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Transformation relations of matrix functions associated to the hypergeometric function of Gauss under modular transformations

Dedicated to Professor Hiroshi Umemura on his sixtieth birthday

By Humihiko Watanabe

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Abstract. Two-by-two matrix functions, which are the lifts of the local solutions of the matrix hypergeometric differential equation of SL type at $0, 1, \infty$ to the upper half plane by the lambda function, are introduced. Each component of these matrix functions is represented by a definite integral with a power product of theta functions as integrand, which we call in this paper Wirtinger integral. Transformations of the matrix functions under some modular transformations are established by exploiting classical formulas of theta functions. These are regarded as formulas of monodromy or connection of the hypergeometric function of Gauss.

Introduction.

In this paper we consider 2×2 matrix functions analytic on the upper half plane associated to the hypergeometric function of Gauss, and establish transformations of these matrix functions under some modular transformations. The matrix functions studied here are obtained as the lifts of the local solutions of the matrix hypergeometric differential equation of SL type (i.e., whose image of monodromy representation is contained in $SL(2, \boldsymbol{C})$) at $0, 1, \infty$ to the upper half plane by the lambda function (Section 2). Each component of the matrix functions is represented by a definite integral with a power product of theta functions as integrand. Such an integral was invented by Wirtinger in order to uniformize the hypergeometric function of Gauss to the upper half plane ([**5**]). In this paper we call it *Wirtinger integral* (cf. (1.3)). As a classical example in which the Wirtinger integral is considered, we can cite Elliott's paper [**7**] (see also Dixon [**6**]), where a generalization of the Legendre relation in the theory of elliptic integrals was established. One of the advantages of exploiting the matrix functions above in the study of the hypergeometric function is that the monodromy property and the connection relations of the hypergeometric function are all translated as transformations of those matrix functions under modular transformations of the independent variable (Section 3). Moreover we can derive such transformations by exploiting classical formulas of theta functions without need to use any monodromy property or connection formula of the hypergeometric

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function. That is to say, this gives another new derivation of the monodromy property and the connection formulas of the hypergeometric function of Gauss.

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1. Wirtinger integral for the hypergeometric function of Gauss.

Following the notation of Chandrasekharan [**1**], we introduce the four theta functions $\theta(u,\tau), \theta_i(u,\tau)$ $(i = 1, 2, 3)$ by

$$\theta(u,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iu},$$

$$\theta_1(u,\tau) = \sum_{n=-\infty}^{+\infty} e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iu},$$

$$\theta_2(u,\tau) = \sum_{n=-\infty}^{+\infty}(-1)^n e^{n^2\pi i\tau}e^{2n\pi iu},$$

$$\theta_3(u,\tau) = \sum_{n=-\infty}^{+\infty} e^{n^2\pi i\tau}e^{2n\pi iu},$$

which are defined for all $(u,\tau) \in \boldsymbol{C}\times H$, where H denotes the upper half plane. Mumford [**2**] (see also Umemura [**3**]) adopts the symbols $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ to denote the theta functions above. The relations between the two notations are as follows: $\theta(u,\tau) = -\theta_{11}(u,\tau)$, $\theta_1(u,\tau) = \theta_{10}(u,\tau)$, $\theta_2(u,\tau) = \theta_{01}(u,\tau)$, $\theta_3(u,\tau) = \theta_{00}(u,\tau)$. The lambda function $\lambda(\tau)$ is defined by $\lambda(\tau) = \frac{\theta_1(0,\tau)^4}{\theta_3(0,\tau)^4}$. It defines a mapping of H to the open set $U = \boldsymbol{P}^1 - \{0, 1, \infty\}$ of $\boldsymbol{P}^1$ the complex projective line, and is invariant under the action of $\Gamma(2)$ the principal congruence subgroup of level 2: $\lambda\bigl(\frac{a\tau+b}{c\tau+d}\bigr) = \lambda(\tau)$ for $\bigl(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\bigr) \in \Gamma(2)$. So the mapping defined by $\lambda(\tau)$ induces an isomorphism of $H/\Gamma(2)$ onto U. We can choose the set $C = \{\tau \in H \mid -1 \le \operatorname{Re}\tau < 1,\ |\tau + \frac{1}{2}| \ge \frac{1}{2},\ |\tau - \frac{1}{2}| > \frac{1}{2}\}$ as a fundamental domain of H for the group $\Gamma(2)$. By the behaviour of $\lambda(\tau)$ near the cusps, the points $\tau = 0, \pm 1, \infty$ correspond to the points $x = 1, \infty, 0$ of $\boldsymbol{P}^1$, respectively. Moreover, by the mapping defined by $\lambda(\tau)$, the positive imaginary axis of H maps to a curve of U with boundary points $x = 0, 1$ homotopic to the real open interval $(0, 1)$ in U, each of the upper semi-circles of H centered at $\tau = \pm\frac{1}{2}$ with radius $\frac{1}{2}$ maps to a curve of U with boundary points $x = 1, \infty$ homotopic to the real ray $(1, +\infty)$ in U, and each of the rays $(-1, -1 + i\infty)$, $(1, 1 + i\infty)$ of H parallel to the positive imaginary axis maps to a curve of U with boundary points $x = -\infty, 0$ homotopic to the real ray $(-\infty, 0)$ in U.

