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Multiplicity-free branching rules for outer automorphisms of simple Lie algebras

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Abstract. We find explicit multiplicity-free branching rules of some series of irreducible finite dimensional representations of simple Lie algebras $\mathfrak{g}$ to the fixed point subalgebras $\mathfrak{g}^\sigma$ of outer automorphisms σ. The representations have highest weights which are scalar multiples of fundamental weights or linear combinations of two scalar ones. Our list of pairs of Lie algebras $(\mathfrak{g}, \mathfrak{g}^\sigma)$ includes an exceptional symmetric pair (E_6, F_4) and also a non-symmetric pair (D_4, G_2) as well as a number of classical symmetric pairs. Some of the branching rules were known and others are new, but all the rules in this paper are proved by a unified method. Our key lemma is a characterization of the "middle" cosets of the Weyl group of $\mathfrak{g}$ in terms of the subalgebras $\mathfrak{g}^\sigma$ on one hand, and the length function on the other hand.

1. Introduction and statement of main theorems.

1.1.

One of the fundamental problems in representation theory is to understand the irreducible decomposition of a given representation. A typical case arises as the decomposition of the restriction $\pi|_H$ for an irreducible representation π of a group G to its subgroup H. Its irreducible decomposition is called a *branching rule*. In general, there may appear complicated multiplicities in branching rules, which would then cause difficult combinatorial problems for actual calculations.

The branching rule is said to be *multiplicity-free*, if any irreducible representation of H occurs in $\pi|_H$ at most one. Multiplicity-free branching rules are not only in good order but also revealing geometric background of the representation [**14**]. As was emphasized in [**13**], the multiplicity-free branching rules are a special class of representations, for which one may expect a simple and detailed study of its own and of which one may hope powerful applications of representation theory to other fields of mathematics.

Multiplicity-free branching rules of *finite* dimensional representations in some classical cases were obtained by Macdonald [**17**], Stembridge [**22**], Okada [**18**] and many others as the case may be. We note that there are combinatorial algorithms to obtain branching rules for finite dimensional representations of classical groups, such as Littlewood-Richardson rule and the algorithm of Koike–Terada [**15**], but such algorithms involve too many cancellations for actual computations in general cases. Instead, Okada uses new combinatorial formulas on minors due to Ishikawa–Wakayama [**7**] to obtain explicit branching rules [**18**].

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Kobayashi recently obtained an abstract theorem of multiplicity-free branching rules for both infinite and finite dimensional representations for a general symmetric pair (G, H) [**11**], [**12**], [**13**]. Multiplicity-free branching rules of *infinite* dimensional discrete series representations $\pi|_H$ were obtained by Kostant and Schmid for compact group H [**21**], and by Kobayashi for general H [**11**].

Kobayashi's method is geometric and gives a conceptual explanation why branching rules become multiplicity-free in a wide setting. It should be interesting to obtain explicit formulas if the representations are known a priori to be multiplicity-free. In this paper, we take up branching problem including some exceptional Lie algebras, which one can tell to be multiplicity-free a priori by [**12**]. It should be noted that this problem does not arise from combinatorial problems, but from a geometric results, although our proof of main results uses combinatorial techniques.

In order to find explicit branching rules, we take a new approach based on combinatorial results of quotients of Weyl groups. That is, we find a characterization of what we call the "middle" cosets of the Weyl group in three ways (Lemma 6.1), by which we find explicit branching rules of certain finite dimensional representations (see Theorem 1.2). Some of these branching rules were predicted by Kobayashi [**12**], [**11**] and a part of the results were found by other mathematicians including Okada, Proctor, Krattentharler, Želobenko [**16**], [**18**], [**19**], [**23**], through combinatorial methods, quite recently.

Our method is also combinatorial, but is different from the previous known methods. Furthermore, its applications include the rules for the exceptional symmetric pair (E_6, F_4) and the non-symmetric pair (D_4, G_2).

1.2. Main results.

Let $\mathfrak{g}$ be a complex simple Lie algebra and σ be an *outer* automorphism of $\mathfrak{g}$, that is, σ is an automorphism of $\mathfrak{g}$ which induces a non-trivial automorphism of the Dynkin diagram of $\mathfrak{g}$ (see Section 2 for the details). Then $\mathfrak{g}^\sigma := \{X \in \mathfrak{g} | \sigma X = X\}$ becomes a Lie subalgebra in $\mathfrak{g}$. In fact, $\mathfrak{g}^\sigma$ becomes a simple Lie algebra as is observed in the classification of $(\mathfrak{g}, \mathfrak{g}^\sigma)$ (see Lemma 2.8). We note that $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is a symmetric pair (i.e. $\sigma^2 = \mathrm{id}$) except for $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathrm{so}(8, \boldsymbol{C}), \mathfrak{g}_2)$.

We denote by $\mathrm{L}(\lambda) \equiv \mathrm{L}(\mathfrak{g}, \lambda)$ the irreducible finite dimensional representation of the complex simple Lie algebra $\mathfrak{g}$ with the highest weight λ, and by $\mathrm{L}(\lambda)|_{\mathfrak{g}^\sigma}$ the restriction to the subalgebra $\mathfrak{g}^\sigma$. Let $\{\varpi_j\}_{j=1}^n$ and $\{\varpi'_{j'}\}_{j'=1}^{n'}$ be the set of fundamental weights, of the complex Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^\sigma$, which are labeled in Table 1 at the end of Section 2, respectively. The labeling follows Bourbaki [**2**].

The main results of this paper is to give the branching rules $\mathrm{L}(\lambda)|_{\mathfrak{g}^\sigma}$ for the above pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ and some special highest weight λ.

Theorem 1.1. *For any $k \in \boldsymbol{N}$, we have the following irreducible decomposition.*

(1) $(\mathfrak{sl}(2m, \boldsymbol{C}) \downarrow \mathfrak{sp}(m, \boldsymbol{C}))$

$$\mathrm{L}(A_{2m-1}, k\varpi_1)|_{C_m} = \mathrm{L}(A_{2m-1}, k\varpi_{2m-1})|_{C_m} = \mathrm{L}(C_m, k\varpi'_1) \quad (m \geq 2). \quad \text{(2A)}$$

(2) $(\mathfrak{so}(2m, \boldsymbol{C}) \downarrow \mathfrak{so}(2m-1, \boldsymbol{C}))$

$$\mathrm{L}(D_m, k\varpi_{m-1})|_{B_{m-1}} = \mathrm{L}(D_m, k\varpi_m)|_{B_{m-1}} = \mathrm{L}\left(B_{m-1}, k\varpi'_{m-1}\right) \quad (m \geq 4). \quad \text{(4A)}$$

THEOREM 1.2. *For any $k, l \in \boldsymbol{N}$, we have the following irreducible decomposition:*

(1) $(\mathfrak{sl}(2m, \boldsymbol{C}) \downarrow \mathfrak{sp}(m, \boldsymbol{C}))$

$$\begin{aligned} \mathrm{L}\,(A_{2m-1}, k\varpi_1 + l\varpi_2)\,|_{C_m} &= \mathrm{L}\,(A_{2m-1}, k\varpi_{2m-1} + l\varpi_{2m-2})\,|_{C_m} \\ &= \bigoplus_{s=0}^{l} \mathrm{L}\,(C_m, k\varpi_1' + s\varpi_2') \quad (m \geq 2). \end{aligned} \tag{2B}$$

(2) $(\mathfrak{sl}(2m+1, \boldsymbol{C}) \downarrow \mathfrak{so}(2m+1, \boldsymbol{C}))$

$$\mathrm{L}\,(A_{2m}, k\varpi_1)\,|_{B_m} = \mathrm{L}\,(A_{2m}, k\varpi_{2m})\,|_{B_m} = \bigoplus_{\substack{0 \leq s \leq k \\ s \equiv k \mod 2}} \mathrm{L}\,(B_m, s\varpi_1') \quad (m \geq 1). \tag{3B}$$

(3) $(\mathfrak{so}(2m, \boldsymbol{C}) \downarrow \mathfrak{so}(2m-1, \boldsymbol{C}))$

$$\mathrm{L}\,(D_m, k\varpi_1)\,|_{B_{m-1}} = \bigoplus_{s=0}^{k} \mathrm{L}\,(B_{m-1}, s\varpi_1') \quad (m \geq 4). \tag{4B}$$

(4) $(\mathfrak{e}_6 \downarrow \mathfrak{f}_4)$

$$\mathrm{L}\,(E_6, k\varpi_1)\,|_{F_4} = \mathrm{L}\,(E_6, k\varpi_6)\,|_{F_4} = \bigoplus_{s=0}^{k} \mathrm{L}\,(F_4, s\varpi_4')\,. \tag{5B}$$

(5) $(\mathfrak{so}(8, \boldsymbol{C}) \downarrow \mathfrak{g}_2)$

$$\mathrm{L}\,(D_4, k\varpi_1)\,|_{G_2} = \mathrm{L}\,(D_4, k\varpi_3)\,|_{G_2} = \mathrm{L}\,(D_4, k\varpi_4)\,|_{G_2} = \bigoplus_{s=0}^{k} \mathrm{L}\,(G_2, s\varpi_1')\,. \tag{6B}$$

REMARK 1.3. Some of these branching rules were preciously known by different methods such as the Borel–Weil theory, eigenvalues of Casimir operators, formulas of minor determinants, and so on (see, for example, [**10**], [**16**], [**18**], [**19**], [**23**]), summarized as follows:

(1) The author learned a geometric proof of Theorem 1.1 via the Borel–Weil theory by Kobayashi [**10**, Example 5.2]. For this, we use a special case (i.e. $q = 0$) of the following isomorphism of homogeneous spaces $U(2p, 2q)/U(2p-1, 2q) \simeq Sp(p, q)/Sp(p-1, q)$ and $SO(2p, 2q)/U(p, q) \simeq SO(2p-1, 2q)/U(p-1, q)$.
(2) The rules (4A) and (4B) are proved by Želobenko [**23**, X Theorem 3, XVIII Theorem 3], via eigenvalues of Casimir operators.
(3) The rule (2A) is proved by Proctor [**19**, Lemma 4].
(4) The rule (4A) is proved by Okada [**18**, Theorem 2.2(1)], using combinatorial formulas of minor determinants.
(5) The rule (2B) is proved by Krattentharler [**16**, Theorem 1].

