $v = [p/q, \overline{p}/\overline{q}, \pi/4]$. Any shortest simplicial path from v to $\tau(\infty) = [\alpha, \beta, \pi/4]$ is of the form

$$v \longrightarrow [p/q, \overline{p}/\overline{q}, 0] = [p/q, \beta, 0] \longrightarrow [p/q, \beta, \pi/2] = [\alpha, \beta, \pi/2] \longrightarrow [\alpha, \beta, \pi/4],$$

or

$$v \longrightarrow [p/q, \overline{p}/\overline{q}, \pi/2] = [\alpha, \overline{p}/\overline{q}, \pi/2] \longrightarrow [\alpha, \overline{p}/\overline{q}, 0] = [\alpha, \beta, 0] \longrightarrow [\alpha, \beta, \pi/4].$$

So we have $Td(v, \tau(\infty)) = \pi$. From (1) in Lemma 6.1, even if τ meets the horoball $\iota(g)\cdot HB(C)$, τ goes outside this horoball later and never meets it again.

From Théorème 13.1 of [**2**], there exists some positive number C such that

$$X = \bigcup_{i=1}^{h} \bigcup_{g\in SL(2,\mathscr{O})} \iota(g)\iota(g_i)\cdot HB(C).$$

Hence the geodesic τ is covered by some of the translates of $HB(C)$ by elements of $\bigcup_{i=1}^{h} SL(2, \mathscr{O})\cdot g_i$. From (1), this means that τ meets infinitely many translates of

$HB(C)$ by elements of $\bigcup_{i=1}^{h} SL(2,\mathscr{O}) \cdot g_i$. □

From the argument in the proof of this proposition we also have the following.

PROPOSITION 6.3.
(1) *Let $\alpha, \beta \in \boldsymbol{R} - \boldsymbol{k}$. Suppose that there exists a fundamental domain, or fundamental set, of $SL(2,\mathscr{O})$ in $\boldsymbol{H} \times \boldsymbol{H}$ contained in $\bigcup_{i=1}^{h} \iota(g_i) \cdot HB(C)$ for some positive number C. Then* (2) *of Theorem* 3 *holds for the same C.*
(2) *Let $\alpha \in \boldsymbol{C} - \boldsymbol{k}'$, $\beta \in \boldsymbol{C} - \widetilde{\boldsymbol{k}'}$. Suppose that there exists a fundamental domain, or fundamental set, of $SL(2,\mathscr{O}')$ in $\mathscr{H} \times \mathscr{H}$ contained in $\bigcup_{i=1}^{h'} \iota(g'_i) \cdot HB'(C)$ for some positive number C. Then* (2) *of Theorem* 4 *holds for the same C.*

The proof of [**2**, Théorème 13.1] is based on the reduction theory of quadratic forms. Hence the process in finding such C corresponds to the argument based on the linear forms theorem of Minkowski in the next section. Let us consider the case of a real quadratic field $\boldsymbol{k}$. In [**17**] C. L. Siegel showed that such a fundamental domain exists for $C = 2\sqrt{\triangle}$ by using Minkowski's theorem. H. Cohn ([**6**]) studied this Siegel's fundamental domain in detail by using the classical result of A. Korkine and G. Zolotareff, and showed that for any $(z_1, z_2) \in \boldsymbol{H} \times \boldsymbol{H}$ there exists $g \in SL(2,\mathscr{O})$ with $\iota(g) \cdot (z_1, z_2) \in \bigcup_{i=1}^{h} \iota(g_i) \cdot \overline{HB}(\sqrt{\triangle/2})$, where $\overline{HB}(C)$ is the closure of $HB(C)$. (The proof in [**6**] also works for the case $h \neq 1$.) Hence there are infinitely many solutions of (1.3) for any $C > \sqrt{\triangle/2}$ when $\alpha, \beta \in \boldsymbol{R} - \boldsymbol{k}$. As a result, we have $C(\boldsymbol{k}) \leqslant \sqrt{\triangle/2}$. It might be possible to obtain a better estimate for $C(\boldsymbol{k})$ if one has more informations on this fundamental domain.

7. Linear forms theorem.

We prove the existence of infinitely many solutions of (1.3), (1.4) with estimates for C by using the linear forms theorem of Minkowski. It suffices to show that there are infinitely many solutions of (2.1), (2.2).

Let $\{\xi_1, \xi_2\}$ be an integral basis of $\boldsymbol{k}$ and $c_3 = \sqrt[4]{\triangle}$. By Minkowski's theorem on linear forms (Theorem 2C of [**16**, II]) there is for every $Q > c_3$ a non-zero rational integral 4-tuple (x_1, x_2, x_3, x_4) with

$$|\xi_1 x_1 + \xi_2 x_2 - \alpha \xi_1 x_3 - \alpha \xi_2 x_4| < \frac{c_3}{Q}, \tag{7.1}$$

$$\left|\overline{\xi}_1 x_1 + \overline{\xi}_2 x_2 - \beta \overline{\xi}_1 x_3 - \beta \overline{\xi}_2 x_4\right| < \frac{c_3}{Q}, \tag{7.2}$$

$$|\xi_1 x_3 + \xi_2 x_4| < c_3 Q, \tag{7.3}$$

$$\left|\overline{\xi}_1 x_3 + \overline{\xi}_2 x_4\right| < c_3 Q. \tag{7.4}$$

If $x_3 = x_4 = 0$, then we have

$$|N(\xi_1 x_1 + \xi_2 x_2)| = |\xi_1 x_1 + \xi_2 x_2|\left|\overline{\xi}_1 x_1 + \overline{\xi}_2 x_2\right| < (c_3)^2/Q^2 < 1$$

and $x_1 = x_2 = 0$. Hence $\xi_1 x_3 + \xi_2 x_4 \neq 0$. Let $p = \xi_1 x_1 + \xi_2 x_2$, $q = \xi_1 x_3 + \xi_2 x_4 \in \mathscr{O}$.