Let $F(\alpha, \beta, \gamma, x)$ denote the hypergeometric series of Gauss or its analytic continuation, and let $E(\alpha, \beta, \gamma)$ denote the hypergeometric differential equation of Gauss:

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0.$$

The following formula of modified Pochhammer type holds (cf. [**4**]):

$$F(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\times\int^{(1++,0++,1--,0--)} t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt, \qquad (1.1)$$

where $\alpha\neq 1,2,3,\ldots$, $\gamma\neq 0,-1,-2,\ldots$, $\gamma-\alpha\neq 1,2,3,\ldots$, and $(1++,0++,1--,0--)$ denotes the cycle with base point $t=\frac{1}{2}$ where $\arg t=\arg(1-t)=0$ turning first around $t=1$ twice anticlockwisely, second around $t=0$ twice anticlockwisely, third around $t=1$ twice clockwisely and lastly around $t=0$ twice clockwisely. Let us make the lift of the function $F(\alpha,\beta,\gamma,x)$ or, strictly speaking, of the analytic continuation of $F(\alpha,\beta,\gamma,x)$ to the upper half plane H. We set $\omega_1=\pi\theta_3(0,\tau)^2$, $\omega_2=\omega_1\tau$, $k^2=\lambda(\tau)$. Jacobi's elliptic functions are defined by

$$\operatorname{sn} v=\operatorname{sn}(v,k)=\frac{\theta_3(0,\tau)\theta\left(\dfrac{v}{\omega_1},\tau\right)}{\theta_1(0,\tau)\theta_2\left(\dfrac{v}{\omega_1},\tau\right)},$$

$$\operatorname{cn} v=\operatorname{cn}(v,k)=\frac{\theta_2(0,\tau)\theta_1\left(\dfrac{v}{\omega_1},\tau\right)}{\theta_1(0,\tau)\theta_2\left(\dfrac{v}{\omega_1},\tau\right)},$$

$$\operatorname{dn} v=\operatorname{dn}(v,k)=\frac{\theta_2(0,\tau)\theta_3\left(\dfrac{v}{\omega_1},\tau\right)}{\theta_3(0,\tau)\theta_2\left(\dfrac{v}{\omega_1},\tau\right)}.$$

Substituting $x=\lambda(\tau)$ and $t=\operatorname{sn}^2 v$ into (1.1), we have

$$F(\alpha,\beta,\gamma,\lambda(\tau))=\frac{2\Gamma(\gamma)}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\times\int^{(\frac{\omega_1}{2}+,0+,\frac{\omega_1}{2}-,0-)}(\operatorname{sn} v)^{2\alpha-1}(\operatorname{cn} v)^{2\gamma-2\alpha-1}(\operatorname{dn} v)^{-2\beta+1}dv, \qquad (1.2)$$

where $(\frac{\omega_1}{2}+,0+,\frac{\omega_1}{2}-,0-)$ denotes a usual Pochhammer cycle. This integral representation is obtained by Elliott [**7**] (see also [**6**]). Rewriting (1.2) with theta functions and setting $u=v/\omega_1$, we have

$$F(\alpha,\beta,\gamma,\lambda(\tau))=\frac{2\pi\Gamma(\gamma)\theta_3(0,\tau)^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\lambda(\tau)^{\frac{1-\gamma}{2}}(1-\lambda(\tau))^{\frac{\gamma-\alpha-\beta}{2}}\times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau)^{2\alpha-1}\theta_1(u,\tau)^{2\gamma-2\alpha-1}\theta_2(u,\tau)^{2\beta-2\gamma+1}\theta_3(u,\tau)^{-2\beta+1}du. \qquad (1.3)$$

We call this representation *Wirtinger integral* for the hypergeometric function of Gauss (see [**5**]). Note that the function $F(\alpha,\beta,\gamma,\lambda(\tau))$ of the variable τ is single-valued and holomorphic on H.

2. Hypergeometric functions of matrix form and their lifts to the upper half plane.

Let $Y=Y(x)$ be a 2×2 matrix-valued analytic function of the complex variable x, and let $A(x)$ denote the matrix-valued function given by

$$\begin{aligned} A(x) = {} & \frac{1}{(\alpha-\beta)x}\begin{bmatrix}\alpha(\beta-\gamma+1) & \alpha(\gamma-\beta-1)\\ \beta(\alpha-\gamma+1) & \beta(\gamma-\alpha-1)\end{bmatrix} \\ & + \frac{1}{(\alpha-\beta)(x-1)}\begin{bmatrix}\alpha(\gamma-\alpha-1) & \alpha(\beta-\gamma+1)\\ \beta(\gamma-\alpha-1) & \beta(\beta-\gamma+1)\end{bmatrix}, \end{aligned}$$

where α,β,γ denote complex parameters. Let us consider the following differential equation of 2×2 matrix form:

$$\frac{d}{dx}Y = A(x)Y. \tag{2.1}$$

This is a hypergeometric differential equation of matrix form. In fact, if we set

$$Y = \begin{bmatrix} y_{11}(x) & y_{12}(x)\\ y_{21}(x) & y_{22}(x)\end{bmatrix},$$

we see that the functions $y_{11}(x)$ and $y_{12}(x)$ satisfy the equation $E(\alpha,\beta+1,\gamma)$, and the functions $y_{21}(x)$ and $y_{22}(x)$ satisfy the equation $E(\alpha+1,\beta,\gamma)$. In what follows, we always assume that the parameters α,β,γ satisfy the conditions

$$\alpha\notin \boldsymbol{Z},\ \beta\notin \boldsymbol{Z},\ \gamma\notin \boldsymbol{Z},\ \gamma-\alpha\notin \boldsymbol{Z},\ \gamma-\beta\notin \boldsymbol{Z},\ \gamma-\alpha-\beta\notin \boldsymbol{Z},\ \text{and}\ \alpha-\beta\notin \boldsymbol{Z}. \tag{2.2}$$

Let $Y_0(x),Y_1(x),Y_\infty(x)$ be the local solutions of (2.1) at $x=0,1,\infty$, respectively, given by

$$Y_0(x) = \begin{bmatrix} F(\alpha,\beta+1,\gamma,x) & \alpha(\beta-\gamma+1)x^{1-\gamma}F(1+\alpha-\gamma,2+\beta-\gamma,2-\gamma,x)\\ F(\alpha+1,\beta,\gamma,x) & \beta(\alpha-\gamma+1)x^{1-\gamma}F(2+\alpha-\gamma,1+\beta-\gamma,2-\gamma,x)\end{bmatrix},$$