Remark 1.4. Koike–Terada [**15**] gave general formulas of the restrictions from $GL(n)$ to $SO(n)$ and from $GL(2n)$ to $Sp(n)$. By using their formulas, we can give an alternative proof of (2A), (2B) and (3B). However, as remarked in [**18**], this approach does not always work for actual computations in some multiplicity-free branching rules, because their general formulas may involve many cancellations.

Remark 1.5. The above equation labels are related to the case labels in Lemma 2.8 as follows. ($\Box$ A) is the branching rules in which an irreducible representation stays irreducible, and ($\Box$ B) the multiplicity-free branching rules in which the decomposition is a sum with one parameter for the Lie algebra pair labeled $\Box$ in Lemma 2.8.

1.3.

Our paper is organized as follows. In Section 2, we give a quick review of basic properties of outer automorphisms and explicit data to use in this paper (Table 1). In Section 3, we review Satake's theorem and Chevalley's lemma without proof and we obtain a map between the some representatives of quotients of the two Weyl groups involved. In Section 4, we give the framework of the proof of Theorems 1.1 and 1.2 by deforming Weyl's character formula, although two facts to be proved will be postponed to the later sections. In Section 5, we prove one of the postponed fact in Section 4.3. The lemma is proved by comparison between $\Delta^+ \setminus \Delta^+_\lambda$ and $\Delta'^+ \setminus \Delta'^+_{\lambda'}$. In Section 6 we prove the other postponed lemma (Lemma 6.1) in Section 4. This lemma is the characterization of the "middle" cosets of the Weyl group involved.

In this paper, $\boldsymbol{N}$ is a set of non-negative integers $\{0, 1, 2, \dots\}$.

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2. Outer automorphisms of simple Lie algebras.

In this section, we give a quick review on basic properties of outer automorphisms of a simple Lie algebra. Most of the results in this section can be found in Helgason [**5**, Chapter X] or Kac [**8**, Sections 7 and 8].

Let $\mathfrak{g}$ be a complex simple Lie algebra and σ be an automorphism of $\mathfrak{g}$ of finite order. Then $\mathfrak{g}^\sigma := \{X \in \mathfrak{g} | \sigma X = X\}$ becomes a Lie subalgebra in $\mathfrak{g}$. We remark that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is a symmetric pair, if the order of σ is two.

Lemma 2.1. *There exists a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}^\sigma := \{X \in \mathfrak{h} | \sigma X = X\}$ becomes a Cartan subalgebra of $\mathfrak{g}^\sigma$.*

Remark 2.2. We refer the proof to Helgason [**5**, Chapter X Lemma 5.3].

For a σ-stable Cartan subalgebra $\mathfrak{h}$, we define the action of σ on $\mathfrak{h}$ by restriction, and also on $\mathfrak{h}^*$ by identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via a non-degenerate invariant bilinear form on $\mathfrak{g}$. Furthermore, we naturally identify $(\mathfrak{h}^\sigma)^*$ with $(\mathfrak{h}^*)^\sigma$.

LEMMA 2.3. *There exists a positive system $\Delta^+ \equiv \Delta^+(\mathfrak{g}, \mathfrak{h})$ such that $\sigma\Delta^+ = \Delta^+$.*

LEMMA 2.4 (Cartan [**3**]). *The group $\mathrm{Aut}_{\boldsymbol{C}}\,\mathfrak{g}/\,\mathrm{Int}_{\boldsymbol{C}}\,\mathfrak{g}$ is isomorphic to the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.*

REMARK 2.5. We refer the proof of Lemma 2.3 and Lemma 2.4 to Knapp [**9**, Theorem 7.8].

Lemma 2.4 leads us to the concept of *outer* automorphisms.

DEFINITION 2.6. We say an automorphism σ of $\mathfrak{g}$ of finite order is *outer*, if σ induces a non-trivial automorphism of the Dynkin diagram of $\mathfrak{g}$.

REMARK 2.7. Lemma 2.4 implies that there exists an outer automorphism of $\mathfrak{g}$, if and only if its Dynkin diagram admits a non-trivial automorphism. This is the case if $\mathfrak{g}$ is of type A_n, D_n or E_6.

LEMMA 2.8 (Helgason [**5**, Chapter X Example 2]). *If σ is an outer automorphism of a simple Lie algebra $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is isomorphic to the one of the followings*:

Case 1 (A_2, A_1) $(\simeq (\mathfrak{sl}(3, \boldsymbol{C}), \mathfrak{so}(3, \boldsymbol{C})))$.

Case 2 (A_{2m-1}, C_m) $(\simeq (\mathfrak{sl}(2m, \boldsymbol{C}), \mathfrak{sp}(m, \boldsymbol{C})))$.

Case 3 (A_{2m}, B_m) $(\simeq (\mathfrak{sl}(2m+1, \boldsymbol{C}), \mathfrak{so}(2m+1, \boldsymbol{C})))$.

Case 4 (D_m, B_{m-1}) $(\simeq (\mathfrak{so}(2m, \boldsymbol{C}), \mathfrak{so}(2m-1, \boldsymbol{C})))$.

Case 5 (E_6, F_4).

Case 6 (D_4, G_2).

We tell from the above list that the Lie subalgebra $\mathfrak{g}^\sigma$ is simple.

For each pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ in Lemma 2.8, we choose a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}^\sigma$ is a Cartan subalgebra of $\mathfrak{g}^\sigma$ (Lemma 2.1).

Let $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Likewise, let $\Delta' \equiv \Delta(\mathfrak{g}^\sigma, \mathfrak{h}^\sigma)$ be that of $\mathfrak{g}^\sigma$ with respect to $\mathfrak{h}^\sigma$. We fix a positive system Δ^+ such that $\sigma\Delta^+ = \Delta^+$ (Lemma 2.3).

LEMMA 2.9. *There is a unique positive system $\Delta'^+ \equiv \Delta^+(\mathfrak{g}^\sigma, \mathfrak{h}^\sigma)$ with the following property*: *if α is a simple root for Δ^+, then $\alpha|_{\mathfrak{h}^\sigma}$ becomes a simple root for Δ'^+.*

PROOF. The positive system Δ'^+ is given in Helgason [**5**, Chapter X Example 2] and Kac [**8**, Sections 7.9 and 7.10]. The necessary data is given in Table 1 at the end of this section. □

We shall take a positive system Δ'^+ as in Lemma 2.9.

In later sections, the following formula of the restriction of $\lambda \in \mathfrak{h}^*$ will be useful. The proof is an easy consequence of linear algebra.

LEMMA 2.10. *Let $\lambda \in \mathfrak{h}^*$ and let m be the order of $\sigma \in GL(\mathfrak{h}^*)$. Then $\lambda|_{\mathfrak{h}^\sigma} = \frac{1}{m}\sum_{j=0}^{m-1} \sigma^j \lambda$.*

We give the data of the restrictions of outer automorphisms in Table 1, which are obtained in Helgason [**5**, Chapter X, Example 2] and Kac [**8**, Sections 7.9 and 7.10].

Let $\{\alpha_j\}_{j=1}^{n}$ and $\{\varpi_j\}_{j=1}^{n}$ denote the simple system and the set of fundamental weights for $\mathfrak{g}$. Likewise, $\{\alpha'_{j'}\}_{j'=1}^{n'}$ and $\{\varpi'_{j'}\}_{j'=1}^{n'}$ denote those for $\mathfrak{g}^\sigma$.

In the first column at Table 1, we write the Dynkin diagrams of $\Delta := \Delta(\mathfrak{g},\mathfrak{h})$ and $\Delta' := \Delta(\mathfrak{g}^\sigma,\mathfrak{h}^\sigma)$, whose labeling follows Bourbaki [**2**], Planche. The second column describes the induced action of σ on simple roots of $\mathfrak{g}$. In the third column, the restriction of simple roots of $\mathfrak{g}$ from $\mathfrak{h}$ to $\mathfrak{h}^\sigma$ is given in terms of simple roots for $\mathfrak{g}^\sigma$. At the fourth column, the restriction of fundamental weights of $\mathfrak{g}$ from $\mathfrak{h}$ to $\mathfrak{h}^\sigma$ is given in terms of fundamental weights for $\mathfrak{g}^\sigma$.

3. Satake's theorem and Chevalley's lemma.

In the rest of this paper, the pair $(\mathfrak{g},\mathfrak{g}^\sigma)$ is one of the pairs listed in Section 2. For the root systems $\Delta \equiv \Delta(\mathfrak{g},\mathfrak{h})$ and $\Delta' \equiv \Delta(\mathfrak{g}^\sigma,\mathfrak{h}^\sigma)$, we choose compatible positive systems Δ^+ and Δ'^+ as in Lemma 2.9.

In this section, we review two theorems on Weyl groups, which are called Satake's theorem and Chevalley's lemma. Satake's theorem gives a connection between the Weyl groups of the root systems attached to the pair $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}^\sigma,\mathfrak{h}^\sigma)$. Chevalley's lemma gives generators of a fixed subgroup of a highest weight in Weyl groups. Using these theorems, we obtain the relationship between the sets of certain right cosets of the Weyl groups of $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}^\sigma,\mathfrak{h}^\sigma)$ (Lemma 3.3).

3.1.

Let W be the Weyl group of the root system $(\mathfrak{h}^*,\Delta)$, and W' be that of $((\mathfrak{h}^\sigma)^*,\Delta')$. The subgroup $\widetilde{W} = \{w \in W | \sigma w = w\sigma\}$ in W is isomorphic to W' by the following Satake's theorem.

Satake's theorem. *$\widetilde{W} \to W', w \mapsto w|_{(\mathfrak{h}^*)^\sigma}$ is a group isomorphism.*

Remark 3.1.

(1) Satake proved the above theorem in a more general context, that is, for a Cartan involution σ [**20**, Appendix Proposition A].
(2) Fuchs–Ray–Schweigert generalized the above theorem for automorphisms of Dynkin diagrams of generalized Kac–Moody Lie algebras [**4**, Proposition 3.3].

3.2.

We recall Chevalley's lemma for a reduced abstract root space (V,Δ) and its Weyl group W. Let Δ^+ be a positive system of (V,Δ). For $\lambda \in V$, we define a subgroup $W_\lambda := \{w \in W | w\lambda = \lambda\}$ in W.

