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Computations of spaces of Siegel modular cusp forms

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Abstract. Simple homomorphisms to elliptic modular forms are defined on the ring of Siegel modular forms and linear relations on the Fourier coefficients of Siegel modular forms are implied by the codomains of these homomorphisms. We use the linear relations provided by these homomorphisms to compute the Siegel cusp forms of degree n and weight k in some new cases: $(n,k)=(4,14)$, $(4,16)$, $(5,8)$, $(5,10)$, $(6,8)$. We also compute enough Fourier coefficients using this method to determine the Hecke eigenforms in the nontrivial cases. We also put the open question of whether our technique always succeeds in a precise form. As a partial converse we prove that the Fourier series of Siegel modular forms are characterized among all formal series by the codomain spaces of these homomorphisms and a certain boundedness condition.

1. Notation.

For a commutative ring R let $M_{m\times n}(R)$ denote the R-module of m-by-n matrices with coefficients in R. For $x\in M_{m\times n}(R)$ let $x'\in M_{n\times m}(R)$ denote the transpose. Let $V_n(R)=\{x\in M_{n\times n}(R) : x'=x\}$ be the symmetric n-by-n matrices over R. For $R\subseteq \boldsymbol{R}$, an element $x\in V_n(R)$ is called positive definite, written $x>0$, when $v'xv>0$ for all $v\in R^n\setminus\{0\}$; $x\in V_n(R)$ is called semidefinite, written $x\ge 0$, when $v'xv\ge 0$ for all $v\in R^n$. We write $x>y$ if $x-y>0$ and $x\ge y$ if $x-y\ge 0$. Let $\mathscr{P}_n(R)=\{x\in V_n(R) : x>0\}$ and $\mathscr{P}_n^{\mathrm{semi}}(R)=\{x\in V_n(R) : x\ge 0\}$. $V_n(\boldsymbol{R})$ is a topological vector space and $\mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{R})$ is the closure of $\mathscr{P}_n(\boldsymbol{R})$ inside $V_n(\boldsymbol{R})$. $V_n(\boldsymbol{R})$ is a euclidean vector space under the inner product $\langle x,y\rangle=\mathrm{tr}(xy)$. The semi-integral lattice is $V_n'(\boldsymbol{Z})=\{T\in V_n(\boldsymbol{Q}) : \forall\, v\in \boldsymbol{Z}^n,\ v'Tv\in\boldsymbol{Z}\}$ and we set $\mathscr{X}_n^{\mathrm{semi}}=V_n'(\boldsymbol{Z})\cap\mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{Q})$ and $\mathscr{X}_n=V_n'(\boldsymbol{Z})\cap\mathscr{P}_n(\boldsymbol{Q})$.

The general linear group is defined by $GL_n(R)=\{x\in M_{n\times n}(R) : \det(x)$ is a unit in $R\}$ and the special linear group by $SL_n(R)=\{x\in GL_n(R) : \det(x)=1\}$. For $x\in GL_n(R)$ let x^* denote the inverse transpose. Let $I_n\in GL_n(R)$ be the identity matrix and set $J_n=\left(\begin{smallmatrix}0 & I_n\\ -I_n & 0\end{smallmatrix}\right)\in SL_{2n}(R)$. The orthogonal group is defined by $O_n(R)=\{g\in GL_n(R) : g'I_ng=I_n\}$. The symplectic group is defined by $\mathrm{Sp}_n(R)=\{x\in GL_n(R) : x'J_nx=J_n\}$. The group $GL_n(\ell)$ is the kernel of the natural map of $GL_n(\boldsymbol{Z})$ to $GL_n(\boldsymbol{Z}/\ell\boldsymbol{Z})$; the group $\mathrm{Sp}_n(\ell)$ is the kernel of the natural map of $\mathrm{Sp}_n(\boldsymbol{Z})$ to $\mathrm{Sp}_n(\boldsymbol{Z}/\ell\boldsymbol{Z})$. Let $t: V_n(R)\to\mathrm{Sp}_n(R)$ be the homomorphism defined by $t(\zeta)=\left(\begin{smallmatrix}I_n & \zeta\\ 0 & I_n\end{smallmatrix}\right)$ and $u: GL_n(R)\to\mathrm{Sp}_n(R)$ be that defined by $u(M)=\left(\begin{smallmatrix}M & 0\\ 0 & M^*\end{smallmatrix}\right)$. Let $\nabla_n(R)=\{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\mathrm{Sp}_n(R) : C=0\}$; let $\Delta_n(\ell)=\nabla_n(\boldsymbol{Z})\cap\mathrm{Sp}_n(\ell)$ and set $\Delta_n=\nabla_n(\boldsymbol{Z})=\Delta_n(1)$. We write $\Gamma_n=\mathrm{Sp}_n(\boldsymbol{Z})$ and $\Gamma(\ell)=\mathrm{Sp}_1(\ell)$. As usual, let $\Gamma_0(\ell)=\{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma_n : C\equiv 0 \mod \ell\}$ and $\Gamma_1(\ell)=\{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma_n : A,D\equiv I_n \mod \ell$ and $C\equiv 0 \mod \ell\}$. The projective rational

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symplectic group is defined by $\mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}} = \mathrm{Sp}_n(\boldsymbol{R}) \cap \boldsymbol{R}^+ M_{2n\times 2n}(\boldsymbol{Q})$; the group $\nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ is defined by $\nabla_n(\boldsymbol{R})^{\mathrm{pr}} = \mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}} \cap \nabla_n(\boldsymbol{R})$.

Define the Siegel upper half space $\mathscr{H}_n = \{\Omega \in V_n(\boldsymbol{C}) : \Im\Omega \in \mathscr{P}_n(\boldsymbol{R})\}$. The group $\mathrm{Sp}_n(\boldsymbol{R})$ acts on $\mathscr{H}_n$ as $g\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1}$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. For any function $f : \mathscr{H}_n \to \boldsymbol{C}$ and any $k \in \boldsymbol{Z}$ define the group action $(f|_k g)(\Omega) = \det(C\Omega + D)^{-k} f(g\langle\Omega\rangle)$. Denote the set of holomorphic functions $f : \mathscr{H}_n \to \boldsymbol{C}$ by $\mathscr{O}(\mathscr{H}_n)$. Let $\Gamma \subseteq \Gamma_n$ be a subgroup of finite index. The complex vector space of Siegel modular forms of degree n and weight k automorphic with respect to Γ is denoted by $M_n^k(\Gamma)$ and is defined as the $f \in \mathscr{O}(\mathscr{H}_n)$ such that $f|_k g = f$ for all $g \in \Gamma$ and such that for all $Y_0 \in \mathscr{P}_n(\boldsymbol{R})$ and for all $g \in \mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}}$, $f|_k g$ is bounded on $\{\Omega \in \mathscr{H}_n : \Im\Omega > Y_0\}$. For the groups $\Delta_n(\ell)$, which are not of finite index in Γ_n, we let $M_n^k(\Delta_n(\ell))$ denote the $f \in \mathscr{O}(\mathscr{H}_n)$ such that $f|_k g = f$ for all $g \in \Delta_n(\ell)$ and such that for all $Y_0 \in \mathscr{P}_n(\boldsymbol{R})$ and for all $g \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$, $f|_k g$ is bounded on $\{\Omega \in \mathscr{H}_n : \Im\Omega > Y_0\}$. For $f \in M_n^k(\Gamma)$ the Siegel Φ-map is defined by $(\Phi f)(\Omega) = \lim_{\lambda \to +\infty} f(\left(\begin{smallmatrix} i\lambda & 0 \\ 0 & \Omega \end{smallmatrix}\right))$ and the space of cusp forms is defined by $S_n^k(\Gamma) = \{f \in M_n^k(\Gamma) : \forall\, g \in \mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}}, \Phi(f|g) = 0\}$. We have, for example, $\Phi : M_n^k(\mathrm{Sp}_n(\ell)) \to M_{n-1}^k(\mathrm{Sp}_{n-1}(\ell))$. For the case of $\Delta_n(\ell)$ we set $S_n^k(\Delta_n(\ell)) = \{f \in M_n^k(\Delta_n(\ell)) : \forall\, g \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}, \Phi(f|g) = 0\}$. Let $e(z) = e^{2\pi i z}$.

By the Koecher principle, an $f \in S_n^k(\Delta_n(\ell))$ has a Fourier expansion

$$f(\Omega) = \sum_{T \in \frac{1}{\ell}\mathscr{X}_n} a(T) e(\langle T, \Omega\rangle)$$

where the Fourier coefficients $a(T)$ satisfy $a(U'TU) = \det(U)^k a(T)$ for all $U \in GL_n(\ell)$. We mainly work with $S_n^k(\Delta_n)$ but some actions we perform on this space increase the level and so we also consider the spaces $M_n^k(\Gamma(\infty)) = \bigcup_\ell M_n^k(\mathrm{Sp}_n(\ell)) \subseteq \mathscr{O}(\mathscr{H}_n)$ and

$$M_n^k(\Delta_n(\infty)) = \bigcup_\ell M_n^k(\Delta_n(\ell)) \subseteq \mathscr{O}(\mathscr{H}_n);\ S_n^k(\Delta_n(\infty)) = \bigcup_\ell S_n^k(\Delta_n(\ell)) \subseteq M_n^k(\Delta_n(\infty)).$$

Let $\boldsymbol{M}_n^k(\Delta_n(\ell))$, the Koecher formal series for $\Delta_n(\ell)$ of degree n and weight k, and the Koecher formal cusp series $\boldsymbol{S}_n^k(\Delta_n(\ell))$ be defined as:

$$\boldsymbol{M}_n^k(\Delta_n(\ell)) = \left\{(a(T)) \in \boldsymbol{C}^{\frac{1}{\ell}\mathscr{X}_n^{\mathrm{semi}}} : \forall U \in GL_n(\ell),\, a(U'TU) = \det(U)^k a(T)\right\}$$
$$\boldsymbol{S}_n^k(\Delta_n(\ell)) = \left\{(a(T)) \in \boldsymbol{C}^{\frac{1}{\ell}\mathscr{X}_n} : \forall U \in GL_n(\ell),\, a(U'TU) = \det(U)^k a(T)\right\}.$$

We also write $\boldsymbol{M}_n^k = \boldsymbol{M}_n^k(\Delta_n(1))$ and $\boldsymbol{S}_n^k = \boldsymbol{S}_n^k(\Delta_n(1))$ for brevity. By the uniqueness of the Fourier expansion, the maps

$$FS_n : M_n^k(\Delta_n(\ell)) \to \boldsymbol{M}_n^k(\Delta_n(\ell)) \text{ and}$$
$$FS_n : S_n^k(\Delta_n(\ell)) \to \boldsymbol{S}_n^k(\Delta_n(\ell)) \text{ given by } FS_n(f) = (a(T))$$

are defined for all ℓ and, if we denote $\boldsymbol{S}_n^k(\Delta_n(\infty)) = \cup_\ell \boldsymbol{S}_n^k(\Delta_n(\ell)) \subseteq \boldsymbol{C}^{\mathscr{P}_n(\boldsymbol{Q})}$, even for $\ell = \infty$. Instead of $(a(T))$ we write the formal q-expansion as $F = \sum_T a(T) q_n^T$, employing an index-keeping variable q_n as a reminder of the ring structure on $\oplus_k \boldsymbol{S}_n^k(\Delta_n(\infty))$.

2. Introduction.

We give a tractable method for producing linear relations among the Fourier coefficients of Siegel modular forms. This is equivalent to the standard problem of giving linear relations among Poincare series. Whether or not our method generates all linear relations among Fourier coefficients remains open but it does generate enough relations for two applications. First, in a theoretical vein, we prove that any convergent Fourier series satisfying these linear relations is the Fourier series of a Siegel modular form. Second, in a computational vein, we compute enough relations to determine $\dim S_n^k$ in several new cases, namely: $(n,k) = (4,14)$, $(4,16)$, $(5,8)$, $(5,10)$ and $(6,8)$. We also give bases of Hecke eigenforms. More information, including many Fourier coefficients, is on our website [**15**]. We consider only even weights k in this Introduction.

The method for producing linear relations is as follows: For $s \in \mathscr{P}_n(\boldsymbol{Z})$ and $\zeta \in V_n(\boldsymbol{Q})$ define $\phi_{s,\zeta} : \mathscr{H}_1 \to \mathscr{H}_n$ by $\phi_{s,\zeta}(\tau) = s\tau + \zeta$. This induces a ring homomorphism of modular forms:

$$\exists \ell : \forall N, \qquad \phi_{s,\zeta}^* : M_n^k(\Gamma_1(N)) \to M_1^{nk}(\Gamma_1(N\ell)).$$

When $\zeta = 0$ we also write $\phi_s^* = \phi_{s,0}^*$ for brevity. One can calculate the effect of $\phi_{s,\zeta}^*$ on the Fourier series of the modular forms. For $f \in S_n^k$ with Fourier series

$$f(\Omega) = \sum_{T \in \mathscr{X}_n} a(T) e(\langle T, \Omega \rangle)$$

we have, setting $q = e(\tau)$ and $\chi_\zeta(T) = e(\langle \zeta, T \rangle)$,

$$\left(\phi_{s,\zeta}^* f\right)(\tau) = \sum_j \left(\sum_{T : \langle s, T \rangle = j} \chi_\zeta(T) a(T) \right) q^j. \tag{2.1}$$

One can calculate the linear relations among the Fourier coefficients of elliptic modular forms in the codomain. Eichler's solution of the Basis Problem, for example, allows one to construct a basis of theta series and from these all the linear relations among the Fourier coefficients may be derived. By using equation (2.1), linear relations on the Fourier coefficients of elliptic forms pullback to linear relations on the Fourier coefficients of Siegel forms. A countable set of relations is obtained in this manner and this is the thrust of the method. However, in order to strengthen the method we use linear relations among the elliptic Fourier coefficients at all the cusps, taken collectively. Thus we also need the Fourier coefficients of $(\phi_{s,\zeta}^* f)|\sigma$ in terms of the $a(T)$. This can be done for $f \in M_n^k$ because

$$\forall\, \sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}, \quad \left(\phi_{s,\zeta}^* f\right)|\sigma = (\mathrm{const}) \phi_{s_1,\zeta_1}^* f \text{ for some } s_1 \text{ and } \zeta_1. \tag{2.2}$$

An intrinsic reformulation of equation (2.2) is proven in Theorem 2.6. This paper aims to advance both theory and computation. For an intrinsic description of these cusp expansions it is better to introduce rational polarized lattices. A polarized lattice $\mathscr{L} \subseteq \boldsymbol{R}^{2n}$ is a lattice for which there exists a basis L in $\nabla_n(\boldsymbol{R})$. Write $L = t(\zeta)u(M)$ for unique choices of $\zeta \in V_n(\boldsymbol{R})$ and $M \in GL_n(\boldsymbol{R})$ so that if we set $s = MM'$ a Gram matrix is

$$\mathrm{Gram}(\mathscr{L}) = LL' = \begin{pmatrix} M & \zeta M^* \\ 0 & M^* \end{pmatrix} \begin{pmatrix} M & \zeta M^* \\ 0 & M^* \end{pmatrix}' = \begin{pmatrix} s + \zeta s^{-1}\zeta & \zeta s^{-1} \\ s^{-1}\zeta' & s^{-1} \end{pmatrix}. \tag{2.3}$$

A rational polarized lattice then has both s and ζ rational. Symplectic matrices are uniquely written in the above concluding form so that any symplectic lattice is a polarized lattice. However, the isometry class of a symplectic lattice Ψ is $\Psi O_{2n}(\boldsymbol{R})$ whereas the appropriate isometry class of a polarized lattice $\mathscr{L}$ is $\mathscr{L}\, u(O_n(\boldsymbol{R}))$ and operations on polarized lattices need only respect this type of isometry. The group $\mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}$ acts on rational polarized lattices via $\mathscr{L}\sigma = \mathscr{L}(\sigma \otimes I_n)$ and the group $\Gamma(\mathscr{L}) = \{\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}} : \mathscr{L}\sigma = \mathscr{L}\}$ is commensurable with Γ_1, see Propositions 7.1 and 7.3. Any two bases of $\mathscr{L}$ from $\nabla_n(\boldsymbol{R})$ differ on the left by an element of $\nabla_n(\boldsymbol{Z})$ and so we may make the following definition.

Definition 2.4. For a rational polarized lattice $\mathscr{L}$, let $\mu(\mathscr{L}, \cdot) : M_n^k \to M_1^{nk}(\Gamma(\mathscr{L}))$ be defined by $\mu(\mathscr{L}, f) = \phi^*_{I_n}(f|L)$ for any $L \in \nabla_n(\boldsymbol{R})$ with $\mathscr{L} = \boldsymbol{Z}^{2n}L$.

It is uncommon to slash a Siegel form f with an element $L \in \nabla_n(\boldsymbol{R})$ that is not necessarily projective rational. The function $f|L$ is automorphic for $L^{-1}\Gamma_n L$ but this group is not necessarily commensurable with Γ_n. However, by descending to degree one, we regain commensurability since $\phi^*_I(f|L)$ is automorphic for the commensurable group $\Gamma(\mathscr{L})$. One easily checks, see Proposition 8.1, that

$$\mu(\mathscr{L}, f) = \det(s)^{k/2}\, \phi^*_{s,\zeta} f \tag{2.5}$$

so that $\mu(\mathscr{L}, \cdot)$ is just a minor modification of $\phi^*_{s,\zeta}$; however, the following intrinsic version of equation (2.2) justifies the change of notation. This theorem is proved in Section 8.

Theorem 2.6. *Let $\mathscr{L}$ be a rational polarized lattice. For $f \in M_n^k$ and for $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{pr}$ we have $\mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$.*

The maps $\phi^*_{s,\zeta}$ and $\mu(\mathscr{L}, \cdot)$ can be defined on formal series $F \in \boldsymbol{M}_n^k$ in such a way that they commute with $FS_n : M_n^k \to \boldsymbol{M}_n^k$; this is essentially done by imitating equations (2.5) and (2.1). If F is the Fourier series of a Siegel modular form f and we set $\Psi_{\mathscr{L}} = \mu(\mathscr{L}, f)$ then the following linear relations hold: For all rational polarized lattices $\mathscr{L}$, there exists a $\Psi_{\mathscr{L}} \in M_1^{nk}(\Gamma(\mathscr{L}))$ such that for all $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}$ we have $FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$. We have not been able to characterize the Fourier expansions of Siegel forms in S_n^k from among all formal series in $\boldsymbol{S}_n^k$ by using only these linear relations. By adding a boundedness condition, however, we can prove a partial converse. Recall that the Hecke bound B of a cusp form f is given by $B = \sup_{\Omega \in \mathscr{H}_n} \det(Y)^{k/2}|f(\Omega)|$. In this case the following elliptic Hecke bounds are uniformly bounded:

$$y^{nk/2}\big|\big(\phi^*_{s,\zeta}f\big)(\tau)\big| = y^{nk/2}|f(s\tau+\zeta)| \le y^{nk/2}B\det(sy)^{-k/2} = B\det(s)^{-k/2}.$$

The following theorem characterizes the Fourier expansions of Siegel cusp forms from *among all formal series* solely in terms of elliptic data $\Psi_{\mathscr{L}}$. This theorem is proved in Section 9.

Theorem 2.7. *Let $F \in \boldsymbol{S}_n^k$. We have $F \in FS_n(S_n^k)$ if and only if*

(1) *For all rational polarized lattices $\mathscr{L}$ with integral s,*

$$\exists\, \Psi_{\mathscr{L}} \in S_1^{nk}(\Gamma(\mathscr{L})) : \forall\,[\sigma] \in \Gamma(\mathscr{L})\backslash\Gamma_1, \quad FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F).$$

(2) *There exists a $B > 0$ such that for all $\mathscr{L}$ as in item (1), we have $y^{\frac{nk}{2}}|\Psi_{\mathscr{L}}(\tau)| \le B$ for all $\tau \in \mathscr{H}_1$.*

Now let us describe the computational application of determining $\dim S_n^k$. A basic example was worked out in detail in [**18**] and this paper is a sequel in that we augment and improve the restriction technique and give a general exposition of the method. There are finite sets $\mathscr{C} \subset \mathscr{X}_n$ for which the map given by $f \mapsto (a(T))_{T\in\mathscr{C}}$ is injective and reasonably practical sets $\mathscr{C}$ have been given in [**17**]. If linear relations on the $a(T)$ for $T \in \mathscr{C}$ can be found then their nullity has been demonstrated to be an upper bound for $\dim S_n^k$. One then constructs Siegel cusp forms to produce a lower bound. If these bounds agree then $\dim S_n^k$ has been found. We improve this method by keeping track of Fourier coefficients in a net $\mathscr{B}$ larger than $\mathscr{C}$. We augment the method by considering Witt homomorphisms as well as the maps $\mu(\mathscr{L},\cdot)$. New maps may be added in the future. Furthermore, in a real computation it is finite truncations of the above maps that actually occur and so we need a flexible general set-up:

Definition 2.8. Let $\pi : (V,S) \to (\pi V, \pi S)$ and $\phi : (V,S) \to (V_1, S_1)$ be morphisms of relative vector spaces. When $\pi\phi^{-1}(S_1) = \pi S$ we say ϕ dominates π. We say that a set of morphisms $\phi_\alpha : (V,S) \to (V_\alpha, S_\alpha)$ for $\alpha \in \mathscr{A}$ dominates π when the product map $\prod_{\alpha\in\mathscr{A}} \phi_\alpha : (V,S) \to \prod_{\alpha\in\mathscr{A}}(V_\alpha, S_\alpha)$ dominates π.