$$Y_1(x) = \begin{bmatrix} (\beta-\gamma+1)F(\alpha,\beta+1,\alpha+\beta-\gamma+2,1-x) & \alpha(1-x)^{\gamma-\alpha-\beta-1}F(\gamma-\alpha,\gamma-\beta-1,\gamma-\alpha-\beta,1-x)\\ (\alpha-\gamma+1)F(\alpha+1,\beta,\alpha+\beta-\gamma+2,1-x) & \beta(1-x)^{\gamma-\alpha-\beta-1}F(\gamma-\alpha-1,\gamma-\beta,\gamma-\alpha-\beta,1-x)\end{bmatrix},$$

$$Y_\infty(x) = \begin{bmatrix} (\alpha-\beta)(\alpha-\beta+1)(1-x)^{-\alpha}F(\alpha,\gamma-\beta-1,\alpha-\beta,(1-x)^{-1}) \\ \beta(\gamma-\alpha-1)(1-x)^{-\alpha-1}F(\gamma-\beta,\alpha+1,\alpha-\beta+2,(1-x)^{-1}) \end{bmatrix.}$$

$$\begin{matrix} \alpha(\gamma-\beta-1)(1-x)^{-\beta-1}F(\gamma-\alpha,\beta+1,\beta-\alpha+2,(1-x)^{-1}) \\ (\beta-\alpha)(\beta-\alpha+1)(1-x)^{-\beta}F(\beta,\gamma-\alpha-1,\beta-\alpha,(1-x)^{-1}) \end{bmatrix}.$$

The image of the monodromy representation of (2.1) is contained in the gerenal linear group $GL(2,\boldsymbol{C})$, but not in the special linear group $SL(2,\boldsymbol{C})$. To obtain from (2.1) a matrix differential equation whose image of the monodromy representation is contained in $SL(2,\boldsymbol{C})$, we introduce a new 2×2 matrix unknown $\tilde{Y}$ by

$$Y = x^{\frac{1-\gamma}{2}}(1-x)^{\frac{\gamma-\alpha-\beta-1}{2}}\tilde{Y}. \tag{2.3}$$

If we eliminate Y from (2.1) and (2.3), we have a new differential equation

$$\frac{d}{dx}\tilde{Y} = \tilde{A}(x)\tilde{Y}, \tag{2.4}$$

where $\tilde{A}(x)$ is given by

$$\tilde{A}(x) = \frac{1}{(\alpha-\beta)x}\begin{bmatrix} \alpha\beta-\dfrac{(\alpha+\beta)(\gamma-1)}{2} & \alpha(\gamma-\beta-1) \\ \beta(\alpha-\gamma+1) & -\alpha\beta+\dfrac{(\alpha+\beta)(\gamma-1)}{2} \end{bmatrix}$$

$$+\frac{1}{(\alpha-\beta)(x-1)}\begin{bmatrix} \alpha\beta-\dfrac{(\alpha+\beta)(\alpha+\beta-\gamma+1)}{2} & \alpha(\beta-\gamma+1) \\ \beta(\gamma-\alpha-1) & -\alpha\beta+\dfrac{(\alpha+\beta)(\alpha+\beta-\gamma+1)}{2} \end{bmatrix}.$$

From $Y_0(x), Y_1(x), Y_\infty(x)$, we can obtain via (2.3) the local solutions $\tilde{Y}_0(x), \tilde{Y}_1(x), \tilde{Y}_\infty(x)$ of Equation (2.4) at $x = 0, 1, \infty$, respectively. In fact, we have

$$\tilde{Y}_0(x) = \begin{bmatrix} x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha,\beta+1,\gamma,x) \\ x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha+1,\beta,\gamma,x) \end{matrix}$$

$$\begin{matrix} \alpha(\beta-\gamma+1)x^{\frac{1-\gamma}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(1+\alpha-\gamma,2+\beta-\gamma,2-\gamma,x) \\ \beta(\alpha-\gamma+1)x^{\frac{1-\gamma}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(2+\alpha-\gamma,1+\beta-\gamma,2-\gamma,x) \end{bmatrix},$$

$$\tilde{Y}_1(x) = \begin{bmatrix} (\beta-\gamma+1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha,\beta+1,\alpha+\beta-\gamma+2,1-x) \\ (\alpha-\gamma+1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha+1,\beta,\alpha+\beta-\gamma+2,1-x) \end{matrix}$$

$$\begin{matrix} \alpha x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\gamma-\alpha-\beta-1}{2}}F(\gamma-\alpha,\gamma-\beta-1,\gamma-\alpha-\beta,1-x) \\ \beta x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\gamma-\alpha-\beta-1}{2}}F(\gamma-\alpha-1,\gamma-\beta,\gamma-\alpha-\beta,1-x) \end{bmatrix},$$

$$\tilde{Y}_\infty(x)=\begin{bmatrix}(\alpha-\beta)(\alpha-\beta+1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{-\alpha+\beta-\gamma+1}{2}}F(\alpha,\gamma-\beta-1,\alpha-\beta,(1-x)^{-1}) \\ \beta(\gamma-\alpha-1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{-\alpha+\beta-\gamma-1}{2}}F(\gamma-\beta,\alpha+1,\alpha-\beta+2,(1-x)^{-1}) \\ \alpha(\gamma-\beta-1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha-\beta-\gamma-1}{2}}F(\gamma-\alpha,\beta+1,\beta-\alpha+2,(1-x)^{-1}) \\ (\beta-\alpha)(\beta-\alpha+1)x^{\frac{\gamma-1}{2}}(1-x)^{\frac{\alpha-\beta-\gamma+1}{2}}F(\beta,\gamma-\alpha-1,\beta-\alpha,(1-x)^{-1})\end{bmatrix}.$$

It is easy to see that the local monodromy matrix of each function $\tilde{Y}_i(x)$ $(i=0,1,\infty)$ has determinant one.