Chevalley's lemma. *If $\lambda \in V$ is dominant, that is $\langle \lambda,\alpha\rangle \geq 0$ for all $\alpha \in \Delta^+$, the subgroup W_λ is generated by the simple reflections s_α such that $\langle\lambda,\alpha\rangle = 0$.*

Remark 3.2. We refer the proof to Knapp [**9**, Proposition 2.72].

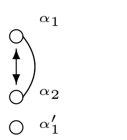
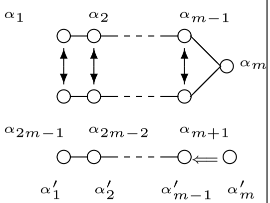
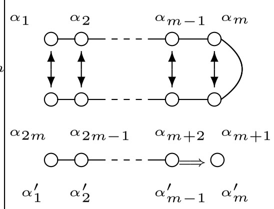
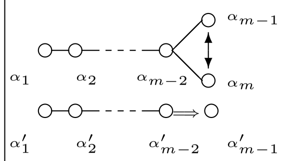
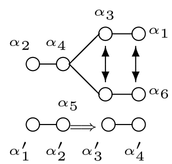
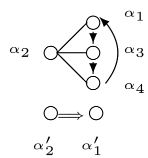
Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
(A_2, A_1)	(A_{2m-1}, C_m)	(A_{2m}, B_m)	(D_m, B_{m-1})	(E_6, F_4)	(D_4, G_2)
					
$\sigma\alpha_1 = \alpha_2$ $\sigma\alpha_2 = \alpha_1$	$\sigma\alpha_i = \alpha_{2m-i}$ $(1 \leq i \leq 2m-1)$	$\sigma\alpha_i = \alpha_{2m-i+1}$ $(1 \leq i \leq 2m)$	$\sigma\alpha_i = \alpha_i$ $(1 \leq i \leq m-2)$ $\sigma\alpha_{m-1} = \alpha_m$ $\sigma\alpha_m = \alpha_{m-1}$	$\sigma\alpha_1 = \alpha_6$ $\sigma\alpha_2 = \alpha_2$ $\sigma\alpha_3 = \alpha_5$ $\sigma\alpha_4 = \alpha_4$ $\sigma\alpha_5 = \alpha_3$ $\sigma\alpha_6 = \alpha_1$	$\sigma\alpha_1 = \alpha_3$ $\sigma\alpha_2 = \alpha_2$ $\sigma\alpha_3 = \alpha_4$ $\sigma\alpha_4 = \alpha_1$
$\alpha_1\|_{A_1} = \alpha_1'$ $\alpha_2\|_{A_1} = \alpha_1'$	$\alpha_i\|_{C_m} = \alpha_i'$ $(1 \leq i \leq m)$ $\alpha_{2m-i}\|_{C_m} = \alpha_i'$ $(1 \leq i \leq m)$	$\alpha_i\|_{B_m} = \alpha_i'$ $(1 \leq i \leq m)$ $\alpha_{2m-i+1}\|_{B_m} = \alpha_i'$ $(1 \leq i \leq m)$	$\alpha_i\|_{B_{m-1}} = \alpha_i'$ $(1 \leq i \leq m-2)$ $\alpha_{m-1}\|_{B_{m-1}} = \alpha_{m-1}'$ $\alpha_m\|_{B_{m-1}} = \alpha_{m-1}'$	$\alpha_1\|_{F_4} = \alpha_4'$ $\alpha_2\|_{F_4} = \alpha_1'$ $\alpha_3\|_{F_4} = \alpha_3'$ $\alpha_4\|_{F_4} = \alpha_2'$ $\alpha_5\|_{F_4} = \alpha_3'$ $\alpha_6\|_{F_4} = \alpha_4'$	$\alpha_1\|_{G_2} = \alpha_1'$ $\alpha_2\|_{G_2} = \alpha_2'$ $\alpha_3\|_{G_2} = \alpha_1'$ $\alpha_4\|_{G_2} = \alpha_1'$
$\varpi_1\|_{A_1} = 2\varpi_1'$ $\varpi_2\|_{A_1} = 2\varpi_1'$	$\varpi_i\|_{C_m} = \varpi_i'$ $(1 \leq i \leq m)$ $\varpi_{2m-i}\|_{C_m} = \varpi_i'$ $(1 \leq i \leq m)$	$\varpi_i\|_{B_m} = \varpi_i'$ $(1 \leq i \leq m-1)$ $\varpi_m\|_{B_m} = \varpi_{m+1}\|_{B_m} = 2\varpi_m'$ $\varpi_{2m-i+1}\|_{B_m} = \varpi_i'$ $(1 \leq i \leq m-1)$	$\varpi_i\|_{B_{m-1}} = \varpi_i'$ $(1 \leq i \leq m-2)$ $\varpi_{m-1}\|_{B_{m-1}} = \varpi_{m-1}'$ $\varpi_m\|_{B_{m-1}} = \varpi_{m-1}'$	$\varpi_1\|_{F_4} = \varpi_4'$ $\varpi_2\|_{F_4} = \varpi_1'$ $\varpi_3\|_{F_4} = \varpi_3'$ $\varpi_4\|_{F_4} = \varpi_2'$ $\varpi_5\|_{F_4} = \varpi_3'$ $\varpi_6\|_{F_4} = \varpi_4'$	$\varpi_1\|_{G_2} = \varpi_1'$ $\varpi_2\|_{G_2} = \varpi_2'$ $\varpi_3\|_{G_2} = \varpi_1'$ $\varpi_4\|_{G_2} = \varpi_1'$

Table 1. Data of outer automorphisms.

3.3.

We obtain Lemma 3.3, by applying Satake's theorem and Chevalley's lemma.

Lemma 3.3. *Let $\lambda \in \mathfrak{h}^*$ be dominant for Δ^+, and put $\lambda' = \lambda|_{\mathfrak{h}^\sigma}$. Then,*

(1) *λ' is dominant for Δ'^+.*
(2) *Put $\widetilde{W_\lambda} := \widetilde{W} \cap W_\lambda$, then we obtain $\widetilde{W_\lambda} = \widetilde{W} \cap W_{\lambda'}$. Therefore, the image of the restriction of $(\widetilde{W} \to W')$ to $\widetilde{W_\lambda}$ is $W'_{\lambda'}$.*
(3) *The following commutative diagram of groups:*

$$\begin{array}{ccccc} W' & \xleftarrow{\sim} & \widetilde{W} & \hookrightarrow & W \\ \uparrow & & \uparrow & & \uparrow \\ W'_{\lambda'} & \xleftarrow{\sim} & \widetilde{W_\lambda} & \hookrightarrow & W_\lambda \end{array}$$

induces a natural injective map on the right cosets:

$$W'/W'_{\lambda'} \hookrightarrow W/W_\lambda.$$

Proof.

(1) We tell, by seeing Table 1, that the restriction of a fundamental weight of $\mathfrak{g}$ from $\mathfrak{h}$ to $\mathfrak{h}^\sigma$ becomes a positive multiple of a fundamental weight of $\mathfrak{g}^\sigma$. Thus, if $\lambda \in \mathfrak{h}^*$ is dominant for Δ^+, $\lambda' = \lambda|_{\mathfrak{h}^\sigma}$ is dominant for Δ'^+.
(2) By Lemma 2.10, we have $\lambda' = \frac{1}{m}\sum_{j=0}^{m-1} \sigma^j \lambda$ where m is the order of σ.
If $w \in \widetilde{W_\lambda}$, then $w\sigma^j\lambda = \sigma^j(w\lambda) = \sigma^j\lambda$. Thus

$$w\lambda' = \frac{1}{m}\sum_{j=0}^{m-1} w(\sigma^j\lambda) = \frac{1}{m}\sum_{j=0}^{m-1} \sigma^j\lambda = \lambda'.$$

We conclude $w \in \widetilde{W} \cap W_{\lambda'}$.
Conversely, if $w \in \widetilde{W} \cap W_{\lambda'}$, we can assume w is a simple reflection $w = s_\alpha$ such that $\langle \alpha, \lambda' \rangle = 0$ by Chevalley's lemma. Then,

$$0 = \langle \alpha, \lambda' \rangle = \left\langle \alpha, \frac{1}{m}\sum_{j=0}^{m-1} \sigma^j\lambda \right\rangle = \frac{1}{m}\sum_{j=0}^{m-1} \langle \sigma^j\alpha, \lambda \rangle.$$

Then $\langle \sigma^j\alpha, \lambda \rangle = 0$ for all j, because λ is dominant and $\sigma^j\alpha \in \Delta^+$. In particular, $\langle \alpha, \lambda \rangle = 0$. Then $w = s_\alpha \in \widetilde{W_\lambda}$ by Chevalley's lemma.
Thus we conclude the claim $\widetilde{W_\lambda} = \widetilde{W} \cap W_{\lambda'}$.
(3) $W' \simeq \widetilde{W}$ and $W'_{\lambda'} \simeq \widetilde{W} \cap W_{\lambda'}$ is obvious by Satake's theorem. We have the commutative diagram, because $\widetilde{W_\lambda} = \widetilde{W} \cap W_{\lambda'}$. Thus we obtain the natural bijective maps

$$W'/W'_{\lambda'} \simeq \widetilde{W}/\widetilde{W_{\lambda'}} \simeq \widetilde{W}/\widetilde{W_\lambda}.$$

Then we obtain the natural injective maps $W'/W'_{\lambda'} \hookrightarrow W/W_\lambda$. $\square$

4. Calculation of character formula.

The aim of this section is to obtain the deformation (8) and (9) of Weyl's character formulas by using the denominator formula and to give the framework of the proof of Theorems 1.1 and 1.2.

4.1.
Let us recall the notation in the previous sections. Let $\mathfrak{g}$ be a complex simple Lie algebra, and σ be an outer automorphism of $\mathfrak{g}$. Then $\mathfrak{g}^\sigma$ is also a simple Lie algebra. As in Lemma 2.9, we fix compatible positive systems Δ^+ and Δ'^+ with respect to Cartan subalgebras $\mathfrak{h} \supset \mathfrak{h}^\sigma$ of $\mathfrak{g} \supset \mathfrak{g}^\sigma$. Let W be the Weyl group of Δ and W' that of Δ'.

For a dominant weight $\lambda \in \mathfrak{h}^*$, we choose a complete system of representatives W^λ of W/W_λ and for $\lambda' = \lambda|_{\mathfrak{h}^\sigma}$ and $W'^{\lambda'}$ of $W'/W'_{\lambda'}$, respectively.

Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and $\rho' := \frac{1}{2}\sum_{\alpha'\in\Delta'^+}\alpha'$. Let d and d' be the Weyl's denominator of $\mathfrak{g}$ and $\mathfrak{g}^\sigma$ respectively and by $\chi_{\mathfrak{g}}(\lambda)$ the character of the irreducible representation $\mathrm{L}(\lambda)$ of $\mathfrak{g}$ with the highest weight λ. Likewise by $\chi_{\mathfrak{g}^\sigma}(\lambda')$ that of $\mathfrak{g}^\sigma$ with λ'.

4.2.
We deform Weyl's character formulas of $\chi_{\mathfrak{g}}(\lambda)$ and $\chi_{\mathfrak{g}^\sigma}(\lambda')$ to obtain (8) and (9).
By Weyl's character formula,

$$\chi_{\mathfrak{g}}(\lambda) = d^{-1}\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}. \tag{1}$$

We obtain by decomposing $w \in W$ to $w = w_1w_2$ where $w_1 \in W^\lambda$ and $w_2 \in W_\lambda$,

$$\chi_{\mathfrak{g}}(\lambda) = d^{-1}\sum_{w_1\in W^\lambda}\Big(\sum_{w_2\in W_\lambda}\varepsilon(w_1w_2)e^{w_1w_2\lambda+w_1w_2\rho}\Big). \tag{2}$$

Because $w_2\lambda = \lambda$ for $w_2 \in W_\lambda$,

$$\chi_{\mathfrak{g}}(\lambda) = d^{-1}\sum_{w_1\in W^\lambda}\Big(\varepsilon(w_1)e^{w_1\lambda}\Big(\sum_{w_2\in W_\lambda}\varepsilon(w_2)e^{w_1w_2\rho}\Big)\Big). \tag{3}$$

By Chevalley's lemma, W_λ is the Weyl group generated by the root reflections of $\Delta^+_\lambda := \{\alpha \in \Delta^+ | \langle\alpha,\lambda\rangle = 0\}$. We can apply Weyl's denominator formula to W_λ and obtain

$$\sum_{w_2\in W_\lambda}\varepsilon(w_2)e^{w_2\rho} = e^\rho\prod_{\alpha\in\Delta^+_\lambda}(1-e^{-\alpha}). \tag{4}$$

We extend the action of W to the ring $\boldsymbol{Z} < \mathfrak{h}^* >$, naturally. By applying $w_1 \in W^\lambda$ to both sides of (4),

$$\sum_{w_2 \in W_\lambda} \varepsilon(w_2) e^{w_1 w_2 \rho} = e^{w_1 \rho} \prod_{\alpha \in \Delta_\lambda^+} (1 - e^{-w_1 \alpha}). \tag{5}$$

The substitution of (5) into (3) yields,

$$\chi_{\mathfrak{g}}(\lambda) = d^{-1} \sum_{w_1 \in W^\lambda} \left(\varepsilon(w_1) e^{w_1 \lambda} \left(e^{w_1 \rho} \prod_{\alpha \in \Delta_\lambda^+} (1 - e^{-w_1 \alpha}) \right) \right). \tag{6}$$

We obtain $w_1 \cdot d = \varepsilon(w_1) d$ because $d = \sum_{w \in W} \varepsilon(w) e^{w\rho}$. Then,

$$\chi_{\mathfrak{g}}(\lambda) = \sum_{w_1 \in W^\lambda} w_1 \cdot \left(d^{-1} e^{\lambda + \rho} \prod_{\alpha \in \Delta_\lambda^+} (1 - e^{-\alpha}) \right). \tag{7}$$

Weyl's denominator formula $d = \prod_{\alpha \in \Delta^+} e^\rho (1 - e^{-\alpha})$ implies

$$\chi_{\mathfrak{g}}(\lambda) = \sum_{w_1 \in W^\lambda} w_1 \cdot \left(e^\lambda \prod_{\alpha \in \Delta^+ \backslash \Delta_\lambda^+} (1 - e^{-\alpha})^{-1} \right). \tag{8}$$

We remark that $\left(e^\lambda \prod_{\alpha \in \Delta^+ \backslash \Delta_\lambda^+} (1 - e^{-\alpha})^{-1}\right)$ is W_λ-invariant and (8) is independent of the choice of a system of representatives W^λ.

In the same way, we calculate $\chi_{\mathfrak{g}^\sigma}(\lambda')$. Then,

$$\chi_{\mathfrak{g}^\sigma}(\lambda') = \sum_{w_1' \in W'^{\lambda'}} w_1' \cdot \left(e^{\lambda'} \prod_{\alpha \in \Delta'^+ \backslash \Delta_{\lambda'}'^+} (1 - e^{-\alpha})^{-1} \right). \tag{9}$$

4.3. Framework of proof of main theorems.

We give the framework of the proof of Theorems 1.1 and 1.2 by using (8) and (9) which are deformation of Weyl's character formulas, although several facts to be proved are postponed to Section 5 and Section 6.

4.3.1

Theorem 1.1 follows from the identity of characters, namely the following lemmas.

Lemma 4.1. *For $k \in \boldsymbol{N}$,*

$$\chi_{A_{2m-1}}(k\varpi_1)\,|_{C_m} = \chi_{C_m}(k\varpi_1') \quad (m \geq 2), \tag{2A$'$-1}$$

$$\chi_{A_{2m-1}}(k\varpi_{2m-1})\,|_{C_m} = \chi_{C_m}(k\varpi_1') \quad (m \geq 2). \tag{2A$'$-2}$$

Lemma 4.2. *For $k \in \boldsymbol{N}$,*

$$\chi_{D_m}(k\varpi_{m-1})\,|_{B_{m-1}} = \chi_{B_{m-1}}\left(k\varpi_{m-1}'\right) \quad (m \geq 4), \tag{4A$'$-1}$$

$$\chi_{D_m}(k\varpi_m)\,|_{B_{m-1}} = \chi_{B_{m-1}}\left(k\varpi_{m-1}'\right) \quad (m \geq 4). \tag{4A$'$-2}$$

We give the framework of the proof of Lemmas 4.1 and 4.2 by several steps.

Step 0:
If $k = 0$, the statements are clear, because both sides of equations of (2A′-1), (2A′-2), (4A′-1) and (4A′-2) are the characters of trivial representations. We assume k is a positive integer. If the equation (2A′-1) is proved, we obtain (2A′-2) by applying σ to (2A′-1). Similarly, if (4A′-1), we obtain (4A′-2). Thus, the remainder to be proved are (2A′-1) and (4A′-1).

We apply the deformation of Weyl's character formulas (8) and (9) to the both side of (2A′-1) and (4A′-1), respectively. Then, the summation parameters of the left hand side are W^λ and those of the right hand side are $W'^{\lambda'}$.

Suppose $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1)$ $(k \in \boldsymbol{N} \setminus \{0\})$ or $(D_m, B_{m-1}, k\varpi_{m-1})$ $(k \in \boldsymbol{N} \setminus \{0\})$ below.

Step 1 (correspondence between the summation parameters):
We prove, in this step, that the number of $W^\lambda \simeq W/W_\lambda$ equals that of $W'^{\lambda'} \simeq W'/W'_{\lambda'}$ on a case basis.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1)$ $(k \in \boldsymbol{N} \setminus \{0\})$,

$$\sharp(W/W_\lambda) = \frac{(2m)!}{(2m-1)!} = 2m, \quad \sharp\big(W'/W'_{\lambda'}\big) = \frac{m! \cdot 2^m}{(m-1)! \cdot 2^{m-1}} = 2m.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_m, B_{m-1}, \lambda = k\varpi_{m-1})$ $(k \in \boldsymbol{N} \setminus \{0\})$,

$$\sharp(W/W_\lambda) = \frac{m! \cdot 2^{m-1}}{m!} = 2^{m-1}, \quad \sharp\big(W'/W'_{\lambda'}\big) = \frac{(m-1)! \cdot 2^{m-1}}{(m-1)!} = 2^{m-1}.$$

Then we obtain $\sharp(W/W_\lambda) = \sharp(W'/W'_{\lambda'})$ in each case.

Step 2 (correspondence of the first term):
The following equality holds:

$$\left(e^\lambda \prod_{\alpha \in \Delta^+ \setminus \Delta^+_\lambda} (1 - e^{-\alpha})^{-1} \right)\Bigg|_{\mathfrak{h}^\sigma} = \left(e^{\lambda'} \prod_{\alpha \in \Delta'^+ \setminus \Delta'^+_{\lambda'}} (1 - e^{-\alpha})^{-1} \right). \tag{10}$$

We shall postpone the proof of (10) to Section 5. The proof is comparison between $\Delta^+ \setminus \Delta^+_\lambda$ and $\Delta'^+ \setminus \Delta'^+_{\lambda'}$ on a case basis.

Step 3:
The equation (10) is equality between the summands for $w_1 = 1$ in (8) and (9).
Applying $w_1 \in \widetilde{W}^\lambda$ to both sides of (10), we obtain the following equation.

$$w_1 \cdot \left(e^\lambda \prod_{\alpha \in \Delta^+ \setminus \Delta^+_\lambda} (1 - e^{-\alpha})^{-1} \right)\Bigg|_{\mathfrak{h}^\sigma} = w_1 \,|_{\mathfrak{h}^\sigma} \cdot \left(e^{\lambda'} \prod_{\alpha \in \Delta'^+ \setminus \Delta'^+_{\lambda'}} (1 - e^{-\alpha})^{-1} \right). \tag{11}$$

Then, we obtain (2A′-1) and (4A′-1) and prove Lemmas 4.1 and 4.2, if the postponed fact (10) is proved. We postpone the proof to Section 5.

4.3.2

Theorem 1.2 follows from the identity of characters, namely the following lemmas.

