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Secondary Novikov-Shubin invariants of groups and quasi-isometry

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Abstract. We define new L^2-invariants which we call secondary Novikov-Shubin invariants. We calculate the first secondary Novikov-Shubin invariants of finitely generated groups by using random walk on Cayley graphs and see in particular that these are invariant under quasi-isometry.

1. Introduction.

In this paper we study secondary Novikov-Shubin invariants. These new L^2-invariants are naturally defined by modifying the original definition of Novikov-Shubin invariants (Section 2), where L^2-invariants mean L^2-Betti numbers, Novikov-Shubin invariants, L^2-torsions and so on. By using secondary Novikov-Shubin invariants, we can study density functions with infinite Novikov-Shubin invariants. It is known that the first Novikov-Shubin invariants of finitely generated groups classify infinite virtually nilpotent groups ([**6**, Lemma 2.46]). By using the first secondary Novikov-Shubin invariants, we would like to study finitely generated groups which are not virtually nilpotent. We prove the following in Section 4.

Theorem 1.1. *Let G be an infinite amenable finitely generated group and $0 < a < 1$. Then,*

(i) $\beta_1(G) = 0$ *if and only if* $p(n) \not\preceq \exp(-n^b)$ *for any* $b \in (0,1)$,
(ii) $\beta_1(G) = \frac{2a}{1-a}$ *if and only if* $p(n) \preceq \exp(-n^b)$ *for any* $b \in (0,a)$ *and* $p(n) \not\preceq \exp(-n^b)$ *for any* $b \in (a,1)$,
(iii) $\beta_1(G) = \infty$ *if and only if* $p(2n) \preceq \exp(-n^b)$ *for any* $b \in (0,1)$.

In particular the first secondary Novikov-Shubin invariants of finitely generated groups are invariant under quasi-isometry.

Here $\beta_1(G)$ is the first secondary Novikov-Shubin invariant of G (Section 3) and $p(n)$ is the asymptotic type of the probability of return after n steps ($n \in 2\boldsymbol{Z}$) for the random walk on the Cayley graph of G (Section 4). Also a relation $\preceq$ is defined in Section 4. By using the above theorem, we can calculate some examples.

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EXAMPLE 1.2. For any non-trivial finite group U and any positive integer d the asymptotic type of the wreath product group $U \wr \boldsymbol{Z}^d$ is $\exp(-n^{d/d+2})$ ([**9**, Theorem 3.5]). Here the wreath product is the semiproduct $(\oplus_{i \in \boldsymbol{Z}^d} U) \rtimes \boldsymbol{Z}^d$, where $\boldsymbol{Z}^d$ acts on $\oplus_{i \in \boldsymbol{Z}^d} U$ by translation. Thus we have

$$\beta_1(U \wr \boldsymbol{Z}^d) = d$$

by the above theorem. In particular any positive integer can occur as the first secondary Novikov-Shubin invariants of finitely generated groups. In the case where $d = 1$, we know the spectral density function of the Laplacian of the Cayley graph of $U \wr \boldsymbol{Z}$ ([**3**, Corollary 3], [**2**, Theorem 5], [**1**, Theorem 1.1]). Hence we can also get

$$\beta_1(U \wr \boldsymbol{Z}) = 1$$

by a direct calculation.

EXAMPLE 1.3. The asymptotic type of the wreath product group $\boldsymbol{Z} \wr \boldsymbol{Z}$ is $\exp(-n^{1/3}(\ln(n))^{2/3})$ ([**9**, Theorem 3.11]). Thus we have

$$\beta_1(\boldsymbol{Z} \wr \boldsymbol{Z}) = 1.$$

Though $\exp(-n^{1/3}(\ln(n))^{2/3})$ and $\exp(-n^{1/3})$ are not asymptotically equivalent, their first secondary Novikov-Shubin invariants are equal.

Gromov indicates that Novikov-Shubin invariants of a certain class of groups may be invariant under quasi-isometry ([**4**, p. 241]). Naturally we can formulate the following conjecture.

CONJECTURE 1.4. *Secondary Novikov-Shubin invariants of groups of finite type are invariant under quasi-isometry.*

The author does not know whether these conjectures hold. Novikov-Shubin invariants of amenable groups are studied by Roman Sauer ([**10**]).

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2. Secondary Novikov-Shubin invariants of density functions.

We will recall the definition of density functions, their L^2-Betti numbers and their Novikov-Shubin invariants. For the details, we refer to [**6**, Chapter 1, 2].