Let us make the lifts of the functions $\tilde{Y}_0(x),\tilde{Y}_1(x),\tilde{Y}_\infty(x)$ to the upper half plane H, using the Wirtinger integral (1.3). Namely, we set $\tau'=-1/\tau$, $\tau''=1/(1-\tau)$, $Z_0(\tau)=\tilde{Y}_0(\lambda(\tau))$, $Z_1(\tau')=\tilde{Y}_1(\lambda(\tau))$ and $Z_\infty(\tau'')=\tilde{Y}_\infty(\lambda(\tau))$. Applying (1.3) to each component of $\tilde{Y}_i(\lambda(\tau))$ $(i=0,1,\infty)$, we have

$$Z_0(\tau)=\begin{bmatrix}\dfrac{2\pi\Gamma(\gamma)\theta_3(0,\tau)^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau)^{2\alpha-1}\theta_1(u,\tau)^{2\gamma-2\alpha-1}\theta_2(u,\tau)^{2\beta-2\gamma+3}\theta_3(u,\tau)^{-2\beta-1}du \\ \dfrac{2\pi\Gamma(\gamma)\theta_3(0,\tau)^2}{(1-e^{4\pi i\beta})(1-e^{4\pi i(\gamma-\beta)})\Gamma(\beta)\Gamma(\gamma-\beta)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau)^{2\beta-1}\theta_1(u,\tau)^{2\gamma-2\beta-1}\theta_2(u,\tau)^{2\alpha-2\gamma+3}\theta_3(u,\tau)^{-2\alpha-1}du \\ \dfrac{2\pi\alpha\Gamma(2-\gamma)\theta_3(0,\tau)^2}{(1-e^{4\pi i(\beta-\gamma)})(1-e^{-4\pi i\beta})\Gamma(-\beta)\Gamma(1+\beta-\gamma)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau)^{2\beta-2\gamma+3}\theta_1(u,\tau)^{-2\beta-1}\theta_2(u,\tau)^{2\alpha-1}\theta_3(u,\tau)^{2\gamma-2\alpha-1}du \\ \dfrac{2\pi\beta\Gamma(2-\gamma)\theta_3(0,\tau)^2}{(1-e^{4\pi i(\alpha-\gamma)})(1-e^{-4\pi i\alpha})\Gamma(-\alpha)\Gamma(1+\alpha-\gamma)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau)^{2\alpha-2\gamma+3}\theta_1(u,\tau)^{-2\alpha-1}\theta_2(u,\tau)^{2\beta-1}\theta_3(u,\tau)^{2\gamma-2\beta-1}du\end{bmatrix},$$

$$Z_1(\tau')=\begin{bmatrix}\dfrac{2\pi\Gamma(\alpha+\beta-\gamma+2)\theta_3(0,\tau')^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\beta-\gamma)})\Gamma(\alpha)\Gamma(\beta-\gamma+1)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau')^{2\alpha-1}\theta_1(u,\tau')^{2\beta-2\gamma+3}\theta_2(u,\tau')^{2\gamma-2\alpha-1}\theta_3(u,\tau')^{-2\beta-1}du \\ \dfrac{2\pi\Gamma(\alpha+\beta-\gamma+2)\theta_3(0,\tau')^2}{(1-e^{4\pi i\beta})(1-e^{4\pi i(\alpha-\gamma)})\Gamma(\beta)\Gamma(\alpha-\gamma+1)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau')^{2\beta-1}\theta_1(u,\tau')^{2\alpha-2\gamma+3}\theta_2(u,\tau')^{2\gamma-2\beta-1}\theta_3(u,\tau')^{-2\alpha-1}du \\ \dfrac{2\pi\alpha\Gamma(\gamma-\alpha-\beta)\theta_3(0,\tau')^2}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{-4\pi i\beta})\Gamma(-\beta)\Gamma(\gamma-\alpha)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau')^{2\gamma-2\alpha-1}\theta_1(u,\tau')^{-2\beta-1}\theta_2(u,\tau')^{2\alpha-1}\theta_3(u,\tau')^{2\beta-2\gamma+3}du \\ \dfrac{2\pi\beta\Gamma(\gamma-\alpha-\beta)\theta_3(0,\tau')^2}{(1-e^{4\pi i(\gamma-\beta)})(1-e^{-4\pi i\alpha})\Gamma(-\alpha)\Gamma(\gamma-\beta)} \\ \times\displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau')^{2\gamma-2\beta-1}\theta_1(u,\tau')^{-2\alpha-1}\theta_2(u,\tau')^{2\beta-1}\theta_3(u,\tau')^{2\alpha-2\gamma+3}du\end{bmatrix},$$

$$Z_\infty(\tau'') = \left[\begin{matrix} \dfrac{2\pi e^{\frac{1}{2}\pi i(\gamma-1)}\Gamma(\alpha-\beta+2)\theta_1(0,\tau'')^2}{(1-e^{4\pi i\alpha})(1-e^{-4\pi i\beta})\Gamma(\alpha)\Gamma(-\beta)} \\ \times \displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(u,\tau'')^{2\alpha-1}\theta_1(u,\tau'')^{-2\beta-1}\theta_2(u,\tau'')^{2\gamma-2\alpha-1}\theta_3(u,\tau'')^{2\beta-2\gamma+3}du \\ \dfrac{-2\pi\beta e^{\frac{1}{2}\pi i(\gamma-1)}\Gamma(\alpha-\beta+2)\theta_1(0,\tau'')^2}{(1-e^{4\pi i(\gamma-\beta)})(1-e^{4\pi i(\alpha-\gamma)})\Gamma(\gamma-\beta)\Gamma(\alpha-\gamma+1)} \\ \times \displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(u,\tau'')^{2\gamma-2\beta-1}\theta_1(u,\tau'')^{2\alpha-2\gamma+3}\theta_2(u,\tau'')^{2\beta-1}\theta_3(u,\tau'')^{-2\alpha-1}du \end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{-2\pi\alpha e^{\frac{1}{2}\pi i(\gamma-1)}\Gamma(\beta-\alpha+2)\theta_1(0,\tau'')^2}{(1-e^{4\pi i(\beta-\gamma)})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\gamma-\alpha)\Gamma(\beta-\gamma+1)} \\ \times \displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(u,\tau'')^{2\gamma-2\alpha-1}\theta_1(u,\tau'')^{2\beta-2\gamma+3}\theta_2(u,\tau'')^{2\alpha-1}\theta_3(u,\tau'')^{-2\beta-1}du \\ \dfrac{2\pi e^{\frac{1}{2}\pi i(\gamma-1)}\Gamma(\beta-\alpha+2)\theta_1(0,\tau'')^2}{(1-e^{4\pi i\beta})(1-e^{-4\pi i\alpha})\Gamma(-\alpha)\Gamma(\beta)} \\ \times \displaystyle\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(u,\tau'')^{2\beta-1}\theta_1(u,\tau'')^{-2\alpha-1}\theta_2(u,\tau'')^{2\gamma-2\beta-1}\theta_3(u,\tau'')^{2\alpha-2\gamma+3}du \end{matrix}\right].$$

Note that the matrix functions $Z_0(\tau),\ Z_1(\tau'),\ Z_\infty(\tau'')$ are single-valued and holomorphic on H.