Lemma 4.3. *For $l \in \boldsymbol{N}$ and $k \in \boldsymbol{N} \setminus \{0\}$,*

$$\chi_{A_{2m-1}}(l\varpi_1)\,|_{C_m} = \chi_{C_m}(l\varpi_1'), \tag{2B′-1}$$

$$\chi_{A_{2m-1}}(l\varpi_{2m-1})\,|_{C_m} = \chi_{C_m}(l\varpi_1'), \tag{2B′-2}$$

$$\chi_{A_{2m-1}}(l\varpi_1 + k\varpi_2) - \chi_{A_{2m-1}}(l\varpi_1 + (k-1)\varpi_2)\,|_{C_m} = \chi_{C_m}(l\varpi_1' + l\varpi_2'), \tag{2B′-3}$$

$$\begin{aligned}&\chi_{A_{2m-1}}(l\varpi_{2m-1} + k\varpi_{2m-2}) - \chi_{A_{2m-1}}(l\varpi_{2m-1} + (k-1)\varpi_{2m-2})\,|_{C_m}\\ &\quad = \chi_{C_m}(l\varpi_1' + l\varpi_2').\end{aligned} \tag{2B′-4}$$

Lemma 4.4. *For $k \in \boldsymbol{N} \setminus \{0, 1\}$,*

$$\chi_{A_{2m}}(0)\,|_{B_m} = \chi_{B_m}(0), \tag{3B′-1}$$

$$\chi_{A_{2m}}(\varpi_1)\,|_{B_m} = \chi_{B_m}(\varpi_1'), \tag{3B′-2}$$

$$\chi_{A_{2m}}(k\varpi_1) - \chi_{A_{2m}}((k-2)\varpi_1)\,|_{B_m} = \chi_{B_m}(k\varpi_1'), \tag{3B′-3}$$

$$\chi_{A_{2m}}(k\varpi_{2m}) - \chi_{A_{2m}}((k-2)\varpi_{2m})\,|_{B_m} = \chi_{B_m}(k\varpi_1'). \tag{3B′-4}$$

Lemma 4.5. *For $k \in \boldsymbol{N} \setminus \{0\}$,*

$$\chi_{D_m}(0)\,|_{B_{m-1}} = \chi_{B_{m-1}}(0), \tag{4B′-1}$$

$$\chi_{D_m}(k\varpi_1) - \chi_{D_m}((k-1)\varpi_1)\,|_{B_{m-1}} = \chi_{B_{m-1}}(k\varpi_1'). \tag{4B′-2}$$

Lemma 4.6. *For $k \in \boldsymbol{N} \setminus \{0\}$,*

$$\chi_{E_6}(0)\,|_{F_4} = \chi_{F_6}(0), \tag{5B′-1}$$

$$\chi_{E_6}(k\varpi_1) - \chi_{E_6}((k-1)\varpi_1)\,|_{F_4} = \chi_{F_4}(k\varpi_1'), \tag{5B′-2}$$

$$\chi_{E_6}(k\varpi_6) - \chi_{E_6}((k-1)\varpi_6)\,|_{F_4} = \chi_{F_4}(k\varpi_1'). \tag{5B′-3}$$

Lemma 4.7. *For $k \in \boldsymbol{N} \setminus \{0\}$,*

$$\chi_{D_4}(0)\,|_{G_2} = \chi_{G_2}(0), \tag{6B′-1}$$

$$\chi_{D_4}(k\varpi_1) - \chi_{D_4}((k-1)\varpi_1)\,|_{G_2} = \chi_{G_2}(k\varpi_1'), \tag{6B′-2}$$

$$\chi_{D_4}(k\varpi_3) - \chi_{D_4}((k-1)\varpi_3)\,|_{G_2} = \chi_{G_2}(k\varpi_1'), \tag{6B′-3}$$

$$\chi_{D_4}(k\varpi_4) - \chi_{D_4}((k-1)\varpi_4)\,|_{G_2} = \chi_{G_2}(k\varpi_1'). \tag{6B′-4}$$

We give the framework of the proof of Lemmas 4.3, 4.4, 4.5, 4.6 and 4.7 by several steps.

STEP 0:
Since the characters of trivial representation are 0, the equations (3B′-1), (4B′-1), (5B′-1) and (6B′-1) are clear. The equations (2B′-1) and (2B′-2) were proved in Lemma 4.1. $\mathrm{L}\,(A_{2m}, \varpi_1)$ and $\mathrm{L}\,(B_m, \varpi_1)$ are the standard representations, respectively; that are the same representation $\boldsymbol{C}^{2m+1}$. Then we obtain $\mathrm{L}\,(A_{2m}, \varpi_1)\,|_{B_m} = \mathrm{L}\,(B_m, \varpi_1)$, that is (3B′-2). If (2B′-3) is proved, we obtain the equation (2B′-4) by applying σ to (2B′-3). Similarly, we need not prove the equations (3B′-4), (5B′-3), (6B′-3) and (6B′-4). Thus, the remainder to be proved are (2B′-3), (3B′-3), (4B′-2), (5B′-2) and (6B′-2).

Suppose $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi)$ is one of the following cases:

$$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = \begin{cases} (A_{2m-1}, C_m, l\varpi_1 + k\varpi_2, \varpi_2) \\ (A_{2m}, B_m, k\varpi_1, \varpi_1) \\ (D_m, B_{m-1}, k\varpi_1, \varpi_1) \\ (E_6, F_4, k\varpi_1, \varpi_1) \\ (D_4, G_2, k\varpi_1, \varpi_1), \end{cases}$$

where $k, l \in \boldsymbol{N} \setminus \{0\}$.

STEP 1 (correspondence between the summation parameters):
Put an integer q as the following:

$$q = \begin{cases} 2 & \text{for} \ \ (A_{2m}, B_m, k\varpi_1, \varpi_1) \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

By the deformation (8) of Weyl's character formula of $\mathfrak{g}$, the left-hand side of any of (2B′-2), (3B′-3), (4B′-2), (5B′-2) and (6B′-2) is the form:

$$\begin{aligned} &\sum_{w_1 \in W^\lambda} w_1 \cdot \left(e^\lambda \prod_{\alpha \in \Delta^+ \setminus \Delta^+_\lambda} (1 - e^{-\alpha})^{-1} \cdot (1 - e^{-q\varpi}) \right) \Bigg|_{\mathfrak{h}^\sigma} \\ &= \sum_{w_1 \in W'^{\lambda'}} w_1 \cdot \left(e^\lambda \prod_{\alpha \in \Delta^+ \setminus \Delta^+_\lambda} (1 - e^{-\alpha})^{-1} \cdot (1 - e^{-q\varpi}) \right) \Bigg|_{\mathfrak{h}^\sigma} \\ &\quad + \sum_{w_1 \in W^\lambda \setminus W'^{\lambda'}} w_1 \cdot \left(e^\lambda \prod_{\alpha \in \Delta^+ \setminus \Delta^+_\lambda} (1 - e^{-\alpha})^{-1} \cdot (1 - e^{-q\varpi}) \right) \Bigg|_{\mathfrak{h}^\sigma} . \end{aligned}$$

Our key technique (Lemma 6.1) is the equality $e^{-w_1\varpi}|_{\mathfrak{h}^\sigma} = 1$ for $w_1 \in W^\lambda \setminus W'^{\lambda'}$ by choosing a specific system of representatives (minimal coset representatives). We postpone the proof of the lemma to Section 6.

Lemma 6.1 implies that the second term vanishes. Thus the left-hand side of any of

(2B′-2), (3B′-3), (4B′-2), (5B′-2) and (6B′-2) is the form:

$$\sum_{w_1\in W'^{\lambda'}} w_1\cdot\left(e^{\lambda}\prod_{\alpha\in\Delta^+\backslash\Delta^+_\lambda}(1-e^{-\alpha})^{-1}\cdot(1-e^{-q\varpi})\right)\bigg|_{\mathfrak{h}^\sigma}. \tag{13}$$

On the other hand, by the deformation (9) of Weyl's character formula of $\mathfrak{g}^\sigma$, the right-hand side of any of (2B′-2), (3B′-3), (4B′-2), (5B′-2) and (6B′-2) is the form:

$$\sum_{w_1\in W'^{\lambda'}} w_1\cdot\left(e^{\lambda'}\prod_{\alpha\in\Delta'^+\backslash\Delta'^+_{\lambda'}}(1-e^{-\alpha})^{-1}\right). \tag{14}$$

Step 2 (correspondence of the first term):
The following equality holds:

$$e^{\lambda}\prod_{\alpha\in\Delta^+\backslash\Delta^+_\lambda}(1-e^{-\alpha})^{-1}\cdot(1-e^{-q\varpi})\bigg|_{\mathfrak{h}\sigma}=e^{\lambda'}\prod_{\alpha\in\Delta'^+\backslash\Delta'^+_{\lambda'}}(1-e^{-\alpha})^{-1}. \tag{15}$$

We shall postpone the proof of (15) to Section 5. The proof is comparison between $\Delta^+\setminus\Delta^+_\lambda$ and $\Delta'^+\setminus\Delta'^+_{\lambda'}$ on a case basis.

Step 3:
The equation (15) is equality between the summands for $w_1=1$ in (13) and (14).
Applying $w_1\in W'^{\lambda'}$ to both sides of (15), we obtain the following equation.

$$w_1\cdot\left(e^{\lambda}\prod_{\alpha\in\Delta^+\backslash\Delta^+_\lambda}(1-e^{-\alpha})^{-1}\cdot(1-e^{-q\varpi})\right)\bigg|_{\mathfrak{h}^\sigma}=w_1|_{\mathfrak{h}^\sigma}\left(e^{\lambda'}\prod_{\alpha\in\Delta'^+\backslash\Delta'^+_{\lambda'}}(1-e^{-\alpha})^{-1}\right). \tag{16}$$

Then, we obtain (2B′-2), (3B′-3), (4B′-2), (5B′-2) and (6B′-2) and Lemmas 4.3, 4.4, 4.5, 4.6 and 4.7, if the postponed facts $e^{-w_1\varpi}|_{\mathfrak{h}^\sigma}=1$ for $w_1\in W^\lambda\smallsetminus W'^{\lambda'}$ (Lemma 6.1) and (15) are proved. We postpone the proofs to Section 5 and Section 6.

5. Comparison of positive roots.

In this section, we prove the equations (10) and (15) which are remained in previous section by comparing the restriction of the elements in $\Delta^+\smallsetminus\Delta^+_\lambda$ with the elements in $\Delta'^+\smallsetminus\Delta'^+_{\lambda'}$ on a case basis. That is, we obtain the following lemma.

Lemma 5.1.