From (7.1) we see that

$$|\alpha - p/q| < \frac{c_3}{Q|q|}.$$

From (7.4) we obtain

$$|\alpha - p/q| < \frac{c_3}{Q}\frac{|\overline{q}|}{|N(q)|} \leqslant \frac{(c_3)^2}{|N(q)|}.$$

Similarly, from (7.2) and (7.3), we have

$$|\beta - \overline{p}/\overline{q}| < \frac{(c_3)^2}{|N(q)|}.$$

Since for fixed Q there are only a finite number of such (x_1, x_2, x_3, x_4), the maximum of $c_3/|\xi_1 x_1 + \xi_2 x_2 - \alpha\xi_1 x_3 - \alpha\xi_2 x_4|$, $c_3/|\overline{\xi}_1 x_1 + \overline{\xi}_2 x_2 - \beta\overline{\xi}_1 x_3 - \beta\overline{\xi}_2 x_4|$ for these tuples is bounded. Hence as $Q \to \infty$, there will be infinitely many distinct pairs $(p, q) \in \mathscr{O} \times \mathscr{O}$ with

$$|\alpha - p/q| + |\beta - \overline{p}/\overline{q}| < \frac{2\sqrt{\triangle}}{|N(q)|}. \tag{7.5}$$

Suppose that these pairs produce only a finite number of distinct solutions $p/q \in \boldsymbol{k}$ of (7.5). Then there exist a sequence $\{(p_k, q_k)\}_{k=0}^{\infty}$ of such pairs and a sequence $\{a_k\}_{k=1}^{\infty}$ of elements of $\boldsymbol{k}$ with $p_k = a_k p_0$, $q_k = a_k q_0$ for all $k \geqslant 1$. From the above construction, Q tends to infinity when k tends to infinity. Hence, from (7.1), (7.2), we have

$$\lim_{k\to\infty} |p_k - \alpha q_k||\overline{p}_k - \beta\overline{q}_k| = 0. \tag{7.6}$$

On the other hand, from (7.5), $|N(q_k)|$ is bounded from above by some positive number D, since p_k/q_k is constant. There are only a finite number of such q_k up to units in $\boldsymbol{k}$ (cf. 5.2 of [**3**, Ch. 2]). Hence we may suppose, by taking a subsequence if necessary, that there exist $b_0 \in \mathscr{O} - \{0\}$ and a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of distinct units in $\boldsymbol{k}$ such that $q_k = b_0\varepsilon_k$. Then we have $a_k = b_0\varepsilon_k/q_0$ and

$$|p_k - \alpha q_k||\overline{p}_k - \beta\overline{q}_k| = |N(a_k)||p_0 - \alpha q_0||\overline{p}_0 - \beta\overline{q}_0| = \frac{|N(b_0)|}{|N(q_0)|}|p_0 - \alpha q_0||\overline{p}_0 - \beta\overline{q}_0|.$$

Since $|p_0 - \alpha q_0| \neq 0$ and $|\overline{p}_0 - \beta\overline{q}_0| \neq 0$, $|p_k - \alpha q_k||\overline{p}_k - \beta\overline{q}_k|$ is a positive constant, which contradicts (7.6). Thus there are infinitely many distinct solutions $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$ of the inequality (7.5).

For the inequality (2.2), let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be an integral basis of $\boldsymbol{k}'$ and $c_4 = \sqrt[8]{\triangle'/16}$. Let f_1, f_2, f_3, f_4 be complex linear forms of 8-variables $\boldsymbol{y} = (y_1, \ldots, y_8)$ defined by

$$
\begin{aligned}
f_1(\boldsymbol{y}) &= \xi_1 y_5 + \xi_2 y_6 + \xi_3 y_7 + \xi_4 y_8,\\
f_2(\boldsymbol{y}) &= \widetilde{\xi}_1 y_5 + \widetilde{\xi}_2 y_6 + \widetilde{\xi}_3 y_7 + \widetilde{\xi}_4 y_8,\\
f_3(\boldsymbol{y}) &= \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 + \xi_4 y_4 - \alpha f_1(\boldsymbol{y}),\\
f_4(\boldsymbol{y}) &= \widetilde{\xi}_1 y_1 + \widetilde{\xi}_2 y_2 + \widetilde{\xi}_3 y_3 + \widetilde{\xi}_4 y_4 - \beta f_2(\boldsymbol{y}).
\end{aligned}
$$

Then, by Minkowski's theorem, there is for any $Q > c_4$ a non-zero rational integral 8-tuple $\boldsymbol{x} = (x_1, \dots, x_8)$ with

$$
\begin{aligned}
&|\mathrm{Re}(f_3(\boldsymbol{x}))|,\ |\mathrm{Im}(f_3(\boldsymbol{x}))|,\ |\mathrm{Re}(f_4(\boldsymbol{x}))|,\ |\mathrm{Im}(f_4(\boldsymbol{x}))| < c_4/Q,\\
&|\mathrm{Re}(f_1(\boldsymbol{x}))|,\ |\mathrm{Im}(f_1(\boldsymbol{x}))|,\ |\mathrm{Re}(f_2(\boldsymbol{x}))|,\ |\mathrm{Im}(f_2(\boldsymbol{x}))| < c_4 Q,
\end{aligned}
$$

where Re and Im denote the real part and the imaginary part, respectively. By a similar argument as above, as $Q \to \infty$, we obtain infinitely many distinct solutions p/q of

$$|\alpha - p/q| + |\beta - \widetilde{p}/\widetilde{q}| < \frac{2\sqrt[4]{\triangle'}}{\sqrt{N(q)}}.$$

8. The exponents.

We prove that the exponents in (1.3), (1.4) are best possible. We first recall the definition of badly approximable numbers.