Let $\pi_{\mathscr{C}} : \boldsymbol{C}^{\mathscr{X}_n} \to \boldsymbol{C}^{\mathscr{C}}$ be the projection map and select a finite set $\mathscr{C}$ such that $\pi_{\mathscr{C}} \circ FS_n : S_n^k \to \boldsymbol{C}^{\mathscr{C}}$ is injective. We want to understand the projection map as a morphism of relative vector spaces $\pi_{\mathscr{C}} : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (\pi_{\mathscr{C}}\boldsymbol{S}_n^k, \pi_{\mathscr{C}}FS_n(S_n^k))$ by studying a set of simpler morphisms $\phi_\alpha : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (V_\alpha, S_\alpha)$, indexed by a certain finite set $\mathscr{A}$, whose codomains (V_α, S_α) are amenable to computation. Use the notation $\Phi_{\mathscr{A}} : (V,S) \to (V_{\mathscr{A}}, S_{\mathscr{A}})$ for the product map $\prod_{\alpha\in\mathscr{A}} \phi_\alpha : (V,S) \to \prod_{\alpha\in\mathscr{A}}(V_\alpha, S_\alpha)$. When $\pi_{\mathscr{C}} \circ FS_n$ is injective then a set of morphisms $\{\phi_\alpha\}_{\alpha\in\mathscr{A}}$ which dominates $\pi_{\mathscr{C}}$ also determines $\dim S_n^k = \dim \pi_{\mathscr{C}} FS_n(S_n^k) = \dim \pi_{\mathscr{C}}\Phi_{\mathscr{A}}^{-1}(S_{\mathscr{A}})$ and as $\mathscr{C}$ increases we compute the initial Fourier expansions from S_n^k in this way. The hope, borne out in some examples, is that for increasingly large sets $\mathscr{A}$ the morphism $\Phi_{\mathscr{A}}$ will dominate any fixed $\pi_{\mathscr{C}}$. The computations in section 13 are presented in this format.

We now frame the relation among sections 10, 11, 12 and 13. As discussed, our philosophy is to study S_n^k by using simpler homomorphisms of relative vector spaces. In section 10, for each polarized lattice $\mathscr{L}$, we define a morphism

$$\phi_{\mathscr{L}} : \big(\boldsymbol{S}_n^k, FS_n\big(S_n^k\big)\big) \to (V_{\mathscr{L}}, S_{\mathscr{L}}),$$

based on the restriction technique. In section 11, for $i+j=n$, we define another type of morphism

$$\psi_{ij}^* : \big(\boldsymbol{S}_n^k, FS_n\big(S_n^k\big)\big) \to \big(\boldsymbol{S}_i^k \otimes \boldsymbol{S}_j^k, (FS_i \otimes FS_j)\big(S_i^k \otimes S_j^k\big)\big),$$

based on the Witt maps. The codomains are amenable to computation because, if one is studying S_n^k, one presumably understands S_i^k for $i<n$. These are the two types of morphisms we use although they are by no means the only possibilities.

Section 12 carefully explains the truncated morphisms used in the computations. We show how to select finite dimensional projections of the domains and codomains so that the maps induced by $\phi_{\mathscr{L}}$ and ψ_{ij}^* are still morphisms of relative vector spaces on these finite dimensional projections. The truncated versions of $\phi_{\mathscr{L}}$ and ψ_{ij}^* are the morphisms used to dominate

$$\pi_{\mathscr{C}} : \pi_{\mathscr{B}}\big(\boldsymbol{S}_n^k, FS_n\big(S_n^k\big)\big) \to \pi_{\mathscr{C}}\big(\boldsymbol{S}_n^k, FS_n\big(S_n^k\big)\big) \tag{2.9}$$

in the computations of $\dim S_n^k$ in section 13. For each example we give the determining set of Fourier coefficients $\mathscr{C}$, the net $\mathscr{B}$ containing $\mathscr{C}$ used to define the truncations, the set of morphisms $\mathscr{A}$ used to dominate the map (2.9) and the constructed subspace of cusp forms $\tilde{S}_n^k$ used in the proof that the set of morphisms $\mathscr{A}$ dominates the map (2.9). We then compute $\dim S_n^k$, $\dim M_n^k$, a minimal set of determining Fourier coefficients for S_n^k, a rational basis of S_n^k and a basis of Hecke eigenforms for S_n^k. We thank the referee for his helpful suggestions, particularly for the expansion of section 13.

3. Formal Series.

Definition 3.1. For $F = \sum_{T \in \mathscr{X}_n^{\text{semi}}} a(T) q_n^T \in \boldsymbol{M}_n^k(\Delta_n)$ define $\Phi F = \sum_{t \in \mathscr{X}_{n-1}^{\text{semi}}} a(0 \oplus t) q_{n-1}^t \in \prod_{t \in \mathscr{X}_{n-1}^{\text{semi}}} \boldsymbol{C}$.

Lemma 3.2. *We have $\Phi : \boldsymbol{M}_n^k(\Delta_n) \to \boldsymbol{M}_{n-1}^k(\Delta_{n-1})$ and $\Phi FS_n(f) = FS_{n-1}\Phi(f)$ for all $f \in M_n^k(\Delta_n)$. The kernel of Φ on $\boldsymbol{M}_n^k$ is $\boldsymbol{S}_n^k$.*

Proof. $\Phi FS_n(f)$ is in the Koecher space because $a(0 \oplus u'tu) = a((1 \oplus u)'(0 \oplus t)(1 \oplus u)) = \det(1 \oplus u)^k a(0 \oplus t) = \det(u)^k a(0 \oplus t)$. The commutativity of Φ and FS follows from the Fourier expansion $(\Phi f)(\Omega) = \sum_{t \in \mathscr{X}_{n-1}^{\text{semi}}} a(0 \oplus t)\, e(\langle t, \Omega \rangle)$, [**11**, page 55]. The vanishing of the $a(T)$ is a class function and so $a(0 \oplus t) = 0$ for all $t \in \mathscr{X}_{n-1}^{\text{semi}}$ implies that the support of $a(T)$ is contained in $\mathscr{X}_n$. □

Lemma 3.3. *Let $F = \sum_T a(T) q_n^T \in \boldsymbol{M}_n^k(\Delta_n(\infty))$ be convergent. Then the function f defined by $f(\Omega) = \sum_T a(T) e(\langle T, \Omega \rangle)$ is a holomorphic function on $\mathscr{H}_n$. We have $F \in \boldsymbol{M}_n^k(\Delta_n(\ell))$ if and only if $f \in M_n^k(\Delta_n(\ell))$.*

Proof. The assumption that F converges means that $\sum_T a(T) e(\langle T, \Omega \rangle)$ converges for every $\Omega \in \mathscr{H}_n$. The series of absolute values converges uniformly on compact subsets of $\mathscr{H}_n$ so that f defines a holomorphic function on $\mathscr{H}_n$ with Fourier series equal to F. If $f \in M_n^k(\Delta_n(\ell))$ then $F = FS_n(f) \in \boldsymbol{M}_n^k(\Delta_n(\ell))$.

On the other hand, f is periodic with respect to $\ell V_n(\boldsymbol{Z})$ and for all $U \in GL_n(\ell)$ we have $(f|_k u(U))(\Omega) = \det(U^*)^{-k} f(U\Omega U') = \det(U)^k \sum_T a(T)e(\langle T, U\Omega U'\rangle)$. Setting $R = U'TU$ this becomes $= \det(U)^k \sum_R a(U^* R U^{-1}) e(\langle R, \Omega\rangle)$ and since $F \in \boldsymbol{M}_n^k(\Delta_n)$ this equals $\det(U)^k \det(U^{-1})^k \sum_R a(R)e(\langle R, \Omega\rangle) = f(\Omega)$ showing that $f \in M_n^k(\Delta_n(\ell))$. □

4. Slashing formal series.

Slashing by $\delta \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ stabilizes $M_n^k(\Delta_n(\infty))$.

LEMMA 4.1. *For all $\delta \in \nabla_n(\boldsymbol{R})^{pr}$, there exists an $N \in \boldsymbol{Z}^+$ such that for all $\ell \in \boldsymbol{Z}^+ \cup \{\infty\}$, the operator $\cdot|_k\delta$ maps $M_n^k(\Delta_n(\ell))$ to $M_n^k(\Delta_n(\ell N))$.*

PROOF. One follows the proof for $\mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}}$ in [**7**, page 128]. Take $t \in \boldsymbol{R}_{>0}$ such that $t\delta \in M_{2n\times 2n}(\boldsymbol{Z}) \cap GL_{2n}(\boldsymbol{Q})$. Take $N \in \boldsymbol{Z}^+$ such that $N(t\delta)^{-1}$ is integral. It suffices to show that $\Delta_n(\ell N) \subseteq \delta^{-1}\Delta_n(\ell)\,\delta$, for then we have $M_n^k(\Delta_n(\ell))|_k\delta \subseteq M_n^k(\Delta_n(\ell N))$. Since $\delta\,\Delta_n(\ell N)\,\delta^{-1} \subseteq \nabla_n(\boldsymbol{Q})$ it suffices to show that $\delta\,\Delta_n(\ell N)\,\delta^{-1} \subseteq I + \ell\, M_{2n\times 2n}(\boldsymbol{Z})$. We have

$$\begin{aligned}\delta\,\Delta_n(\ell N)\,\delta^{-1} &= (t\delta)\Delta_n(\ell N)(t\delta)^{-1} \subseteq I + (t\delta)\ell N\, M_{2n\times 2n}(\boldsymbol{Z})(t\delta)^{-1}\\ &\subseteq I + \ell\, M_{2n\times 2n}(\boldsymbol{Z}).\end{aligned}$$ □

The action of $\nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ on $M_n^k(\Delta_n(\ell))$ is transparent on Fourier expansions and we define an action of $\nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ on formal series to mimic the action on functions. For the next definition note that if $t(\zeta)u(M) \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ then $M^*\mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{Q})M^{-1} = \mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{Q})$.

DEFINITION 4.2. Let $F = \sum_T a(T)q_n^T \in \boldsymbol{M}_n^k(\Delta_n(\infty))$. Let $\delta \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$. Write $\delta = t(\zeta)u(M)$ uniquely for $M \in GL_n(\boldsymbol{R})^{\mathrm{pr}}$ and $\zeta \in V_n(\boldsymbol{Q})$. Define

$$(F|_k\delta) = \sum_{T \in \mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{Q})} \det(M)^k\, \chi_\zeta(M^*TM^{-1}) a(M^*TM^{-1}) q_n^T.$$

Notice that if we denote, for a group $\Delta \subseteq \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$,

$$\boldsymbol{M}_n^k(\Delta) = \left\{ F \in \prod_{T \in \mathscr{P}_n^{\mathrm{semi}}(\boldsymbol{Q})} \boldsymbol{C} : \forall\, \delta \in \Delta,\ F|_k\delta = F \right\}$$

we have an equivalent alternate description of $\boldsymbol{M}_n^k(\Delta_n(\ell))$ for $\ell \in \boldsymbol{Z}^+$.

LEMMA 4.3. *For all $\delta_1, \delta_2 \in \nabla_n(\boldsymbol{R})^{pr}$ and $F \in \boldsymbol{M}_n^k(\Delta_n(\infty))$ we have $(F|_k\delta_1)|_k\delta_2 = F|_k(\delta_1\delta_2)$. For all $\delta \in \nabla_n(\boldsymbol{R})^{pr}$, there exists an $N \in \boldsymbol{Z}^+$ such that for all $\ell \in \boldsymbol{Z}^+ \cup \{\infty\}$, the operator $\cdot|_k\delta$ maps $\boldsymbol{M}_n^k(\Delta_n(\ell))$ to $\boldsymbol{M}_n^k(\Delta_n(\ell N))$.*

PROOF. First we check that $(F|_k\delta_1)|_k\delta_2 = F|_k(\delta_1\delta_2)$. Then $\boldsymbol{M}_n^k(\Delta_n(\ell))|_k\delta \subseteq$

$\boldsymbol{M}_n^k(\Delta_n(\ell N))$ follows from $\Delta_n(\ell N) \subseteq \delta^{-1}\Delta_n(\ell)\,\delta$ as in the proof of Lemma 4.1 by the above alternate description. □

The action of $\nabla_n(\boldsymbol{R})^{\mathrm{pr}}$ on $\boldsymbol{M}_n^k(\Delta_n(\infty))$ has simply been defined to make slashing commute with the map to Fourier series, as the proof below illustrates.

Lemma 4.4. *Let $\delta \in \nabla_n(\boldsymbol{R})^{pr}$. Let $f \in M_n^k(\Delta_n(\infty))$. We have $FS_n(f|\delta) = FS_n(f)|\delta$.*

Proof. Let $f(\Omega) = \sum_T a(T)e(\langle T, \Omega\rangle) \in M_n^k(\Delta_n(\ell))$ and let $\delta = t(\zeta)u(M) \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$. We have $FS_n(f) = \sum_T a(T)q_n^T$ and by the definition of $\cdot|_k\delta$ on formal series we have $FS_n(f)|_k\delta = \sum_T \det(M)^k e(\langle \zeta, M^*TM^{-1}\rangle)a(M^*TM^{-1})q_n^T$. On the other hand,

$$
\begin{aligned}
(f|_k\delta)(\Omega) &= \det(M^*)^{-k} f(M\Omega M' + \zeta) \\
&= \det(M)^k \sum_T a(T)e(\langle T, M\Omega M'\rangle)e(\langle T, \zeta\rangle) \\
&= \det(M)^k \sum_R a(M^*RM^{-1})e(\langle R, \Omega\rangle)e\big(\langle M^*RM^{-1}, \zeta\rangle\big); \\
\therefore\ FS_n(f|_k\delta) &= \det(M)^k \sum_R a(M^*RM^{-1})e\big(\langle M^*RM^{-1}, \zeta\rangle\big)q_n^R.
\end{aligned}
$$
□

5. Restrictions to modular curves.

Lemma 5.1. *Let $L \in \nabla_n(\boldsymbol{R})$ be written as $L = t(\zeta)u(M)$ for $\zeta \in V_n(\boldsymbol{R})$ and $M \in GL_n(\boldsymbol{R})$. Set $s = MM'$. For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\boldsymbol{R})$ we have*

$$
L \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix} L^{-1} = \begin{pmatrix} aI_n + c\zeta s^{-1} & (d-a)\zeta + bs - c\zeta s^{-1}\zeta \\ cs^{-1} & dI_n - cs^{-1}\zeta \end{pmatrix}.
$$

Definition 5.2. The homomorphism $\alpha_{s,\zeta} : \mathrm{Sp}_1(\boldsymbol{R}) \to \mathrm{Sp}_n(\boldsymbol{R})$ is given by, for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$
\alpha_{s,\zeta}(\sigma) = \begin{pmatrix} aI_n + c\zeta s^{-1} & (d-a)\zeta + bs - c\zeta s^{-1}\zeta \\ cs^{-1} & dI_n - cs^{-1}\zeta \end{pmatrix}.
$$

We write α_s for $\alpha_{s,0}$. We have the following important relation, a minor modification of [**7**, page 301].

Lemma 5.3. *Let $f \in \mathscr{O}(\mathscr{H}_n)$, $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})$ and $(s, \zeta) \in \mathscr{P}_n(\boldsymbol{R}) \times V_n(\boldsymbol{R})$. We have*

$$
\big(\phi_{s,\zeta}^* f\big)|_{nk}\sigma = \phi_{s,\zeta}^*(f|_k\alpha_{s,\zeta}(\sigma)).
$$

Proof. This is a direct calculation.

$$\begin{aligned}
\big[(\phi^*_{s,\zeta}f)|_{nk}\sigma\big](\tau) &= (c\tau+d)^{-nk}(\phi^*_{s,\zeta}f)(\sigma\langle\tau\rangle) = (c\tau+d)^{-nk}f(s\sigma\langle\tau\rangle+\zeta)\\
&= (c\tau+d)^{-nk}f(MI\sigma\langle\tau\rangle M'+\zeta) = (c\tau+d)^{-nk}\det(M^*)^k\left(f|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}\right)(I\sigma\langle\tau\rangle)\\
&= (c\tau+d)^{-nk}\det(M^*)^k\det(cI\tau+dI)^k\left(f|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}|\begin{pmatrix} aI & bI\\ cI & dI\end{pmatrix}\right)(I\tau)\\
&= \det(M^*)^k\left(f|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}|\begin{pmatrix} aI & bI\\ cI & dI\end{pmatrix}|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}^{-1}|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}\right)(I\tau)\\
&= \det(M^*)^k\left(f|\alpha_{s,\zeta}(\sigma)|\begin{pmatrix} M & \zeta M^*\\ 0 & M^*\end{pmatrix}\right)(I\tau)\\
&= (f|\alpha_{s,\zeta}(\sigma))(MI\tau+\zeta M^*)M') = (f|\alpha_{s,\zeta}(\sigma))(s\tau+\zeta) = \phi^*_{s,\zeta}(f|\alpha_{s,\zeta}(\sigma))(\tau). \qquad \Box
\end{aligned}$$

Proposition 5.4. *For all $(s,\zeta)\in\mathscr{P}_n(\boldsymbol{Q})\times V_n(\boldsymbol{Q})$, there exists an $N\in\boldsymbol{Z}^+$ such that for all $\ell\in\boldsymbol{Z}^+$ we have*

$$\phi^*_{s,\zeta}: M^k_n(\Delta_n(\ell))\to M^{nk}_1(\Delta_1(\ell N)) \ \text{ and } \ \phi^*_{s,\zeta}: M^k_n(\mathrm{Sp}_n(\ell))\to M^{nk}_1(\Gamma(\ell N)).$$

The map takes cusp forms to cusp forms and in the latter case N need only satisfy s, ζ, s^{-1}, ζs^{-1}, $\zeta s^{-1}\zeta\in\frac{1}{N}M_{n\times n}(\boldsymbol{Z})$. In the former case N need only satisfy $s,\zeta,\in\frac{1}{N}V_n(\boldsymbol{Z})$.

Proof. Let f be in $M^k_n(\Delta_n(\ell))$ or $M^k_n(\mathrm{Sp}_n(\ell))$. For $\sigma\in\mathrm{Sp}_1(\boldsymbol{R})$ we have $(\phi^*_{s,\zeta}f)|\sigma=\phi^*_{s,\zeta}(f|\alpha_{s,\zeta}(\sigma))$. Hence if $\alpha_{s,\zeta}(\sigma)$ is in $\Delta_n(\ell)$ or $\mathrm{Sp}_n(\ell)$ we have $(\phi^*_{s,\zeta}f)|\sigma=\phi^*_{s,\zeta}f$. From the explicit form of the definition of $\alpha_{s,\zeta}$ we see that $\sigma\in\Gamma(\ell N)$ suffices to guarantee $\alpha_{s,\zeta}(\sigma)\in\mathrm{Sp}_n(\ell)$ under the conditions stated. When $c=0$ we have the simpler set of conditions. To see that the latter map takes cusp forms to cusp forms note that $\Phi(\phi^*_{s,\zeta}f|\sigma)=\Phi(\phi^*_{s,\zeta}(f|_k\alpha_{s,\zeta}(\sigma)))$ is the constant term in the Fourier series of $(f|_k\alpha_{s,\zeta}(\sigma))$ which is zero, $\alpha_{s,\zeta}(\sigma)$ being projective rational when σ is. $\Box$

Lemma 5.5. *For $s\in\mathscr{P}_n(\boldsymbol{R})$ and $f\in\mathscr{O}(\mathscr{H}_n)$ we have $\phi^*_{s^{-1}}f=\det(s)^k\phi^*_s(f|_k\,u(s^{-1}))$.*

Proof. We have $\det(s)^k(\phi^*_s(f|\,u(s^{-1})))(\tau)=\det(s)^k(f|\,u(s^{-1}))(s\tau)=f(s^{-1}\,s\tau\,s^{-1})$. We also have $(\phi^*_{s^{-1}}f)(\tau)=f(s^{-1}\tau)=f(s^{-1}\,s\tau\,s^{-1})$. $\Box$

6. The action of $\phi^*_{s,\zeta}$ on formal series.

We now want to define the map $\phi^*_{s,\zeta}$ on formal series in $\boldsymbol{M}^k_n(\Delta_n(\ell))$ in such a way that it will commute with the map to Fourier series.

Definition 6.1. For $(s,\zeta)\in\mathscr{P}_n(\boldsymbol{Q})\times V_n(\boldsymbol{Q})$ and for a formal series $F=\sum_T a(T)q_n^T\in\boldsymbol{M}^k_n(\Delta_n(\ell))$, define $\phi^*_{s,\zeta}F\in\prod_{T\in\mathscr{P}_1^{\mathrm{semi}}(\boldsymbol{Q})}\boldsymbol{C}$ via

$$\phi^*_{s,\zeta}F=\sum_{j\in\boldsymbol{Q}_{\ge 0}}\left(\sum_{T:\langle s,T\rangle=j}\chi_\zeta(T)a(T)\right)q_1^j.$$

LEMMA 6.2. *For all $(s,\zeta) \in \mathscr{P}_n(\boldsymbol{Q}) \times V_n(\boldsymbol{Q})$ we have $FS_1(\phi^*_{s,\zeta} f) = \phi^*_{s,\zeta}(FS_n(f))$ for all $f \in M^k_n(\Delta_n(\infty))$. Furthermore, for each (s,ζ) there is an $N \in \boldsymbol{Z}^+$ such that for all $\ell \in \boldsymbol{Z}^+$ the following diagram commutes.*

$$\begin{array}{ccc} M^k_n(\Delta_n(\ell)) & \xrightarrow{\phi^*_{s,\zeta}} & M^{nk}_1(\Gamma(N\ell)) \\ \Big\downarrow{\scriptstyle FS_n} & & \Big\downarrow{\scriptstyle FS_1} \\ \boldsymbol{M}^k_n(\Delta_n(\ell)) & \xrightarrow{\phi^*_{s,\zeta}} & \boldsymbol{M}^{nk}_1(\Delta_1(N\ell)) \end{array}$$

PROOF. It suffices to show the diagram commutes. First we check that $\phi^*_{s,\zeta}(\boldsymbol{M}^k_n(\Delta_n(\ell))) \subseteq \boldsymbol{M}^{nk}_1(\Delta_1(\ell N))$: if $\ell T \in \mathscr{X}^{\text{semi}}_n$ and $Ns \in \mathscr{P}_n(\boldsymbol{Z})$ then $j = \langle s, T\rangle$ is in $\frac{1}{\ell N}\boldsymbol{Z}$.