DEFINITION 2.1. A function $F : [0, \infty) \to [0, \infty]$ is called a density function if it is monotone non-decreasing and right-continuous. We say that F is Fredholm if there exists $\lambda > 0$ such that $F(\lambda) < \infty$, in which case we denote its L^2-Betti number as $b^{(2)}(F) := F(0)$ and its Novikov-Shubin invariant as

$$\alpha(F) := \liminf_{\lambda \to 0+} \frac{\ln\big(F^{\perp}(\lambda)\big)}{\ln(\lambda)},$$

provided that $F(\lambda) > b^{(2)}(F)$ holds for all $\lambda > 0$ and otherwise, $\alpha(F) := \infty^{+}$. Here we write $F^{\perp}(\lambda) := F(\lambda) - F(0)$ and ∞^{+} is a formal symbol.

For two density functions F and F' we write $F \preceq F'$ if there exist $C > 0$ and $\epsilon > 0$ such that $F(\lambda) \leq F'(C\lambda)$ holds for all $\lambda \in [0, \epsilon]$. We say that F and F' are dilatationally equivalent (denoted by $F \simeq F'$) if $F \preceq F'$ and $F' \preceq F$ and also we say that F and F' are dilatationally equivalent up to L^2-Betti numbers if $F^{\perp} \simeq F'^{\perp}$.

It is clear that L^2-Betti numbers of density functions are invariant under dilatational equivalence and Novikov-Shubin invariants of density functions are invariant under dilatational equivalence up to L^2-Betti numbers ([**6**, Chapter 2]). If F and F' are two Fredholm density functions which are dilatationally equivalent, then F and F' are certainly dilatationally equivalent up to L^2-Betti numbers.

Here we will define secondary Novikov-Shubin invariants of density functions.

Definition 2.2. Let F be a Fredholm density function. Its secondary Novikov-Shubin invariant of F is

$$\beta(F) := \liminf_{\lambda \to 0+} \frac{-\ln\big(-\ln(F^{\perp}(\lambda))\big)}{\ln(\lambda)},$$

provided that $F(\lambda) > b^{(2)}(F)$ holds for all $\lambda > 0$ and otherwise, we put $\beta(F) := \infty^{+}$.

The above definition is well-defined under dilatational equivalence up to L^2-Betti numbers. Indeed we can confirm the following.

Lemma 2.3. *Let F and F' be two Fredholm density functions. Then, $F^{\perp} \preceq F'^{\perp}$ implies $\beta(F) \geq \beta(F')$.*

In particular secondary Novikov-Shubin invariants of density functions are invariant under dilatational equivalence up to L^2-Betti numbers.

Proof. Since $F^{\perp} \preceq F'^{\perp}$, there exist $C > 0$ and $\epsilon > 0$ such that $F^{\perp}(\lambda) \leq F'^{\perp}(C\lambda)$ holds for all $\lambda \in [0, \epsilon]$. Hence we have

$$\frac{\ln\big(-\ln(F^{\perp}(\lambda))\big)}{-\ln(\lambda)} \geq \frac{\ln\big(-\ln(F'^{\perp}(C\lambda))\big)}{-\ln(\lambda)} = \frac{\ln\big(-\ln(F'^{\perp}(C\lambda))\big)}{-\ln(C\lambda)} \cdot \frac{-\ln(C\lambda)}{-\ln(C\lambda) + \ln(C)}.$$

Thus we get

$$\beta(F) \geq \beta(F'). \qquad \square$$

Surely we can define other Novikov-Shubin type invariants which are invariant under dilatational equivalence up to L^2-Betti numbers, but we do not deal with them in this paper.

The following relationship between Novikov-Shubin invariants and secondary Novikov-Shubin invariants is valid.

Lemma 2.4. *Let F be a Fredholm density function. Then,*

(i) $\alpha(F)=\infty^+$ *if and only if* $\beta(F)=\infty^+$,
(ii) $\alpha(F)<\infty$ *only if* $\beta(F)=0$.