Definition 8.1 (cf. Section 5 of [**16**, I]). An irrational number $\theta \in \boldsymbol{R}$ is badly approximable if there is a positive constant c depending only on θ such that $|\theta - p/q| > c/q^2$ for every rational p/q.

In this section we use the following generalization of this notion for convenience.

Definition 8.2. A pair $(\alpha, \beta) \in (\boldsymbol{R} - \boldsymbol{k})^2$ is a badly approximable pair for $\boldsymbol{k}$ if there is a positive constant c depending only on α and β such that

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| > \frac{c}{|N(q)|}$$

for every $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$. A pair $(\alpha, \beta) \in (\boldsymbol{C} - \boldsymbol{k}') \times (\boldsymbol{C} - \widetilde{\boldsymbol{k}'})$ is a badly approximable pair for $\boldsymbol{k}'$ if there is a positive constant c depending only on α and β such that

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\widetilde{p}}{\widetilde{q}}\right| > \frac{c}{\sqrt{N(q)}}$$

for every $p/q \in \boldsymbol{k}'$ with $p, q \in \mathscr{O}'$.

Proposition 8.3. *If there exists a badly approximable pair for* $\boldsymbol{k}$ (*resp.* $\boldsymbol{k}'$), *then the exponent in the denominator of the right-hand side of the inequality* (1.3) (*resp.* (1.4)) *is best possible.*

Proof. Let $(\alpha_0, \beta_0) \in (\boldsymbol{R} - \boldsymbol{k})^2$ be a badly approximable pair for $\boldsymbol{k}$. Then there exists a positive number c depending only on α_0 and β_0 such that

$$\left|\alpha_0 - \frac{p}{q}\right| + \left|\beta_0 - \frac{\overline{p}}{\overline{q}}\right| > \frac{c}{|N(q)|} \tag{8.1}$$

for every $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$. Suppose that the exponent in the denominator of the right-hand side of (1.3) is not best possible. Then there exist $\delta > 0$ and $C' > 0$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| < \frac{C'}{|N(q)|^{1+\delta}} \tag{8.2}$$

has infinitely many solutions $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$ for any $(\alpha, \beta) \in (\boldsymbol{R} - \boldsymbol{k})^2$. From Proposition 2.2, this also means that there are infinitely many solutions of (8.2) with $|N(q)| \geqslant (C'/c)^{1/\delta}$. For such solutions, we have

$$\frac{C'}{|N(q)|^{1+\delta}} = \frac{C'}{|N(q)|^{\delta}} \frac{1}{|N(q)|} \leqslant \frac{c}{|N(q)|},$$

and this contradicts (8.1) when $\alpha = \alpha_0$, $\beta = \beta_0$. Therefore, the exponent in (1.3) is best possible. The case of $\boldsymbol{k}'$ and (1.4) is proved in the same way. □

Recall that $\Pi : \boldsymbol{H} \times \boldsymbol{H} \longrightarrow V = \iota(SL(2, \mathscr{O}))\backslash \boldsymbol{H} \times \boldsymbol{H}$ and $\Pi' : \mathscr{H} \times \mathscr{H} \longrightarrow V' = \iota'(SL(2, \mathscr{O}'))\backslash \mathscr{H} \times \mathscr{H}$ are the natural projections to the quotient spaces.

Lemma 8.4. *A pair* $(\alpha, \beta) \in (\boldsymbol{R} - \boldsymbol{k})^2$ *is a badly approximable pair for* $\boldsymbol{k}$ *if and only if the projection* $\Pi \circ \tau(\alpha, \beta)$ *of the geodesic* $\tau(\alpha, \beta)$ *is contained in some compact subset of* V. *Similarly,* $(\alpha, \beta) \in (\boldsymbol{C} - \boldsymbol{k}') \times (\boldsymbol{C} - \widetilde{\boldsymbol{k}'})$ *is a badly approximable pair for* $\boldsymbol{k}'$ *if and only if the projection* $\Pi' \circ \tau'(\alpha, \beta)$ *of the geodesic* $\tau'(\alpha, \beta)$ *is contained in some compact subset of* V'.

Proof. For any positive number C, let

$$W(C) = \boldsymbol{H} \times \boldsymbol{H} - \bigcup_{i=1}^{h} \bigcup_{g \in SL(2, \mathscr{O})} \iota(g)\iota(g_i) \cdot HB(C).$$

Since $\iota(SL(2, \mathscr{O}))$ is a $\boldsymbol{Q}$-rank 1 lattice of $SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R})$, it is known ([**13**, Proposition 2.1], see also [**15**, Chapter XIII]) that there exists a positive number C_0 such that the following hold: for any positive number $C \leqslant C_0$, $\Pi(W(C))$ is a compact submanifold with boundary and the inverse image under Π of each connected component of the complement $V - \Pi(W(C))$ coincides with $\bigcup_{g \in SL(2, \mathscr{O})} \iota(g)\iota(g_i) \cdot HB(C)$ for some i.

Moreover,

$$\boldsymbol{H} \times \boldsymbol{H} = \bigcup_{0<C\leqslant C_0} W(C)$$

and $V = \bigcup_{0<C\leqslant C_0} \Pi(W(C))$ is an exhaustion of V by compact submanifolds with boundary.