Take $f \in M^k_n(\Delta_n(\ell))$ with $f(\Omega) = \sum_{T \in \frac{1}{\ell}\mathscr{X}^{\text{semi}}_n} a(T) e(\langle T, \Omega\rangle)$ so that the formal series is $FS_n(f) = \sum_T a(T) q_n^T$, we have

$$\phi^*_{s,\zeta}(FS_n(f)) = \sum_{j \in \boldsymbol{Q}_{\geq 0}} \left(\sum_{T : \langle s,T\rangle = j} \chi_\zeta(T) a(T) \right) q_1^j.$$

On the other hand, $(\phi^*_{s,\zeta} f)(\tau) = f(s\tau + \zeta) = \sum_T a(T) e(\langle T, s\tau + \zeta\rangle) = \sum_T a(T) e(\langle T, \zeta\rangle) e(\tau)^{\langle T, s\rangle}$ so that

$$FS_1(\phi^*_{s,\zeta} f) = \sum_{j \in \frac{1}{\ell N}\boldsymbol{Z}_{\geq 0}} \left(\sum_{T : \langle s,T\rangle = j} \chi_\zeta(T) a(T) \right) q_1^j.$$

These are the same since $a(T) = 0$ off $\frac{1}{\ell}\mathscr{X}^{\text{semi}}_n$. □

7. Polarized Lattices.

Let $\boldsymbol{R}^n$ be the n-dimensional euclidean inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $\Lambda \subseteq \boldsymbol{R}^n$ be a rank n lattice. If $M \in GL_n(\boldsymbol{R})$ is a basis of row vectors for $\Lambda = \boldsymbol{Z}^n M$ then the quadratic form $s = MM'$ is called a Gram matrix for Λ. A lattice Λ yields an equivalence class of quadratic forms $[s] = \{U' s U : U \in GL_n(\boldsymbol{Z})\}$. By factoring $s = MM'$, the class $[s]$ determines the isometry class of Λ, $\{\Lambda g : g \in O_n(\boldsymbol{R})\}$. Any operation on lattices which does not commute with isometries should be carefully pointed out. The dual lattice is given by $\Lambda^* = \{\xi \in \boldsymbol{R}^n : \forall\, x \in \Lambda,\ \langle x, \xi\rangle \in \boldsymbol{Z}\}$. A lattice is called rational when the Gram matrices are rational or equivalently, letting $\Lambda^*_{\boldsymbol{Q}} = \Lambda^* \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, when $\Lambda \subseteq \Lambda^*_{\boldsymbol{Q}}$. Occasionally we consider an oriented lattice, (ϵ, Λ), for $\epsilon = \pm 1$, where Λ has a distinguished class of bases M with $\epsilon = \operatorname{sign}(\det(M))$. The equivalence on oriented quadratic forms becomes $(\epsilon, s) \sim (\det(U)\epsilon, U' s U)$.

A *polarized* lattice $\mathscr{L} \subseteq \boldsymbol{R}^{2n}$ is a lattice for which there exists a basis L in $\nabla_n(\boldsymbol{R})$. Let $\pi_1 : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the projection onto the first factor and let $i_2 : \boldsymbol{R}^n \to \boldsymbol{R}^n \times \boldsymbol{R}^n$ be the injection to the second factor. Now consider a polarized lattice $\mathscr{L} \subseteq \boldsymbol{R}^n \oplus \boldsymbol{R}^n$ with basis $L = t(\zeta) u(M)$ and set $\Lambda = \boldsymbol{Z}^n M$. From the form of the basis the sequence $0 \to \Lambda^* \xrightarrow{i_2} \mathscr{L} \xrightarrow{\pi_1} \Lambda \to 0$ is exact. We also have $\mathscr{L}^* = \boldsymbol{Z}^{2n} L^* = \boldsymbol{Z}^{2n} J_n L J_n = \boldsymbol{Z}^{2n} L J_n = \mathscr{L} J_n = \tau(\mathscr{L})$ where $\tau : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n \times \boldsymbol{R}^n$ is defined by $\tau((x,\xi)) = (\xi, x)$. By the properties of Ext, the class of $\mathscr{L}$ as an extension is given by some $X \in \mathrm{Ext}^{\mathrm{sym}}(\Lambda, \Lambda^*; \boldsymbol{R}) = \mathrm{Sym}(\Lambda^* \otimes \Lambda^*)_{\boldsymbol{R}} / \mathrm{Sym}(\Lambda^* \otimes \Lambda^*)_{\boldsymbol{Z}}$. We refer to Λ as the lattice and to X as the polarization. The condition that X be symmetric expresses itself as $\mathscr{L}^* = \tau(\mathscr{L})$, see [**19**]. The polarized lattice can also be described by giving quadratic form information (s, χ_ζ) where $s \in \mathscr{P}_n(\boldsymbol{R})$ is a Gram matrix $s = MM'$ of Λ. The pair (s, χ_ζ) gives the same polarized lattice as $(U'sU, \chi_{U'\zeta U})$ for any $U \in GL_n(\boldsymbol{Z})$ and this characterizes the equivalence as well. The polarization is *trivial* when $\mathscr{L} = \Lambda \oplus \Lambda^*$, $X = 0$ or $\chi_\zeta = 1$. The natural definitions of rationality for $\mathscr{L}$, (s, χ_ζ) and (Λ, X) all correspond. A rational Gram matrix for $\mathscr{L}$ was given in equation (2.3).

Moving among the three descriptions of a rational polarized lattice is important but not difficult. For $\Lambda = \boldsymbol{Z}^n M$, take $\bar{X} \in \mathrm{Sym}(\Lambda^* \otimes \Lambda^*)_{\boldsymbol{Q}}$ with $X = [\bar{X}]$. Let $j : \boldsymbol{R}^n \otimes \boldsymbol{R}^n \to M_{n\times n}(\boldsymbol{R})$ be given by $j(x \otimes \xi) = x'\xi$; then $\zeta = Mj\bar{X}M'$ is in $V_n(\boldsymbol{Q})$ and another choice of $\bar{X}$ changes ζ by an element of $V_n(\boldsymbol{Z})$ so that the character $\chi_\zeta : V_n'(\boldsymbol{Z}) \to e(\boldsymbol{Q})$, like s, depends only upon the choice of M. From (Λ, X) or (s, χ_ζ) we set $\mathscr{L} = \boldsymbol{Z}^{2n} u(M) t(j\bar{X}) = \boldsymbol{Z}^{2n} t(\zeta) u(M)$. From $\mathscr{L}$ we set $\Lambda = \pi_1 \mathscr{L}$ and the exactness of $0 \to \Lambda^* \xrightarrow{i_2} \mathscr{L} \xrightarrow{\pi_1} \Lambda \to 0$ tells us that $\mathscr{L}$ has a basis of the form $L = t(\zeta)u(M)$ where M gives a basis of $\Lambda = \boldsymbol{Z}^n M$. The symmetry of ζ is equivalent to L being symplectic. It bears repetition that an equivalence class $[(s, \chi_\zeta)]$ corresponds to an $O_n(\boldsymbol{R})$-isometry class for (Λ, X) but that the lattices $\mathscr{L}$ orbit in a restricted $u(O_n(\boldsymbol{R}))$-isometry class. When these isometry classes correspond we write $\mathscr{L} \sim (\Lambda, X) \sim (s, \chi_\zeta)$ to denote the correspondence between these three ways of describing a polarized lattice.

We have an action of $\mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}$ on the rational polarized lattices.

Proposition 7.1. *For $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{pr}$ and for any rational polarized lattice $\mathscr{L}$ we define $\mathscr{L}\sigma = \mathscr{L}(\sigma \otimes I_n)$. Then $\mathscr{L}\sigma$ is also a rational polarized lattice.*

Proof. Let $\mathscr{L} = \boldsymbol{Z}^{2n} L$ for $L \in \nabla_n(\boldsymbol{R})$. By Lemma 5.1 we see that $L(\sigma \otimes I_n)L^{-1}$ is projective rational and so we have $L(\sigma \otimes I_n)L^{-1} = g\delta$ for $g \in \Gamma_n$ and $\delta \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$. Then $\mathscr{L}\sigma = \boldsymbol{Z}^{2n} L(\sigma \otimes I_n) = \boldsymbol{Z}^{2n} g\delta L = Z^{2n}\delta L$ so that $\mathscr{L}\sigma$ has a basis $\delta L \in \nabla_n(\boldsymbol{R})$ and $\mathscr{L}\sigma$ is a polarized lattice.

To see that $\mathscr{L}\sigma$ is a rational lattice note that $L(\sigma \otimes I_n)$ is a basis of $\mathscr{L}\sigma$ and we have $(L(\sigma \otimes I_n))(L(\sigma \otimes I_n))' = (L(\sigma\sigma' \otimes I_n)L^{-1})(LL')$. The Gram matrix LL' is rational. The matrix $\sigma\sigma'$ is rational because σ is projective rational, hence $L(\sigma\sigma' \otimes I_n)L^{-1}$ is also rational by Lemma 5.1 and $(L(\sigma\sigma' \otimes I_n)L^{-1})(LL')$ is rational too. □

We now translate this action using the other descriptions $\mathscr{L}\sigma \sim (\Lambda_\sigma, X_\sigma) \sim (s_\sigma, \chi_{\zeta_\sigma})$. If $L = t(\zeta)u(M)$ is a basis for $\mathscr{L}$ then $L(\sigma \otimes I_n)$ is a symplectic basis for $\mathscr{L}\sigma$ and we proceed by factoring the element $L(\sigma \otimes I_n)L^{-1}$ of $\mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}}$:

$$L(\sigma \otimes I_n)L^{-1} = \begin{pmatrix} aI_n + c\zeta s^{-1} & (d-a)\zeta + bs - c\zeta s^{-1}\zeta \\ cs^{-1} & dI_n - cs^{-1}\zeta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} K & \beta K^* \\ 0 & K^* \end{pmatrix} \quad (7.2)$$

for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\boldsymbol{Z})$ and $\left(\begin{smallmatrix} K & \beta K^* \\ 0 & K^* \end{smallmatrix}\right) \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$. We then obtain

$$\mathscr{L}\sigma = \boldsymbol{Z}^{2n}\begin{pmatrix} K & \beta K^* \\ 0 & K^* \end{pmatrix}\begin{pmatrix} M & \zeta M^* \\ 0 & M^* \end{pmatrix} = \boldsymbol{Z}^{2n}\begin{pmatrix} KM & (K\zeta K' + \beta)(KM)^* \\ 0 & (KM)^* \end{pmatrix}$$

so that $\Lambda_\sigma = \boldsymbol{Z}^n KM$, $s_\sigma = KsK'$ and $\zeta_\sigma = K\zeta K' + \beta$. Finally, from

$$L(\sigma \otimes I_n) = \begin{pmatrix} aM + c\zeta M^* & bM + d\zeta M^* \\ cM^* & dM^* \end{pmatrix} = \begin{pmatrix} aM + cMj\bar{X} & bM + dMj\bar{X} \\ cM^* & dM^* \end{pmatrix}$$

we see that

$$\Lambda(aI + c\,j\bar{X}) + c\Lambda^* = \Lambda_\sigma \text{ and } \Lambda(bI + d\,j\bar{X}) + d\Lambda^* = \Lambda_\sigma j\bar{X}_\sigma + \Lambda_\sigma^*.$$

When the polarization is trivial we write $\Lambda_\sigma = \Lambda\Box\sigma = a\Lambda + c\Lambda^*$. In this context we also write $s_\sigma = s\Box\sigma = KsK'$ for some Gram matrix of $\Lambda\Box\sigma$. From the computation of $L\sigma L^{-1}$ we can make some remarks about the pollattice group $\Gamma(\mathscr{L}) = \{\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}} : \mathscr{L}\sigma = \mathscr{L}\}$.

PROPOSITION 7.3. *Let $\mathscr{L} \sim (\Lambda, X) \sim (s, \chi_\zeta)$ be a rational polarized lattice. We have $\Gamma(\mathscr{L}) \supseteq \Gamma(\ell)$ for s, ζ, s^{-1}, ζs^{-1}, $\zeta s^{-1}\zeta \in \frac{1}{\ell}M_{n\times n}(\boldsymbol{Z})$. For integral Λ we have $\Gamma(\mathscr{L}) \supseteq \Gamma_1(\ell)$ for $\zeta, s^{-1}, \zeta s^{-1}, \zeta s^{-1}\zeta \in \frac{1}{\ell}M_{n\times n}(\boldsymbol{Z})$. For integral Λ with a trivial polarization we have $\Gamma(\mathscr{L}) = \Gamma(\Lambda \oplus \Lambda^*) \supseteq \Gamma_0(\ell)$ for $\ell \in \boldsymbol{Z}^+$ with $\ell s^{-1} \in \mathscr{P}_n(\boldsymbol{Z})$. In general, we have $\Delta_1 \cap \Gamma(\mathscr{L}) = \Delta_1(\ell)$ for the minimal $\ell \in \boldsymbol{Z}^+$ with $\ell s \in \mathscr{P}_n(\boldsymbol{Z})$.*

PROOF. Pick a basis $L = t(\zeta)u(M)$ for $\mathscr{L}$. The condition $\mathscr{L}\sigma = \mathscr{L}$ becomes the condition $L(\sigma \otimes I_n) \in GL_{2n}(\boldsymbol{Z})L$ and from Lemma 5.1 we can read off the integrality conditions in the Proposition. □

8. The pairing of rational polarized lattices with $M_n^k(\Delta_n)$.

PROPOSITION 8.1. *Let $\mathscr{L}$ be a rational polarized lattice. Let $f \in M_n^k(\Delta_n)$. For $L_1, L_2 \in \nabla_n(\boldsymbol{R})$ such that $\mathscr{L} = \boldsymbol{Z}^{2n}L_1 = \boldsymbol{Z}^{2n}L_2$ we have $f|L_1 = f|L_2$. Define $\mu(\mathscr{L}, f) = \phi_I^*(f|L)$ for any $L \in \nabla_n(\boldsymbol{R})$ such that $\mathscr{L} = \boldsymbol{Z}^{2n}L$.*

For an oriented polarized rational lattice $\mathscr{L} \sim (\epsilon, s, \chi_\zeta)$ we have

$$\mu(\mathscr{L}, f) = \epsilon^k \det(s)^{\frac{k}{2}} \phi_{s,\zeta}^*(f).$$

PROOF. Since $L_1 = uL_2$ for $u \in \nabla_n(\boldsymbol{Z})$ we have $f|L_1 = f|u|L_2 = f|L_2$ showing that $\mu(\mathscr{L}, f)$ is well-defined. For all $\tau \in \mathscr{H}_1$ we have

$$\mu(\mathscr{L}, f)(\tau) = [\phi_I^*(f|L)](\tau) = (f|L)(I\tau) = \left(f|\begin{pmatrix} M & \zeta M^* \\ 0 & M^* \end{pmatrix}\right)(I\tau)$$

$$= \det(M^*)^{-k} f(MI\tau M' + \zeta) = \det(M)^k f(s\tau + \zeta) = \epsilon^k \det(s)^{k/2}(\phi_{s,\zeta}^* f)(\tau)$$

where ϵ is the orientation of Λ. □

We now come to the Theorem previously stated in Section 2; it is the raison d'être of our present formalism.

THEOREM 2.6. *Let $\mathscr{L}$ be a rational polarized lattice. Let $f \in M_n^k$. Let $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{pr}$. We have $\mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$.*

PROOF. We have $\mu(\mathscr{L}, f)|\sigma = \phi_I^*(f|L)|\sigma$ for any $L \in \nabla_n(\boldsymbol{R})$ such that $\mathscr{L} = \boldsymbol{Z}^{2n}L$. By Lemma 5.3, $\phi_I^*(f|L)|\sigma = \phi_I^*(f|L\alpha_I(\sigma))$. The matrix $L\alpha_I(\sigma)$ is a basis for $\mathscr{L}\sigma$ but $L\alpha_I(\sigma)$ is not necessarily in $\nabla_n(\boldsymbol{R})$. We note that $L\alpha_I(\sigma)L^{-1} \in \mathrm{Sp}_n(\boldsymbol{R})^{\mathrm{pr}}$ and proceed by factoring $L\alpha_I(\sigma)L^{-1} = g\delta$ for $g \in \Gamma_n$ and $\delta \in \nabla_n(\boldsymbol{R})^{\mathrm{pr}}$. From $\mathscr{L}\sigma = \boldsymbol{Z}^{2n}L\alpha_I(\sigma) = \boldsymbol{Z}^{2n}\delta L$ we see that δL is a basis for $\mathscr{L}\sigma$ with $\delta L \in \nabla_n(\boldsymbol{R})$; therefore $\phi_I^*(f|L\alpha_I(\sigma)) = \phi_I^*(f|g\delta L) = \phi_I^*(f|\delta L) = \mu(\mathscr{L}\sigma, f)$. □

COROLLARY 8.2. *Let $\mathscr{L}$ be a rational polarized lattice. We have*

$$\mu(\mathscr{L}, \cdot) : M_n^k \to M_1^{nk}(\Gamma(\mathscr{L})),$$
$$\mu(\mathscr{L}, \cdot) : S_n^k \to S_1^{nk}(\Gamma(\mathscr{L})).$$

We wish to let $\mu(\mathscr{L}, \cdot)$ act on formal series and commute with the map to Fourier series. We imitate Proposition 8.1.

PROPOSITION 8.3. *Let $\mathscr{L}$ be an oriented rational polarized lattice. Let $F \in \boldsymbol{M}_n^k$. If we have $\mathscr{L} \sim (\epsilon_1, s_1, \chi_{\zeta_1}) \sim (\epsilon_2, s_2, \chi_{\zeta_2})$ then $\epsilon_1^k \det(s_1)^{k/2}\phi_{s_1,\zeta_1}^* F = \epsilon_2^k \det(s_2)^{k/2}\phi_{s_2,\zeta_2}^* F$. Define*

$$\mu(\mathscr{L}, F) = \epsilon^k \det(s)^{\frac{k}{2}} \phi_{s,\zeta}^* F \textit{ for any } (\epsilon, s, \chi_\zeta) \sim \mathscr{L}.$$

For all $\sigma \in \Gamma_1$, if $F \in \boldsymbol{S}_n^k$ then we have $\mu(\mathscr{L}\sigma, F) \in \boldsymbol{S}_1^{nk}(\Delta_1 \cap \Gamma(\mathscr{L})^\sigma)$.

PROOF. Let $U \in GL_n(\boldsymbol{Z})$ and $B \in V_n(\boldsymbol{Z})$ with $(\epsilon_2, s_2, \chi_{\zeta_2}) = (\det(U)\epsilon_1, Us_1U', \chi_{U(\zeta_1+B)U'})$. Since $\det(s_2) = \det(Us_1U') = \det(s_1)$ it suffices to show $\epsilon_1^k\phi_{s_1,\zeta_1}^* F = \epsilon_2^k\phi_{s_2,\zeta_2}^* F$. We have

$$\begin{aligned}
\epsilon_2^k\phi_{s_2,\zeta_2}^* F &= \epsilon_2^k \sum_j \left(\sum_{T:\langle s_2, T\rangle = j} \chi_{\zeta_2}(T)a(T) \right) \\
&= \epsilon_2^k \sum_j \left(\sum_{T:\langle Us_1U', T\rangle = j} \chi_{U(\zeta_1+B)U'}(T)a(T) \right) \\
&= \epsilon_2^k \sum_j \left(\sum_{T:\langle s_1, U'TU\rangle = j} \chi_{(\zeta_1+B)}(U'TU)a(T) \right) \\
&= \epsilon_2^k \sum_j \left(\sum_{R:\langle s_1, R\rangle = j} \chi_{\zeta_1}(R)a(U^*RU^{-1}) \right)
\end{aligned}$$

$$= \epsilon_2^k \sum_j \left(\sum_{R:\langle s_1, R\rangle = j} \chi_{\zeta_1}(R) \det(U^{-1})^k a(R) \right)$$

$$= \epsilon_1^k \sum_j \left(\sum_{R:\langle s_1, R\rangle = j} \chi_{\zeta_1}(R) a(R) \right) = \epsilon_1^k \phi^*_{s_1,\zeta_1} F.$$

For the final statement it suffices to show $\mu(\mathscr{L}, F) \in \boldsymbol{S}_1^{nk}(\Delta_1 \cap \Gamma(\mathscr{L}))$ because $\Gamma(\mathscr{L}\sigma) = \sigma^{-1}\Gamma(\mathscr{L})\sigma = \Gamma(\mathscr{L})^\sigma$. By Proposition 7.3 we see that for $\Delta_1 \cap \Gamma(\mathscr{L}) = \Delta_1(\ell)$ that $j \in \langle s, \mathscr{X}_n\rangle$ implies $\ell j \in \boldsymbol{Z}^+$ so that

$$\mu(\mathscr{L}, F) = \epsilon^k \det(s)^{k/2} \sum_{j \in \boldsymbol{Q}^+} \left(\sum_{T:\langle s, T\rangle = j} \chi_\zeta(T) a(T) \right) q_1^j \in \boldsymbol{S}_1^{nk}(\Delta_1(\ell)). \qquad \square$$

Lemma 8.4. *Let $\mathscr{L}$ be an oriented rational polarized lattice. Let $f \in M_n^k(\Delta_n)$. We have $FS_1(\mu(\mathscr{L}, f)) = \mu(\mathscr{L}, FS_n(f))$.*

Proof. Since $\mu(\mathscr{L}, f) = \epsilon^k \det(s)^{\frac{k}{2}} \phi^*_{s,\zeta} f$ and $\mu(\mathscr{L}, FS_n(f)) = \epsilon^k \det(s)^{\frac{k}{2}} \phi^*_{s,\zeta} FS_n(f)$, this is a corollary of Lemma 6.2. $\square$

