J. Math. Soc. Japan
Vol. 59, No. 1 (2007) pp. 97–112

Stable suspension order of universal phantom maps and some stably indecomposable loop spaces

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(Received Jan. 16, 2006)

Abstract. We study a stable suspension order of a universal phantom map out of a space. We prove that it is infinite if X is a non-trivial finite Postnikov space, a classifying space of connected Lie group or a loop space on a connected Lie group with torsion. We also show that the loop spaces on the exceptional Lie groups E_6 and E_7 are stably indecomposable.

1. Introduction.

Throughout this paper all spaces have basepoints, all maps and homotopies preserve them. p denotes a fixed prime and $X_{(p)}$ denotes the localization at the prime p of a nilpotent space X.

A map out of a CW-complex X is called a *phantom map* if its restriction to each n-skeleton X_n is null homotopic. The *universal phantom map* out of X is a based map

$$\Theta : X \to \bigvee_{n=1}^{\infty} \Sigma X_n$$

through which all other phantom maps out of X factor. This map is a part of the extended cofiber sequence

$$\bigvee_{n=1}^{\infty} X_n \xrightarrow{F} X \xrightarrow{\Theta} \bigvee_{n=1}^{\infty} \Sigma X_n \to \bigvee_{n=1}^{\infty} \Sigma X_n \to \cdots ,$$

where $F : \bigvee_{n=1}^{\infty} X_n \to X$ is the folding map, that is, $F|_{X_n} : X_n \to X$ is the inclusion map. For a map $f : X \to Y$ by the *stable suspension order* of f we mean the order of the class $[f]$ in $\varinjlim [\Sigma^n X, \Sigma^n Y]$.

For a CW-complex X by $\Sigma^\infty X$ we denote the suspension spectrum. For a map $f : X \to Y$ between CW-complexes by the *strong stable suspension order* of f we mean the order of the class $\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$ in $\{\Sigma^\infty X, \Sigma^\infty Y\}$.

Since the natural map

$$\varinjlim [\Sigma^n X, \Sigma^n Y] \to \{\Sigma^\infty X, \Sigma^\infty Y\}$$

2000 *Mathematics Subject Classification.* Primary 55P99; Secondary 55P35, 55S05.
Key Words and Phrases. phantom map, loop space, exceptional Lie group.
This work is partially supported by Grant-in-Aid for Scientific Research (No. 16540076), Ministry of Education, Culture, Sports, Science and Technology.

is not necessarily monomorphic for an infinite dimensional complex X, for a map $f : X \to Y$ it is necessary to distinguish between its stable suspension order and strong stable suspension order.

First we study the (strong) stable suspension order of the universal phantom maps out of $K(\pi, n)_{(p)}$ and $BG_{(p)}$. These spaces satisfy the assumption of the following theorem.

Theorem 1.1. *Let X be a connected p-local CW-complex of finite type over the ring $\boldsymbol{Z}_{(p)}$. If $\tilde{H}^*(X; \boldsymbol{F}_p)$ has an element of infinite height, then the strong stable suspension order of the universal phantom map out of X is infinite.*

As a corollary we have the following partial answer to Question 18 of McGibbon [**12**]. Needless to say, if the strong stable suspension order of a map is infinite, then so is its stable suspension order.

Corollary 1.2. *Let X be a connected nilpotent finite Postnikov system of finite type with finite $\pi_1(X)$. Then the strong stable suspension order of the universal phantom map out of $X_{(p)}$ is infinite unless its mod p homology groups are trivial.*

Let G be a connected Lie group. Then the strong stable suspension order of the universal phantom map out of $BG_{(p)}$ is infinite unless its mod p homology groups are trivial.

Next we study the (strong) stable suspension order of the universal phantom map out of a loop space on a simply connected Lie group.

In [**7**] we proved that for almost all Lie groups G the universal phantom maps out of ΩG are essential. More precisely we proved the following theorem.

Theorem 1.3. *Let G be a simply connected Lie group. The universal phantom map out of $\Omega G_{(p)}$ is trivial if and only if G is p-equivalent to a product of spheres.*

By the Mitchell-Richter splitting of $\Omega SU(n)$ [**1**], it is stably homotopy equivalent to a bouquet of finite complexes. The identity map $id : \Sigma^\infty \Omega SU(n) \to \Sigma^\infty \Omega SU(n)$, therefore, factors throught the folding map $\Sigma^\infty F : \vee \Sigma^\infty (\Omega SU(n))_i \to \Sigma^\infty \Omega SU(n)$, that is, the universal phantom map out of $\Omega SU(n)$ is stably trivial. Thus the strong stable suspension order of the universal phantom map out of $\Omega SU(n)$ is zero. But we do not know whether the stable suspension order of the universal phantom map out of $\Omega SU(n)$ is zero.