Suppose that (α, β) is a badly approximable pair for $\boldsymbol{k}$. Then there exists a positive number c such that the inequality

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| < \frac{c}{|N(q)|}$$

has no solutions $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$. From Theorem 3 and Proposition 2.1, the geodesic $\tau(\alpha, \beta)$ meets at most a finite number of translates of $HB(c)$ by elements of $\bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$. From Proposition 6.2 (1), we can find a positive number $c_0 \leqslant C_0$ such that $\tau(\alpha, \beta)$ does not meet any translates of $HB(c_0)$ by elements of $\bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$. This means that the geodesic $\tau(\alpha, \beta)$ is contained in $W(c_0)$ and that $\Pi \circ \tau(\alpha, \beta)$ is contained in the compact subset $\Pi(W(c_0))$ of V.

Conversely, suppose that $\Pi(\tau(\alpha, \beta))$ is contained in some compact subset of V. Then we can find a positive number c such that $\Pi(\tau(\alpha, \beta))$ is contained in $\Pi(W(c))$. Then the geodesic $\tau(\alpha, \beta)$ does not meet any translate of $HB(c)$ by elements of $\bigcup_{i=1}^{h} SL(2, \mathscr{O}) \cdot g_i$. From Theorem 3, there are at most a finite number of solutions p/q of

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| < \frac{c}{H_{\boldsymbol{k}}(q)} \leqslant \frac{c}{|N(q)|}$$

with $p, q \in \mathscr{O}$. Then we can find a positive number $c_0 \leqslant c$ such that

$$\left|\alpha - \frac{p}{q}\right| + \left|\beta - \frac{\overline{p}}{\overline{q}}\right| > \frac{c_0}{|N(q)|}$$

for every $p/q \in \boldsymbol{k}$ with $p, q \in \mathscr{O}$. Hence (α, β) is a badly approximable pair for $\boldsymbol{k}$.

The case of $\boldsymbol{k}'$ is proved in the same way. □

By using this lemma we show the existence of badly approximable pairs, which completes the proofs of Theorems 1, 2 by Proposition 8.3.

Proposition 8.5. *There exist uncountably many badly approximable pairs for* $\boldsymbol{k}$ *(resp.* $\boldsymbol{k}'$*).*

Proof. There are uncountably many badly approximable number which are not in $\boldsymbol{k} \cup \boldsymbol{k}'$ because there exist continuum many badly approximable numbers (Corollary 5G of [**16**, I]). Let θ_0 be any one of such numbers. We define a geodesic $\gamma_0 : [0, \infty) \longrightarrow \boldsymbol{H}$ by

$$\gamma_0(t) = \theta_0 + \sqrt{-1}e^{-t/\sqrt{2}}.$$

Let $\Pi_0 : \boldsymbol{H} \longrightarrow SL(2,\boldsymbol{Z})\backslash\boldsymbol{H}$ be the natural projection to the quotient space. The horoball $\{x + y\sqrt{-1} \mid y > 1/(2C)\}$ in the upper half-plane is mapped by $g = \begin{pmatrix} p & s \\ q & r \end{pmatrix} \in SL(2,\boldsymbol{Z})$ $(q \neq 0)$ onto the interior of the circle $(x - p/q)^2 + (y - C/q^2)^2 = C^2/q^4$ tangent to the real axis at p/q. Consequently the geodesic γ_0 intersects this image if and only if $|\theta_0 - p/q| < C/q^2$ ([**8**]). From this and the well-known description of the fundamental domain of the modular group, $\Pi_0(\gamma_0([0,\infty)))$ is contained in some compact set in $SL(2,\boldsymbol{Z})\backslash\boldsymbol{H}$ and hence the closure of $\Pi_0(\gamma_0([0,\infty)))$ is compact.

Let $\Delta : \boldsymbol{H} \longrightarrow \boldsymbol{H} \times \boldsymbol{H}$ and $\iota_0 : SL(2,\boldsymbol{R}) \longrightarrow SL(2,\boldsymbol{R}) \times SL(2,\boldsymbol{R})$ be the diagonal embeddings. We define a geodesic ray by

$$\gamma(t) = (\gamma_0(t), \gamma_0(t)) = \Delta(\gamma_0(t)).$$

Let $\Pi : \boldsymbol{H}\times\boldsymbol{H} \longrightarrow \iota(SL(2,\mathscr{O}))\backslash\boldsymbol{H}\times\boldsymbol{H}$ and $\Pi_1 : \boldsymbol{H}\times\boldsymbol{H} \longrightarrow \iota_0(SL(2,\boldsymbol{Z}))\backslash\boldsymbol{H}\times\boldsymbol{H}$ be the natural projections to the quotient spaces. Since $\iota_0(SL(2,\boldsymbol{Z}))$ is contained in $\iota(SL(2,\mathscr{O}))$, we also have the projection $\Pi_2 : \iota_0(SL(2,\boldsymbol{Z}))\backslash\boldsymbol{H} \times \boldsymbol{H} \longrightarrow \iota(SL(2,\mathscr{O}))\backslash\boldsymbol{H} \times \boldsymbol{H}$ as in the following diagram.

$$\begin{tikzcd}
\boldsymbol{H} \arrow[r, "\Delta"] & \boldsymbol{H}\times\boldsymbol{H} \arrow[r, "\Pi_1"] \arrow[dr, "\Pi"'] & \iota_0(SL(2,\boldsymbol{Z}))\backslash\boldsymbol{H}\times\boldsymbol{H} \arrow[d, "\Pi_2"] \\
& & \iota(SL(2,\mathscr{O}))\backslash\boldsymbol{H}\times\boldsymbol{H}
\end{tikzcd}$$

Then Π_2 is continuous and $\Pi = \Pi_2 \circ \Pi_1$. Since $(\Pi_1 \circ \Delta)(\boldsymbol{H})$ is homeomorphic to $SL(2,\boldsymbol{Z})\backslash\boldsymbol{H}$ and

$$\Pi \circ \gamma = \Pi \circ \Delta \circ \gamma_0 = \Pi_2 \circ (\Pi_1 \circ \Delta) \circ \gamma_0,$$

the closure of $\Pi(\gamma([0,\infty)))$ is also compact. From Lemma 8.4, (θ_0, θ_0) is a badly approximable pair for $\boldsymbol{k}$.