Proposition 8.5. *Let $\mathscr{L}$ be an rational polarized lattice. Let $F \in \boldsymbol{M}_n^k$. Let $\delta \in \nabla_n(\boldsymbol{R})^{pr}$. We have*

$$\mu(\mathscr{L}, F)|\delta = \mu(\mathscr{L}\delta, F).$$

Proof. We first argue this for convergent F. In this case we have $F = FS_n(f)$ for some $f \in M_n^k(\Delta_n)$ by Lemma 3.3. Then $\mu(\mathscr{L}\delta, F) = FS_1(\mu(\mathscr{L}\delta, f))$ by Lemma 8.4. We have $FS_1(\mu(\mathscr{L}\delta, f)) = FS_1(\mu(\mathscr{L}, f)|\delta)$ by Theorem 2.6 and this is $FS_1(\mu(\mathscr{L}, f))|\delta$ by Lemma 4.4 with $n = 1$. Finally, by Lemma 8.4 again, this last term is $\mu(\mathscr{L}, F)|\delta$. But any counterexample would give disagreement on some index T_0 and the computation of any coefficient involves only a finite number of other coefficients. By omitting the uninvolved classes we would obtain a counterexample with support on a finite number of classes, hence a convergent counterexample. $\square$

9. Characterizations of Siegel modular q-expansions.

Theorem 9.1. *The following three conditions hold for any $F \in FS_n(M_n^k)$.*

1. *$F \in \boldsymbol{M}_n^k$ is a convergent series and the Fourier coefficients $a(T)$ are bound by polynomial growth in* $\mathrm{tr}(T)$.
2. *For all oriented rational polarized lattices $\mathscr{L}$, there exists a $\Psi_{\mathscr{L}} \in M_1^{nk}(\Gamma(\mathscr{L}))$ such that for all $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{pr}$ we have $FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$.*
3. *If $F \in FS_n(S_n^k)$ then the $\Psi_{\mathscr{L}}$ from item (2) are in $S_1^{nk}(\Gamma(\mathscr{L}))$ and there is a $B > 0$ such that for all $\mathscr{L}$ we have $y^{\frac{nk}{2}}|\Psi_{\mathscr{L}}(\tau)| \le B$ for all $\tau \in \mathscr{H}_1$.*

Proof. The item (1) is a well-known estimate for the growth of the Fourier coefficients of Siegel modular forms, see [**1**]. We show that item (2) is a consequence of Theorem 2.6. Let $F = FS_n(f)$ for $f \in M_n^k$ so that $\mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$. Define $\Psi_{\mathscr{L}} = \mu(\mathscr{L}, f)$ so that $\Psi_{\mathscr{L}} \in M_1^{nk}(\Gamma(\mathscr{L}))$ by Corollary 8.2. Thus we have $\Psi_{\mathscr{L}}|\sigma = \mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$ and

$$\begin{aligned} FS_1(\Psi_{\mathscr{L}}|\sigma) = FS_1(\mu(\mathscr{L}\sigma, f)) &= \mu(\mathscr{L}\sigma, FS_n(f)) \text{ by Lemma 8.4} \\ &= \mu(\mathscr{L}\sigma, F). \end{aligned}$$

For item (3) we additionally assume that $f \in S_n^k$ in the proof of item (2) so that $\Psi_{\mathscr{L}} \in S_1^{nk}(\Gamma(\mathscr{L}))$ by Corollary 8.2. Let B be the Hecke bound for the cusp form f so that $\det(Y)^{\frac{k}{2}}|f(\Omega)| \le B$ for all $\Omega \in \mathscr{H}_n$. Let $\mathscr{L} \sim (\epsilon, s, \chi_\zeta)$; then we have by Proposition 8.1

$$\begin{aligned} y^{\frac{nk}{2}}|\Psi_{\mathscr{L}}(\tau)| = y^{\frac{nk}{2}}|\mu(\mathscr{L}, f)(\tau)| &= \det(s)^{\frac{k}{2}} y^{\frac{nk}{2}} \left|(\phi_{s,\zeta}^* f)(\tau)\right| \\ &= \det(sy)^{\frac{k}{2}}|f(s\tau + \zeta)| \le B. \end{aligned}$$ □

Theorem 9.1 gives properties of Fourier expansions of Siegel modular forms. We can prove that judicious subsets of these suffice to characterize Fourier expansions of Siegel modular forms among all Koecher formal series. For a finite abelian group G, let $\exp(G) = \min\{\ell \in \boldsymbol{Z}^+ : \forall g \in G, \ell g = \mathrm{Id}_G\}$.

THEOREM 9.2. *Let $F \in \boldsymbol{M}_n^k$. We have $F \in FS_n(M_n^k)$ if and only if*

(1) *F is a convergent series and*
(2) *For all trivially polarized positively oriented lattices $\mathscr{L} = \Lambda \oplus \Lambda^*$, with Λ integral and $\ell = \exp(\Lambda^*/\Lambda)$,*

$$\exists\, \Psi_{\mathscr{L}} \in M_1^{nk}(\Gamma_0(\ell)) : \forall\, [\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1, FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F).$$

PROOF. The "only if" part follows from Theorem 9.1, so we only need to prove the "if" part. So assume items (1) and (2) hold and we will prove that $F \in FS_n(M_n^k)$. First we show that the restrictions that Λ be integral and that $[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1$ are simply normalizations and that the condition $\exists\, \Psi_{\mathscr{L}} \in M_1^{nk}(\Gamma(\infty)) : \forall\, \sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}$, $FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$, holds for all rational trivially polarized $\mathscr{L}$. First consider the case Λ is integral and $\sigma \in \mathrm{Sp}_1(\boldsymbol{R})^{\mathrm{pr}}$, and write $\sigma = \gamma\delta$ for $[\gamma] \in \Gamma_0(\ell)\backslash\Gamma_1$ and $\delta \in \nabla_1(\boldsymbol{R})^{\mathrm{pr}}$. We have:

$$\begin{aligned} FS_1(\Psi_{\mathscr{L}}|\sigma) = FS_1(\Psi_{\mathscr{L}}|\gamma\delta) &= (FS_1(\Psi_{\mathscr{L}}|\gamma))|\delta \\ &= \mu(\mathscr{L}\gamma, F)|\delta = \mu(\mathscr{L}\gamma\delta, F) = \mu(\mathscr{L}\sigma, F). \end{aligned}$$

The second equality follows from Lemma 4.4, and the fourth equality follows from Proposition 8.5. Now let Λ be rational and choose α so that $\alpha\Lambda$ is integral. In terms of $s = \mathrm{Gram}(\Lambda)$ we take $N \in \boldsymbol{Z}^+$ such that Ns is integral and set $\alpha = \sqrt{N}$. Define $\delta = u(\alpha) \in \nabla_1(\boldsymbol{R})^{\mathrm{pr}}$ and note that $\pi_1(\mathscr{L}\delta) = \alpha\Lambda$ is integral. Define $\Psi_{\mathscr{L}} = \Psi_{\mathscr{L}\delta}|\delta^{-1} \in M_1^{nk}(\delta\Gamma_0(\ell)\delta^{-1}) \subseteq M_1^{nk}(\Gamma_0(\ell N))$. We have

$$FS_1(\Psi_{\mathscr{L}}|\sigma) = FS_1(\Psi_{\mathscr{L}\delta}|\delta^{-1}|\sigma) = \mu((\mathscr{L}\delta)\delta^{-1}\sigma, F) = \mu(\mathscr{L}\sigma, F).$$

Now we prove $F \in FS_n(M_n^k)$. The assumption that F converges implies that $f(\Omega) = \sum_{T \in \mathscr{X}_n^{\text{semi}}} a(T)e(\langle T, \Omega \rangle)$ is in $M_n^k(\Delta_n)$ and has $FS_n(f) = F$, see Lemma 3.3. Since J_n and integral translations generate Γ_n and since f is periodic with respect to $V_n(\boldsymbol{Z})$ it suffices to show that $f|J_n = f$ to prove that $f \in M_n^k$. Because f and $f|J_n$ are holomorphic, the assertion $\forall\, \Omega \in \mathscr{H}_n$, $(f|J_n)(\Omega) = f(\Omega)$ would follow if it were true for all $\Omega = iY$, $Y \in \mathscr{P}_n(\boldsymbol{R})$. By continuity it suffices to prove this for all $Y \in \mathscr{P}_n(\boldsymbol{Q})$. So it suffices to prove it for $\Omega = s\tau$ for all $\tau \in \mathscr{H}_1$ and all $s \in \mathscr{P}_n(\boldsymbol{Z})$. Thus showing $\phi_s^*(f|J_n) = \phi_s^*(f)$ for all $s \in \mathscr{P}_n(\boldsymbol{Z})$ will complete the proof.

Take any $s \in \mathscr{P}_n(\boldsymbol{Z})$. Let Λ be a positively oriented lattice with Gram matrix s. Since $J_n = u(s^{-1})\alpha_s(J_1)$ we have $\phi_s^*(f|J_n) = \phi_s^*(f|u(s^{-1})\alpha_s(J_1)) = \phi_s^*(f|u(s^{-1}))|J_1$ by Lemma 5.3 and $\phi_s^*(f|u(s^{-1}))|J_1 = \det(s)^{-k}(\phi_{s^{-1}}^* f)|J_1$ by Lemma 5.5. Now accordingly Λ^* is a positively oriented lattice with Gram matrix s^{-1}. Proposition 8.1 gives us $\mu(\Lambda^* \oplus \Lambda, f) = \det(s^{-1})^{k/2}\phi_{s^{-1}}^* f$ so that we may conclude

$$\phi_s^*(f|J_n) = \det(s)^{-k/2}\mu(\Lambda^* \oplus \Lambda, f)|J_1.$$

So far we have not made essential use of the main hypothesis (2); we do now to show $\mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$ for all $\sigma \in \Gamma_1$. For $\sigma = I$ the equality $FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$ gives $FS_1(\Psi_{\mathscr{L}}) = \mu(\mathscr{L}, F) = \mu(\mathscr{L}, FS_n(f)) = FS_1(\mu(\mathscr{L}, f))$ by Lemma 8.4. Therefore $\Psi_{\mathscr{L}} = \mu(\mathscr{L}, f)$ since they are holomorphic functions with the same Fourier series. So $\mu(\mathscr{L}, f)|\sigma \in M_1^{nk}(\Gamma(\mathscr{L})^\sigma) \subseteq M_1^{nk}(\Gamma(\infty))$ has a formal series and it satisfies $FS_1(\mu(\mathscr{L}, f)|\sigma) = FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$ by the main hypothesis (2). We have

$$FS_1(\mu(\mathscr{L}, f)|\sigma) = \mu(\mathscr{L}\sigma, F) = \mu(\mathscr{L}\sigma, FS_n(f)) = FS_1(\mu(\mathscr{L}\sigma, f)),$$

where the last equality is from Lemma 8.4. So $\mu(\mathscr{L}, f)|\sigma = \mu(\mathscr{L}\sigma, f)$ as they are holomorphic functions with equal Fourier series.

We use Proposition 8.1 and $\mu(\mathscr{L}^*, f) = \mu(\mathscr{L}J_1, f) = \mu(\mathscr{L}, f)|J_1$ to deduce the following,

$$\begin{aligned}\phi_s^* f &= \det(s)^{-k/2}\mu((\Lambda \oplus \Lambda^*), f) \\ &= \det(s)^{-k/2}\mu((\Lambda^* \oplus \Lambda)J_1, f) = \det(s)^{-k/2}\mu(\Lambda^* \oplus \Lambda, f)|J_1.\end{aligned}$$

By comparing their expressions, we see that we have $\phi_s^*(f|J_n) = \phi_s^* f$. □

The question of whether Theorem 9.2 remains true if the convergence condition (1) is omitted will be reformulated in section 10. We now prove Theorem 2.7, restated here to include odd as well as even weights.

THEOREM 2.7. *Let $F \in \boldsymbol{S}_n^k$. We have $F \in FS_n(S_n^k)$ if and only if*

(1) *For all positively oriented rational polarized lattices $\mathscr{L}$, with Λ integral and $\Gamma(\mathscr{L}) \supseteq \Gamma_1(\ell)$,*

$$\exists\, \Psi_{\mathscr{L}} \in S_1^{nk}(\Gamma_1(\ell)) : \forall\, [\sigma] \in \Gamma_1(\ell)\backslash\Gamma_1, FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F).$$

(2) *There exists a* $B > 0$ *such that for all* $\mathscr{L}$ *as in item* (1), *we have* $y^{\frac{nk}{2}}|\Psi_{\mathscr{L}}(\tau)| \le B$ *for all* $\tau \in \mathscr{H}_1$.

Proof. We only need to prove the "if" part. So assume items (1) and (2) hold. As in the proof of Theorem 9.2, the restriction that Λ is integral is just a normalization and we have the same result for Λ rational,

$$\exists\, \Psi_{\mathscr{L}} \in S_1^{nk}(\Gamma(\infty)) : \forall\, \sigma \in \Gamma_1, FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F).$$

Pick any Λ_0 and let $\mathscr{L} \sim (+1, \Lambda_0, X) \sim (+1, s_0, \chi_\zeta)$ with a varying polarization χ_ζ. Let B be the uniform Hecke bound from item (2). The j^{th} Fourier coefficient of $\Psi_{\mathscr{L}}$ satisfies $|a(\Psi_{\mathscr{L}}; j)| \le (\frac{4\pi}{nke})^{nk/2} B j^{\frac{nk}{2}}$. Let $B_1 = (\frac{4\pi}{nke})^{nk/2} B$. From $FS_1(\Psi_{\mathscr{L}}) = \mu(\mathscr{L}, F)$ we have

$$a(\Psi_{\mathscr{L}}; j) = \det(s_0)^{\frac{k}{2}} \sum_{T \in \mathscr{X}_n : \langle s_0, T\rangle = j} a(T)\chi_\zeta(T).$$

Now pick any $T_0 \in \mathscr{X}_n$. We will show that $|a(T_0)| \le B_1 n^{\frac{nk}{2}} \det(T_0)^{\frac{k}{2}}$, proving that $a(T) \in O(\det(T)^{\frac{k}{2}})$ and that F is convergent. From Theorem 9.2 it then follows that $F \in FS_n(M_n^k)$. Since $F \in \boldsymbol{S}_n^k$ it follows that $F \in FS_n(S_n^k)$ by the commutativity of the Φ map from Lemma 3.2.

For $j = \langle s_0, T_0\rangle \in \boldsymbol{Z}^+$ we have

$$a(\Psi_{\mathscr{L}}; \langle s_0, T_0\rangle) = \det(s_0)^{\frac{k}{2}} \sum_{T \in \mathscr{X}_n : \langle s_0, T\rangle = \langle s_0, T_0\rangle} a(T)\chi_\zeta(T).$$

The set $\{T \in \mathscr{X}_n : \langle s_0, T\rangle = \langle s_0, T_0\rangle\}$ is finite and so there exists a $p \in \boldsymbol{Z}^+$ such that $\{T \in \mathscr{X}_n : \langle s_0, T\rangle = \langle s_0, T_0\rangle$ and $T \equiv T_0 \mod pV_n'(\boldsymbol{Z})\} = \{T_0\}$. Multiplying $a(\Psi_{\mathscr{L}}; \langle s_0, T_0\rangle)$ by $\chi_{-\zeta}(T_0)$ and averaging over $[\zeta] \in \frac{1}{p}V_n(\boldsymbol{Z})/V_n(\boldsymbol{Z})$ we obtain for $d = n(n+1)/2$,

$$\begin{aligned}
&\frac{1}{p^d} \sum_{[\zeta] \in \frac{1}{p}V_n(\boldsymbol{Z})/V_n(\boldsymbol{Z})} a(\Psi_{\mathscr{L}}; \langle s_0, T_0\rangle)\chi_{-\zeta}(T_0) \\
&= \frac{1}{p^d} \sum_{[\zeta] \in \frac{1}{p}V_n(\boldsymbol{Z})/V_n(\boldsymbol{Z})} \det(s_0)^{\frac{k}{2}} \sum_{T \in \mathscr{X}_n : \langle s_0, T\rangle = \langle s_0, T_0\rangle} a(T)\chi_\zeta(T - T_0) \\
&= \det(s_0)^{\frac{k}{2}} \sum_{T \in \mathscr{X}_n : \langle s_0, T\rangle = \langle s_0, T_0\rangle} a(T)\mathbf{1}_{pV_n'(\boldsymbol{Z})}(T - T_0) = \det(s_0)^{\frac{k}{2}} a(T_0).
\end{aligned}$$

An estimate of $a(T_0)$ is given by

$$\det(s_0)^{\frac{k}{2}}|a(T_0)| \le \frac{1}{p^d}\sum_\zeta \left|a(\Psi_{\mathscr{L}}; \langle s_0, T_0\rangle)\right| \le \frac{1}{p^d}\sum_\zeta B_1\langle s_0, T_0\rangle^{\frac{nk}{2}} = B_1\langle s_0, T_0\rangle^{\frac{nk}{2}}.$$

Pick $s_0 = T_0^{-1}$. Then we have $|a(T_0)| \le B_1 n^{\frac{nk}{2}} \det(T_0)^{\frac{k}{2}}$. $\square$

It is the convergence and uniform boundedness assumptions in our characterizations that would need to be removed in order to provide a purely linear characterization of the Fourier series of Siegel forms from among Koecher formal series. Even as it stands, the reduction of Siegel forms to elliptic forms given by Theorems 9.2 and 2.7 is encouraging as we are so much more the masters of functions of a single variable. We do not have any deep applications of these characterizations but do give a simple example.

Let $\mathscr{G}$ be an even unimodular rank $2k$ lattice with Gram matrix $Q \in \mathscr{P}_{2k}(\boldsymbol{Z})$. The usual Siegel theta series is

$$\vartheta_{\mathscr{G}}(\Omega) = \sum_{L \in \mathscr{G}^n} e\left(\frac{1}{2}\langle LL', \Omega\rangle\right) = \sum_{N \in \boldsymbol{Z}^{2k\times n}} e\left(\frac{1}{2}\langle N'QN, \Omega\rangle\right).$$

We will use Theorem 9.2 to show that $\vartheta_{\mathscr{G}}$ defines a Siegel form by reducing the question to elliptic theta series instead of directly addressing the multivariable transformation $\vartheta_{\mathscr{G}}|J_n$. Let $F = \sum_{T\in\mathscr{X}_n} \#\{N \in \boldsymbol{Z}^{2k\times n} : N'QN = 2T\} q_n^T \in \boldsymbol{M}_n^k$. The series converges because the growth of the coefficients is $O(\mathrm{tr}(T)^{nk})$. For all integral trivially polarized positively oriented lattices $\mathscr{L} = \Lambda \oplus \Lambda^*$ we will show that $\mu(\mathscr{L}, F) = \det(s)^{k/2} FS_1(\vartheta_{\mathscr{G}\otimes\Lambda})$ for any choice of Gram matrix s for Λ. We have, by Definition 6.1 and Proposition 8.1,

$$\begin{aligned}
\mu(\mathscr{L}, F) &= \det(s)^{k/2}\phi_s^* F \\
&= \det(s)^{k/2} \sum_{j\in\boldsymbol{Q}^+} \left(\sum_{T:\langle s,T\rangle = j} \#\big\{N \in \boldsymbol{Z}^{2k\times n} : N'QN = 2T\big\}\right) q_1^j \\
&= \det(s)^{k/2} \sum_{j\in\boldsymbol{Q}^+} \#\big\{N \in \boldsymbol{Z}^{2k\times n} : \langle s, N'QN\rangle = 2j\big\} q_1^j \\
&= \det(s)^{k/2} \sum_{j\in\boldsymbol{Q}^+} \#\big\{\vec{N} \in \boldsymbol{Z}^{2kn} : \vec{N}'(Q\otimes s)\vec{N} = 2j\big\} q_1^j \\
&= \det(s)^{k/2} FS_1(\vartheta_{\mathscr{G}\otimes\Lambda}).
\end{aligned}$$

If we let $\ell = \exp(\Lambda^*/\Lambda)$ then we may take $\psi_{\mathscr{L}} = \det(s)^{k/2}\vartheta_{\mathscr{G}\otimes\Lambda} \in M_1^{nk}(\Gamma_0(\ell))$ since $Q \otimes s$ has level ℓ and $\det(Q\otimes s) = \det(Q)^n \det(s)^{2k}$ is a square integer. To conclude that $F \in FS_n(M_n^k)$ it now suffices to show that the elliptic forms $\psi_{\mathscr{L}}$ transform correctly: $\forall\,[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1, FS_1(\Psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F)$. Thus our problem has been transferred to the elliptic level. The full transformation can be checked but an inspection of the proof of Theorem 9.2 reveals that it is enough to check the case $\sigma = J_1$. Since $\mathscr{L}\sigma = (\Lambda\oplus\Lambda^*)J_1 = \Lambda^* \oplus \Lambda$ and Λ^* has Gram matrix s^*, we need to check:

$$\begin{gathered}
FS_1(\psi_{\mathscr{L}}|\sigma) = \mu(\mathscr{L}\sigma, F), \text{ or} \\
FS_1(\det(s)^{k/2}\vartheta_{\mathscr{G}\otimes\Lambda}|J_1) = \mu(\Lambda^*\oplus\Lambda, F) = \det(s^*)^{k/2} FS_1(\vartheta_{\mathscr{G}\otimes\Lambda^*}), \text{ or} \\
\det(s)^{k/2}\vartheta_{\mathscr{G}\otimes\Lambda}|J_1 = \det(s^*)^{k/2}\vartheta_{\mathscr{G}\otimes\Lambda^*}.
\end{gathered}$$

This last equation follows from the inversion of the one variable theta series [**7**, pg. 14], $\vartheta_{\mathscr{G}}|J_1 = i^k \det(Q)^{-1/2}\vartheta_{\mathscr{G}^*}$, upon substitution of $\mathscr{G}\otimes\Lambda$ for $\mathscr{G}$, $Q\otimes s$ for Q and nk for k.