3. Transformations of $Z_0(\tau)$, $Z_1(\tau')$, $Z_\infty(\tau'')$.

The translation of the local monodromies of the matrix functions $\tilde{Y}_0(x), \tilde{Y}_1(x), \tilde{Y}_\infty(x)$ into $Z_0(\tau), Z_1(\tau'), Z_\infty(\tau'')$ is as follows:

$$Z_0(\tau+2) = Z_0(\tau)\begin{bmatrix} e^{\pi i(\gamma-1)} & 0 \\ 0 & e^{\pi i(1-\gamma)} \end{bmatrix}; \tag{3.1}$$

$$Z_1(\tau'+2) = Z_1(\tau')\begin{bmatrix} e^{\pi i(\alpha+\beta-\gamma+1)} & 0 \\ 0 & e^{\pi i(\gamma-\alpha-\beta-1)} \end{bmatrix}; \tag{3.2}$$

$$Z_\infty(\tau''+2) = Z_\infty(\tau'')\begin{bmatrix} e^{\pi i(\alpha-\beta)} & 0 \\ 0 & e^{\pi i(\beta-\alpha)} \end{bmatrix}. \tag{3.3}$$

Without need to use any formula for $\tilde{Y}_0(x), \tilde{Y}_1(x), \tilde{Y}_\infty(x)$, we can easily verify these formulas directly by transformation rules of theta functions. The translation of the connection formulas of $\tilde{Y}_0(x), \tilde{Y}_1(x), \tilde{Y}_\infty(x)$ into $Z_0(\tau), Z_1(\tau'), Z_\infty(\tau'')$ is as follows:

THEOREM (Gauss-Riemann). *Assume the conditions* (2.2). *Then we have*

$$Z_0(\tau) = Z_1(\tau')\begin{bmatrix} -\dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & -\dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \\ \dfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \end{bmatrix}; \tag{3.4}$$

$$Z_0(\tau) = Z_\infty(\tau'') \begin{bmatrix} \dfrac{\Gamma(\gamma)\Gamma(\beta-\alpha-1)}{\Gamma(\gamma-\alpha)\Gamma(\beta+1)} & \dfrac{\Gamma(2-\gamma)\Gamma(\beta-\alpha-1)}{\Gamma(-\alpha)\Gamma(1+\beta-\gamma)}e^{-\pi i\gamma} \\ \dfrac{\Gamma(\gamma)\Gamma(\alpha-\beta-1)}{\Gamma(\alpha+1)\Gamma(\gamma-\beta)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha-\beta-1)}{\Gamma(1+\alpha-\gamma)\Gamma(-\beta)}e^{-\pi i\gamma} \end{bmatrix}. \tag{3.5}$$

Our proof is given in the next section. Formulas (3.1)–(3.5) determine the monodromy of the hypergeometric function of Gauss completely. For example, combining (3.4) with (3.2), we have immediately

Corollary. *We have*

$$Z_0\left(\frac{\tau}{-2\tau+1}\right) = Z_0(\tau) \begin{bmatrix} \dfrac{-\cos\pi(\alpha-\beta)+e^{-\pi i\gamma}\cos\pi(\gamma-\alpha-\beta)}{i\sin\pi\gamma} & \dfrac{2\pi i\Gamma(1-\gamma)\Gamma(2-\gamma)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)} \\ \dfrac{2\pi i\Gamma(\gamma-1)\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & \dfrac{\cos\pi(\alpha-\beta)-e^{\pi i\gamma}\cos\pi(\alpha+\beta-\gamma)}{i\sin\pi\gamma} \end{bmatrix}. \tag{3.6}$$

This is the translation of the monodromy of $\tilde{Y}_0(x)$ along a curve with base point near $x=0$ turning around $x=1$ in the anticlockwise direction.

4. Proof of Theorem.

We set

$$Z_0(\tau) = \begin{bmatrix} z_{11}(\tau) & z_{12}(\tau) \\ z_{21}(\tau) & z_{22}(\tau) \end{bmatrix},$$

$$Z_1(\tau') = \begin{bmatrix} \zeta_{11}(\tau') & \zeta_{12}(\tau') \\ \zeta_{21}(\tau') & \zeta_{22}(\tau') \end{bmatrix},$$

$$Z_\infty(\tau'') = \begin{bmatrix} Z_{11}(\tau'') & Z_{12}(\tau'') \\ Z_{21}(\tau'') & Z_{22}(\tau'') \end{bmatrix}.$$

Let us first prove Formula (3.4) by exploiting transformation rules of theta functions.