The case of $\boldsymbol{k}'$ is proved in the same way, because the action of $SL(2,\boldsymbol{Z}) \subset SL(2,\boldsymbol{C})$ on $\mathscr{H}$ preserves the totally geodesically embedded isometric copy $\{(z,\lambda) \mid z \in \boldsymbol{R}, \lambda > 0\}$ of $\boldsymbol{H}$. $\square$

9. Generalization to other number fields.

For any number field $\boldsymbol{k}''$, the special linear group $SL(2,\boldsymbol{k}'')$ acts on some product of hyperbolic spaces, and it is possible to obtain an inequality by a similar argument as in Section 7. Suppose that $\boldsymbol{k}''$ is a number field with l real places and m complex places. Let $\mathscr{O}''$ be the ring of integers of $\boldsymbol{k}''$ and $\triangle''$ the discriminant of $\boldsymbol{k}''$. We denote by $\sigma_1,\ldots,\sigma_l : \boldsymbol{k}'' \longrightarrow \boldsymbol{R}$ the real embeddings and $\sigma_{l+1},\ldots,\sigma_{l+m} : \boldsymbol{k}'' \longrightarrow \boldsymbol{C}$ the complex embeddings which are not complex conjugate to each other. For $\xi \in \boldsymbol{k}''$, let $\xi^{(i)} = \sigma_i(\xi)$ for each i, $N(\xi)$ the norm of ξ in $\boldsymbol{k}''$, and $H_{\boldsymbol{k}''}(\xi)$ the field height of ξ. We define an embedding $\sigma'' : \boldsymbol{k}'' \longrightarrow \boldsymbol{R}^l \times \boldsymbol{C}^m$ by $\sigma''(\xi) = (\xi^{(1)}, \ldots, \xi^{(l)}, \xi^{(l+1)}, \ldots, \xi^{(l+m)})$. For $\beta = (\beta_1, \ldots, \beta_l, \beta_{l+1}, \ldots, \beta_{l+m}) \in \boldsymbol{R}^l \times \boldsymbol{C}^m$, we put

$$\|\beta_i\| = \begin{cases} |\beta_i| & \text{if } 1 \leqslant i \leqslant l, \\ |\beta_i|^2 & \text{if } l+1 \leqslant i \leqslant l+m. \end{cases}$$

From the linear forms theorem and a similar argument as in Section 7, we have the following.

THEOREM 9.1.

(1) *Suppose that $l+m$ is even and $l+m=2n$. Then there exists a positive number $C \leqslant \sqrt{|\triangle''|}$ depending only on $\boldsymbol{k}''$ such that the following holds. Let $\alpha = (\alpha_1, \ldots, \alpha_{l+m}) \in \boldsymbol{R}^l \times \boldsymbol{C}^m$ with $\alpha_i \notin \sigma_i(\boldsymbol{k}'')$ for all i. Then there are infinitely many solutions $p/q \in \boldsymbol{k}''$ with $p, q \in \mathscr{O}''$ of the inequality*

$$\sum_{1 \leqslant i_1 < \cdots < i_n \leqslant l+m} \left\| \alpha_{i_1} - \frac{p^{(i_1)}}{q^{(i_1)}} \right\| \cdots \left\| \alpha_{i_n} - \frac{p^{(i_n)}}{q^{(i_n)}} \right\| < \binom{l+m}{n} \frac{C}{|N(q)|}. \tag{9.1}$$

(2) *Suppose that $l+m$ is odd and $l+m=2n-1$. Then there exists a positive number $C \leqslant |\triangle''|^{\frac{n}{l+m}}$ depending only on $\boldsymbol{k}''$ such that the following holds. Let $\alpha = (\alpha_1, \ldots, \alpha_{l+m}) \in \boldsymbol{R}^l \times \boldsymbol{C}^m$ with $\alpha_i \notin \sigma_i(\boldsymbol{k}'')$ for all i. Then there are infinitely many solutions $p/q \in \boldsymbol{k}''$ with $p, q \in \mathscr{O}''$ of the inequality*

$$\sum_{1 \leqslant i_1 < \cdots < i_n \leqslant l+m} \left\| \alpha_{i_1} - \frac{p^{(i_1)}}{q^{(i_1)}} \right\| \cdots \left\| \alpha_{i_n} - \frac{p^{(i_n)}}{q^{(i_n)}} \right\| < \binom{l+m}{n} \frac{C}{|N(q)|^{\frac{2n}{l+m}}}. \tag{9.2}$$

Let $e_1, \ldots, e_{l+m-1}$ be fundamental units in $\boldsymbol{k}''$. We denote by $\mathscr{E}$ the set consisting of the absolute values of e_j $(j = 1, \ldots, l+m-1)$ and their conjugates as well as the absolute values of $1/e_j$ $(j = 1, \ldots, l+m-1)$ and their conjugates. Let

$$C'' = (\sup \mathscr{E})^{(l+m-1)/2}$$

and

$$C' = \begin{cases} (C'')^{l+2m-1} \dbinom{l+2m}{\frac{l+2m}{2}} & \text{if } l+2m \text{ is even}, \\ (C'')^{l+2m-1} \dbinom{l+2m}{\frac{l+2m-1}{2}} & \text{if } l+2m \text{ is odd}. \end{cases}$$

Then, from a similar argument as in the proof of Proposition 2.1, we can replace (9.1) (resp. (9.2)) with the following inequality (9.3) (resp. (9.4)).