Proof. (i) is clear by definition. We prove (ii), that is, $\beta(F)>0$ and $\beta(F)\neq\infty^+$ only if $\alpha(F)=\infty$. Since

$$\beta(F)=\liminf_{\lambda\to 0+}\frac{\ln\big(-\ln(F^{\perp}(\lambda))\big)}{-\ln(\lambda)},$$

for any $\epsilon>0$ there exists $\lambda_0\in(0,1)$ such that

$$\beta(F)-\epsilon\leq\inf_{\lambda\in(0,\lambda_0]}\frac{\ln\big(-\ln(F^{\perp}(\lambda))\big)}{-\ln(\lambda)}.$$

Hence we have for all $\lambda\in(0,\lambda_0]$

$$\beta(F)-\epsilon\leq\frac{\ln\big(-\ln(F^{\perp}(\lambda))\big)}{-\ln(\lambda)}.$$

When we take

$$\epsilon=\frac{1}{2}\beta(F),$$

then we have for all $\lambda\in(0,\lambda_0]$

$$\frac{1}{2}\beta(F)(-\ln(\lambda))\leq\ln\big(-\ln(F^{\perp}(\lambda))\big).$$

Thus we have

$$\frac{\exp\big(\frac{1}{2}\beta(F)(-\ln(\lambda))\big)}{-\ln(\lambda)}\leq\frac{-\ln\big(F^{\perp}(\lambda)\big)}{-\ln(\lambda)}.$$

Since

$$\frac{\exp\big(\frac{1}{2}\beta(F)(-\ln(\lambda))\big)}{-\ln(\lambda)}\to\infty\ (\lambda\to 0+),$$

we get

$$\alpha(F)=\infty. \qquad\square$$

The next example shows that any possible value can occur as the secondary Novikov-Shubin invariant of a density function.

EXAMPLE 2.5. Let us define density functions F_s for $s \in [0,\infty] \sqcup \{\infty^+\}$ by $F_s(0) = 0$ and for $\lambda > 0$ by

$$\begin{aligned} F_0(\lambda) &= \lambda, \\ F_s(\lambda) &= \exp\left(-\frac{1}{\lambda^s}\right), \\ F_\infty(\lambda) &= \exp\left(-\exp\left(\frac{1}{\lambda}\right)\right), \\ F_{\infty^+}(\lambda) &= 0. \end{aligned}$$

Then we can check for $s \in [0,\infty] \sqcup \{\infty^+\}$

$$\beta(F_s) = s.$$

3. Secondary Novikov-Shubin invariants of groups.

In this section and the next, we will concentrate on secondary Novikov-Shubin invariants (see [**6**] for other L^2-invariants).

Let G be a discrete group. The Hilbert space with orthonormal basis G is denoted by $l^2(G)$. Then we have the left and right regular representations of the group ring $\boldsymbol{C}G$ on $l^2(G)$ by extending linearly the left and right regular representations of G. The bounded operators on $l^2(G)$ which are equivariant with respect to the left regular representation of $\boldsymbol{C}G$ on $l^2(G)$ form a von Neumann algebra $\mathscr{N}(G) := B(l^2(G))^G$, called the group von Neumann algebra. Equivalently, $\mathscr{N}(G)$ can be defined as the weak closure of the right regular representation of $\boldsymbol{C}G$ in $B(l^2(G))$. The group von Neumann algebra $\mathscr{N}(G)$ is equipped with its standard trace $\mathrm{tr}_{\mathscr{N}(G)} : \mathscr{N}(G) \to \boldsymbol{C}$ given by $\mathrm{tr}_{\mathscr{N}(G)}(T) := \langle T(1_G), 1_G \rangle_{l^2(G)}$. The trace of an element in the right regular representation of $\boldsymbol{C}G$ is just the coefficient of the unit element. Moreover for an n-dimensional square matrix $T \in M_n(\mathscr{N}(G)) = B(l^2(G)^n)^G$, we have

$$\mathrm{tr}_{\mathscr{N}(G)}\left((T_{ij})_{1\le i,j\le n}\right) := \sum_{i=1}^{n} \mathrm{tr}_{\mathscr{N}(G)}(T_{ii}).$$

The spectral density function up to the L^2-Betti number of an operator $T \in M_{m,n}(\mathscr{N}(G)) = B(l^2(G)^n, l^2(G)^m)^G$ is defined as $F(T)^\perp(\lambda) := \mathrm{tr}_{\mathscr{N}(G)}\, \chi_{(0,\lambda^2]}(T^*T)$ by using spectral calculus. Here $\chi_{(0,\lambda^2]}$ is the characteristic function of the interval $(0,\lambda^2]$. Also we can write $F(T)^\perp(\lambda) = \mathrm{tr}_{\mathscr{N}(G)}\, E_{\lambda^2}^{T^*T} - \mathrm{tr}_{\mathscr{N}(G)}\, E_0^{T^*T}$, where $\{E_\mu^{T^*T}\}_{\mu\in[0,\infty)}$ is the spectral family of the positive operator T^*T. Its secondary Novikov-Shubin invariant $\beta(T) \in [0,\infty] \cup \infty^+$ is defined as $\beta(F(T)^\perp)$.