For a nilpotent CW-complex X of finite type, by Theorem 3.3 of [**3**], the stable suspension order of the universal phantom map out of $X_{(p)}$ is zero if and only if $\Sigma^n X_{(p)}$ is homotopy equivalent to a bouquet of finite dimensional complexes for some n.

Question 1.4. *Let $n > 2$. Is $\Sigma^m \Omega SU(n)$ homotopy equivalent to a bouquet of finite complexes for sufficiently large m?*

Hopkins [**5**] proved that $\Omega Sp(2)$ and $\Omega Sp(3)$ are stably indecomposable. Later Hubbuck [**6**] added ΩG_2 and ΩF_4 to the list of such spaces. Their results imply that the stable suspension order of the universal phantom maps out of ΩG are non-zero for $G = Sp(2),\ Sp(3),\ G_2,\ F_4$. We extend this result to loop spaces on Lie groups as follows:

THEOREM 1.5. *Let G be a simply connected, simple Lie group.*

If $H_*(G;\boldsymbol{Z})$ has a p-torsion, then the strong stable suspension order of the universal phantom map out of $\Omega G_{(p)}$ is infinite.

If $G = Sp(n)$ with $n > 1$, then the strong stable suspension order of the universal phantom map out of $\Omega G_{(2)}$ is non-zero.

Theorem 1.5 and the fact that ΩG_2 and ΩF_4 are stably indecomposable suggest that if a simply connected simple Lie group G has a p-torsion, then ΩG is stably indecomposable. Partially we can prove this suggestion.

THEOREM 1.6. *ΩE_6 and ΩE_7 are stably indecomposable at the prime 2.*

As for $\Omega Sp(n)$, although $Sp(n)$ is torsion free, Hubbuck conjectured that they are all stably indecomposable at the prime 2 unless $n = 1$. For $n \leq 10$ it is not difficult to show that his conjecture is true.

For a connected space X of finite type we associate a graph $G(X)$ as follows. The vertices of $G(X)$ are non-zero elements of $\tilde{H}_*(X;\boldsymbol{F}_2)$ and a pair of vertices $\{x,\ y\}$ is an edge of $G(X)$ if and only if $Sq^i x = y$ or $Sq^i y = x$ for some $i > 0$, where Sq^i is the dual Steenrod operation of degree i. If X is stably homotopy equivalent to a wedge of non-trivial spaces or spectra, then $G(X)$ is not connected. To prove Theorem 1.6 we will show that the graphs associated with ΩE_6 and ΩE_7 are connected. Unfortunately, the graphs associated with loop spaces on other Lie groups are not connected.

This paper is organized as follows: In Section 2 we study a stable suspension order of a universal phantom map and prove Theorem 1.1, Corollary 1.2 and Theorem 1.5. In Section 3 we prove that ΩE_6 and ΩE_7 are stably indecomposable by assuming technical theorems. In Section 4 and Section 5 we compute the sets $\{x \in H_*(X;\boldsymbol{F}_2) | Sq^i x = 0 \text{ for all } i > 0\}$ for $X = \Omega E_6$ and ΩE_7.

The author would like to thank N. Minami. He kindly told the author that the natural map $\varinjlim[\Sigma^n X, \Sigma^n Y] \to \{\Sigma^\infty X, \Sigma^\infty Y\}$ is not necessarily monomorphic for an infinite dimensional complex X.

2. Stable suspension order of universal phantom map.

In this section we prove Theorem 1.1, Corollary 1.2 and Theorem 1.5.

PROOF OF THEOREM 1.1. In this proof $H_*(X)$ stands for $H_*(X;\boldsymbol{F}_p)$. Since X is connected, we can assume that each n-skeleton of X is also connected.