$$\sum_{1\leqslant i_1<\cdots<i_n\leqslant l+m}\left\|\alpha_{i_1}-\frac{p^{(i_1)}}{q^{(i_1)}}\right\|\cdots\left\|\alpha_{i_n}-\frac{p^{(i_n)}}{q^{(i_n)}}\right\|<\binom{l+m}{n}\frac{CC'}{H_{\boldsymbol{k}''}(q)}. \tag{9.3}$$

$$\sum_{1\leqslant i_1<\cdots<i_n\leqslant l+m}\left\|\alpha_{i_1}-\frac{p^{(i_1)}}{q^{(i_1)}}\right\|\cdots\left\|\alpha_{i_n}-\frac{p^{(i_n)}}{q^{(i_n)}}\right\|<\binom{l+m}{n}\frac{CC'}{H_{\boldsymbol{k}''}(q)^{\frac{2n}{l+m}}}. \tag{9.4}$$

We have extra multiplicative constants in (9.3) and (9.4) because it is not clear whether there are only finitely many (distinct) solutions $p/q \in \boldsymbol{k}''$ of these inequalities with $p, q \in \mathscr{O}''$ such that $|N(q)| \leqslant D$ for a given positive constant D.

Let

$$B=\left(\frac{4}{\pi}\right)^m \frac{(l+2m)!}{(l+2m)^{l+2m}}\sqrt{|\triangle''|}.$$

For any $\xi \in \boldsymbol{k}''$ we put

$$d_{\boldsymbol{k}''}(\xi)=|\xi^{(1)}|+\cdots+|\xi^{(l)}|+2|\xi^{(l+1)}|+\cdots+2|\xi^{(l+m)}|.$$

R. Quême showed ([**14**, Theorem 1]) that for any $\alpha=(\alpha_1,\dots,\alpha_{l+m})\in \boldsymbol{R}^l\times\boldsymbol{C}^m-\sigma''(\boldsymbol{k}'')$ and for any real number $m>0$, there are infinitely many different p/q with $p, q\in\mathscr{O}''$ such that $d_{\boldsymbol{k}''}(q)>m$ and

$$\begin{aligned}0 &< \left|\alpha_1 q^{(1)}-p^{(1)}\right|+\cdots+\left|\alpha_l q^{(l)}-p^{(l)}\right| \\ &\quad +2\left|\alpha_{l+1}q^{(l+1)}-p^{(l+1)}\right|+\cdots+2\left|\alpha_{l+m}q^{(l+m)}-p^{(l+m)}\right| \\ &< \frac{(l+2m)^2B^{2/(l+2m)}}{d_{\boldsymbol{k}''}(q)}. \end{aligned} \tag{9.5}$$

Note that this inequality does not necessarily imply that each $|\alpha_i - p^{(i)}/q^{(i)}|$ is also small. From (9.5) one obtains ([**14**, Corollary 2]) that for any $\alpha=(\alpha_1,\dots,\alpha_{l+m})\in\boldsymbol{R}^l\times\boldsymbol{C}^m$ with $\alpha_i\notin\sigma_i(\boldsymbol{k}'')$ for all i, there are infinitely many p/q with $p,q\in\mathscr{O}''$ such that

$$0<\prod_{i=1}^{l+m}\left\|\alpha_i-\frac{p^{(i)}}{q^{(i)}}\right\|<\frac{B^2}{N(q)^2}. \tag{9.6}$$

This is a special case of the inequalities of E. Burger ([**5**]), who generalised Dirichlet's theorem to the setting of an arbitrary number field in the context of the ring of S-integers. We remark that in the case of $\boldsymbol{k}$, for any given $m>0$, one can find a solution p/q of (9.6) such that not only $d_{\boldsymbol{k}}(q)=|q|+|\overline{q}|$ but also $|N(q)|$ is larger than m (cf. [**14**, p. 281]): as mentioned at the end of Section 6, for any $\delta>0$, there exists an infinite sequence $\{p_k/q_k\}_{k=1}^{\infty}$ of (distinct) solutions of (1.3) with $p_k, q_k\in\mathscr{O}$ and $C=\sqrt{|\triangle|/2}(1+\delta)$. From Proposition 2.2, we have $\lim_{k\to\infty}|N(q_k)|=\infty$ and hence $\lim_{k\to\infty}(|q_k|+|\overline{q}_k|)\geqslant \lim_{k\to\infty}2\sqrt{|N(q_k)|}=\infty$. For any sufficiently small δ, we have

$$\left|\alpha - \frac{p_k}{q_k}\right|\left|\beta - \frac{\overline{p}_k}{\overline{q}_k}\right| \leqslant \frac{1}{4}\left(\left|\alpha - \frac{p_k}{q_k}\right| + \left|\beta - \frac{\overline{p}_k}{\overline{q}_k}\right|\right)^2 \leqslant \frac{|\triangle|(1+\delta)^2}{16|N(q_k)|^2} < \frac{B^2}{|N(q_k)|^2}.$$

If one may replace B with a larger constant, namely the square root of the absolute value of the discriminant, one can show the similar thing for $\boldsymbol{k}'$ (and also for cubic fields with one real place and one complex places).