DEFINITION 3.1. Let X be a free G-CW-complex of finite type. Define its cellular

L^2-chain complex by $C_*^{(2)}(X) := l^2(G) \otimes_{\mathbf{Z}G} C_*(X)$, where $C_*(X)$ is the cellular chain complex. Moreover denote the cellular p-th L^2-boundary map by c_p. When we fix a cellular basis for $C_p(X)$, we can regard $C_p^{(2)}(X)$ as the Hilbert space $l^2(G)^{n_p}$ with the natural left action of $\mathbf{C}G$ and $c_p \in B(l^2(G)^{n_p}, l^2(G)^{n_{p-1}})^G$, where n_p is the number of the p-dimensional G-cells. We define its cellular p-th spectral density function up to the L^2-Betti number and its cellular p-th secondary Novikov-Shubin invariant of X as follows:

$$F_p^{\perp}(X) := F^{\perp}(c_p)$$
$$\beta_p(X) := \beta\big(F_p^{\perp}(X)\big).$$

It is known that the dilatational equivalence class of $F_p^{\perp}(X)$ is invariant under G-homotopy equivalence ([**6**, Theorem 2.55(1)]). Hence so is $\beta_p(X)$.

Remark 3.2. In the case when X is a cocompact free proper G-manifold without boundary and with G-Riemannian metric, we can define its analytic spectral density function and its analytic secondary Novikov-Shubin invariant by using L^2-de Rham complex. In this paper we do not deal with the analytic spectral density functions. However when we regard X as a free G-CW-complex of finite type, its cellular spectral density function and its analytic one are dilatationally equivalent ([**6**, Theorem 2.68]) so that its cellular secondary Novikov-Shubin invariant is the same as its analytic one.

We will deal with the case of groups.

Definition 3.3. Let n be a non-negative integer or $n = \infty$. Define $\mathscr{F}_n$ as the class of groups for which BG are CW-complexes with a finite number of p-dimensional cells for $p \leq n$. Here BG are the classifying spaces of G.

We note that $\mathscr{F}_1$ is the class of finitely generated groups and that $\mathscr{F}_\infty$ is the class of finite type groups.

Definition 3.4. Let n be a non-negative integer or $n = \infty$ and $G \in \mathscr{F}_n$. Then for $1 \leq p \leq n$ we set

$$F_p^{\perp}(G) := F_p^{\perp}(EG),$$
$$\beta_p(G) := \beta_p(EG),$$

where EG is the universal covering of BG.

4. The first secondary Novikov-Shubin invariants of groups.

In this section we calculate the first secondary Novikov-Shubin invariants of finitely generated groups by using the random walk on the Cayley graphs.

Let G be a finitely generated group with a finite set S of generators. The Cayley graph $C_S(G)$ of (G, S) is the following connected one-dimensional free G-CW-complex. Its 0-skeleton is G. For each element $s \in S$ we attach a free equivalent G-cells $G \times [-1, 1]$

by the attaching map $G \times \{-1,1\} \to G$ which sends $(g,-1)$ to g and $(g,1)$ to gs. We will study the first secondary Novikov-Shubin invariant of $C_S(G)$. We can identify the first L^2-boundary map of $C_S(G)$

$$c_S : C_1^{(2)}(C_S(G)) \to C_0^{(2)}(C_S(G))$$

with

$$\bigoplus_{s\in S} r_{s^{-1}-1} : \bigoplus_{s\in S} l^2(G) \to l^2(G),$$

where r is the right regular representation of $\boldsymbol{C}G$ on $l^2(G)$.

LEMMA 4.1. *Let G be a finitely generated group and let X be a connected free G-CW-complex of finite type. Then for any finite set S of generators of G, we have*

$$\beta_1(X) = \beta_1(C_S(G)).$$

In particular $\beta_1(C_S(G))$ is independent of the choice of a finite set S of generators and we have

$$\beta_1(G) = \beta_1(C_S(G)).$$

PROOF. This is clear since it is known that $F_1^{\perp}(X)$ and $F_1^{\perp}(C_S(G))$ are dilatationally equivalent ([**6**, Lemma 2.45, Theorem 2.55(1)]). □

We can assume that S is symmetric, that is, $s \in S$ implies $s^{-1} \in S$ and S does not contain the unit element of G. We will recall simple random walk on $C_S(G)$. The probability distribution is

$$p : G \to [0,1],\ g \mapsto \begin{cases} |S|^{-1} & \text{if } g \in S, \\ 0 & \text{if } g \notin S. \end{cases}$$