By way of a contradiction, we assume that the strong stable suspension order of the universal phantom map out of X is finite. Since spaces are p-local, the order is p^m for some non-negative integer m. Since in the cofiber sequence

$$\bigvee_{n=1}^{\infty} \Sigma^\infty X_n \xrightarrow{\Sigma^\infty F} \Sigma^\infty X \xrightarrow{\Sigma^\infty \Theta} \bigvee_{n=1}^{\infty} \Sigma^\infty \Sigma X_n$$

$p^m \Sigma^\infty \Theta \simeq *$, there is a map $g : \Sigma^\infty X \to \bigvee_{n=1}^{\infty} \Sigma^\infty X_n$ such that $\Sigma^\infty F \circ g \simeq p^m id_{\Sigma^\infty X}$. By taking adjoint we have the following (homotopy) commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Q(X) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle p^m} \\ Q(\bigvee_{n=1}^{\infty} X_n) & \xrightarrow{Q(F)} & Q(X) \end{array}$$

where $Q(X) = \varinjlim \Omega^k \Sigma^k X$, $\eta : X \to Q(X)$ is the adjoint map to the identity $id_{\Sigma^\infty X} : \Sigma^\infty X \to \Sigma^\infty X$, $p : Q(X) \to Q(X)$ is the p-th power map, and $g' : X \to Q(\bigvee_{n=1}^{\infty} X_n)$ is the adjoint map to $g : \Sigma^\infty X \to \bigvee_{n=1}^{\infty} \Sigma^\infty X_n$. Now we apply the mod p homology theory to the diagram above and we will obtain a contradiction.

For a space K of finite type over the ring $\boldsymbol{Z}_{(p)}$ by $\xi_* : H_{p*}(K) \to H_*(K)$ we denote the p-th root map, which is the dual of the p-th power map in $H^*(K)$. If K is an H-space, then the p-th power map $p : K \to K$ induces a map given by $p_*(x) = \xi_*(x)^p$ in homology and $\xi_* : H_{p*}(K) \to H_*(K)$ is a homomorphism of algebras.

Thus we have $(p^m \circ \eta)_* = \xi_*^m(\ )^{p^m} \circ \eta_* : H_*(X) \to H_*(Q(X))$. Let $\bar{x} \in H^*(X)$ be an element of infinite height and $x_j \in H_*(X)$ be the dual of $\bar{x}^{p^j}$ for $j \geq 0$ and its image in $H_*(Q(X))$ will be denoted by the same letter since $H_*(X)$ is a submodule of $H_*(Q(X))$. Then $\xi_*(x_{j+1}) = x_j$ for $j \geq 0$.

We choose a positive integer M such that $x_m \in \mathrm{Im}(H_*(X_M) \to H_*(X))$. Since X_M is finite dimensional, we may assume that $g'(X_M) \subset Q(\bigvee_{n=1}^{N} X_n)$ for some N and that g' is the composite of $g'|_{X_M} : X_M \to Q(\bigvee_{n=1}^{N} X_n)$ and the natural inclusion map $Q(\bigvee_{n=1}^{N} X_n) \to Q(\bigvee_{n=1}^{\infty} X_n)$. We consider the composite $h : X \to Q(X_N)$:

$$h : X \xrightarrow{g'} Q\left(\bigvee_{n=1}^{\infty} X_n\right) \xrightarrow{Q(q)} Q\left(\bigvee_{n=1}^{N} X_n\right) \xrightarrow{Q(F)} Q(X_N)$$

where $q : \bigvee_{n=1}^{\infty} X_n = \bigvee_{n=1}^{N} X_n \vee \bigvee_{n=N+1}^{\infty} X_n \to \bigvee_{n=1}^{N} X_n$ is the map which collapses the second factor to the base point.

Now we recall the following two facts to complete the proof.

(1) Any even dimensional element x in $\tilde{H}_*(Q(X))$ has an infinite height since $H_*(Q(X))$ is a free commutative algebra. This is well known, see Section 4 of [**10**]. In particular, $x^{p^m} \neq 0$.

(2) There is no infinite sequence $\{y_j \in \widetilde{H}_*(Q(X_N))|\ \xi_*(y_{j+1}) = y_j$ for all $j \geq 0$ and $y_j \neq 0$ for some $j\}$. This can be proved by using the Nishida relation, see e.g., Lemma 3.5 of [**13**]. In fact, if $\xi_*^k = 0$ on $\widetilde{H}_*(K)$ for a connected space K, then so is on $\widetilde{H}_*(Q(K))$.