10. Computing vector subspaces by homomorphisms.

The restriction technique uses morphisms $\phi_{\mathscr{L}} : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (V_{\mathscr{L}}, S_{\mathscr{L}})$, indexed by rational polarized lattices $\mathscr{L}$. On the level of modular forms we define the map

$$\begin{aligned} \phi_{\mathscr{L}} : S_n^k &\to \prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} S_1^{nk}(\Gamma(\mathscr{L})^\sigma) \\ f &\mapsto (\mu(\mathscr{L}\sigma, f)) \end{aligned} \tag{10.1}$$

and if we define

$$\begin{aligned} \mathrm{Cusp}(\Gamma, K) : S_1^K(\Gamma) &\to \prod_{[\sigma]\in\Gamma\backslash\Gamma_1} S_1^K(\Gamma^\sigma) \\ f &\mapsto (f|\sigma) \end{aligned} \tag{10.2}$$

then $\phi_{\mathscr{L}}$ factors as $\phi_{\mathscr{L}}(f) = \mathrm{Cusp}(\Gamma(\mathscr{L}), nk)(\mu(\mathscr{L}, f))$ by Theorem 2.6. We attempt to lift the map $\phi_{\mathscr{L}}$ to the level of formal series. Use Proposition 8.3 to define

$$\begin{aligned} \phi_{\mathscr{L}} : \boldsymbol{S}_n^k &\to \prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} \boldsymbol{S}_1^{nk}(\Delta_1\cap\Gamma(\mathscr{L})^\sigma) \\ F &\mapsto (\mu(\mathscr{L}\sigma, F)) \end{aligned} \tag{10.3}$$

and obtain the commutative diagram:

$$\begin{array}{ccccc} \boldsymbol{S}_n^k & & \xrightarrow{\quad\phi_{\mathscr{L}}\quad} & & \prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} \boldsymbol{S}_1^{nk}(\Delta_1\cap\Gamma(\mathscr{L})^\sigma) \\ {\scriptstyle FS_n}\uparrow & & & & \uparrow{\scriptstyle \prod_\sigma FS_1} \\ S_n^k & \xrightarrow{\mu(\mathscr{L},\cdot)} & S_1^{nk}(\Gamma(\mathscr{L})) & \xrightarrow{\mathrm{Cusp}(\Gamma(\mathscr{L}),nk)} & \prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} S_1^{nk}(\Gamma(\mathscr{L})^\sigma) \end{array} \tag{10.4}$$

Thus we have constructed a morphism of relative vector spaces

$$\begin{aligned} &\phi_{\mathscr{L}} : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (V_{\mathscr{L}}, S_{\mathscr{L}}), \text{ with} \\ &V_{\mathscr{L}} = \prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} \boldsymbol{S}_1^k(\Delta_1\cap\Gamma(\mathscr{L})^\sigma) \text{ and} \\ &S_{\mathscr{L}} = \left(\prod_{[\sigma]\in\Gamma(\mathscr{L})\backslash\Gamma_1} FS_1\right)\big(\mathrm{Cusp}(\Gamma(\mathscr{L}), nk)(S_1^{nk}(\Gamma(\mathscr{L})))\big) \subseteq V_{\mathscr{L}}. \end{aligned} \tag{10.5}$$

It is the condition $(\mu(\mathscr{L}\sigma, F))_\sigma \in S_{\mathscr{L}}$ that imposes the nontrivial linear relations on the coefficients of $F = \sum a(T) q_n^T \in \boldsymbol{S}_n^k$ that hold when F is the formal series of a Siegel modular cusp form. In the computations performed to date we have only used polarized lattices of the form $\mathscr{L} = \Lambda \oplus \Lambda^*$ for integral Λ. For $\mathscr{C} = \mathscr{X}_n$, $\pi_{\mathscr{C}}$ is the identity map and the question of whether these homomorphisms $\phi_{\mathscr{L}}$ with codomain $(V_{\mathscr{L}}, S_{\mathscr{L}})$ carry enough information to characterize $FS_n(S_n^k)$ inside $\boldsymbol{S}_n^k$ amounts to the following "infinite" computation:

Open Problem 10.6. (Version A) Let $\mathrm{Id} : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (\boldsymbol{S}_n^k, FS_n(S_n^k))$ be given and let $\phi_{\mathscr{L}} : (\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (V_{\mathscr{L}}, S_{\mathscr{L}})$ be as in equation (10.5). Does the set of homomorphisms $\{\phi_{\mathscr{L}}\}$ with $\mathscr{L} = \Lambda \oplus \Lambda^*$ and Λ integral dominate Id?

11. Witt Homomorphisms.

There are other homomorphisms arising from symplectic embeddings one might consider besides the restriction homomorphisms $\phi_{s,\zeta}^*$. We augment our technique by including the Witt maps in our set of homomorphisms. The Witt maps are given by restriction to the reducible locus, see [**21**].

Proposition 11.1. *For $i + j = n$ there are homomorphisms $\psi_{ij}^* : S_n^k \to S_i^k \otimes S_j^k$ for $\psi_{ij} : \mathscr{H}_i \times \mathscr{H}_j \to \mathscr{H}_n$ defined by $\psi_{ij}(\Omega_1, \Omega_2) = \Omega_1 \oplus \Omega_2$. Furthermore, if $n = 2i$ we have $\psi_{ii}^* : S_n^k \to \mathrm{Sym}(S_i^k \otimes S_i^k)$.*

Definition 11.2.

$$S_n^{k,\mathrm{red}} = \bigcap_{i,j:\, i+j=n,\ i\geq 1,\ j\geq 1} \ker \psi_{ij}^*.$$

We extend the Witt map to a map on formal series so that it commutes with the map to Fourier series.

Definition 11.3. For $F = \sum_{T \in \mathscr{X}_n} a(T) q_n^T \in \boldsymbol{S}_n^k$ define $\psi_{ij}^* F \in \boldsymbol{S}_i^k \otimes \boldsymbol{S}_j^k$ by

$$\psi_{ij}^* F = \sum_{T_1 \in \mathscr{X}_i,\ T_2 \in \mathscr{X}_j} \left(\sum_{T \in \mathscr{X}_n:\, \pi_{i\times i}^{\mathrm{upper}}(T) = T_1,\, \pi_{j\times j}^{\mathrm{lower}}(T) = T_2} a(T) \right) q_i^{T_1} \otimes q_j^{T_2}.$$

Then the following is essentially due to Ozeki, compare [**14**].

Proposition 11.4. *For all $f \in S_n^k$ we have $\psi_{ij}^* FS_n(f) = (FS_i \otimes FS_j)\psi_{ij}^*(f)$*

We then have the relative morphism

$$\psi_{ij}^* : \big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) \to \big(\boldsymbol{S}_i^k \otimes \boldsymbol{S}_j^k, (FS_i \otimes FS_j)(S_i^k \otimes S_j^k)\big),$$

but instead we ease the computations by using

$$\psi_{ij}^* : \big(\boldsymbol{S}_n^k, FS_n\big(S_n^{k,\mathrm{red}}\big)\big) \to \big(\boldsymbol{S}_i^k \otimes \boldsymbol{S}_j^k, 0\big).$$

We then recover S_n^k from $S_n^{k,\mathrm{red}}$ by ad hoc arguments.

12. Finitization.

The objects of theoretical interest have been introduced but we still have not explained how to perform the actual computations. We use the following elementary Proposition.

PROPOSITION 12.1. *Let* $\pi : (V, S) \to (\pi V, \pi S)$ *and* $\phi : (V, S) \to (V_1, S_1)$ *be morphisms of relative vector spaces with* $\dim \pi S < +\infty$. *We have* $\dim \pi S \le \dim \pi\phi^{-1}(S_1)$. *If there is a subspace* $\tilde{S} \subseteq S$ *with* $\dim \pi\phi^{-1}(S_1) = \dim \pi\tilde{S}$ *then* ϕ *dominates* π *and* $\pi S = \pi\tilde{S}$.

PROOF. We have $\tilde{S} \subseteq S \subseteq \phi^{-1}(S_1)$ and $\pi\tilde{S} \subseteq \pi S \subseteq \pi\phi^{-1}(S_1)$. We also have $\dim \pi\tilde{S} \le \dim \pi S \le \dim \pi\phi^{-1}(S_1)$ and so $\dim \pi\phi^{-1}(S_1) = \dim \pi\tilde{S}$ implies $\dim \pi\tilde{S} = \dim \pi S = \dim \pi\phi^{-1}(S_1)$. These spaces are all of equal finite dimension so that $\pi\tilde{S} \subseteq \pi S \subseteq \pi\phi^{-1}(S_1)$ implies $\pi\tilde{S} = \pi S = \pi\phi^{-1}(S_1)$. □

Recall the discussion following Definition 2.8. We will take $\pi = \pi_{\mathscr{C}}$ for a $\mathscr{C}$ with $\pi_{\mathscr{C}}|_{FS_n(S_n^k)}$ injective. For example, let $\mathscr{C}$ represent each class in $\{T \in \mathscr{X}_n : w(T) \le \frac{2}{\sqrt{3}} n \frac{k}{4\pi}\}$, where $w : \mathscr{P}_n(\boldsymbol{R}) \to \boldsymbol{R}_{>0}$ is the dyadic trace, see [**17**]. The dyadic trace $w : \mathscr{P}_n(\boldsymbol{R}) \to \boldsymbol{R}_{>0}$ is defined by $w(s) = \inf_{Y>0} \langle s, Y\rangle / m(Y)$ where $m(Y) = \min_{v \in \boldsymbol{Z}^n \setminus \{0\}} v'Yv$ is the Minimum function. In order to prove that $\Phi_{\mathscr{A}}$ dominates $\pi_{\mathscr{C}}$ we must construct a subspace $\tilde{S}_n^k \subseteq S_n^k$ with $\dim \tilde{S}_n^k = \dim \pi_{\mathscr{C}} \Phi_{\mathscr{A}}^{-1}(S_{\mathscr{A}})$. Consequently we will have $\pi_{\mathscr{C}} FS_n(S_n^k) = \pi_{\mathscr{C}} FS_n(\tilde{S}_n^k)$ and for the $\mathscr{C}$ just given this would prove $FS_n(\tilde{S}_n^k) = FS_n(S_n^k)$ and hence $\tilde{S}_n^k = S_n^k$. Constructing modular forms is the enjoyable part, whereas showing that a given set of forms spans the entire space has always been the difficulty.

In computer computations we map from and into finite dimensional vector spaces. The issue is that the finitization must be done with care to ensure that the morphisms on truncations of Fourier series of Siegel cusp forms still have images that are truncations of elliptic forms. Describing actual computations is always a bit messy. It is accomplished as follows: Let $\mathscr{B} \supseteq \mathscr{C}$ be the net of Fourier coefficients that we are going to keep track of, $\mathscr{B}$ should be the union of a finite number of equivalence classes. For trivially polarized rational lattices $\mathscr{L} = \Lambda \oplus \Lambda^*$ with Λ integral and $\ell = \exp(\Lambda^*/\Lambda)$ we have $\Gamma(\mathscr{L}) \supseteq \Gamma_0(\ell)$. Also, for all $\sigma \in \Gamma_1$ we have $\Gamma_0(\ell)^\sigma \cap \nabla_1(\boldsymbol{Z}) = \Delta_1(\mathrm{w}_\ell(\sigma))$, where $\mathrm{w}_\ell(\sigma) = \ell/(\ell, c^2)$ for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ is the width of σ. We now make these substitutions into the general set-up of section 10. In general, nothing essential is changed if a subgroup of finite index in $\Gamma(\mathscr{L})$ is used in place of $\Gamma(\mathscr{L})$. We need to define some projections.

DEFINITION 12.2. Given a $j_1 \in \boldsymbol{Z}_{\ge 0}$ and $N \in \boldsymbol{Z}^+$, let $\pi(j_1, N)$ be the projection

$$\pi(j_1, N) : \prod_{j \in \frac{1}{N}\boldsymbol{Z}_{\ge 0}} \boldsymbol{C} \to \prod_{j \in \frac{1}{N}\boldsymbol{Z}_{\ge 0} :\, j \le j_1} \boldsymbol{C}.$$

Since $\boldsymbol{S}_1^k(\Delta_1(N)) \subseteq \prod_{j \in \frac{1}{N}\boldsymbol{Z}_{>0}} \boldsymbol{C}$, $\pi(j_1, N)$ is defined on $\boldsymbol{S}_1^k(\Delta_1(N))$. Let $(\Lambda \oplus$

$\Lambda^*)\sigma \sim (\Lambda\Box\sigma, X[\Lambda,\sigma]) \sim (s\Box\sigma, \zeta[s,\sigma])$ denote the action of Γ_1 on trivially polarized lattices in each of the three descriptions, where $s\Box\sigma$ is a Gram matrix of $\Lambda\Box\sigma = \pi_1((\Lambda \oplus \Lambda^*)\sigma)$, see section 7.

DEFINITION 12.3. For $s \in \mathscr{P}_n(\boldsymbol{Q})$, $\ell \in \boldsymbol{Z}^+$ and $\mathscr{B} \subseteq \mathscr{X}_n$ define

$$J(s,\ell;\mathscr{B}) = \sup\left\{ j \in \frac{1}{\ell}\boldsymbol{Z} : \{T \in \mathscr{X}_n : \langle s, T\rangle \le j\} \subseteq \mathscr{B} \right\}.$$

Let π^J be the projection on $V_{\mathscr{L}} = \prod_{[\sigma]\in\Gamma_0(\ell)\backslash\Gamma_1} \boldsymbol{S}_1^k(\Delta_1(\mathrm{w}_\ell(\sigma)))$ defined by

$$\pi^J : V_{\mathscr{L}} \to \prod_{[\sigma]\in\Gamma_0(\ell)\backslash\Gamma_1} \pi\big(J(s\Box\sigma, \mathrm{w}_\ell(\sigma);\mathscr{B}), \mathrm{w}_\ell(\sigma)\big)\big(\boldsymbol{S}_1^k(\Delta_1(\mathrm{w}_\ell(\sigma)))\big).$$

PROPOSITION 12.4. *We have the following commutative diagram,*

$$\begin{array}{ccc} \big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) & \xrightarrow{\ \phi_{\mathscr{L}}\ } & (V_{\mathscr{L}}, S_{\mathscr{L}}) \\ \Big\downarrow{\scriptstyle \pi_{\mathscr{B}}} & & \Big\downarrow{\scriptstyle \pi^J} \\ \pi_{\mathscr{B}}\big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) & \xrightarrow{\ \pi^J\circ\phi_{\mathscr{L}}\ } & \pi^J(V_{\mathscr{L}}, S_{\mathscr{L}}) \end{array}$$

so that the bottom row is a morphism of relative finite dimensional vector spaces when $\mathscr{B}$ represents finitely many classes.

PROOF. For $F = \sum a(T)q_n^T$ we have $\phi_{\mathscr{L}}F = \prod_{[\sigma]\in\Gamma_0(\ell)\backslash\Gamma_1} \mu(\mathscr{L}\sigma, F)$ and

$$\begin{aligned} \pi^J\phi_{\mathscr{L}} &= \prod_{[\sigma]\in\Gamma_0(\ell)\backslash\Gamma_1} \pi\big(J(s\Box\sigma, \mathrm{w}_\ell(\sigma);\mathscr{B}), \mathrm{w}_\ell(\sigma)\big)\mu(\mathscr{L}\sigma, F) \\ &= \prod_{[\sigma]\in\Gamma_0(\ell)\backslash\Gamma_1} \sum_{j=1/\mathrm{w}_\ell(\sigma)}^{J(s\Box\sigma,\mathrm{w}_\ell(\sigma);\mathscr{B})} \left(\sum_{T\in\mathscr{X}_n : \langle s\Box\sigma, T\rangle = j} \chi_{\zeta[s,\sigma]}(T)a(T) \right) q_1^j. \end{aligned}$$

The formula for $\pi^J \circ \phi_{\mathscr{L}} \circ \pi_{\mathscr{B}}$ will be the same except that the characteristic function $\mathbf{1}_{\mathscr{B}}(T)$ will multiply $a(T)$. For $j \le J(s\Box\sigma, \mathrm{w}_\ell(\sigma);\mathscr{B})$ we have $\{T \in \mathscr{X}_n : \langle s\Box\sigma, T\rangle \le j\} \subseteq \mathscr{B}$ by Definition 12.3 so that $\mathbf{1}_{\mathscr{B}}(T) = 1$ in this summand. □

OPEN PROBLEM 12.5. (Version B) For all $\mathscr{C} \subseteq \mathscr{X}_n$ representing a finite number of classes, does there exist a finite set of integral lattices, $\mathscr{A}$, and a $\mathscr{B} \subseteq \mathscr{X}_n$ representing a finite number of classes with $\mathscr{C} \subseteq \mathscr{B}$ such that for:

$$\pi_{\mathscr{C}} : \pi_{\mathscr{B}}\big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) \to \pi_{\mathscr{C}}\big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) \text{ and}$$
$$\pi^J\phi_{\Lambda\oplus\Lambda^*} : \pi_{\mathscr{B}}\big(\boldsymbol{S}_n^k, FS_n(S_n^k)\big) \to \pi^J(V_{\Lambda\oplus\Lambda^*}, S_{\Lambda\oplus\Lambda^*}),$$

the set of morphisms $\{\pi^J\phi_{\Lambda\oplus\Lambda^*}\}$ with $\Lambda \in \mathscr{A}$ dominates $\pi_{\mathscr{C}}$?

Version B of the Open Problem is equivalent to Version A (10.6) by general nonsense. It is Version B that we have some data to support. For some sets $\mathscr{C}$ we have been able to find finite sets $\mathscr{A}$ and $\mathscr{B}$ such that the set of homomorphisms $\{\pi^J \phi_{\Lambda\oplus\Lambda^*}\}$ with $\Lambda \in \mathscr{A}$ dominates $\pi_{\mathscr{C}}$. Since our determining sets of Fourier coefficients $\mathscr{C}$ are not optimal and since our sets of lattices $\mathscr{A}$ were found in no inspired manner, we have come to believe that this method of computing S_n^k will always work.

To finitize the Witt maps we use nets $\mathscr{B}_i \subseteq \mathscr{X}_i$ and $\mathscr{B}_j \subseteq \mathscr{X}_j$ to construct

$$(\pi_{\mathscr{B}_i} \otimes \pi_{\mathscr{B}_j})\, \psi_{ij}^* : \pi_{\mathscr{B}}\big(\boldsymbol{S}_n^k, FS_n(S_n^{k,\mathrm{red}})\big) \to \big(\pi_{\mathscr{B}_i}\boldsymbol{S}_i^k \otimes \pi_{\mathscr{B}_j}\boldsymbol{S}_j^k, 0\big)$$

where $\mathscr{B}_i$ and $\mathscr{B}_j$ satisfy $\{T \in \mathscr{X}_n : \pi_{i\times i}^{\mathrm{upper}}(T) \subseteq \mathscr{B}_i$ and $\pi_{j\times j}^{\mathrm{lower}}(T) \subseteq \mathscr{B}_j\} \subseteq \mathscr{B}$. In our examples we will have this condition for $\mathscr{B}_i = \pi_{i\times i}^{\mathrm{upper}}\{T \in \mathscr{B} : T = \pi_{i\times i}^{\mathrm{upper}}(T) \oplus \pi_{j\times j}^{\mathrm{lower}}(T)\}$ and for $\mathscr{B}_j = \pi_{j\times j}^{\mathrm{lower}}\{T \in \mathscr{B} : T = \pi_{i\times i}^{\mathrm{upper}}(T) \oplus \pi_{j\times j}^{\mathrm{lower}}(T)\}$. One reason for this is that we prefer to order our quadratic forms by the dyadic trace w and it is easy to check that $w(T) \le w(\pi_{i\times i}^{\mathrm{upper}}(T) \oplus \pi_{j\times j}^{\mathrm{lower}}(T))$. Note this notation is consistent because $\mathscr{B}$ is a set consisting of classes of forms. We view the Witt maps as a limited resource useful for accelerating the computation but not essential to it.

13. Examples.

The seventeen examples computed in this section prove some new results and reprove some old results for Siegel modular cusp forms of degree $n \ge 4$. The new results determine S_n^k for $(n,k) = (4,14)$, $(4,16)$, $(5,6)$, $(5,10)$ and $(6,8)$. For $n \ge 4$, the only known dimension results for level one and nonsingular even weight not yet reproven by our method are $\dim S_5^{12} = 2$ from [**12**] and $S_8^8 = \{0\}$, $S_n^6 = \{0\}$ for $7 \le n \le 12$ from [**3**]. In the examples we dominate the projection morphism of relative vector spaces $\pi_{\mathscr{C}} : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^k)) \to \pi_{\mathscr{C}}(\boldsymbol{S}_n^k, FS_n(S_n^k))$ by the indicated morphisms of the type $\pi^J\phi_{\Lambda\oplus\Lambda^*} : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^k)) \to \pi^J(V_{\Lambda\oplus\Lambda^*}, S_{\Lambda\oplus\Lambda^*})$. In some cases, we dominate the morphism $\pi_{\mathscr{C}} : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^{k,\mathrm{red}})) \to \pi_{\mathscr{C}}(\boldsymbol{S}_n^k, FS_n(S_n^{k,\mathrm{red}}))$ by a combination of morphisms of the type $\pi^J\phi_{\Lambda\oplus\Lambda^*} : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^{k,\mathrm{red}})) \to \pi^J(V_{\Lambda\oplus\Lambda^*}, S_{\Lambda\oplus\Lambda^*})$ and by the truncated Witt maps $(\pi_{\mathscr{B}_i} \otimes \pi_{\mathscr{B}_j})\,\psi_{ij}^* : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^{k,\mathrm{red}})) \to (\pi_{\mathscr{B}_i}\boldsymbol{S}_i^k \otimes \pi_{\mathscr{B}_j}\boldsymbol{S}_j^k, 0)$.