(1) *Suppose* $(\mathfrak{g},\mathfrak{g}^\sigma,\lambda)$ *is one of the following cases*:

$$(\mathfrak{g},\mathfrak{g}^\sigma,\lambda)=\begin{cases}(A_{2m-1},C_m,k\varpi_1)\\(D_m,\ B_{m-1},k\varpi_{m-1}),\end{cases}$$

where $k \in \boldsymbol{N} \setminus \{0\}$. *Then the restriction map*

$$(\cdot)|_{\mathfrak{h}^\sigma} : \left(\Delta^+ \setminus \Delta^+_\lambda\right) \to \left(\Delta'^+ \setminus \Delta'^+_{\lambda'}\right) \tag{17}$$

is bijective.

(2) *Suppose*

$$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m}, B_m, k\varpi_1, \varpi_1),$$

where $k \in \boldsymbol{N} \setminus \{0\}$ *and* $l \in \boldsymbol{N}$. *Then the restriction map*

$$(\cdot)|_{\mathfrak{h}^\sigma} : \left(\Delta^+ \setminus \Delta^+_\lambda\right) \to \left((\Delta'^+ \setminus \Delta'^+_{\lambda'}) \sqcup \{2\varpi|_{\mathfrak{h}^\sigma}\}\right) \tag{18}$$

is bijective.

(3) *Suppose* $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda)$ *is one of the following cases*:

$$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = \begin{cases} (A_{2m-1}, C_m, l\varpi_1 + k\varpi_2, \varpi_2), \\ (D_m, B_{m-1}, k\varpi_1, \varpi_1), \\ (E_6, F_4, k\varpi_1, \varpi_1), \\ (D_4, G_2, k\varpi_1, \varpi_1), \end{cases}$$

where $k \in \boldsymbol{N} \setminus \{0\}$. *Then the restriction map*

$$(\cdot)|_{\mathfrak{h}^\sigma} : \left(\Delta^+ \setminus \Delta^+_\lambda\right) \to \left(\Delta'^+ \setminus \Delta'^+_{\lambda'}\right) \tag{19}$$

is onto. Furthermore, we have $\varpi|_{\mathfrak{h}^\sigma} \in \Delta'^+ \setminus \Delta'^+_{\lambda'}$ *and the map is one-to-one to* $(\Delta'^+ \setminus \Delta'^+_{\lambda'}) \setminus \{\varpi|_{\mathfrak{h}^\sigma}\}$ *and two-to-one to* $\{\varpi|_{\mathfrak{h}^\sigma}\}$.

Proof. Let us prove Lemma 5.1 by describing the subsets $\Delta^+ \setminus \Delta^+_\lambda$ and $\Delta'^+ \setminus \Delta'^+_{\lambda'}$. The labeling of the simple roots here (see Table 1) follows that of Bourbaki [**2**].

(1) We give the list of the elements of $\Delta^+ \setminus \Delta^+_\lambda$ and their restriction to $\mathfrak{h}^\sigma$ for $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1)$ or $(D_m, B_{m-1}, k\varpi_{m-1})$.

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1)$	
elements of $\Delta^+ \setminus \Delta^+_\lambda$	restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=1}^{j-1} \alpha_\nu \quad (2 \leq j \leq m+1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu$
$\sum_{\nu=1}^{j-1} \alpha_\nu \quad (m+2 \leq j \leq 2m)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu + \sum_{\nu=j}^{m-1} 2\alpha' + \alpha'_m$

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_m, B_{m-1}, k\varpi_{m-1})$	
elements of $\Delta^+ \setminus \Delta_\lambda^+$	restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=i}^{m-1} \alpha_\nu \quad (1 \le i \le m-1)$	$\sum_{\nu=i}^{m-1} \alpha'_\nu$
$\sum_{\nu=i}^{m} \alpha_\nu \quad (1 \le i \le m-2)$	$\sum_{\nu=i}^{m-2} \alpha'_\nu + 2\alpha'_m$
$\sum_{\nu=i}^{j-1} \alpha_\nu + \sum_{\nu=j}^{m-2} 2\alpha_\nu + \alpha_{m-1} + \alpha_m \quad (1 \le i < j \le m-2)$	$\sum_{\nu=i}^{j-1} \alpha'_\nu + \sum_{\nu=j}^{m-2} 2\alpha'_\nu + 2\alpha'_{m-1}$

Hence we obtain Lemma 5.1(1), because the restriction to $\mathfrak{h}^\sigma$ of the elements of $\Delta^+ \setminus \Delta_\lambda^+$ correspond to the elements of $\Delta'^+ \setminus \Delta'^+_{\lambda'}$.

(2) We give the list of the elements of $\Delta^+ \setminus \Delta_\lambda^+$ and their restriction to $\mathfrak{h}^\sigma$ for $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m}, B_m, k\varpi_1, \varpi_1)$. In the following list, the element with frame means that it is a scalar multiple of $\varpi|_{\mathfrak{h}^\sigma}$. We have $\varpi|_{\mathfrak{h}^\sigma} = \sum_{\nu=1}^m \alpha_\nu$ because $\varpi|_{\mathfrak{h}^\sigma} = \varpi'_2$ (see Table 1) and an easy computation implies $\varpi'_2 = \sum_{\nu=1}^m \alpha_\nu$.

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m}, B_m, k\varpi_1, \varpi_1)$	
elements of $\Delta^+ \setminus \Delta_\lambda^+$	restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=1}^{j-1} \alpha_\nu \quad (2 \le j \le m+1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu$
$(m+2 \le j \le 2m)$	$\sum_{\nu=1}^{2m-j+1} \alpha_\nu + \sum_{\nu=2m-j+2}^{m} 2\alpha'_\nu$
$(j = 2m+1)$	$\boxed{\sum_{\nu=1}^{m} 2\alpha'_\nu}$

The element with frame is $2\varpi|_{\mathfrak{h}^\sigma}$.

Thus, the above table completes the proof of Lemma 5.1(2).

(3) We give the lists of the elements of $\Delta^+ \setminus \Delta_\lambda^+$ and their restriction to $\mathfrak{h}^\sigma$ according to $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m-1}, C_m, k\varpi_2, \varpi_2)$, $(A_{2m-1}, C_m, l\varpi_1 + k\varpi_2, \varpi_2)$, $(D_m, B_{m-1}, k\varpi_1, \varpi_1)$, $(E_6, F_4, k\varpi_1, \varpi_1)$ and $(D_4, G_2, k\varpi_1, \varpi_1)$. In the following lists, the element with frame means that it is a scalar multiple of $\varpi|_{\mathfrak{h}^\sigma}$. The explicit form of the restriction $\varpi|_{\mathfrak{h}^\sigma}$ (e.g. $\alpha'_1 + \sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m$) can be found by using Table 1 and by a computation of fundamental weights of $\mathfrak{g}^\sigma$ for each case.

We abbreviate $k_1\alpha_1 + \cdots + k_n\alpha_n$ to $k_1 \cdots k_n$ for $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (E_6, F_4, k\varpi_1, \varpi_1)$ and $(D_4, G_2, k\varpi_1, \varpi_1)$ for convenient.

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m-1}, C_m, k\varpi_2, \varpi_2)$	
elements of $\Delta^+ \setminus \Delta^+_\lambda$	restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=1}^{j-1} \alpha_\nu \quad (3 \leq j \leq m+1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu$
$(m+2 \leq j \leq 2m-2)$	$\sum_{\nu=1}^{2m-j} \alpha'_\nu + \sum_{\nu=2m-j+1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m-1)$	$\boxed{\alpha'_1 + \sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m}$
$(j = 2m)$	$\sum_{\nu=1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$\sum_{\nu=2}^{j-1} \alpha_\nu \quad (3 \leq j \leq m+1)$	$\sum_{\nu=2}^{j-1} \alpha'_\nu$
$(m+2 \leq j \leq 2m-2)$	$\sum_{\nu=2}^{2m-j} \alpha'_\nu + \sum_{\nu=2m-j+1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m-1)$	$\sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m)$	$\boxed{\alpha'_1 + \sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m}$

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (A_{2m-1}, C_m, l\varpi_1 + k\varpi_2, \varpi_2)$	
elements of $\Delta^+ \setminus \Delta^+_\lambda$	restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=1}^{j-1} \alpha_\nu \quad (2 \leq j \leq m+1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu$
$(m+2 \leq j \leq 2m-2)$	$\sum_{\nu=1}^{2m-j} \alpha_\nu + \sum_{\nu=2m-j+1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m-1)$	$\boxed{\alpha'_1 + \sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m}$
$(j = 2m)$	$\sum_{\nu=1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$\sum_{\nu=2}^{j-1} \alpha_\nu \quad (3 \leq j \leq m+1)$	$\sum_{\nu=2}^{j-1} \alpha'_\nu$
$(m+2 \leq j \leq 2m-2)$	$\sum_{\nu=2}^{2m-j} \alpha'_\nu + \sum_{\nu=2m-j+1}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m-1)$	$\sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m$
$(j = 2m)$	$\boxed{\alpha_1 + \sum_{\nu=2}^{m-1} 2\alpha'_\nu + \alpha'_m}$

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (D_m, B_{m-1}, k\varpi_1, \varpi_1)$		
elements of $\Delta^+ \setminus \Delta^+_\lambda$		restriction to $\mathfrak{h}^\sigma$
$\sum_{\nu=1}^{j-1} \alpha_\nu$	$(2 \le j \le m-1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu$
	$(j = m+1)$	$\sum_{\nu=1}^{m-2} \alpha'_\nu + 2\alpha'_{m-1}$
	$(j = m)$	$\boxed{\sum_{\nu=1}^{m-1} \alpha'_\nu}$
$\sum_{\nu=1}^{j-1} \alpha_\nu + \sum_{\nu=j}^{m-2} 2\alpha_\nu + \alpha_{m-1} + \alpha_m$	$(2 \le j \le m-1)$	$\sum_{\nu=1}^{j-1} \alpha'_\nu + \sum_{\nu=j}^{m-1} 2\alpha'_\nu$
$\sum_{\nu=1}^{m-2} \alpha_\nu + \alpha_m$		$\boxed{\sum_{\nu=1}^{m-1} \alpha'_\nu}$

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (E_6, F_4, k\varpi_1, \varpi_1)$	
elements of $\Delta^+ \setminus \Delta^+_\lambda$	restriction to $\mathfrak{h}^\sigma$
100000	0001
101000	0011
101100	0111
101110	0121
101111	0122
111100	1111
111110	1121
111111	1122
111210	1221
111211	1222
112210	1231
112211	$\boxed{1232}$
111221	$\boxed{1232}$
112221	1242
112321	1342
122321	2342

$(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda, \varpi) = (D_4, G_2, k\varpi_1, \varpi_1)$	
elements of $\Delta^+ \setminus \Delta^+_\lambda$	restriction to $\mathfrak{h}^\sigma$
1000	10
1100	11
1110	$\boxed{21}$
1111	31
1211	32
1101	$\boxed{21}$

By definition, the elements with frame means that it is a scalar multiple of $\varpi|_{\mathfrak{h}^\sigma}$. It turns out that this is exactly $\varpi|_{\mathfrak{h}^\sigma}$ itself for all the cases in (3). Moreover the framed element $\varpi|_{\mathfrak{h}^\sigma}$ appears twice. We have that $\Delta'^+ \setminus \Delta'^+_{\lambda'}$ contains the fundamental weight $\varpi|_{\mathfrak{h}^\sigma}$ of $\mathfrak{g}^\sigma$ in each case. Thus, the above tables complete the proof of Lemma 5.1(3). □

Thus we have proved (10) and (15) which were postponed in Section 4.