Thus the transition probability operator is

$$P = \sum_{s\in S} \frac{1}{|S|} r_{s^{-1}} : l^2(G) \to l^2(G),$$

in particular, we can confirm

$$P = id - \frac{1}{2|S|} c_S c_S^*.$$

Then for $n \in \boldsymbol{Z}_{\geq 0}$

$$p(n) := \operatorname{tr}_{\mathscr{N}(G)} P^n$$

is the probability of return after n steps for the random walk on the Cayley graph. It is clear that $p(n)$ for $n \in 2\boldsymbol{Z}_{\geq 0}$ is a non-increasing function.

In the following, we regard $p(n)$ as a function defined only on the even numbers.

Definition 4.2. Let u, v be two positive non-increasing functions defined on the positive real axis. We write $u \preceq v$ if there exists $C \geq 1$ such that for any $t > 0$

$$u(t) \leq Cv(t/C).$$

We say that u and v are asymptotically equivalent (denoted by $u \simeq v$) if $u \preceq v$ and $v \preceq u$, and we call this equivalence class the asymptotic type. When a function is defined only on the even numbers, we extend it to the positive real axis by linear interpolation. We will use the same notation for the original function and its extension.

Remark 4.3. The asymptotic type of $p(n)$ is invariant under quasi-isometry ([**8**, Theorem 1.2]).

In particular the asymptotic type of $p(n)$ is independent of the choice of a finite symmetric set S of generators of G.

Here we will prove the main theorem.

Theorem 4.4. *Let G be a finitely generated group and $0 < b < 1$. Then,*

(i) *G is non-amenable or finite if and only if $\beta_1(G) = \infty^+$,*
(ii) *If G is infinite amenable and $p(n) \preceq \exp(-n^b)$, then $\beta_1(G) \geq \frac{2b}{1-b}$,*
(iii) *If G is infinite amenable and $p(n) \not\preceq \exp(-n^b)$, then $\beta_1(G) \leq \frac{2b}{1-b}$.*

Proof. If G is finite, then obviously we have $\beta_1(G) = \infty^+$. Hence we can assume that G is infinite. We have

$$F(\lambda) := \operatorname{tr}_{\mathscr{N}(G)} \big(\chi_{[1-\lambda,1]}(P)\big) = F_1^{\perp}(C_S(G))\big(\sqrt{2|S|\lambda}\big).$$

Indeed, because $\operatorname{tr}_{\mathscr{N}(G)} E_0^{c_S c_S^*} = 0$ for G infinite ([**6**, Theorem 1.35(8)]), we observe

$$\begin{aligned} F_1^{\perp}(C_S(G))\big(\sqrt{2|S|\lambda}\big) &= \operatorname{tr}_{\mathscr{N}(G)} E_{2|S|\lambda}^{c_S^* c_S} - \operatorname{tr}_{\mathscr{N}(G)} E_0^{c_S^* c_S} = \operatorname{tr}_{\mathscr{N}(G)} E_{2|S|\lambda}^{c_S c_S^*} - \operatorname{tr}_{\mathscr{N}(G)} E_0^{c_S c_S^*} \\ &= \operatorname{tr}_{\mathscr{N}(G)} E_{\lambda}^{\frac{1}{2|S|} c_S c_S^*} = \operatorname{tr}_{\mathscr{N}(G)} \left(\chi_{[0,\lambda]}\left(\frac{1}{2|S|} c_S c_S^*\right)\right) \\ &= \operatorname{tr}_{\mathscr{N}(G)} \big((\chi_{[0,\lambda]} \circ f)(P)\big) = \operatorname{tr}_{\mathscr{N}(G)} \big(\chi_{[1-\lambda,1]}(P)\big) \end{aligned}$$

with $f(\mu) := 1 - \mu$. Then we get

$$\beta_1(G) = 2\beta(F).$$

(i) is clear since it is well known that the spectrum of P contains 1 if and only if G is amenable ([**5**], [**11**]). We have for $n \in 2\boldsymbol{Z}_{>0}$

$$(1-\lambda)^n \big(\chi_{[-1,-1+\lambda]} + \chi_{[1-\lambda,1]}\big)(P) \le P^n.$$

Hence we have

$$(1-\lambda)^n F(\lambda) \le p(n). \tag{1}$$

Also we have

$$P^n \le (1-\lambda)^n \chi_{(-1+\lambda,1-\lambda)}(P) + \big(\chi_{[-1,-1+\lambda]} + \chi_{[1-\lambda,1]}\big)(P).$$

Here we need the following claim.