To prove that the set of morphisms chosen in each case dominates the projection $\pi_{\mathscr{C}}$ we rely on Proposition 12.1. Using the notation following Proposition 12.1 and Definition 2.8, our task is twofold: First we provide a set of morphisms in the hope that the product map $\Phi_{\mathscr{A}} : \pi_{\mathscr{B}}(\boldsymbol{S}_n^k, FS_n(S_n^k)) \to (V_{\mathscr{A}}, S_{\mathscr{A}})$ will dominate $\pi_{\mathscr{C}}$. Second, we provide a subspace of cusp forms $\tilde{S}_n^k$ in the hope that it is all of S_n^k. Since $\pi_{\mathscr{C}} \circ FS_n$ is injective on S_n^k we have $\dim \tilde{S}_n^k = \dim \pi_{\mathscr{C}} FS_n(\tilde{S}_n^k)$. If computations show that $\dim \tilde{S}_n^k = \dim \pi_{\mathscr{C}} \Phi_{\mathscr{A}}^{-1}(S_{\mathscr{A}})$ then $\Phi_{\mathscr{A}}$ dominates $\pi_{\mathscr{C}}$ by Proposition 12.1 and we may conclude that $\pi_{\mathscr{C}} FS_n(\tilde{S}_n^k) = \pi_{\mathscr{C}} FS_n(S_n^k)$ and $\tilde{S}_n^k = S_n^k$. This precise description may make the computations seem remote. We give Example 6 in greater detail and recommend it to the reader interested in performing similar computations.

We construct Siegel modular forms in various ways. If it exists, let $E_k \in M_n^k$ be the Eisenstein series normalized to have constant term 1. Let I denote the Ikeda lift, see [**9**], which creates a nontrivial Hecke eigenform in S_n^k from one in S_1^{2k-n} when n, k are even. Let $\Lambda \subseteq \boldsymbol{R}^N$ be an even unimodular lattice of rank m and let $Q : M_{n\times N}(\boldsymbol{C}) \to$

$\boldsymbol{C}$ be a pluri-harmonic polynomial [**7**, p. 161], of degree ν and define $\vartheta_{\Lambda,Q} : \mathscr{H}_n \to \boldsymbol{C}$ by

$$\vartheta_{\Lambda,Q}(\Omega) = \sum_{L \in \Lambda^n} Q(L) e\left(\frac{1}{2}\langle LL', \Omega\rangle\right).$$

The function $\vartheta_{\Lambda,Q}$ is then a Siegel modular cusp form of weight $\frac{m}{2} + \nu$ and degree n. Furthermore for $V, X \in M_{n\times N}(\boldsymbol{C})$, the polynomial $Q(X) = \det(VX')^\nu$ is pluri-harmonic whenever V satisfies $VM'(MM')^{-1}MV' = 0$ where M is a basis of row vectors for the lattice Λ. When $Q = 1$ we obtain the basic theta series $\vartheta_\Lambda = \vartheta_{\Lambda,1}$. For $\Omega \in \mathscr{H}_n$, $z \in \boldsymbol{C}^n$ and $a, b \in \boldsymbol{R}^n$ the theta function is defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \Omega) = \sum_{m \in \boldsymbol{Z}^n} e\left(\frac{1}{2}(m+a)'\Omega(m+a) + (m+a)'(z+b)\right).$$

In the examples that follow, we use the above methods to construct enough linearly independent Siegel modular forms to get a lower bound on $\dim S^k_n$. However, aside from the Ikeda lifts, it is computationally not feasible to compute directly from their definitions the Fourier coefficients $a(T)$ when the trace of T gets much bigger than that of a root lattice. But such Fourier coefficients are needed to compute the action of Hecke operators, as in our examples of S^{14}_4 and S^{16}_4. We should emphasize that our technique gives relations on the Fourier coefficients, and in the examples that follow it is by solving these relations that we compute enough Fourier coefficients to determine the action of the Hecke operators. The action of the Hecke operator T_p is normalized by $T_p f = \sum f|M_i$ for the finite double coset decomposition $\Gamma_n \left(\begin{smallmatrix} E_n & 0 \\ 0 & pE_n \end{smallmatrix}\right)\Gamma_n = \cup_i \Gamma_n M_i$; the definition of the slash operator is extended by $f|M = f|\big(\frac{1}{\sqrt[2n]{\det(M)}}M\big)$ if $\frac{1}{\sqrt[2n]{\det(M)}}M \in \mathrm{Sp}_n(\boldsymbol{R})$.

In the examples that follow, we always choose $\mathscr{C} = \{[T] : T \in \mathscr{X}_n, w(T) \le \frac{2}{\sqrt{3}} n \frac{k}{4\pi}\}$.

Example 1: $\dim S^2_4 = 0$. This is a result of Christian, as a special case of $\dim M^2_n = 0$. We can reprove this because $\mathscr{C} = \varnothing$ and so $\dim S^2_4 = 0$.

Example 2: $\dim S^4_4 = 0$. This is a result of M. Eichler, along with $M^4_4 = \boldsymbol{C}\vartheta_{E_8} = \boldsymbol{C}E_4$. W. Duke and Ö Imamoḡlu have shown $M^4_n = \boldsymbol{C}\vartheta_{E_8}$ for all n. We can reprove this because $\mathscr{C} = \varnothing$ and so $\dim S^4_4 = 0$.

Example 3: $\dim S^6_4 = 0$. This follows from the work of R. Salvati-Manni [**20**] who showed that in degree four cusp forms have minimal slope 8. Proofs are also given in [**16**] and [**3**]. W. Duke and Ö. Imamoḡlu show that $\dim S^6_n = 0$ for all n and that $M^6_n = \boldsymbol{C}E_6$ if $n < 9$ and that $M^6_n = \{0\}$ for $n \ge 9$. We can reprove $S^6_4 = \{0\}$ by choosing $\mathscr{B} = \mathscr{C}$, where $\mathscr{C} = \{T \in \mathscr{X}_4 : w(T) \le 2\} = \{\frac{1}{2}D_4\}$ and using the morphism $\pi^J \phi_{D_4 \oplus D_4^*}$ to dominate $\pi_{\mathscr{C}}$. Specifically, the morphism $\pi^J \phi_{D_4 \oplus D_4^*}$ implies that the $\frac{1}{2}D_4$ Fourier coefficient must be zero and hence $\dim S^6_4 = 0$.

Example 4: $\dim S^8_4 = 1$. This also follows from the work of R. Salvati-Manni [**20**] and was proven in [**16**] and [**3**]. We have $S^8_4 = \boldsymbol{C}J_8$ where J_8 is, up to a normalizing constant, the Schottky modular form studied by F. Schottky and J. I. Igusa [**8**]. Some

Fourier coefficients of J_8 are given in Table 2. We have

$$\begin{aligned}\vartheta_{E_8\oplus E_8}-\vartheta_{D_{16}^+}&=5160960\ J_8,\\ \vartheta_{E_8,Q}&=5529600\ J_8,\\ r_{00}^2+r_{0\frac12}^2+r_{\frac12 0}^2-2\big(r_{00}r_{0\frac12}+r_{00}r_{\frac12 0}+r_{0\frac12}r_{\frac12 0}\big)&=2^{16}\ J_8,\\ I(\Delta)&=-120\ J_8,\end{aligned}$$

where $r_{\mu\nu}=\prod_{\alpha,\beta,\gamma\in\{0,\frac12\}}^{8}\theta\left[\begin{smallmatrix}\mu&0&0&0\\ \nu&\alpha&\beta&\gamma\end{smallmatrix}\right](0,\Omega)$ is a theta null for $\mu,\nu\in\{0,\frac12\}$, and where $I(\Delta)$ is the Ikeda lift of $\Delta\in S_1^{12}$ and where $Q(X)=\det(U_1X')^4$ with $U_1=[I_4\ iI_4]$. The form J_8 vanishes on the Jacobian locus and $\psi_{13}^*J_8=0$ and $\psi_{22}^*J_8=0$. We have $\dim M_4^8=2$. We can reprove this result by using $\mathscr{B}=\mathscr{C}=\{T\in\mathscr{X}_4: w(T)\le 2.5\}=\{\frac12 D_4,\frac12 A_4\}$ and by showing that the morphism $\pi^J\phi_{D_4\oplus D_4^*}$ dominates $\pi_{\mathscr{C}}$. Specifically, the morphism $\pi^J\phi_{D_4\oplus D_4^*}$ implies one relation between the Fourier coefficients for $\{\frac12 D_4,\frac12 A_4\}$ and so $\dim S_4^8\le 1$. Then $\dim S_4^8=1$ follows from the existence of J_8.

Example 5: $\dim S_4^{10}=1$. This was proven in [**18**] by the earlier version of the method of this paper. In the current language, we take $\mathscr{B}=\mathscr{C}=\{T\in\mathscr{X}_4: w(T)\le 3.5\}=\{B_0,\dots,B_9\}$ (see Table 1) and show that $\pi_{\mathscr{C}}$ is dominated by the morphisms $\pi^J\phi_{\Lambda\oplus\Lambda^*}$ for $\Lambda\in\mathscr{A}=\left\{D_4,\left(\begin{smallmatrix}2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&4\end{smallmatrix}\right)\right\}$. Specifically, the two morphisms imply 9 linearly independent relations (see [**18**]) among the Fourier coefficients for $\mathscr{C}$, and so $\dim S_4^{10}\le 1$. Then $\dim S_4^{10}=1$ follows from the existence of G_{10}; we have $S_4^{10}=\boldsymbol{C}G_{10}$ where G_{10} may be given by any of the following:

$$\begin{aligned}\vartheta_{E_8,Q_1}&=-5529600\ G_{10},\quad Q_1(X)=\det(U_1X')^6\text{ where }U_1=[I_4\ iI_4],\\ \vartheta_{E_8\oplus E_8,Q_2}&=106168320\ G_{10},\quad Q_2(X)=\det(U_2X')^2\text{ where }U_2=[I_4\ 0\ iI_4\ 0],\\ \sum_F^{13\,056}\prod_{\zeta\in F}^{10}\theta[\zeta]^2&=2^{25}\cdot 3\ G_{10},\quad\text{the sum is over even fundamental systems }F\text{, see [17]},\\ I(E_4\Delta)&=-168\ G_{10}.\end{aligned}$$

We have $\psi_{13}^*G_{10}=0$ and $\psi_{22}^*G_{10}=720\,X_{10}\otimes X_{10}$. We have $\dim M_4^{10}=3$.

Example 6: $\dim S_4^{12}=2$. This follows from the work of Erokhin [**5**], [**6**] on theta series and was proven in [**16**]. We have $\mathscr{B}=\mathscr{C}=\{T\in\mathscr{X}_4: w(T)\le 4\}=\{B_0,\dots,B_{22}\}$ (see Table 1) and the morphisms $(\pi_{\mathscr{B}_1}\otimes\pi_{\mathscr{B}_3})\psi_{13}^*$, $(\pi_{\mathscr{B}_2}\otimes\pi_{\mathscr{B}_2})\psi_{22}^*$ and the $\pi^J\phi_{\Lambda\oplus\Lambda^*}$ for $\Lambda\in\mathscr{A}=\{$ 5 forms $\}$ dominate $\pi_{\mathscr{C}}$. The $\boldsymbol{C}$-vector space $\pi_{\mathscr{B}}\boldsymbol{S}_4^{12}$ is 23 dimensional. We elaborate more on the details of this example to illustrate the technique. Table 3 gives a listing of the 5 forms in $\mathscr{A}$ and the number of relations on Fourier coefficients resulting from each corresponding morphism; the relations themselves are listed in Table 4. Table 5 gives a listing of the relations on Fourier coefficients implied by the Witt homomorphisms $(\pi_{\mathscr{B}_1}\otimes\pi_{\mathscr{B}_3})\psi_{13}^*$ and $(\pi_{\mathscr{B}_2}\otimes\pi_{\mathscr{B}_2})\psi_{22}^*$; these relations are satisfied by forms in $S_4^{12,\mathrm{red}}$.

We now compute some of the entries in Table 5. For $F = \sum_{T \in \mathscr{X}_4} a(T) q_4^T \in S_4^{12}$, we have, by Definition 11.3,

$$\psi_{13}^* F = \sum_{T_1 \in \mathscr{X}_1, T_3 \in \mathscr{X}_3} \left(\sum_{v \in (\frac{1}{2}\boldsymbol{Z})^3} a \begin{pmatrix} T_1 & v \\ v' & T_2 \end{pmatrix} \right) q_1^{T_1} \otimes q_3^{T_3}.$$

We finitize the Witt map with $\mathscr{B}_1 = \{1\}$ and $\mathscr{B}_3 = \{\frac{1}{2}A_3, \frac{1}{2}(A_1 \oplus A_2), I_3\}$, noting that any $T \in \mathscr{X}_4$ with $\pi_{1\times 1}^{\text{upper}}(T) = 1$ and $\pi_{3\times 3}^{\text{lower}}(T) \in \mathscr{B}_3$ has dyadic trace less than or equal to 4 and hence has a class representative in $\mathscr{B}$. We compute, letting $a_i = a(B_i)$,

$$\begin{aligned}(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\, \psi_{13}^* F = {}& (6a_0 + 8a_1 + a_2) q_1^1 \otimes q_3^{\frac{1}{2}A_3} \\ & + (12a_1 + 6a_2 + 2a_3 + a_5) q_1^1 \otimes q_3^{\frac{1}{2}(A_1 \oplus A_2)} \\ & + (8a_0 + 12a_2 + 6a_5 + a_{10}) q_1^1 \otimes q_3^{I_3}. \qquad (13.1)\end{aligned}$$

For example, to obtain the leading coefficient $6a_0 + 8a_1 + a_2$, classify the $(a, b, c) \in \{-1, 0, 1\}^3$ for which $\begin{pmatrix} 2 & a & b & c \\ a & 2 & 1 & 1 \\ b & 1 & 2 & 0 \\ c & 1 & 0 & 2 \end{pmatrix}$ is even: 3 pairs $\pm(1, 1, 1)$, $\pm(1, 0, 0)$ and $\pm(0, 1, -1)$ give the class of D_4; 4 pairs $\pm(1, 1, 0)$, $\pm(1, 0, 1)$, $\pm(0, 1, 0)$ and $\pm(0, 0, 1)$ give the class of A_4; only $(0, 0, 0)$ gives the class of $A_1 \oplus A_3$. The rest give nondefinite forms so that

$$\sum_{(a,b,c) \in \{-1,0,1\}^3} a\left(\frac{1}{2} \begin{pmatrix} 2 & a & b & c \\ a & 2 & 1 & 1 \\ b & 1 & 2 & 0 \\ c & 1 & 0 & 2 \end{pmatrix} \right) = 6a(B_0) + 8a(B_1) + a(B_2) = 6a_0 + 8a_1 + a_2.$$

For the morphism $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\, \psi_{13}^* : \pi_{\mathscr{B}}(\boldsymbol{S}_4^{12}, FS_4(S_4^{12,\text{red}})) \to (\pi_{\mathscr{B}_1} \boldsymbol{S}_1^{12} \otimes \pi_{\mathscr{B}_3} \boldsymbol{S}_3^{12}, 0)$, when we stipulate that $F \in ((\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\, \psi_{13}^*)^{-1}(0)$ we are simply requiring that all the Fourier coefficients of $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\, \psi_{13}^* F$ be zero. The three coefficients from Equation (13.1) give the first, third, and fourth relations in Table 5. The other relations in Table 5 come from setting the coefficients of $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_2}) \psi_{22}^*$ to zero. These relations hold for the Fourier coefficients of cusp forms in $S_4^{12,\text{red}}$.

The relations in Table 4 arising from the restriction technique require more labor. Consider the first entry in Table 3, $\mathscr{L} = D_4 \oplus D_4^*$, where $D_4 = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ has level $\ell = 2$ and $\Gamma_0(2) \subseteq \Gamma(\mathscr{L})$. Using the definition of $\phi_{\mathscr{L}}$ in (10.3) we have

$$\phi_{\mathscr{L}} F = (\mu(\mathscr{L}, F), \mu(\mathscr{L} J_1, F)) \in \boldsymbol{S}_1^{48}(\Delta_1 \cap \Gamma_0(2)) \times \boldsymbol{S}_1^{48}(\Delta_1 \cap \Gamma_0(2)^{J_1}).$$

From Proposition 8.3 and Definition 6.1 we have

$$\mu(\mathscr{L}, F) = \det(D_4)^6 \phi_{D_4}^* F = \det(D_4)^6 \sum_{j \in \boldsymbol{Q}^+} \left(\sum_{T : \langle D_4, T \rangle = j} a(T) \right) q_1^j.$$

To finitize this map recall that the cusps have widths $\mathrm{w}_2(e) = 1$ and $\mathrm{w}_2(J_1) = 2$ so that

$J(D_4 \square e, \mathrm{w}_2(e); \mathscr{B}) = J(D_4, 1; \mathscr{B}) = 8$ and $J(D_4 \square J_1, \mathrm{w}_2(J_1); \mathscr{B}) = J(D_4^*, 2; \mathscr{B}) = 4$. The truncated map therefore has image $\pi^J \phi_{\mathscr{L}} F = (\pi(8,1)\mu(\mathscr{L}, F), \pi(4,2)\mu(\mathscr{L} J_1, F))$ by Definitions 12.2 and 12.3, where

$$\begin{aligned}
&\pi(8,1)\mu(\mathscr{L}, F) \\
&\quad = 4^6 \big(a_0 q_1^4 + (16a_0 + 48a_1) q_1^5 + (144a_0 + 288a_1 + 216a_2 + 48a_3 + 12a_4) q_1^6 \\
&\qquad\quad + (384a_0 + 1488a_1 + 864a_2 + 288a_3 + 144a_4 \\
&\qquad\qquad + 432a_5 + 240a_6 + 288a_7 + 48a_8 + 16a_9) q_1^7 \\
&\qquad\quad + (\text{an integral linear combination stopping at } a_{22}) q_1^8 \big),
\end{aligned}$$

$$\begin{aligned}
&\pi(4,2)\mu(\mathscr{L} J_1, F) = \pi(4,2)\mu(\mathscr{L}^*, F) \\
&\quad = \left(\frac{1}{4}\right)^6 \big(a_0 q_1^2 + (16a_0 + 48a_1) q_1^{5/2} + (144a_0 + 288a_1 + 216a_2 + 48a_3 + 12a_4) q_1^3 \\
&\qquad\quad + (384a_0 + 1488a_1 + 864a_2 + 288a_3 + 144a_4 \\
&\qquad\qquad + 432a_5 + 240a_6 + 288a_7 + 48a_8 + 16a_9) q_1^{7/2} \\
&\qquad\quad + (\text{the same integral linear combination stopping at } a_{22}) q_1^4 \big).
\end{aligned}$$

In this example slashing with $|_{48} \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ changes the first formal series to the second; this is because $D_4^* \sim \frac{1}{2} D_4$. We computed these coefficients by enumerating all possibilities. For example, to find the q_1^5-coefficient $(16a(B_0) + 48a(B_1))$ in the first series we enumerated all $T \in \mathscr{X}_4$ with $\langle D_4, T \rangle = 5$ and found 16 in the class of $\frac{1}{2} D_4 = B_0$ and 48 in the class of $\frac{1}{2} A_4 = B_1$. For the morphism $\pi^J \phi_{D_4 \oplus D_4^*} : \pi_{\mathscr{B}}(\boldsymbol{S}_4^{12}, FS_4(S_4^{12,\mathrm{red}})) \to \pi^J(V_{D_4 \oplus D_4^*}, S_{D_4 \oplus D_4^*})$, when we stipulate that $F \in (\pi^J \phi_{D_4 \oplus D_4^*})^{-1}(S_{D_4 \oplus D_4^*})$ we are simply requiring, from the definition of $S_{D_4 \oplus D_4^*}$ in (10.5), that there exists a $\psi_{\mathscr{L}} \in S_1^{48}(\Gamma_0(2))$ such that

$$\begin{aligned}
\pi^J \phi_{\mathscr{L}} F &= \big(\pi(8,1)\mu(\mathscr{L}, F), \pi(4,2)\mu(\mathscr{L} J_1, F) \big) \\
&= \big((\pi(8,1) FS_1(\psi_{\mathscr{L}}), \pi(4,2) FS_1(\psi_{\mathscr{L}} | J_1) \big).
\end{aligned}$$

The ring $M_1(\Gamma_0(2))$ is generated by $E_{2,2}^- \in M_1^2(\Gamma_0(2))$ and $E_{4,2}^- \in M_1^4(\Gamma_0(2))$ and the ring of cusp forms is principally generated by $C_{8,2}^+ \in S_1^8(\Gamma_0(2))$. The $\pm$ superscript indicates an eigenvalue of ± 1 under the Fricke operator, F_2. The Fourier expansions of these generators are given by

$$E_{2,2}^-(\tau) = 1 + 24 \sum_{n=1}^{\infty} (\sigma_1(n) - 2\sigma_1(n/2)) q^n = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \cdots$$

$$E_{4,2}^-(\tau) = 1 - 80 \sum_{n=1}^{\infty} (\sigma_3(n) - 4\sigma_3(n/2)) q^n = 1 - 80q - 400q^2 - 2240q^3 - 2960q^4 - \cdots$$

$$C_{8,2}^+(z) = \frac{1}{256} \big(E_{2,2}^-(\tau)^4 - E_{4,2}^-(\tau)^2 \big) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 - \cdots$$

Write the general element of weight 48 as $\psi_{\mathscr{L}} = \sum_{i=0}^{12} \alpha_i (E^-_{2,2})^{24-2i}(E^-_{4,2})^i$. The action of the $J_1 = F_2\left(\begin{smallmatrix} \frac12 & 0 \\ 0 & 1\end{smallmatrix}\right)$ operator is a change of sign of the α_i with odd i followed by $|_{48}\left(\begin{smallmatrix} \frac12 & 0 \\ 0 & 1\end{smallmatrix}\right)$; thus $\mu(\mathscr{L}, F) = FS_1(\psi_{\mathscr{L}})$ and $\mu(\mathscr{L}J_1, F) = FS_1(\psi_{\mathscr{L}}|J_1)$ imply that $\alpha_i = 0$ for odd i. The same conclusion follows from the projected equalities $\pi(8,1)\mu(\mathscr{L}, F) = \pi(8,1)FS_1(\psi_{\mathscr{L}})$ and $\pi(4,2)\mu(\mathscr{L}J_1, F) = \pi(4,2)FS_1(\psi_{\mathscr{L}}|J_1)$ because no weight 48 cusp form in the Fricke minus-space vanishes to order 9. Thus we may rewrite $\psi_{\mathscr{L}} = \sum_{i=0}^{6} \beta_i (E^-_{2,2})^{24-4i}(C^+_{8,2})^i$. We have $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$ since the leading term in $\mu(\mathscr{L}, F)$ is q_1^4. Thus we have deduced that $\psi_{\mathscr{L}}$ may be written

$$\begin{aligned}\psi_{\mathscr{L}} &= 4^6\big(\alpha(E^-_{2,2})^8(C^+_{8,2})^4 + \beta(E^-_{2,2})^4(C^+_{8,2})^5 + \gamma(C^+_{8,2})^6\big)\\ &= 4^6\big(\alpha q^4 + (160\alpha+\beta)q^5 + (10608\alpha + 56\beta + \gamma)q^6\\ &\qquad + (364928\alpha + 412\beta - 48\gamma)q^7 + (6354904\alpha - 19008\beta + 1032\gamma)q^8 + \cdots\big).\end{aligned}$$

Using $\pi(8,1)\mu(\mathscr{L}, F) = \pi(8,1)FS_1(\psi_{\mathscr{L}})$ we must have $\alpha = a_0$, $160\alpha + \beta = 16a_0 + 48a_1$, $10608\alpha + 56\beta + \gamma = 144a_0 + 288a_1 + 216a_2 + 48a_3 + 12a_4$ and $364928\alpha + 412\beta - 48\gamma = 384a_0 + 1488a_1 + 864a_2 + 288a_3 + 144a_4 + 432a_5 + 240a_6 + 288a_7 + 48a_8 + 16a_9$. We solve the first three equations for $\alpha = a_0$, $\beta = 48a_1 - 144a_0$, $\gamma = -2400a_0 - 2400a_1 + 216a_2 + 48a_3 + 12a_4$ and obtain relations by substituting α, β and γ into the fourth equation, which becomes: $16(-26276a_0 - 8343a_1 + 702a_2 + 162a_3 + 45a_4 + 27a_5 + 15a_6 + 18a_7 + 3a_8 + a_9) = 0$, the second relation in the first box of Table 4. The relations in Table 4 are satisfied by the Fourier coefficients of all cusp forms in S_4^{12}.