6. Characterization of some representative of a quotient of the Weyl groups involved.

In this section, we obtain the key lemma (Lemma 6.1) of this paper which was postponed in the proof of Theorem 1.2 in Section 4.3.2(Step 1) after reviewing the definitions of the minimal coset representatives and the weak Bruhat order for Weyl groups.

6.1.

Let us review what is called minimal coset representatives of W/W_λ. For a dominant weight $\lambda \in V$, we define a subset of W by $W^\lambda =: \{w \in W | l(ws) > l(w) \text{ for all } s \in W_\lambda\}$, where l is the length function on W. Then the composition of $W^\lambda \hookrightarrow W \rightarrow W/W_\lambda$ becomes a bijective map $W^\lambda \xrightarrow{\sim} W/W_\lambda$, so that W^λ gives a complete set of representatives of W/W_λ. Moreover, $w \in W^\lambda$ is the unique element of smallest length in the coset wW_λ and we may call the set W^λ minimal coset representatives (see Humphreys [**6**, Section 1.10]).

6.2.

We review the definition of the *weak Bruhat order* on the Weyl group W and that of the Hasse diagram of a partial ordered set $(\mathscr{P}, \leq)$.

We recall the weak Bruhat order $\leq$ on W. For $u, w \in W$, we shall say $u \leq w$, if there exist simple reflections $s_{j_1}, \ldots, s_{j_t}$ such that $l(s_{j_1} \cdots s_{j_i} u) = j_i + l(u)$ for $1 \leq j_i \leq j_t$ and $s_{j_1} \cdots s_{j_t} u = w$, where l is the length function on W. (The usual Bruhat order is defined by changing simple reflections into root reflections in the definition of the weak Bruhat order.)

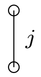
We restrict the weak Bruhat order to the minimal coset representatives W^λ and also call the weak Bruhat order.

6.3.

Below in Lemma 6.1, we take a specific highest weight λ and a fundamental weight ϖ for each simple Lie algebra $\mathfrak{g}$ of type A_{2m-1}, A_{2m}, D_m and E_6:

$$\lambda = \begin{cases} l\varpi_2 \text{ or } k\varpi_1 + l\varpi_2 & (A_{2m-1}) \\ k\varpi_1 & (A_{2m}) \\ k\varpi_1 & (D_m) \\ k\varpi_1 & (E_6) \end{cases} \qquad \varpi = \begin{cases} \varpi_2 & (A_{2m-1}) \\ \varpi_1 & (A_{2m}) \\ \varpi_1 & (D_m) \\ \varpi_1 & (E_6). \end{cases} \tag{20}$$

We draw the Hasse diagram of W^λ for the highest weight λ above (20). The cover relation of x and y in W^λ with the weak Bruhat order is that $y = s_j x$ with s_j a simple reflection s_{α_j} and $l(y) = l(x) + 1$. Moreover, we label the edge j, as following:



The Hasse diagram of $(W^\lambda, \leq)$ in each case is drawn at the end and some of the vertices are marked with filled black nodes, which we call *black circles*. The black circles represent the elements that satisfy the conditions in Lemma 6.1. We call the corresponding cosets in W/W_λ the "middle cosets".

6.4.

We draw the Hasse diagram of W^λ at the end of this section by studying the lexicographic order on the orbits $W \cdot \lambda$. In particular, the Hasse diagrams of W^λ have been studied as the weight diagrams of minuscule weights, except Figure 3.

We remark that the top vertex is the identity element of the Weyl group.

In Figure 3, there are two sheets of the one of Figure 1. We do not draw edges between two sheets to be easy to see, although corresponding elements of each sheet are connected with labeled edge. The top vertex of the left sheet is the identity element and the top of the right is s_1.

6.5.

LEMMA 6.1. *For the fundamental weight ϖ in* (20) *and the highest weight λ in* (20) *in each case, the following three conditions for $w \in W^\lambda$ are equivalent.*

(1) $w \notin \widetilde{W^\lambda}$.
(2) $w\varpi|_{\mathfrak{h}^\sigma} = 0$.
(3) *w is a black circle in Figures at the end of this section.*

REMARK 6.2. The black circles appeared in the condition (3) mean the vertices marked with filled black nodes in Figures at the end of this section.

PROOF OF ((2) $\Longrightarrow$ (1)). By Lemma 2.10, the condition (2) is equivalent to $\sum_{j=0}^{m-1} \sigma^j(w\varpi) = 0$, where m is the order of σ. If $w \in \widetilde{W^\lambda}$, we obtain the following equation, because $w\sigma = \sigma w$:

$$0 = \sum_{j=0}^{m-1} \sigma^j(w\varpi) = w\biggl(\sum_{j=0}^{m-1} \sigma^j \varpi\biggr).$$

Then $\varpi|_{\mathfrak{h}^\sigma} = \frac{1}{m}\sum_{j=0}^{m-1} \sigma^j \varpi = 0$.

On the other hand, $\varpi|_{\mathfrak{h}^\sigma}$ is a fundamental weight of $\mathfrak{g}^\sigma$ by Table 1. In particular, $\varpi|_{\mathfrak{h}^\sigma} \neq 0$. This is contradiction. $\square$

PROOF OF ((3) $\Longrightarrow$ (2)). The proof is studied on a case basis. The black circles are the vertices marked with filled black nodes in at the end of this section.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, l\varpi_2)(l \geq 1)$, if we put $t = s_{2m-1}s_{2m-2}\dots s_3 s_2$, then the black circles in Figure 1 are

$$t_1 := t \quad \text{and} \quad t_j := s_{j-1}s_{2m-j+1}t_{j-1} \ \ (2 \leq j \leq m),$$

and the fundamental weight ϖ in (20) is

$$\varpi_2 = \frac{m-1}{m}\alpha_1 + \frac{2m-2}{m}\alpha_2 + \cdots + \frac{m+1}{m}\alpha_{m-1} + \frac{m}{m}\alpha_m + \frac{m-1}{m}\alpha_{m+1} + \cdots + \frac{2}{m}\alpha_{2m-2} + \frac{1}{m}\alpha_{2m-1}.$$

Then

$$t\varpi_2 = \frac{m-1}{m}\alpha_1 + \frac{m-2}{m}\alpha_2 + \cdots + \frac{1}{m}\alpha_{m-1} + 0\cdot\alpha_m + \frac{-1}{m}\alpha_{m+1} + \cdots + \frac{-(m-1)}{m}\alpha_{2m-1}.$$

We obtain $t\varpi_2 + \sigma t\varpi_2 = 0$. We obtain the conclusion by the following claim.

Claim 6.3. *If $\mu + \sigma\mu = 0$, then $s_j s_{2m-j}\mu + \sigma s_j s_{2m-j}\mu = 0 \quad (j = 1, \dots, m)$.*

Because $s_{2m-j} = \sigma s_j \sigma$ and $s_j s_{2m-j} = s_{2m-j} s_j$, we obtain

$$\sigma s_j s_{2m-j} = \sigma s_j(\sigma s_j \sigma) = (\sigma s_j \sigma) s_j \sigma = s_{2m-j} s_j \sigma.$$

Then

$$s_j s_{2m-j}\mu + \sigma s_j s_{2m-j}\mu = s_j s_{2m-j}(\mu + \sigma\mu) = 0.$$

Thus we proved the claim.

Then we obtain $t_j\varpi_2 + \sigma t_j\varpi_2 = 0 \quad (j = 1, \dots, m)$, inductively.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1 + l\varpi_2) \quad (k, l \geq 1)$, put $t = s_{2m-1}s_{2m-2}\dots s_2$ and $t' = ts_1$, then the black circles in Figure 3 are:

$$t_1 := t, \quad t_j := s_{j-1}s_{2m-1}t_{j-1} \;\; (2 \leq j \leq m), \quad \text{and} \quad t'_j := t_j s_1 \quad (1 \leq j \leq m).$$

By the previous case, $t_j\varpi_2 + \sigma t_j\varpi_2 = 0$. Then $t'_j\varpi_2 + \sigma t'_j\varpi_2 = 0$, because $s_1\varpi_2 = \varpi_2$.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m}, B_m, k\varpi_1) \quad (k \geq 1)$, the black circle in Figure 4 is

$$t_1 = s_m \dots s_2 s_1,$$

and the fundamental weight ϖ in (20) is

$$\varpi_1 = \frac{2m}{2m+1}\alpha_1 + \cdots + \frac{m+1}{2m+1}\alpha_m + \frac{m}{2m+1}\alpha_{m+1} + \cdots + \frac{1}{2m+1}\alpha_{2m}.$$

Then,

$$t_1\varpi_1 = \frac{-1}{2m+1}\alpha_1 + \cdots + \frac{-m}{2m+1}\alpha_m + \frac{m}{2m+1}\alpha_{m+1} + \cdots + \frac{1}{2m+1}\alpha_{2m}.$$

We obtain $t_1\varpi_1 + \sigma t_1\varpi_1 = 0$.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_m, B_{m-1}, k\varpi_1) \quad (k \geq 1)$, the black circles in Figure 5 are:

$$t_1 = s_{m-1}s_{m-2}\dots s_2s_1 \quad \text{and} \quad t_2 = s_m s_{m-2}\dots s_2s_1,$$

and the fundamental weight ϖ in (20) is

$$\varpi_1 = \alpha_1 + \cdots + \alpha_{m-2} + \frac{1}{2}\alpha_{m-1} + \frac{1}{2}\alpha_m.$$