Lemma 1. *We have*

$$z_{11}(\tau) = \frac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)e^{-\pi i\alpha}}{\Gamma(\alpha+\beta-\gamma+2)\Gamma(\gamma-\alpha)}\zeta_{11}(\tau') + \frac{\Gamma(\gamma)\Gamma(-\beta)e^{\pi i(\gamma-\alpha)}}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha-\beta)}\zeta_{12}(\tau') - \frac{\Gamma(\gamma)\Gamma(1+\beta-\gamma)\Gamma(-\beta)e^{\pi i(\gamma-\alpha-\beta)}}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha)\Gamma(2-\gamma)}z_{12}(\tau). \tag{4.1}$$

PROOF. Applying Jacobi transformations of theta functions to the expression of $z_{11}(\tau)$, we have

$$
\begin{aligned}
z_{11}(\tau) = {} & \frac{2\pi\Gamma(\gamma)}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)} \frac{i}{\tau}\theta_3\left(0,-\frac{1}{\tau}\right)^2(-i)e^{\pi i\alpha} \\
& \times \int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta\left(\frac{u}{\tau},-\frac{1}{\tau}\right)^{2\alpha-1} \theta_1\left(\frac{u}{\tau},-\frac{1}{\tau}\right)^{2\beta-2\gamma+3} \\
& \times \theta_2\left(\frac{u}{\tau},-\frac{1}{\tau}\right)^{2\gamma-2\alpha-1} \theta_3\left(\frac{u}{\tau},-\frac{1}{\tau}\right)^{-2\beta-1} du. \qquad (4.2)
\end{aligned}
$$

Substituting $v=-u/\tau$ into the definite integral of (4.2), we have

$$
\begin{aligned}
z_{11}(\tau) = {} & \frac{2\pi e^{-\pi i\alpha}\Gamma(\gamma)\theta_3(0,\tau')^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)} \\
& \times \int^{(\frac{\tau'}{2}+,0+,\frac{\tau'}{2}-,0-)} \theta(v,\tau')^{2\alpha-1}\theta_1(v,\tau')^{2\beta-2\gamma+3} \\
& \times \theta_2(v,\tau')^{2\gamma-2\alpha-1}\theta_3(v,\tau')^{-2\beta-1}dv, \qquad (4.3)
\end{aligned}
$$

where we chose in the integral of (4.3) the branch of $\theta(v,\tau')^{2\alpha-1}$ satisfying $\theta(-v,\tau')^{2\alpha-1} = -e^{-2\pi i\alpha}\theta(v,\tau')^{2\alpha-1}$. Applying Cauchy's theorem to the integration of the integrand of (4.3) along the parallelogram with vertices $0, \frac{1}{2}\tau', \frac{1}{2}(1+\tau'), \frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \\
& \quad \times \int^{(\frac{\tau'}{2}+,0+,\frac{\tau'}{2}-,0-)} \theta(v,\tau')^{2\alpha-1}\theta_1(v,\tau')^{2\beta-2\gamma+3}\theta_2(v,\tau')^{2\gamma-2\alpha-1}\theta_3(v,\tau')^{-2\beta-1}dv \\
& = \frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\beta-\gamma)})} \\
& \quad \times \int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(v,\tau')^{2\alpha-1}\theta_1(v,\tau')^{2\beta-2\gamma+3}\theta_2(v,\tau')^{2\gamma-2\alpha-1}\theta_3(v,\tau')^{-2\beta-1}dv \\
& \quad + \frac{e^{\pi i\gamma}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})} \\
& \quad \times \int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)} \theta(v,\tau')^{2\gamma-2\alpha-1}\theta_1(v,\tau')^{-2\beta-1}\theta_2(v,\tau')^{2\alpha-1}\theta_3(v,\tau')^{2\beta-2\gamma+3}dv \\
& \quad - \frac{e^{2\pi i(\gamma-\beta)}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})} \\
& \quad \times \int^{(\frac{\tau'}{2}+,0+,\frac{\tau'}{2}-,0-)} \theta(v,\tau')^{2\beta-2\gamma+3}\theta_1(v,\tau')^{2\alpha-1}\theta_2(v,\tau')^{-2\beta-1}\theta_3(v,\tau')^{2\gamma-2\alpha-1}dv, \qquad (4.4)
\end{aligned}
$$

where we chose the branch of $\theta_1(v,\tau')^{2\beta-2\gamma+3}$ satisfying $\theta_1(v+\frac{1}{2},\tau')^{2\beta-2\gamma+3} = -e^{2\pi i(\gamma-\beta)}\theta(v,\tau')^{2\beta-2\gamma+3}$. Substituting (4.4) into (4.3), we have the desired formula (4.1), which proves Lemma 1. □

Lemma 2. *We have*

$$z_{12}(\tau) = \frac{\Gamma(2-\gamma)\Gamma(\alpha+1)e^{\pi i(\gamma-\beta)}}{\Gamma(\alpha+\beta-\gamma+2)\Gamma(-\beta)}\zeta_{11}(\tau') + \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)e^{-\pi i\beta}}{\Gamma(1+\beta-\gamma)\Gamma(\gamma-\alpha-\beta)}\zeta_{12}(\tau') - \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\alpha+1)e^{\pi i(\gamma-\alpha-\beta)}}{\Gamma(-\beta)\Gamma(1+\beta-\gamma)\Gamma(\gamma)}z_{11}(\tau). \tag{4.5}$$

The proof is similar to that of Lemma 1. We omit it.

The system of linear equations (4.1) and (4.5) is unified as the single matrix equation:

$$(z_{11}(\tau), z_{12}(\tau))\begin{bmatrix} 1 & \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\alpha+1)e^{\pi i(\gamma-\alpha-\beta)}}{\Gamma(1+\beta-\gamma)\Gamma(-\beta)\Gamma(\gamma)} \\ \dfrac{\Gamma(\gamma)\Gamma(1+\beta-\gamma)\Gamma(-\beta)e^{\pi i(\gamma-\alpha-\beta)}}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha)\Gamma(2-\gamma)} & 1 \end{bmatrix}$$
$$= (\zeta_{11}(\tau'), \zeta_{12}(\tau'))\begin{bmatrix} \dfrac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)e^{-\pi i\alpha}}{\Gamma(\gamma-\alpha)\Gamma(\alpha+\beta-\gamma+2)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+1)e^{\pi i(\gamma-\beta)}}{\Gamma(\alpha+\beta-\gamma+2)\Gamma(-\beta)} \\ \dfrac{\Gamma(\gamma)\Gamma(-\beta)e^{\pi i(\gamma-\alpha)}}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha-\beta)} & \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)e^{-\pi i\beta}}{\Gamma(\gamma-\alpha-\beta)\Gamma(\beta-\gamma+1)} \end{bmatrix},$$

from which it follows that

$$(z_{11}(\tau), z_{12}(\tau)) = (\zeta_{11}(\tau'), \zeta_{12}(\tau'))\begin{bmatrix} -\dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & -\dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \\ \dfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \end{bmatrix}. \tag{4.6}$$

Exchanging the variables α and β in (4.6), we have immediately

$$(z_{21}(\tau), z_{22}(\tau)) = (\zeta_{21}(\tau'), \zeta_{22}(\tau'))\begin{bmatrix} -\dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & -\dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \\ \dfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \end{bmatrix}. \tag{4.7}$$

The system of equations (4.6) and (4.7) is equivalent to the matrix equality (3.4), which proves the first half of the theorem.