The 15 linearly independent relations from Table 4 and the 8 linearly independent relations from Table 5 together yield 22 linearly independent relations implied on the Fourier coefficients of cusp forms from $S_4^{12,\mathrm{red}}$. Since $|\mathscr{C}| = 23$, this implies $\dim S_4^{12,\mathrm{red}} \le 1$. Note that this is reassuring because any generic errors in the calculations would have caused the $15+8$ relations to be totally linearly independent. Using $\tilde{S}_4^{12,\mathrm{red}} = \boldsymbol{C}E_4J_8 \subseteq S_4^{12,\mathrm{red}}$ we conclude that $\dim S_4^{12,\mathrm{red}} = 1$. The equality $\dim S_4^{12} = 1 + \dim S_4^{12,\mathrm{red}}$ follows from the next Lemma.

Lemma 13.2. *The following sequence is exact*:

$$\begin{aligned}0 \to S_4^{12,red} \to S_4^{12} &\xrightarrow{\psi^*_{13}\oplus\psi^*_{22}} \big(S_1^{12}\otimes S_3^{12}\big) \oplus \mathrm{Sym}\,\big(S_2^{12}\otimes S_2^{12}\big)\\ &\xrightarrow{I\otimes\psi^*_{12} - \psi^*_{11}\otimes I} S_1^{12}\otimes S_1^{12}\otimes S_2^{12} \to 0.\end{aligned}$$

Proof. By Definition 11.2 we know $S_4^{12,\mathrm{red}} = \ker\psi^*_{13} \cap \ker\psi^*_{22} = \ker(\psi^*_{13}\oplus\psi^*_{22})$. Using Proposition 11.1 we see that $\mathrm{Im}(\psi^*_{13}\oplus\psi^*_{22}) \subseteq \ker(I_1\otimes\psi^*_{12} - \psi^*_{11}\otimes I_2)$ because

$$\big((I_1\otimes\psi^*_{12})\psi^*_{13}f\big)(\tau_1,\tau_2,\Omega_2) = f(\tau_1\oplus\tau_2\oplus\Omega_2) = \big((\psi^*_{11}\otimes I_2)\psi^*_{22}f\big)(\tau_1,\tau_2,\Omega_2).$$

We have $S_1^{12} = \boldsymbol{C}\Delta$ and $S_2^{12} = \boldsymbol{C}\Phi_{12}$ where $\Psi^*_{11}\Phi_{12} = \Delta\otimes\Delta$, see [**16**, pp. 311–312]. Thus $S_1^{12}\otimes S_1^{12}\otimes S_2^{12}$ is spanned by $\Delta\otimes\Delta\otimes\Phi_{12}$ and this element is the image of $(0, -\Phi_{12}\otimes\Phi_{12})$ under $I_1\otimes\psi^*_{12} - \psi^*_{11}\otimes I_2$. There is an element $\beta \in S_4^{12}$ constructed from

ϑ-series (called ψ_{12} in [**16**]) such that $\Psi^*_{22}\beta = (\text{unit})\Phi_{12} \otimes \Phi_{12}$ and so $\operatorname{Im}(\psi^*_{13} \oplus \psi^*_{22})$ is at least one dimensional. Finally, $\dim S_3^{12} = 1$ so that $S_1^{12} \otimes S_3^{12} \oplus \operatorname{Sym}(S_2^{12} \otimes S_2^{12})$ is two dimensional. These containments and dimensions show that the sequence is exact. □

We have $S_4^{12,\mathrm{red}} = \boldsymbol{C} E_4 J_8$ and S_4^{12} is spanned by $I(E_4^2\Delta)$ and $E_4 J_8$. G. Nebe and B. Venkov [**12**] have computed the Hecke eigenforms in this space and using the Fourier coefficients provided by T. Ikeda [**10**] we have eigenforms

$$f_5 = \frac{\vartheta(d_5)}{-76569927069081600} = \frac{1}{360} I(E_4^2\Delta)$$

$$f_6 = \frac{\vartheta(d_6)}{3605023870156800} = \frac{1}{240} I(E_4^2\Delta) + \frac{13}{2} E_4 J_8.$$

Here d_5 and d_6 are certain linear sums of Niemeier lattices, and $\vartheta(d_5), \vartheta(d_6)$ are the corresponding linear sums of theta series, as given by Nebe and Venkov [**12**]. Initial Fourier coefficients of f_5, f_6 are shown in Table 2. We have $\dim M_4^{12} = 6$. The Fourier coefficients for $\{\frac{1}{2}D_4, \frac{1}{2}A_4\}$ determine S_4^{12} and those for $\{\frac{1}{2}D_4, \frac{1}{2}A_4, \frac{1}{2}A_3, \frac{1}{2}A_2, I_1, 0\}$ determine M_4^{12}.

Example 7: $\dim S_4^{14} = 3$ (new result). We have $\mathscr{C} = \{T \in \mathscr{X}_4 : w(T) \leq 5\} = \{B_0, \ldots, B_{84}\}$, and we take $\mathscr{B} = \{T \in \mathscr{X}_4 : w(T) \leq 6\} \cup \left\{ \left(\begin{smallmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 10 \end{smallmatrix}\right) \right\} = \{B_0, \ldots, B_{275}\}$, and the morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\psi^*_{13}$, $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_2})\psi^*_{22}$ and the $\pi^J \phi_{\Lambda \oplus \Lambda^*}$ for $\Lambda \in \mathscr{A} = \{$ 23 forms$\}$ dominate $\pi_{\mathscr{C}}$. See the authors' website [**15**] for a precise listing of $\mathscr{C}$, $\mathscr{B}$ and $\mathscr{A}$ and for further information on the restriction method using forms from $\mathscr{A}$. Specifically, it turns out the morphisms $\pi^J \phi_{\Lambda \oplus \Lambda^*}$ for $\Lambda \in \mathscr{A}$ imply 145 relations on the Fourier coefficients for $\mathscr{B}$, and the morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\psi^*_{13}$ and $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_2})\psi^*_{22}$ imply 69 relations on the Fourier coefficients for $\mathscr{B}$ for cusp forms in $S_4^{14,\mathrm{red}}$. Together, it turns out these $145 + 69 = 214$ relations give 209 linearly independent relations. Eliminating the terms involving $\mathscr{B} \setminus \mathscr{C}$, we get 83 linearly independent relations on the Fourier coefficients for $\mathscr{C}$. This implies $\dim S_4^{14,\mathrm{red}} \leq 2$.

The equality $\dim S_4^{14} = 1 + \dim S_4^{14,\mathrm{red}}$ follows from the next Lemma. Then $\dim S_4^{14} \leq 3$. Since we can construct 3 linearly independent cusp forms in S_4^{14}, then $\dim S_4^{14} = 3$, and hence also $\dim S_4^{14,\mathrm{red}} = 2$.

Lemma 13.3. *The following sequence is exact*:

$$0 \to S_4^{14,red} \to S_4^{14} \xrightarrow{\psi^*_{22}} \operatorname{Sym}\left(S_2^{14} \otimes S_2^{14}\right) \to 0.$$

Proof. We refer to the proof of Lemma 13.2 for generalities. Since $S_1^{14} = \{0\}$ we have $S_1^{14} \otimes S_3^{14} = \{0\}$ and $S_4^{14,\mathrm{red}} = \ker \psi^*_{22}$. Since S_2^{14} is spanned by $X_{10}E_4$ we see that $\operatorname{Sym}(S_2^{14} \otimes S_2^{14})$ is spanned by $X_{10}E_4 \otimes X_{10}E_4$. We conclude the sequence is exact by noting that $\psi^*_{22}(G_{10}E_4) = 720\, X_{10}E_4 \otimes X_{10}E_4$. □

Let $\alpha = \sqrt{144169}$. We have $\dim S_4^{14,\mathrm{red}} = 2$ and $I(E_4^3\Delta + (540 + 12\alpha)\Delta^2) - I(E_4^3\Delta + (540 - 12\alpha)\Delta^2)$ and $E_6 J_8$ span $S_4^{14,\mathrm{red}}$; S_4^{14} is spanned by $I(E_4^3\Delta + (540 \pm 12\alpha)\Delta^2)$ and $E_6 J_8$. Enough Fourier coefficients to compute the action of the Hecke operator T_2 can be found at the authors' website [**15**], and the three Hecke eigenforms are:

$$\begin{aligned}
12f_7 &= I\big(E_4^3\Delta + (540 + 12\alpha)\Delta^2\big)\\
12f_8 &= I\big(E_4^3\Delta - (540 - 12\alpha)\Delta^2\big)\\
\frac{21}{85}f_9 &= \left(\frac{144169 + 37\alpha}{2941047600}\right) I\big(E_4^3\Delta{+}(540 + 12\alpha)\Delta^2\big)\\
&\quad + \left(\frac{144169 - 37\alpha}{2941047600}\right) I\big(E_4^3\Delta{+}(540 - 12\alpha)\Delta^2\big) + E_6 J_8.
\end{aligned}$$

The Fourier coefficients for $\{\frac{1}{2}D_4, \frac{1}{2}A_4, \frac{1}{2}(A_1 \oplus A_3)\}$ determine S_4^{14}. Initial Fourier expansions of these are provided in Table 2. We have

$$\begin{aligned}
E_4G_{10} &= \frac{3\,(-144169 + 303\alpha)}{539192060} f_7 + \frac{3\,(-144169 - 303\alpha)}{539192060} f_8 - \frac{2}{17} f_9\\
\vartheta_{E_8,Q_3} &= 5529600\left(\frac{144169 + 93\alpha}{6343436} f_7 + \frac{144169 - 93\alpha}{6343436} f_8\right)\\
\vartheta_{6D_4,Q_4} &= \frac{768(-4469239 + 5653\alpha)}{673990075} f_7 + \frac{768(-4469239 - 5653\alpha)}{673990075} f_8 + \frac{110592}{85} f_9\\
\vartheta_{6A_4,Q_5} &= \frac{4(-144169 + 2173\alpha)}{26959603} f_7 + \frac{4(-144169 - 2173\alpha)}{26959603} f_8 - \frac{960}{17} f_9
\end{aligned}$$

where $Q_3(X) = \det(U_1X')^{10}$ where $U_1 = [I_4\ iI_4]$, $Q_4(X) = \det(U_4X')^2$ where $U_4 = [I_4\ 0\cdots 0\ iI_4]$, and $Q_5(X) = \det(U_5X')^2$ where $U_5 = [I_4\ 0\cdots 0\ iI_4]$. The symbols $6D_4$ and $6A_4$ represent the Niemeier lattices with these root systems. We have $\dim M_4^{14} = 6$.

Example 8: $\dim S_4^{16} = 7$ (new result). We have $\mathscr{C} = \{T \in \mathscr{X}_4 : w(T) \le 5.5\} = \{B_0, \ldots, B_{147}\}$. For this calculation, it seems that taking $\mathscr{B}$ to be just a shell of forms with a certain dyadic trace bound may not be the best approach. Instead we take $\mathscr{B}$ to be the set of forms that arise in our calculations, in an ad hoc fashion, in the following way. Denote $\hat{\mathscr{I}}(s,t) = \inf\langle [s],[t]\rangle$. Then we take

$$\begin{aligned}
\mathscr{B} = \Bigg\{ T \in \mathscr{X}_4 : w(T) \le 7 \text{ or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&4\end{pmatrix}\right) \le 18\\
\text{or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&-1&-1&1\\-1&5&2&-2\\-1&2&5&1\\1&-2&1&5\end{pmatrix}\right) \le 31 \text{ or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&1&1&1\\1&2&0&1\\1&0&4&2\\1&1&2&4\end{pmatrix}\right) \le 22\\
\text{or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&1&1&1\\1&8&3&3\\1&3&8&-2\\1&3&-2&8\end{pmatrix}\right) \le 44 \text{ or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&1&0&1\\1&4&2&2\\0&2&4&2\\1&2&2&4\end{pmatrix}\right) \le 26\\
\text{or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&0&1&-1\\0&2&1&1\\1&1&3&0\\-1&1&0&7\end{pmatrix}\right) \le 24 \text{ or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&0&1&1\\0&6&3&3\\1&3&8&2\\1&3&2&8\end{pmatrix}\right) \le 41\\
\text{or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 2&0&0&1\\0&2&0&1\\0&0&4&2\\1&1&2&5\end{pmatrix}\right) \le 24 \text{ or } \hat{\mathscr{I}}\left(T, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}\right) \le 11 \Bigg\}.
\end{aligned}$$

This makes $|\mathscr{B}| = 2249$. The morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\psi^*_{13}$, $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_2})\psi^*_{22}$ and the $\pi^J \phi_{\Lambda\oplus\Lambda^*}$ for $\Lambda \in \mathscr{A} = \{$ 58 forms$\}$ dominate $\pi_{\mathscr{C}}$. See the authors' website [**15**] for a precise listing of $\mathscr{C}$, $\mathscr{B}$ and $\mathscr{A}$ and for further information on the restriction method using forms from $\mathscr{A}$. Specifically, it turns out the morphisms $\pi^J \phi_{\Lambda\oplus\Lambda^*}$ for $\Lambda \in \mathscr{A}$ imply 1260 relations on the Fourier coefficients for $\mathscr{B}$, and the morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_3})\psi^*_{13}$ and $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_2})\psi^*_{22}$ imply 527 relations on the Fourier coefficients for $\mathscr{B}$ for cusp forms in $S_4^{16,\mathrm{red}}$. Together, it turns out these $1260 + 527 = 1787$ relations give 1685 linearly independent relations. Eliminating the terms involving $\mathscr{B}\backslash\mathscr{C}$, we get 145 linearly independent relations on the Fourier coefficients for $\mathscr{C}$. This implies $\dim S_4^{14,\mathrm{red}} \le 3$.

The equality $\dim S_4^{16} = 4 + \dim S_4^{16,\mathrm{red}}$ follows from the next Lemma. Then $\dim S_4^{16} \le 7$. Since we can construct 7 linearly independent cusp forms in S_4^{16}, then $\dim S_4^{16} = 7$, and hence also $\dim S_4^{16,\mathrm{red}} = 3$.

LEMMA 13.4. *The following sequence is exact*:

$$0 \to S_4^{16,red} \to S_4^{16} \xrightarrow{\psi^*_{13}\oplus\psi^*_{22}} \left(S_1^{16} \otimes S_3^{16}\right) \oplus \mathrm{Sym}\left(S_2^{16} \otimes S_2^{16}\right)$$
$$\xrightarrow{I\otimes\psi^*_{12}-\psi^*_{11}\otimes I} S_1^{16} \otimes S_1^{16} \otimes S_2^{16} \to 0.$$

PROOF. That the image of each map is contained in the kernel of the next is clear from the proof of Lemma 13.2. We know that $S_1^{16} = \boldsymbol{C}\Delta E_4$ and that $S_2^{16} = \boldsymbol{C}[X_{10}E_6, \Phi_{12}E_4]$ so that $S_1^{16} \otimes S_1^{16} \otimes S_2^{16}$ is two dimensional and $\mathrm{Sym}(S_2^{16} \otimes S_2^{16})$ is three dimensional. From $(\psi^*_{11} \otimes I_2)(\Phi_{12}E_4 \otimes \Phi_{12}E_4) = \Delta E_4 \otimes \Delta E_4 \otimes \Phi_{12}E_4$ and from $(\psi^*_{11} \otimes I_2)(\Phi_{12}E_4 \otimes X_{10}E_6 + X_{10}E_6 \otimes \Phi_{12}E_4) = \Delta E_4 \otimes \Delta E_4 \otimes X_{10}E_6$ we see that $I_1 \otimes \psi^*_{12} - \psi^*_{11} \otimes I_2$ is surjective. We will prove exactness by showing that the image of $\psi^*_{13} \oplus \psi^*_{22}$ is at least four dimensional.

For each of the six theta series $\vartheta_{\Lambda,P}$ listed below, compute the Fourier coefficients of $\Psi^*_{13}\vartheta_{\Lambda,P}$ corresponding to $q^{T_1} \otimes q^{T_2}$ for $T_1 \otimes T_2 \in \left\{2\otimes\left(\begin{smallmatrix}2&1&1\\1&2&0\\1&0&2\end{smallmatrix}\right), 2\otimes\left(\begin{smallmatrix}2&1&0\\1&2&0\\0&0&2\end{smallmatrix}\right), 2\otimes\left(\begin{smallmatrix}2&0&0\\0&2&0\\0&0&2\end{smallmatrix}\right)\right\}$ and of $\Psi^*_{22}\vartheta_{\Lambda,P}$ for $T_1 \otimes T_2 \in \left\{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\right\}$. Let $a(T; f)$ denote the coefficient of q^T in the formal series of f. Proposition 11.4 and Definition 11.3 say that

$$a\big(T_1 \otimes T_2; \Psi^*_{ij}\vartheta_{\Lambda,P}\big) = \sum a\left(\begin{pmatrix}T_1 & *\\ * & T_2\end{pmatrix}; \vartheta_{\Lambda,P}\right) \text{ summed over } \begin{pmatrix}T_1 & *\\ * & T_2\end{pmatrix} \in \mathscr{X}_4,$$

so that we need only compute Fourier coefficients $a(T; \vartheta_{\Lambda,P})$ for T with all 2s on the diagonal, which is not difficult. The resulting 6-by-6 matrix, with rows indexed by the theta series and columns by the Fourier coefficients, has the following row reduced form after the content has been factored out of the rows

$$\begin{pmatrix}1&0&0&0&6&-8\\0&1&0&0&1&2\\0&0&1&0&0&1\\0&0&0&1&-2&4\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

so that the image of $\psi_{13}^* \oplus \psi_{22}^*$ is at least four dimensional. □

We have $\dim S_4^{16,\mathrm{red}} = 3$ and $S_4^{16,\mathrm{red}}$ is spanned by $E_4^2 J_8$, J_8^2 and the linear combination $\frac{-17}{1536}\vartheta_{12A_2,P_1} + \frac{287}{512}\vartheta_{24A_1,P_2} - \frac{79}{61440}\vartheta_{3A_8,P_2} + \frac{13}{2048}\vartheta_{4A_6,P_4} - \frac{1}{1728}\vartheta_{4E_6,P_5} - \frac{41}{4096}\vartheta_{6A_4,P_6}$, see below for definitions of these theta series. We have that S_4^{16} is spanned by these three forms and $I(E_4^4\Delta + (-5076 \pm 108\sqrt{18209})E_4\Delta^2)$, $E_6 G_{10}$, and $E_4 I(E_4^2\Delta)$.