Claim 4.5. *When $-1 \in \sigma(P)$, we have*

$$\mathrm{tr}_{\mathscr{N}(G)} \big(\chi_{[-1,-1+\lambda]}(P)\big) = \mathrm{tr}_{\mathscr{N}(G)} \big(\chi_{[1-\lambda,1]}(P)\big).$$

We will prove this after the proof of the theorem.
By Claim 4.5, when $-1 \in \sigma(P)$, we have for $\lambda \in [0,1]$

$$p(n) \le (1-\lambda)^n + 2F(\lambda).$$

When $-1 \notin \sigma(P)$, we have for $\lambda \in [0, 1+\inf \sigma(P))$

$$p(n) \le (1-\lambda)^n + 2F(\lambda).$$

Hence if $\lambda > 0$ is sufficiently small, we have

$$p(n) \le (1-\lambda)^n + 2F(\lambda). \tag{2}$$

(ii) By (1) and $p(n) \preceq \exp(-n^b)$, there exists $C \in (0,1]$ such that for any sufficiently large even number n

$$\begin{aligned} F(\lambda) &\le \frac{p(n)}{(1-\lambda)^n} \ \big(\forall n \in 2\boldsymbol{Z}_{>0}\big) \\ &\le \frac{C^{-1}\exp(-Cn^b)}{(1-\lambda)^n}. \end{aligned}$$

Hence we have for any sufficiently large even number n

$$CF(\lambda) \le \frac{\exp(-Cn^b)}{(1-\lambda)^n}. \tag{3}$$

For any $\epsilon \in (0,b)$, we put

$$n_\lambda := \left[\!\left[\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)} \right]\!\right],$$

where $[[v]]$ is the greatest even number not greater than v. Then we have

$$\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)} - 2 < n_\lambda \leq \left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}.$$

When $\lambda > 0$ is sufficiently small, then we have

$$\frac{1}{2^{1/b}}\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)} < n_\lambda \leq \left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}. \tag{4}$$

By (3) and (4), when $\lambda > 0$ is sufficiently small, then we observe

$$\begin{aligned}
CF(\lambda) &\leq \frac{\exp(-Cn_\lambda^b)}{(1-\lambda)^{n_\lambda}} \\
&< \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\frac{1}{(1-\lambda)^{\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}}} \\
&= \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\frac{1}{\left\{(1-\lambda)^{\frac{1}{\lambda}}\right\}^{\left(\frac{1}{\lambda}\right)^{(b-\epsilon)/(1-b+\epsilon)}}} \\
&\leq \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\exp\left(2\left(\frac{1}{\lambda}\right)^{(b-\epsilon)/(1-b+\epsilon)}\right) \\
&= \exp\left\{\left(-\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\left(\frac{C}{2} - 2\left(\frac{1}{\lambda}\right)^{-\epsilon/(1-b+\epsilon)}\right)\right\} \\
&\leq \exp\left\{\frac{C}{4}\left(-\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{-\ln(-\ln CF(\lambda))}{\ln(\lambda)} &\geq \frac{-\ln\left\{\frac{C}{4}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right\}}{\ln(\lambda)} \\
&= \frac{\frac{b}{1-b+\epsilon}\ln\left(\frac{1}{\lambda}\right) + \ln\left(\frac{C}{4}\right)}{-\ln(\lambda)} \\
&\to \frac{b}{1-b+\epsilon}\ (\lambda \to 0+).
\end{aligned}$$

Thus we have for any $\epsilon \in (0, b)$

$$\beta(F) = \beta(CF) \geq \frac{b}{1-b+\epsilon}.$$

Hence we get

$$\beta(F) \geq \frac{b}{1-b}.$$

(iii) Since $p(n) \not\precsim \exp(-n^b)$, for any $C \in (0,1]$ and any $N > 0$ there exists $n \geq N (n \in 2\boldsymbol{Z}_{>0})$ such that

$$p(n) > \frac{1}{C} \exp(-Cn^b).$$

By fixing $C \in (0,1]$, we put $\Lambda_C := \left\{n \in 2\boldsymbol{Z}_{>0} | p(n) > \frac{1}{C} \exp(-Cn^b)\right\}$. By (2), we have for $n \in \Lambda_C$ and $\lambda > 0$ sufficiently small

$$2F(\lambda) \geq p(n) - (1-\lambda)^n > \frac{1}{C} \exp(-Cn^b) - (1-\lambda)^n.$$