In general, we do not know whether there exist close relationships between the inequalities (9.1), (9.2) (or (9.3), (9.4)) and geometry of products of hyperbolic spaces. However, we can show partial results in some cases. For $g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \boldsymbol{k}'')$, we denote by $g^{(i)}$ the matrix $\begin{pmatrix} p^{(i)} & r^{(i)} \\ q^{(i)} & s^{(i)} \end{pmatrix}$. Let $\iota'' : SL(2, \boldsymbol{k}'') \longrightarrow (SL(2, \boldsymbol{R}))^l \times (SL(2, \boldsymbol{C}))^m$ be the embedding given by

$$\iota''(g) = \big(g^{(1)}, \dots, g^{(l)},\ g^{(l+1)}, \dots, g^{(l+m)}\big).$$

We consider the product space $\boldsymbol{H}^l \times \mathscr{H}^m$, where each $\mathscr{H} = \{(x + y\sqrt{-1}, \lambda) \in \boldsymbol{C} \times \boldsymbol{R} \mid \lambda > 0\}$ is equipped with the metric $2(dx^2 + dy^2 + d\lambda^2)/\lambda^2$ instead of the Poincaré metric. Then the group $SL(2, \boldsymbol{k}'')$ acts isometrically on $\boldsymbol{H}^l \times \mathscr{H}^m$ through this embedding. Let ω'' be the geodesic ray in $\boldsymbol{H}^l \times \mathscr{H}^m$ defined by

$$\omega''(t) = \Big(\sqrt{-1}e^{t/\sqrt{l+2m}}, \dots, \sqrt{-1}e^{t/\sqrt{l+2m}},\ \big(0, e^{t/\sqrt{l+2m}}\big), \dots, \big(0, e^{t/\sqrt{l+2m}}\big)\Big). \qquad (9.7)$$

Then the Busemann function with respect to ω'' is given by

$$\begin{aligned} &b(\omega'')\big((x_1 + y_1\sqrt{-1}, \dots, x_l + y_l\sqrt{-1}, (z_1, \lambda_1), \dots, (z_m, \lambda_m))\big) \\ &\quad = -\frac{1}{\sqrt{l+2m}} \log\big\{(y_1 \cdots y_l)(\lambda_1 \cdots \lambda_m)^2\big\}. \end{aligned}$$

For any positive number C we put

$$\begin{aligned} HB''(C) &= \big\{\big(x_1 + y_1\sqrt{-1}, \dots, x_l + y_l\sqrt{-1}, (z_1, \lambda_1), \dots, (z_m, \lambda_m)\big) \\ &\qquad \in \boldsymbol{H}^l \times \mathscr{H}^m \mid (y_1 \cdots y_l)(\lambda_1 \cdots \lambda_m)^2 > 1/C^2\big\} \\ &= b(\omega'')^{-1}\big((-\infty, \sqrt{4/(l+2m)}\,\log C)\big) \end{aligned}$$

and for any $(\alpha_1, \dots, \alpha_{l+m}) \in \boldsymbol{R}^l \times \boldsymbol{C}^m$ we define a geodesic ray $\tau'' = \tau''(\alpha_1, \dots, \alpha_{l+m})$ in $\boldsymbol{H}^l \times \mathscr{H}^m$ by

$$\begin{aligned} \tau''(t) = \Big(&\alpha_1 + \sqrt{-1}e^{-2t/\sqrt{4l+2m}}, \dots, \alpha_l + \sqrt{-1}e^{-2t/\sqrt{4l+2m}}, \\ &\big(\alpha_{l+1}, e^{-t/\sqrt{4l+2m}}\big), \dots, \big(\alpha_{l+m}, e^{-t/\sqrt{4l+2m}}\big)\Big). \end{aligned} \qquad (9.8)$$

We have some partial results in the following 4 cases.

- Let $\boldsymbol{k}''$ be a totally real number field of degree $2n$ over $\boldsymbol{Q}$. Suppose that τ'' meets $\iota''(g)\cdot HB''(C)$, where $g=\begin{pmatrix} p & r\\ q & s\end{pmatrix}\in SL(2,\boldsymbol{k}'')$ with $p,q\in\mathscr{O}''$, $q\neq 0$. Then the inequality (9.1) holds from a similar argument as in Section 5.
- Let $\boldsymbol{k}''$ be a number field with exactly $2n$ complex places. Suppose that τ'' meets $\iota''(g)\cdot HB''(C)$, where $g=\begin{pmatrix} p & r\\ q & s\end{pmatrix}\in SL(2,\boldsymbol{k}'')$ with $p,q\in\mathscr{O}''$, $q\neq 0$. Then the following inequality holds:

$$\sum_{1\leqslant i_1<\cdots<i_n\leqslant 2n}\left\|\alpha_{i_1}-\frac{p^{(i_1)}}{q^{(i_1)}}\right\|\cdots\left\|\alpha_{i_n}-\frac{p^{(i_n)}}{q^{(i_n)}}\right\|<\frac{C}{N(q)}.$$

- Let $\boldsymbol{k}''$ be a totally real number field of degree 3 over $\boldsymbol{Q}$. Suppose that τ'' meets $\iota''(g)\cdot HB''(C)$, where $g=\begin{pmatrix} p & r\\ q & s\end{pmatrix}\in SL(2,\boldsymbol{k}'')$ with $p,q\in\mathscr{O}''$, $q\neq 0$. Then the following inequality holds:

$$\sum_{1\leqslant i<j\leqslant 3}\left\|\alpha_i-\frac{p^{(i)}}{q^{(i)}}\right\|\left\|\alpha_j-\frac{p^{(j)}}{q^{(j)}}\right\|<\frac{3\max\{3C^{4/3},\,C^2\}}{4\,|N(q)|^{4/3}}.$$

- Let $\boldsymbol{k}''$ be a number field with exactly 3 complex places. Suppose that τ'' meets $\iota''(g)\cdot HB''(C)$, where $g=\begin{pmatrix} p & r\\ q & s\end{pmatrix}\in SL(2,\boldsymbol{k}'')$ with $p,q\in\mathscr{O}''$, $q\neq 0$. Then the following inequality holds:

$$\sum_{1\leqslant i<j\leqslant 3}\left\|\alpha_i-\frac{p^{(i)}}{q^{(i)}}\right\|\left\|\alpha_j-\frac{p^{(j)}}{q^{(j)}}\right\|<\frac{3\max\{3C^{4/3},\,C^2\}}{16\,N(q)^{4/3}}.$$