Next, let us prove Formula (3.5).

Lemma 3. *We have*

$$z_{11}(\tau)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha-1)}{\Gamma(1+\beta)\Gamma(\gamma-\alpha)}Z_{11}(\tau'')+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta-1)}{\Gamma(\alpha+1)\Gamma(\gamma-\beta)}Z_{12}(\tau''). \tag{4.8}$$

Proof. By transformation rules of theta functions, we have

$$\begin{aligned} z_{11}(\tau)=&\frac{2\pi e^{\frac{1}{2}\pi i(\gamma-1)}\Gamma(\gamma)\theta_2(0,\tau-1)^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\\ &\times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(u,\tau-1)^{2\alpha-1}\theta_1(u,\tau-1)^{2\gamma-2\alpha-1}\\ &\times\theta_2(u,\tau-1)^{-2\beta-1}\theta_3(u,\tau-1)^{2\beta-2\gamma+3}du. \end{aligned}\tag{4.9}$$

Applying Jacobi transformation formulas to (4.9), we have

$$\begin{aligned} z_{11}(\tau)=&\frac{2\pi e^{\frac{1}{2}\pi i(\gamma-1)}e^{\frac{1}{2}\pi i(2\alpha-1)}\Gamma(\gamma)}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\frac{i}{\tau-1}\theta_1\left(0,\frac{1}{1-\tau}\right)^2\\ &\times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta\left(\frac{u}{\tau-1},\frac{1}{1-\tau}\right)^{2\alpha-1}\theta_1\left(\frac{u}{\tau-1},\frac{1}{1-\tau}\right)^{-2\beta-1}\\ &\times\theta_2\left(\frac{u}{\tau-1},\frac{1}{1-\tau}\right)^{2\gamma-2\alpha-1}\theta_3\left(\frac{u}{\tau-1},\frac{1}{1-\tau}\right)^{2\beta-2\gamma+3}du. \end{aligned}\tag{4.10}$$

Substituting $v=u/(1-\tau)$ into the integral of (4.10), we have

$$\begin{aligned} z_{11}(\tau)=&\frac{2\pi e^{\frac{1}{2}\pi i(\gamma-1)}e^{-\pi i\alpha}\Gamma(\gamma)\theta_1(0,\tau'')^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})\Gamma(\alpha)\Gamma(\gamma-\alpha)}\\ &\times\int^{(\frac{\tau''}{2}+,0+,\frac{\tau''}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\\ &\times\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv, \end{aligned}\tag{4.11}$$

where we chose in the integral of (4.11) the branch of $\theta(v,\tau'')^{2\alpha-1}$ satisfying $\theta(-v,\tau'')^{2\alpha-1}=-e^{-2\pi i\alpha}\theta(v,\tau'')^{2\alpha-1}$. Applying Cauchy's theorem to the integration of the integrand of (4.11) along the parallelogram with vertices 0, $\frac{1}{2}\tau''$, $\frac{1}{2}(1+\tau'')$, $\frac{1}{2}$, we have

$$\begin{aligned} &\frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})}\\ &\quad\times\int^{(0+,\frac{\tau''}{2}+,0-,\frac{\tau''}{2}-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(1-e^{4\pi i\alpha})(1-e^{-4\pi i\beta})}\\
&\quad\times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv\\
&-\frac{e^{2\pi i\beta}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})}\\
&\quad\times\int^{(\frac{\tau''}{2}+,0+,\frac{\tau''}{2}-,0-)}\theta(v,\tau'')^{-2\beta-1}\theta_1(v,\tau'')^{2\alpha-1}\theta_2(v,\tau'')^{2\beta-2\gamma+3}\theta_3(v,\tau'')^{2\gamma-2\alpha-1}dv\\
&-\frac{e^{\pi i\gamma}}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{4\pi i(\beta-\gamma)})}\\
&\quad\times\int^{(0+,\frac{1}{2}+,0-,\frac{1}{2}-)}\theta(v,\tau'')^{2\gamma-2\alpha-1}\theta_1(v,\tau'')^{2\beta-2\gamma+3}\theta_2(v,\tau'')^{2\alpha-1}\theta_3(v,\tau'')^{-2\beta-1}dv\\
&=0, \qquad (4.12)
\end{aligned}
$$

where we chose the branch of $\theta_1(v,\tau'')^{-2\beta-1}$ satisfying $\theta_1(v+\frac{1}{2},\tau'')^{-2\beta-1} = -e^{2\pi i\beta}\theta(v,\tau'')^{-2\beta-1}$. Moreover, applying Cauchy's theorem to the integration of the function $\theta(v,\tau'')^{-2\beta-1}\theta_1(v,\tau'')^{2\alpha-1}\theta_2(v,\tau'')^{2\beta-2\gamma+3}\theta_3(v,\tau'')^{2\gamma-2\alpha-1}$ along the same parallelogram, we have