Hence we have

$$2F(\lambda) \geq \exp(-Cn^b) - (1-\lambda)^n. \tag{5}$$

For any $r > \frac{b}{1-b}$ we have

$$n_\lambda := \left[\left[\frac{1}{\lambda^{r/b}}\right]\right] + 2.$$

Then we have for λ sufficiently small

$$\frac{1}{\lambda^{r/b}} < n_\lambda \leq \frac{1}{\lambda^{r/b}} + 2 \leq \frac{2^{1/b}}{\lambda^{r/b}}. \tag{6}$$

By (5) and (6), when $\lambda > 0$ is sufficiently small, then we observe

$$\begin{aligned}
2F(\lambda) &\geq \exp\big(-Cn_\lambda^b\big) - (1-\lambda)^{n_\lambda} \\
&> \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - (1-\lambda)^{\frac{1}{\lambda^{r/b}}} \\
&= \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - \left\{(1-\lambda)^{\frac{1}{\lambda}}\right\}^{\frac{1}{\lambda^{(r-b)/b}}} \\
&\geq \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - \exp\left(-\left(\frac{1}{\lambda}\right)^{\frac{r-b}{b}}\right) \\
&= \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right)\left(1 - \exp\left\{2C\left(\frac{1}{\lambda}\right)^r - \left(\frac{1}{\lambda}\right)^{(r-b)/b}\right\}\right) \\
&= \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right)\left(1 - \exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C - \left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right) \\
&> 0.
\end{aligned}$$

Hence we have

$$-\ln(2F(\lambda)) \leq 2C\left(\frac{1}{\lambda}\right)^r - \ln\left(1-\exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C-\left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right). \tag{7}$$

Let $1>\delta>0$. Then we have for $\lambda>0$ sufficiently small

$$1-\exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C-\left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right] \geq 1-\delta>0.$$

Since we have

$$-\ln\left(1-\exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C-\left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right) \leq -\ln(1-\delta) =: D$$

and (7), we get

$$-\ln(2F(\lambda)) \leq 2C\left(\frac{1}{\lambda}\right)^r + D.$$

Thus we observe

$$\begin{aligned}\frac{-\ln(-\ln(F(\lambda)))}{\ln(\lambda)} &\leq \frac{\ln\left(2C\left(\frac{1}{\lambda}\right)^r + D + \ln(2)\right)}{-\ln(\lambda)} \\ &\to r\ (\lambda \to 0+).\end{aligned}$$

Because $\beta(F)$ is defined by using "lim inf", we have

$$\beta(F) \leq r.$$

Thus we get

$$\beta(F) \leq \frac{b}{1-b}. \qquad \square$$

Now Theorem 1.1 is clear.

Finally we will prove claim 4.5.

Lemma 4.6. *Let G be a finitely generated group and S be a finite symmetric set of generators of G where $e \notin S$. If $-1 \in \sigma(P)$, then there exists a group homomorphism $f : G \to S^1$ such that $f(s) = -1$ for any $s \in S$, where $P := \frac{1}{|S|}\sum_{s\in S} r_{s^{-1}}$.*

Proof. Since $-1 \in \sigma(P)$, there exists $(\xi_n)_{n\in\boldsymbol{N}} \subset l^2(G)$ such that $\|\xi_n\| = 1$ and $\|P\xi_n + \xi_n\| \to 0 (n \to \infty)$. Then we observe

$$\begin{aligned}\big|\langle P\xi_n+\xi_n,\xi_n\rangle\big| &\le \|P\xi_n+\xi_n\|\|\xi_n\|\\ &\to 0\ (n\to\infty).\end{aligned}$$

Since we have

$$\begin{aligned}2|S|\langle P\xi_n+\xi_n,\xi_n\rangle &= 2\sum_{s\in S}\langle r_s\xi_n+\xi_n,\xi_n\rangle\\ &= \sum_{s\in S}\big\langle (r_s+r_{s^{-1}}+2)\xi_n,\xi_n\big\rangle\end{aligned}$$

and $r_s+r_{s^{-1}}+2$ is a positive operator, we have

$$\big\langle (r_s+r_{s^{-1}}+2)\xi_n,\xi_n\big\rangle \to 0\ (n\to\infty).$$

Because $r_{s^{-1}}$ is a unitary, we observe

$$\begin{aligned}\big\langle (r_s+r_{s^{-1}}+2)\xi_n,\xi_n\big\rangle &= \big\langle (r_s+1)\xi_n,\xi_n\big\rangle + \big\langle (r_{s^{-1}}+1)\xi_n,\xi_n\big\rangle\\ &= \big\langle (r_s+1)\xi_n,\xi_n\big\rangle + \big\langle \xi_n,(r_s+1)\xi_n\big\rangle\\ &= 2\,\mathrm{Re}\,\big\langle (r_s+1)\xi_n,\xi_n\big\rangle\\ &= 2\,\mathrm{Re}\,\big\langle (1+r_{s^{-1}})\xi_n,r_{s^{-1}}\xi_n\big\rangle.\end{aligned}$$