Then,

$$t_1\varpi_1 = \frac{-1}{2}\alpha_{m-1} + \frac{1}{2}\alpha_m \quad \text{and} \quad t_2\varpi_1 = \frac{1}{2}\alpha_{m-1} - \frac{1}{2}\alpha_m.$$

We obtain $t_1\varpi_1 + \sigma t_1\varpi_1 = 0$ and $t_2\varpi_1 + \sigma t_2\varpi_1 = 0$.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (E_6, F_4, k\varpi_1) \quad (k \geq 1)$, the black circles in Figure 6 are:

$$t_1 = s_1s_3s_4s_2s_5s_4s_3s_1, \quad t_2 = s_6s_3s_4s_2s_5s_4s_3s_1 \quad \text{and} \quad t_3 = s_5s_6s_4s_2s_5s_4s_3s_1,$$

and the fundamental weight ϖ in (20) is

$$\varpi_1 = \frac{4}{3}\alpha_1 + \alpha_2 + \frac{5}{3}\alpha_3 + 2\alpha_4 + \frac{4}{3}\alpha_5 + \frac{2}{3}\alpha_6.$$

Then,

$$t_1\varpi_1 = \frac{-2}{3}\alpha_1 + \frac{-1}{3}\alpha_3 + \frac{1}{3}\alpha_5 + \frac{2}{3}\alpha_6.$$
$$t_2\varpi_1 = \frac{1}{3}\alpha_1 + \frac{-1}{3}\alpha_3 + \frac{1}{3}\alpha_5 + \frac{-1}{3}\alpha_6.$$
$$t_3\varpi_1 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3 + \frac{-2}{3}\alpha_5 + \frac{-1}{3}\alpha_6.$$

We obtain $t_1\varpi_1 + \sigma t_1\varpi_1 = t_2\varpi_1 + \sigma t_2\varpi_1 = t_3\varpi_1 + \sigma t_3\varpi_1 = 0$.

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_4, G_2, k\varpi_1) \quad (k \geq 1)$, the black circles in Figure 5 at the end of this section are:

$$t_1 = s_3s_2s_1 \text{ and } t_2 = s_4s_2s_1,$$

and the fundamental weight in (20) is

$$\varpi_1 = \alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_4.$$

Then,

$$t_1\varpi_1 = \frac{-1}{2}\alpha_3 + \frac{1}{2}\alpha_4, \quad t_2\varpi_1 = \frac{1}{2}\alpha_3 - \frac{1}{2}\alpha_4.$$

We obtain $t_1\varpi_1 + \sigma t_1\varpi_1 + \sigma^2 t_1\varpi_1 = 0$ and $t_2\varpi_1 + \sigma t_2\varpi_1 + \sigma^2 t_2\varpi_1 = 0$.

Thus we obtain the conclusion in all the cases. □

PROOF OF ((1) $\Longrightarrow$ (3)). We prove that the number of $W^\lambda \setminus W'^{\lambda'}$ equals the number of black circles on a case basis. That is, we calculate

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \sharp(W/W_\lambda) - \sharp\bigl(W'/W'^{\lambda'}\bigr),$$

for each λ in (20).

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, l\varpi_2) \quad (l \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{(2m)!}{2 \cdot (2m-2)!} - \frac{m! \cdot 2^m}{2 \cdot (m-2)! \cdot 2^{m-2}} = m.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m-1}, C_m, k\varpi_1 + l\varpi_2) \quad (k, l \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{(2m)!}{(2m-2)!} - \frac{m! \cdot 2^m}{(m-2)! \cdot 2^{m-2}} = 2m.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (A_{2m}, B_m, k\varpi_1) \quad (k \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{(2m+1)!}{(2m)!} - \frac{m! \cdot 2^m}{(m-1)! \cdot 2^{m-1}} = 1.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_m, B_{m-1}, k\varpi_1) \quad (k \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{m! \cdot 2^{m-1}}{(m-1)! \cdot 2^{m-2}} - \frac{(m-1)! \cdot 2^{m-2}}{(m-2)! \cdot 2^{m-2}} = 2.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (E_6, F_4, k\varpi_1) \quad (k \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{2^7 \cdot 3^4 \cdot 5}{5! \cdot 2^4} - \frac{2^7 \cdot 3^2}{3! \cdot 2^3} = 3.$$

For $(\mathfrak{g}, \mathfrak{g}^\sigma, \lambda) = (D_4, G_2, k\varpi_1) \quad (k \geq 1)$,

$$\sharp\bigl(W^\lambda \setminus W'^{\lambda'}\bigr) = \frac{4! \cdot 2^3}{3! \cdot 2^2} - \frac{2^2 \cdot 3}{2} = 2.$$

□

We have proved all the facts postponed in Section 4. Thus we complete the proof of Theorems 1.1 and 1.2.

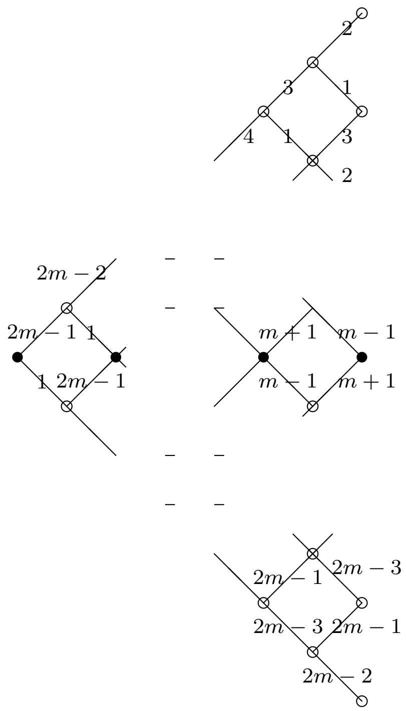


Figure 1. $W^{k\varpi_2}$ ($\mathfrak{g} = A_{2m-1}$).

We draw the diagram of $W^{k\varpi_2}$ ($\mathfrak{g} = A_7$) for convenience.

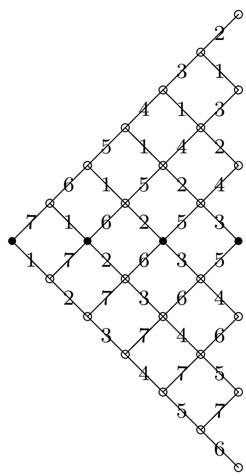


Figure 2. $W^{k\varpi_2}$ ($\mathfrak{g} = A_7$).

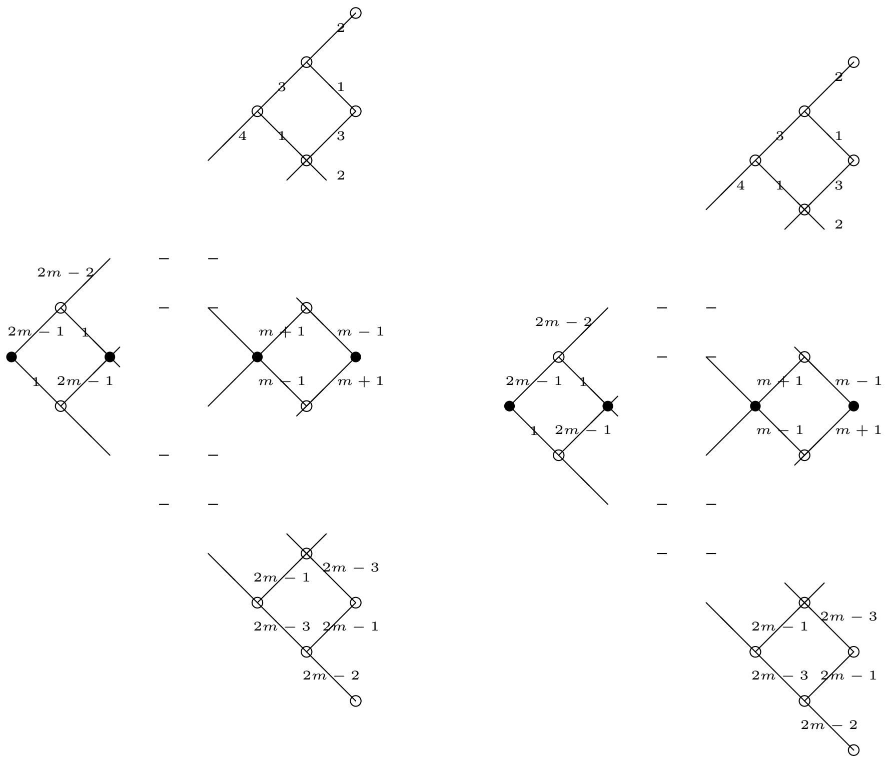


Figure 3. $W^{k\varpi_1+l\varpi_2}$ ($\mathfrak{g} = A_{2m-1}$). The top of the left is 1 and that of the right is s_1.

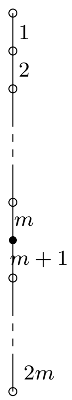


Figure 4. $W^{k\varpi_1}$ ($\mathfrak{g} = A_{2m}$).

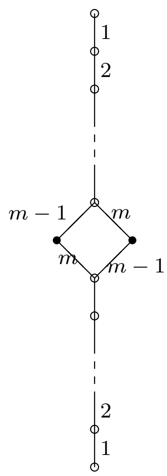


Figure 5. $W^{k\varpi_1}$ ($\mathfrak{g} = D_m$).

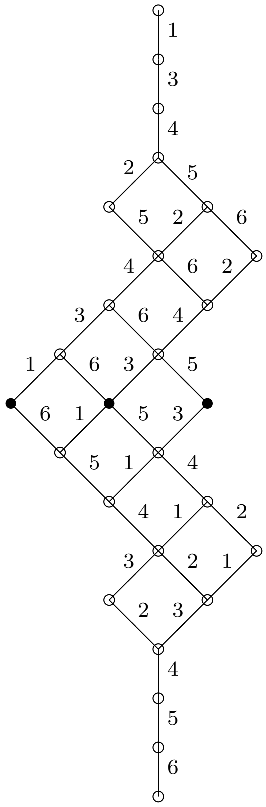


Figure 6. $W^{k\varpi_1}$ ($\mathfrak{g} = E_6$).

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