$$
\begin{aligned}
&\frac{1}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})}\\
&\quad\times\int^{(0+,\frac{\tau''}{2}+,0-,\frac{\tau''}{2}-)}\theta(v,\tau'')^{-2\beta-1}\theta_1(v,\tau'')^{2\alpha-1}\theta_2(v,\tau'')^{2\beta-2\gamma+3}\theta_3(v,\tau'')^{2\gamma-2\alpha-1}dv\\
&+\frac{1}{(1-e^{4\pi i\alpha})(1-e^{-4\pi i\beta})}\\
&\quad\times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{-2\beta-1}\theta_1(v,\tau'')^{2\alpha-1}\theta_2(v,\tau'')^{2\beta-2\gamma+3}\theta_3(v,\tau'')^{2\gamma-2\alpha-1}dv\\
&-\frac{e^{-2\pi i\alpha}}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})}\\
&\quad\times\int^{(\frac{\tau''}{2}+,0+,\frac{\tau''}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv\\
&-\frac{e^{-\pi i\gamma}}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{4\pi i(\beta-\gamma)})}\\
&\quad\times\int^{(0+,\frac{1}{2}+,0-,\frac{1}{2}-)}\theta(v,\tau'')^{2\beta-2\gamma+3}\theta_1(v,\tau'')^{2\gamma-2\alpha-1}\theta_2(v,\tau'')^{-2\beta-1}\theta_3(v,\tau'')^{2\alpha-1}dv\\
&=0, \qquad (4.13)
\end{aligned}
$$

where we chose the branch of $\theta_1(v,\tau'')^{2\alpha-1}$ satisfying $\theta_1(v+\frac{1}{2},\tau'')^{2\alpha-1} =$

$-e^{-2\pi i\alpha}\theta(v,\tau'')^{2\alpha-1}$. From (4.12) and (4.13) it follows that

$$\frac{1-e^{2\pi i(\beta-\alpha)}}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \times\int^{(\frac{\tau''}{2}+,0+,\frac{\tau''}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv$$
$$=\frac{1-e^{2\pi i\beta}}{(1-e^{4\pi i\alpha})(1-e^{-4\pi i\beta})} \times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv$$
$$+\frac{e^{\pi i\gamma}-e^{2\pi i\beta}e^{-\pi i\gamma}}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{4\pi i(\beta-\gamma)})} \times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{2\gamma-2\alpha-1}\theta_1(v,\tau'')^{2\beta-2\gamma+3}\theta_2(v,\tau'')^{2\alpha-1}\theta_3(v,\tau'')^{-2\beta-1}dv, \tag{4.14}$$

$$\frac{1-e^{2\pi i(\beta-\alpha)}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})} \times\int^{(\frac{\tau''}{2}+,0+,\frac{\tau''}{2}-,0-)}\theta(v,\tau'')^{-2\beta-1}\theta_1(v,\tau'')^{2\alpha-1}\theta_2(v,\tau'')^{2\beta-2\gamma+3}\theta_3(v,\tau'')^{2\gamma-2\alpha-1}dv$$
$$=\frac{1-e^{-2\pi i\alpha}}{(1-e^{4\pi i\alpha})(1-e^{-4\pi i\beta})} \times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{2\alpha-1}\theta_1(v,\tau'')^{-2\beta-1}\theta_2(v,\tau'')^{2\gamma-2\alpha-1}\theta_3(v,\tau'')^{2\beta-2\gamma+3}dv$$
$$-\frac{e^{-2\pi i\alpha}e^{\pi i\gamma}+e^{-\pi i\gamma}}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{4\pi i(\beta-\gamma)})} \times\int^{(\frac{1}{2}+,0+,\frac{1}{2}-,0-)}\theta(v,\tau'')^{2\gamma-2\alpha-1}\theta_1(v,\tau'')^{2\beta-2\gamma+3}\theta_2(v,\tau'')^{2\alpha-1}\theta_3(v,\tau'')^{-2\beta-1}dv. \tag{4.15}$$

Substituting (4.14) into (4.11), we have the desired formula (4.8), which proves Lemma 3. □

LEMMA 4. *We have*

$$z_{12}(\tau)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha-1)e^{-\pi i\gamma}}{\Gamma(1+\beta-\gamma)\Gamma(-\alpha)}Z_{11}(\tau'')+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta-1)e^{-\pi i\gamma}}{\Gamma(-\beta)\Gamma(\alpha-\gamma+1)}Z_{12}(\tau''). \tag{4.16}$$

The proof is similar to that of Lemma 3. We omit it.

Exchanging α and β in (4.8) and (4.16), we have immediately

$$z_{21}(\tau) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha-1)}{\Gamma(1+\beta)\Gamma(\gamma-\alpha)} Z_{21}(\tau'') + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta-1)}{\Gamma(\alpha+1)\Gamma(\gamma-\beta)} Z_{22}(\tau''), \tag{4.17}$$

$$z_{22}(\tau) = \frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha-1)e^{-\pi i\gamma}}{\Gamma(1+\beta-\gamma)\Gamma(-\alpha)} Z_{21}(\tau'') + \frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta-1)e^{-\pi i\gamma}}{\Gamma(-\beta)\Gamma(\alpha-\gamma+1)} Z_{22}(\tau''). \tag{4.18}$$

Formulas (4.8), (4.16)–(4.18) are unified to the single matrix equality (3.5), which proves the second half of the theorem.

References

[1] K. Chandrasekharan, Elliptic functions, Springer, 1985.

[2] D. Mumford, Tata lectures on theta I, Birkhäuser, 1983.

[3] H. Umemura, Theory of elliptic functions: Analysis of elliptic curves, Univ. of Tokyo Press, 2000 (in Japanese).

[4] H. Watanabe, Sur la dégénérescence de quelques formules de connexion pour les fonctions hypergéométriques de Gauss, to appear in Aequationes Math.

[5] W. Wirtinger, Zur Darstellung der hypergeometrischen Function durch bestimmte Integrale, Akad. Wiss. Wien. S.-B. IIa, III (1902), 894–900.

[6] A. L. Dixon, Generalisations of Legendre's formula $KE' - (K-E)K' = \frac{1}{2}\pi$, Proc. London Math. Soc. Ser. 2, **3** (1905), 206–224.

[7] E. B. Elliott, A formula including Legendre's $EK' + KE' - KK' = \frac{1}{2}\pi$, Messenger of Math., **33** (1904), 31–32.

Humihiko Watanabe

Kitami Institute of Technology
165, Koencho, Kitami
Hokkaido, 090-8507, Japan
E-mail: hwatanab@cs.kitami-it.ac.jp