Thus we observe for any $s\in S$

$$\begin{aligned}\|r_s\xi_n+\xi_n\|^2 &= \mathrm{Re}\,\|r_s\xi_n+\xi_n\|^2\\ &= \mathrm{Re}\,\big\langle (r_s+1)\xi_n,(r_s+1)\xi_n\big\rangle\\ &= \mathrm{Re}\,\big\langle (r_s+1)\xi_n,r_s\xi_n\big\rangle + \mathrm{Re}\,\big\langle (r_s+1)\xi_n,\xi_n\big\rangle\\ &\to 0\ (n\to\infty).\end{aligned}$$

Here we define for any $s_{i_1},s_{i_2},\dots,s_{i_m}\in S$

$$f\big(s_{i_1}s_{i_2}\cdots s_{i_m}\big) := \lim_{n\to\infty}\big\langle r_{s_{i_1}s_{i_2}\cdots s_{i_m}}\xi_n,\xi_n\big\rangle.$$

This is well-defined since we have for any $s\in S$

$$\lim_{n\to\infty}\langle r_s\xi_n,\xi_n\rangle = -1$$

and for any $s_{i_1},s_{i_2},\dots,s_{i_{m+1}}\in S$

$$\begin{aligned}
&\left|\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n,\xi_n\rangle + \langle r_{s_{i_1}s_{i_2}\cdots s_{i_{m+1}}}\xi_n,\xi_n\rangle\right| \\
&\quad = \left|\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n,\xi_n\rangle + \langle r_{s_{i_1}} r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n,\xi_n\rangle\right| \\
&\quad = \left|\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n,\xi_n\rangle + \langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, r_{s_{i_1}^{-1}}\xi_n\rangle\right| \\
&\quad = \left|\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, (1+r_{s_{i_1}^{-1}})\xi_n\rangle\right| \\
&\quad \le \|r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n\| \left\|(1+r_{s_{i_1}^{-1}})\xi_n\right\| \\
&\quad \to 0\ (n\to\infty). \qquad \square
\end{aligned}$$

Proof of Claim 4.5. For any $\xi = \sum_{g\in G}\xi_g g \in l^2(G)$ we define

$$U(\xi) := \sum_{g\in G} f(g)\xi_g g.$$

This is a unitary on $l^2(G)$ and $U(e) = e$ $(e \in l^2(G))$. Moreover we have $Ur_s = -r_sU$. Indeed by Lemma 4.6 we have

$$\begin{aligned}
Ur_s(\xi) &= \sum_{g\in G} f(gs^{-1})\xi_g gs^{-1} \\
&= \sum_{g\in G} f(g)f(s^{-1})\xi_g gs^{-1} \\
&= -\sum_{g\in G} f(g)\xi_g gs^{-1} \\
&= -r_sU(\xi).
\end{aligned}$$

Hence we have

$$UPU^{-1} = -P.$$

Since U is a unitary, we have

$$\begin{aligned}
U\chi_{[-1,-1+\lambda]}(P)U^{-1} &= UE^P_\lambda U^{-1} \\
&= E^{UPU^{-1}}_\lambda \\
&= \chi_{[-1,-1+\lambda]}(UPU^{-1}) \\
&= \chi_{[-1,-1+\lambda]}(-P) \\
&= \chi_{[1-\lambda,1]}(P).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\operatorname{tr}_{\mathscr{N}(G)}\big(\chi_{[-1,-1+\lambda]}(P)\big) &= \langle \chi_{[-1,-1+\lambda]}(P)e, e\rangle \\
&= \langle \chi_{[-1,-1+\lambda]}(P)U^{-1}e, U^{-1}e\rangle \\
&= \langle U\chi_{[-1,-1+\lambda]}(P)U^{-1}e, e\rangle \\
&= \langle \chi_{[1-\lambda,1]}(P)e, e\rangle \\
&= \operatorname{tr}_{\mathscr{N}(G)}(\chi_{[1-\lambda,1]}(P)). \qquad \square
\end{aligned}$$

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