In the sequel, we let $\boldsymbol{k}''$ be a cubic field with one real place and one complex place. Let $\xi\in\boldsymbol{k}''\longmapsto\widehat{\xi}$ be one of the two complex embeddings and let $\widehat{\boldsymbol{k}}''$ be the image of $\boldsymbol{k}''$ under this embedding. The embedding $\sigma'':\boldsymbol{k}''\longrightarrow\boldsymbol{R}\times\boldsymbol{C}$ is given by $\sigma''(\xi)=(\xi,\widehat{\xi})$. The inequality corresponding to (9.1) is

$$\left|\alpha-\frac{p}{q}\right|+\left|\beta-\frac{\widehat{p}}{\widehat{q}}\right|^2<\frac{C}{|N(q)|}. \tag{9.9}$$

Since there are only finitely many (distinct) solutions $p/q\in\boldsymbol{k}''$ of (9.9) with $p,q\in\mathscr{O}''$ such that $|N(q)|$ is bounded from above by some constant, for $(\alpha,\beta)\in\boldsymbol{R}\times\boldsymbol{C}-\sigma''(\boldsymbol{k}'')$, there are infinitely many solutions of (9.9) if and only if there are infinitely many solutions of

$$\left|\alpha-\frac{p}{q}\right|+\left|\beta-\frac{\widehat{p}}{\widehat{q}}\right|^2<\frac{C}{H_{\boldsymbol{k}''}(q)}. \tag{9.10}$$

From the linear forms theorem of Minkowski, there exists a positive constant $C \leqslant 2\sqrt{|\triangle''|}$ depending only on $\boldsymbol{k}''$, such that for any $(\alpha, \beta) \in (\boldsymbol{R} - \boldsymbol{k}'') \times (\boldsymbol{C} - \widehat{\boldsymbol{k}}'')$ the inequality (9.9) has infinitely many solutions $p/q \in \boldsymbol{k}''$ with $p, q \in \mathscr{O}''$. The geodesic ray in $\boldsymbol{H} \times \mathscr{H}$ corresponding to (9.7), where $\mathscr{H}$ is equipped with the metric $2(dx^2 + dy^2 + d\lambda^2)/\lambda^2$ instead of the Poincaré metric, is

$$\omega''(t) = \Big(\sqrt{-1}e^{t/\sqrt{3}},\ \big(0, e^{t/\sqrt{3}}\big)\Big).$$

Then the Busemann function with respect to ω'' is given by

$$b(\omega'')\big((x + y\sqrt{-1},\ (z, \lambda)\big) = -\frac{1}{\sqrt{3}} \log(y\lambda^2).$$

For any positive number C we have

$$\begin{aligned} HB''(C) &= \big\{(x + y\sqrt{-1},\ (z, \lambda)) \in \boldsymbol{H} \times \mathscr{H} \mid y\lambda^2 > 1/C^2\big\} \\ &= b(\omega'')^{-1}\big((-\infty, \sqrt{4/3}\log C)\big) \end{aligned}$$

and for any $(\alpha, \beta) \in \boldsymbol{R} \times \boldsymbol{C}$ the geodesic ray in $\boldsymbol{H} \times \mathscr{H}$ corresponding to (9.8) is

$$\tau''(t) = \tau''(\alpha, \beta)(t) = \Big(\alpha + \sqrt{-1}e^{-2t/\sqrt{6}},\ \big(\beta, e^{-t/\sqrt{6}}\big)\Big). \tag{9.11}$$

Let h'' be the class number of $\boldsymbol{k}''$. We choose in the h'' ideal classes, fixed integral ideals $\mathfrak{a}''_1 = \langle a''_1, b''_1\rangle, \ldots, \mathfrak{a}''_{h''} = \langle a''_{h''}, b''_{h''}\rangle$ with $a''_i, b''_i \in \mathscr{O}''$, so that each $\mathfrak{a}''_i$ is of minimum norm among all the integral ideals of its class. Let c''_i, d''_i be elements of $(\mathfrak{a}''_i)^{-1}$ with $a''_i d''_i - b''_i c''_i = 1$ and $g''_i = \begin{pmatrix} a''_i & c''_i \\ b''_i & d''_i \end{pmatrix}$ for each $i = 1, \ldots, h''$. By a similar argument as one in the proof of Theorem 3 based on Propositions 2.2, 4.5, we obtain the following.

Theorem 9.2. *Let $(\alpha, \beta) \in \boldsymbol{R} \times \boldsymbol{C} - \sigma''(\boldsymbol{k}'')$.*

(1) *If $\tau'' = \tau''(\alpha, \beta)$ intersects infinitely many translates of $HB''(C/2)$ by elements of $\bigcup_{i=1}^{h''} SL(2, \mathscr{O}'') \cdot g''_i$, then there are infinitely many solutions $p/q \in \boldsymbol{k}''$ of* (9.10) *with $p, q \in \mathscr{O}''$.*

(2) *If there are infinitely many solutions $p/q \in \boldsymbol{k}''$ with $p, q \in \mathscr{O}''$ of the inequality* (9.10)*, then τ'' intersects infinitely many translates of $HB''(3C)$ by elements of $\bigcup_{i=1}^{h''} SL(2, \mathscr{O}'') \cdot g''_i$.*

Hence, by a similar argument as in Section 8, the exponent in (9.10) is best possible if there is a geodesic ray of the form (9.11) which is contained in $\boldsymbol{H} \times \mathscr{H} - \bigcup_{i=1}^{h''} \iota''(SL(2, \mathscr{O}''))\iota''(g''_i) \cdot HB''(C'')$ for some $C'' > 0$.

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