We can alternatively take as a basis of S_4^{16} the following six theta series with spherical characters, along with J_8^2. Here the number of columns in $V_1, \dots, V_6$ is 36, 48, 27, 28, 32, 30, respectively.

$$\vartheta_{12A_2,P_1}, \text{ where } P_1(X) = \det(V_1 X')^4 \text{ and } V_1 = \begin{bmatrix} 0&1&0&0&0&0&0&i&0&0&0&0& &0\\ 0&0&1&0&0&0&0&0&i&0&0&0&\cdots&0\\ 0&0&0&1&0&0&0&0&0&i&0&0& &0\\ 0&0&0&0&1&0&0&0&0&0&i&0& &0 \end{bmatrix}$$

$$\vartheta_{24A_1,P_2}, \text{ where } P_2(X) = \det(V_2 X')^4 \text{ and } V_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&i&0&0&0&0&0&0&0& &0\\ 0&0&1&0&0&0&0&0&0&0&i&0&0&0&0&0&\cdots&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&i&0&0&0& &0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&i&0& &0 \end{bmatrix}$$

$$\vartheta_{3A_8,P_3}, \text{ where } P_3(X) = \det(V_3 X')^4 \text{ and } V_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&i&0&0&0&0& &0\\ 0&1&0&0&0&0&0&0&0&0&i&0&0&0&\cdots&0\\ 0&0&1&0&0&0&0&0&0&0&0&i&0&0& &0\\ 0&0&0&1&0&0&0&0&0&0&0&0&i&0& &0 \end{bmatrix}$$

$$\vartheta_{4A_6,P_4}, \text{ where } P_4(X) = \det(V_4 X')^4 \text{ and } V_4 = \begin{bmatrix} 1&0&0&0&0&0&0&i&0&0&0&0& &0\\ 0&1&0&0&0&0&0&0&i&0&0&0&\cdots&0\\ 0&0&1&0&0&0&0&0&0&i&0&0& &0\\ 0&0&0&1&0&0&0&0&0&0&i&0& &0 \end{bmatrix}$$

$$\vartheta_{4E_6,P_5}, \text{ where } P_5(X) = \det(V_5 X')^4 \text{ and } V_5 = \begin{bmatrix} 1&0&0&0&0&0&0&0&i&0&0&0&0& &0\\ 0&1&0&0&0&0&0&0&0&i&0&0&0&\cdots&0\\ 0&0&1&0&0&0&0&0&0&0&i&0&0& &0\\ 0&0&0&1&0&0&0&0&0&0&0&i&0& &0 \end{bmatrix}$$

$$\vartheta_{6A_4,P_6}, \text{ where } P_6(X) = \det(V_6 X')^4 \text{ and } V_6 = \begin{bmatrix} 1&0&0&0&0&i&0&0&0&0& &0\\ 0&1&0&0&0&0&i&0&0&0&\cdots&0\\ 0&0&1&0&0&0&0&i&0&0& &0\\ 0&0&0&1&0&0&0&0&i&0& &0 \end{bmatrix}.$$

A cusp form in S_4^{16} is determined by its Fourier coefficients $a(T)$ with

$$T \in \left\{ \frac{1}{2}D_4, \frac{1}{2}A_4, \frac{1}{2}(A_1 \oplus A_3), \frac{1}{2}(A_2 \oplus A_2), \frac{1}{2}(A_1 \oplus A_1 \oplus A_2), I_4, D_4 \right\}.$$

Enough Fourier coefficients to compute the action of the Hecke operator T_2 can be found at the authors' website [**15**], and the 7 Hecke eigenforms $h_1, \dots, h_7$ are listed in Table 6 by giving their seven determining Fourier coefficients.

EXAMPLES 9, 10 AND 11: $n = 5$ AND WEIGHTS $2, 4, 6$. By the known results mentioned we have $\dim S_5^2 = 0$, $\dim M_5^2 = 0$ and $\dim S_5^4 = 0$, $\dim M_5^4 = 1$ and $\dim S_5^6 = 0$, $\dim M_5^6 = 1$. These results can be reproduced by our methods in each case as follows. For S_5^2 and S_5^4, we have $\mathscr{C} = \varnothing$ and so $\dim S_5^2 = 0$ and $\dim S_5^4 = 0$ follow automatically. For S_5^6, we take $\mathscr{C} = \mathscr{B} = \{\frac{1}{2}D_5\}$ and find $\pi^J \phi_{A_5 \oplus A_5^*}$ dominates $\pi_{\mathscr{C}}$.

EXAMPLE 12: $\dim S_5^8 = 0$ (NEW RESULT). We have $\mathscr{B} = \mathscr{C} = \{T \in \mathscr{X}_5 : w(T) \leq$

$3.5\} = \{8 \text{ forms}\}$ (see the authors' website [**15**] for a listing of $\mathscr{B}$) and the morphism $\pi^J \phi_{A_5 \oplus A_5^*}$ dominates $\pi_{\mathscr{C}}$. That is, this one morphism implies that all 8 Fourier coefficients for $\mathscr{C}$ are zero; hence $\dim S_5^8 = 0$.

Example 13: $\dim S_5^{10} = 0$ (new result). We have $S_5^{10} = S_5^{10,\mathrm{red}}$. We have $\mathscr{C} = \{T \in \mathscr{X}_5 : w(T) \le 4.5\} = \{54 \text{ forms}\}$. We take $\mathscr{B} = \{T \in \mathscr{X}_5 : w(T) \le 5.25\} = \{135 \text{ forms}\}$, and the homomorphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_4})\psi_{14}^*$, $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_3})\psi_{23}^*$ and the $\pi^J \phi_{\Lambda \oplus \Lambda^*}$ for $\Lambda \in \mathscr{A} = \{9 \text{ forms}\}$ dominate $\pi_{\mathscr{C}}$. Specifically, the morphisms $\pi^J \phi_{\Lambda \oplus \Lambda^*}$ for $\Lambda \in \mathscr{A}$ generate 103 relations and the morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_4})\psi_{14}^*$ and $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_3})\psi_{23}^*$ generate 34 relations on the Fourier coefficients for $\mathscr{B}$, for a total of what turns out to be 93 linearly independent relations. When the terms involving $\mathscr{B} \setminus \mathscr{C}$ are eliminated, we get 54 linearly independent relations on the Fourier coefficients for $\mathscr{C}$, implying they are all zero, and hence $\dim S_5^{10,\mathrm{red}} = 0$. See the authors' website [**15**] for detailed listings of sets $\mathscr{C}, \mathscr{B}, \mathscr{A}$.

Remark: $\dim S_5^{12} = 2$. This dimension and the Hecke eigenforms were computed by G. Nebe and B. Venkov [**12**]. We have $\dim M_5^{12} = 8$.

Examples 14, 15 and 16: $n = 6$ and weights $2, 4, 6$. By the known results mentioned we have $\dim S_6^2 = 0$, $\dim M_6^2 = 0$ and $\dim S_6^4 = 0$, $\dim M_6^4 = 1$ and $\dim S_6^6 = 0$, $\dim M_6^6 = 1$. These results can be reproduced by our methods in each case as follows. For S_6^2 and S_6^4, we have $\mathscr{C} = \varnothing$ and so $\dim S_6^2 = 0$ and $\dim S_6^4 = 0$ follow automatically. For S_6^6, $\mathscr{C} = \mathscr{B} = \{\frac{1}{2}E_6, \frac{1}{2}D_6\}$ and $\pi^J \phi_{A_6 \oplus A_6^*}$ dominates $\pi_{\mathscr{C}}$; that is, this morphism implies that both Fourier coefficients for $\{\frac{1}{2}E_6, \frac{1}{2}D_6\}$ must be zero and so $\dim S_6^6 = 0$.

Example 17: $\dim S_6^8 = 0$ (new result). We have $S_6^8 = S_6^{8,\mathrm{red}}$. We have $\mathscr{B} = \mathscr{C} = \{T \in \mathscr{X}_6 : w(T) \le 4.25\} = \{ \text{ 26 forms } \}$ and the morphisms $(\pi_{\mathscr{B}_1} \otimes \pi_{\mathscr{B}_5})\psi_{15}^*$, $(\pi_{\mathscr{B}_2} \otimes \pi_{\mathscr{B}_4})\psi_{24}^*$, $(\pi_{\mathscr{B}_3} \otimes \pi_{\mathscr{B}_3})\psi_{33}^*$ and the $\pi^J \phi_{\Lambda \oplus \Lambda^*}$ for $\Lambda \in \mathscr{A} = \{ \text{ three forms } \}$ dominate $\pi_{\mathscr{C}}$. That is, these six morphisms imply that the Fourier coefficients for $\mathscr{C}$ are all zero and so $\dim S_6^8 = 0$. The three forms in $\mathscr{A}$ are E_6, A_6, and the form

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 & 4 & 2 \\ 1 & 1 & 0 & 1 & 2 & 4 \end{pmatrix}.$$

14. Tables.

Table 1 gives the semi-integral quaternary forms with dyadic trace less than or equal to 4. The quadratic form $\begin{pmatrix} a & e & f & h \\ e & b & g & i \\ f & g & c & j \\ h & i & j & d \end{pmatrix}$ is presented as $a\ b\ c\ d\ 2e\ 2f\ 2g\ 2h\ 2i\ 2j$. We use GṄipp's tables [**13**] as a reference.

Table 2 gives the Fourier coefficients $a(B_i)$ for some Siegel modular cusp forms we have mentioned.

Here $\alpha = \sqrt{144169}$ for simplicity.

Table 1. Semi-integral quaternary forms with dyadic trace ≤ 4.

name	$w(t)$	$16\det(t)$	t
B_0	2	4	1 1 1 1 0 0 0 1 1 1
B_1	2.5	5	1 1 1 1 1 0 0 1 0 1
B_2	3	8	1 1 1 1 0 0 0 1 1 0
B_3	3	9	1 1 1 1 1 0 0 0 0 1
B_4	3	12	1 1 1 2 1 1 0 1 0 0
B_5	3.5	12	1 1 1 1 0 0 0 1 0 0
B_6	3.5	13	1 1 1 2 1 1 0 0 1 0
B_7	3.5	17	1 1 1 2 1 0 0 1 0 1
B_8	3.5	20	1 1 1 2 0 0 0 1 1 1
B_9	3.5	25	1 1 2 2 1 1 0 1 1 2
B_{10}	4	16	1 1 1 1 0 0 0 0 0 0
B_{11}	4	16	1 1 1 2 1 1 0 0 0 0
B_{12}	4	20	1 1 1 3 1 1 0 1 0 0
B_{13}	4	20	1 1 1 2 1 0 0 1 0 0
B_{14}	4	21	1 1 1 2 1 0 0 0 0 1
B_{15}	4	24	1 1 1 2 0 0 0 1 1 0
B_{16}	4	28	1 1 2 2 1 1 0 0 1 1
B_{17}	4	32	1 1 2 2 0 0 0 1 1 2
B_{18}	4	32	1 1 2 2 1 1 0 1 0 0
B_{19}	4	33	1 1 2 2 0 1 1 1 0 2
B_{20}	4	36	1 1 2 2 0 1 1 1 1 1
B_{21}	4	48	1 1 2 2 0 0 0 1 2 2
B_{22}	4	64	2 2 2 2 0 0 0 2 2 2

Table 2. (Fourier Coefficients $n = 4$ for Hecke Eigenforms, weights 8 through 14)

		S_4^8	S_4^{10}	S_4^{12}		S_4^{14}	
		J_8	G_{10}	f_5	f_6	f_7, f_8	f_9
$\frac{1}{2}D_4$	B_0	1	1	-3	2	$-467 \pm \alpha$	2
$\frac{1}{2}A_4$	B_1	-1	2	1	-5	$-274 \pm 2\alpha$	-5
$\frac{1}{2}(A_1 \oplus A_3)$	B_2	2	-22	-38	-44	$-(-4994 \pm 22\alpha)$	28
$\frac{1}{2}(A_2 \oplus A_2)$	B_3	6	72	-78	-78	$-(44904 \pm 48\alpha)$	-198
	B_4	-12	-36	-492	744	$63132 \pm 204\alpha$	-1752
$\frac{1}{2}(A_1 \oplus A_1 \oplus A_2)$	B_5	-12	-36	-492	-816	$63132 \pm 204\alpha$	288
	B_6	11	26	741	-377	$-238282 \pm 506\alpha$	1039
	B_7	2	-232	1462	646	$-(869176 \pm 2992\alpha)$	-2618
	B_8	-72	1200	1992	4080	$1628640 \pm 10080\alpha$	7200
	B_9	116	2480	-39156	24700	$-(21369520 \pm 15040\alpha)$	81700
I_4	B_{10}	40	472	-4440	-6400	$-(434984 \pm 968\alpha)$	-2080
T_2 eigenvalue		552960	30412800	351866880	42762240	$92160(59701 \pm 137\alpha)$	2180874240

Table 3. ($\mathscr{A}$ for calculation of S_4^{12})

ℓ	s	cusp σ	$J(s\square\sigma, \mathrm{w}_\ell(\sigma); \mathscr{B})$	terms	$\dim M_1^{48}(\Gamma_0(\ell))$	rels
2	$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$	1/0 1/1	8/1 8/2	18	13	2
5	$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$	1/0 1/1	8/1 17/5	27	25	2
6	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$	1/0 1/1 1/2 1/3	10/1 14/6 17/3 9/2	54	49	5
8	$\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix}$	1/0 1/1 1/2 1/4	13/1 18/8 11/2 7/1	53	49	4
9	$\begin{pmatrix} 4 & -1 & 2 & -2 \\ -1 & 4 & 1 & 2 \\ 2 & 1 & 4 & -1 \\ -2 & 2 & -1 & 4 \end{pmatrix}$	1/0 1/1 1/3 2/3	17/1 17/9 8/1 8/1	54	49	2

Table 3 gives the set $\mathscr{A}$ used in the calculation of S_4^{12}. Here, a cusp $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ will be denoted simply by the symbol a/c. Note that the J value is written as an unreduced fraction with the denominator being the width of the corresponding cusp. The column headed "terms" is $\sum_{\text{cusps }\sigma}(\mathrm{w}_\ell(\sigma)J(s\square\sigma, \mathrm{w}_\ell(\sigma), \mathscr{B})+1)$, the sum of the numerators and the number of cusps in the previous column. It is the total number of q-terms remaining after we truncate each $\mu(\mathscr{L}\sigma, F)$ to order $q^{J(s\square\sigma, \mathrm{w}_\ell(\sigma), \mathscr{B})}$.

The next column in Table 3 gives $\dim M_1^{48}(\Gamma_0(\ell))$. This is the number of parameters we must eliminate to obtain relations. That is, suppose $g_1, \dots, g_d$ is a basis of $M_1^{48}(\Gamma_0(\ell))$. For some parameters $c_1, \dots, c_d$ we have $\mu(\mathscr{L}\sigma, F) = \sum_i c_i FS_1(g_i|\sigma)$ to order $q^{J(s\square\sigma, \mathrm{w}_\ell(\sigma), \mathscr{B})}$. By eliminating these d parameters from the number of equations in the (terms)-column we get linear relations in the Fourier coefficients of F. The column headed "rels" gives the number of linearly independent relations; hence in all cases the (rels)-column is less than or equal to the difference of the previous two columns.

Table 4. (Relations on Fourier Coefficients in S_4^{12} alluded to in Table 3)

$0 = -4091488a_0 + 4192992a_1 - 286752a_2 - 64512a_3 - 15552a_4 - 1440a_5 - 288a_6 - 288a_7 + 12a_8 + 312a_{10} + 216a_{11} + 12a_{12} + 864a_{13} + 288a_{14} + 432a_{15} + 288a_{16} + 180a_{17} + 48a_{18} + 288a_{19} + 18a_{20} + 36a_{21} + a_{22},$
$0 = -26276a_0 - 8343a_1 + 702a_2 + 162a_3 + 45a_4 + 27a_5 + 15a_6 + 18a_7 + 3a_8 + a_9$
$0 = -2282672a_0 - 1851544a_1 + 29168a_2 + 68949a_3 - 2160a_4 + 5724a_5 - 5466a_6 + 3929a_7 + 810a_8 + 7a_9 - 60a_{10} - 6a_{11} - 12a_{13} + 9a_{14} + 12a_{15} + 3a_{17} + 6a_{19},$
$0 = 331024a_0 + 260160a_1 - 3324a_2 - 10692a_3 + 459a_4 - 765a_5 + 960a_6 - 564a_7 - 120a_8 + 12a_{10} + 2a_{11} + 6a_{13} + a_{16}$
$0 = -337048a_0 - 292142a_1 - 26660a_2 + 4686a_3 - 6912a_4 + 954a_5 - 4098a_6 + 958a_7 + 86a_8 + 58a_9 - 12a_{10} + 10a_{11} + 16a_{13} - 8a_{14} - 14a_{15} + 16a_{16} + a_{17} + 4a_{18} + a_{20} + a_{21},$
$0 = 86400a_0 + 80562a_1 + 3948a_2 - 2485a_3 + 875a_4 - 321a_5 + 764a_6 - 177a_7 - 40a_8 - 3a_9 + 2a_{11} + 6a_{13} + 5a_{14} + 4a_{15} + a_{16} + 2a_{19},$
$0 = -255036a_0 - 218485a_1 - 3624a_2 + 7233a_3 - 896a_4 + 1035a_5 - 987a_6 + 489a_7 + 132a_8 + 3a_9 + 12a_{10} + a_{12} + 6a_{15} + 3a_{17},$
$0 = -23648a_0 - 23216a_1 + 2968a_2 - 444a_3 + 644a_4 + 372a_5 + 624a_6 + 84a_7 - 9a_8 + 8a_{10} + 18a_{11} + a_{12} + 24a_{13},$
$0 = -9460a_0 - 10239a_1 - 964a_2 + 39a_3 - 228a_4 + 21a_5 - 209a_6 + 15a_7 + a_8 + a_9 - 12a_{11} - 2a_{12}$
$0 = 3976544a_0 + 2481732a_1 - 87690a_2 - 55440a_3 - 10629a_4 - 10608a_5 - 1988a_6 - 7296a_7 + 162a_8 - 32a_9 - 39a_{10} + 26a_{11} + 3a_{12} + 96a_{14} + 114a_{15} + 60a_{16} - 30a_{17} + 12a_{18} + 48a_{19} + 3a_{21},$
$0 = -722992a_0 + 29976a_1 - 19718a_2 + 10156a_3 - 241a_4 - 1232a_5 - 3872a_6 + 1680a_7 - 398a_8 + 55a_{10} + 56a_{11} + a_{12} + 68a_{13} + 16a_{14} + 16a_{15} + 8a_{16} + 30a_{17} + 8a_{19} + a_{20},$
$0 = -277064a_0 - 384a_1 - 7872a_2 + 3924a_3 - 204a_4 - 288a_5 - 1400a_6 + 696a_7 - 144a_8 + 21a_{10} + 2a_{11} + 12a_{13} + 12a_{17},$
$0 = -206596a_0 - 141776a_1 + 8318a_2 + 1908a_3 + 678a_4 + 756a_5 + 608a_6 + 256a_7 + 54a_8 + 6a_9 + 3a_{10} + a_{11} + 10a_{13}$
$0 = 4265928a_0 + 2564422a_1 - 149188a_2 - 36086a_3 - 11990a_4 - 10914a_5 - 6440a_6 - 5471a_7 - 448a_8 - 127a_9 + 2a_{11} + a_{12} + 13a_{14} + 10a_{15} + a_{17} + 3a_{18} + 5a_{19} + a_{20} + a_{21},$
$0 = -1204176a_0 - 657918a_1 + 38988a_2 + 9350a_3 + 3090a_4 + 2814a_5 + 1692a_6 + 1479a_7 + 138a_8 + 39a_9 + 12a_{10} + 6a_{11} + 24a_{13} + 5a_{14} + 6a_{15} + 6a_{16} + 3a_{17} + 3a_{19}$

Table 5. (Relations on Fourier Coefficients in $S_4^{12,\mathrm{red}}$ implied by Witt Homomorphisms)

$0 = 6a_0 + 8a_1 + a_2$
$0 = 12a_0 + 18a_1 + a_3,$
$0 = 12a_1 + 6a_2 + 2a_3 + a_5,$
$0 = 8a_0 + 12a_2 + 6a_5 + a_{10},$
$0 = 8a_0 + 24a_1 + 12a_2 + 6a_4 + 8a_6 + a_{11},$
$0 = 6a_1 + 8a_2 + 6a_4 + 6a_6 + 2a_7 + a_{13},$
$0 = 42a_1 + 24a_2 + 6a_3 + 12a_4 + 12a_6 + 6a_7 + a_{14},$
$0 = 10a_2 + 8a_3 + 4a_4 + 8a_7 + 2a_8 + a_{15},$

In Table 4 and Table 5, for simplicity, we write a_i for the Fourier Coefficient $a(B_i)$.

Table 6. (Fourier Coefficients for the Hecke Eigenforms in S_4^{16})

	h_1, h_2	h_3, h_4
$\frac{1}{2}D_4$	4672	2848
$\frac{1}{2}A_4$	$5(-2165 \pm 27\beta)$	$45(-63 \pm \beta)$
$\frac{1}{2}(A_1 \oplus A_3)$	$16(-5911 \pm 81\beta)$	$24(1235 \pm 3\beta)$
$\frac{1}{2}(A_2 \oplus A_2)$	$54(22727 \pm 23\beta)$	$1782(-63 \pm \beta)$
$\frac{1}{2}(A_1 \oplus A_1 \oplus A_2)$	$-864(-3151 \pm 25\beta)$	$11664(-63 \pm \beta)$
I_4	$18688(3061 \pm 27\beta)$	$-1280(-7931 \pm 27\beta)$
D_4	$149504(21785273 \pm 137943\beta)$	$-32768(40825807 \pm 167265\beta)$
T_2 eigenvalue	$829440(67989 \pm 443\beta)$	$-14745600(1703 \pm 9\beta)$

	h_5, h_6	h_7
$\frac{1}{2}D_4$	15628	9
$\frac{1}{2}A_4$	$25(189 \pm \gamma)$	10
$\frac{1}{2}(A_1 \oplus A_3)$	$-8(-82620 \pm 121\gamma)$	-513
$\frac{1}{2}(A_2 \oplus A_2)$	$-6(63161 \pm 913\gamma)$	-696
$\frac{1}{2}(A_1 \oplus A_1 \oplus A_2)$	$48(119861 \pm 1213\gamma)$	-5962
I_4	$640(-109424 \pm 1881\gamma)$	-77120
D_4	$2048(5158515713 \pm 1743525\gamma)$	2719260672
T_2 eigenvalue	$75202560(557 \pm \gamma)$	14745600000

Here $\beta = \sqrt{18209}$ and $\gamma = \sqrt{51349}$ for simplicity. We remark that h_1, h_2 are multiples of Ikeda lifts, namely, $h_1, h_2 = \frac{-2393-9\beta}{10920} I(E_4^4\Delta + (-5076 \pm 108\beta)E_4\Delta^2)$.

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