Put $x = x_0$ and consider an infinite sequence $\{h_*(x_j) \in H_*(Q(X_N))\}_{j=0,1,2,\ldots}$. This sequence contradicts the fact (2) above as follows. Let $i_N : X_N \to X$ be the inclusion map. Since

$$\begin{aligned} Q(i_N) \circ h \circ i_M &= Q(i_N) \circ Q(F) \circ Q(q) \circ g' \circ i_M \\ &\simeq Q(i_N) \circ Q(F) \circ g'|_{X_M} \simeq Q(F) \circ g' \circ i_M \end{aligned}$$

and $x_m \in \mathrm{Im}(H_*(X_M) \to H_*(X))$, we have

$$Q(i_N)_* \circ h_*(x_m) = Q(F)_* \circ g'_*(x_m) = p^m_* \circ \eta_*(x_m) = (\xi^m_*(x_m))^{p^m} = x^{p^m} \neq 0,$$

that is, $h_*(x_m) \neq 0$. On the other hand we have $\xi_* h_*(x_{j+1}) = h_*(\xi_* x_{j+1}) = h_*(x_j)$. □

Proof of Corollary 1.2. The fact that $\tilde{H}^*(X; \boldsymbol{F}_p)$ has an element of infinite height is proved by Grodal, [**4**, Theorem 1.1]. For BG this fact is also known, see e.g., p. 385 of [**3**]. □

Similarly to Theorem 3.3 of [**3**] it is easy to prove the following theorem, which we need to prove Theorem 1.5.

Theorem 2.1. *For a nilpotent CW-complex X the strong suspension order of the universal phantom map out of $X_{(p)}$ is zero if and only if $X_{(p)}$ is stably homotopy equivalent to a bouquet of p-localization of finite complexes.*

Proof of Theorem 1.5. First we show the second statement.

For $n = 2$ the statement follows Theorem 2.1 and the fact that $\Omega Sp(2)$ is stably indecomposable.

Let $n > 2$. By Kono and Kozima [**8**], $H_*(\Omega Sp(n); \boldsymbol{F}_2)$ is isomorphic to $\boldsymbol{F}_2[x_2, x_6, \ldots, x_{4n-2}]$ and the action of the Steenrod algebra on x_6 and x_{10} are given by $Sq^2 x_6 = x_2^2$, $Sq^2 x_{10} = x_2^4$ and $Sq^4 x_{10} = x_6$. Since $G(\Omega Sp(n))$ has the following path

$$\cdots \mapsto x_6 x_2^{2i+2} \overset{Sq^2}{\mapsto} x_2^{2i+4} \overset{Sq^2}{\hookleftarrow} x_{10} x_2^{2i} \overset{Sq^4}{\mapsto} x_6 x_2^{2i} \overset{Sq^2}{\mapsto} x_2^{2i+2} \hookleftarrow \cdots \mapsto x_2^2,$$

$G(\Omega Sp(n))$ has a connected component which has elements with arbitrary large dimension. Thus in a stable category $\Omega Sp(n)$ is not homotopy equivalent to a bouquet of finite dimensional complexes, which implies the second statement by Theorem 2.1.

To prove the first statement we use complex $\boldsymbol{Z}/2$-graded K-homology theory. We know that $K_0(\Omega G)$ is free $\boldsymbol{Z}$-module and $K_1(\Omega G) = 0$. We give $K_0(\Omega G)$ the ascending filtration corresponding to the CW-filtration of ΩG. Since the Atiyah-Hirzebruch spectral sequence collapses, the natural map $K_0(\Omega G)_{2n} \to H_{2n}(\Omega G; \boldsymbol{Z})$ is epimorphic with kernel $K_0(\Omega G)_{2n-2}$, see [**2**] and [**6**].

Let $\xi_2 \in \tilde{K}_0(\Omega G)_2 \cong \boldsymbol{Z}$ be a generator. Then there is an indecomposable element $\xi_{2p} \in \tilde{K}_0(\Omega G)_{2p}$ such that $\xi_2^p = p\xi_{2p} + \xi_2$ by [**2**]. The $Spin(n)$ case is treated similarly.

From now on until the end of this proof we assume that all spaces are localized at the prime p. If the strong stable suspension order of the universal phantom map out of ΩG is finite, say p^m, then there is a stable map

$$g : \Sigma^\infty \Omega G \to \vee \Sigma^\infty (\Omega G)_{2i}$$

such that $p^m \simeq \Sigma^\infty F \circ g : \Sigma^\infty \Omega G \to \Sigma^\infty \Omega G$. We take sufficiently large N so that the map $h : \Sigma^\infty \Omega G \to \Sigma^\infty (\Omega G)_{2N}$ defined by

$$h = \Sigma^\infty F \circ \Sigma^\infty q \circ g : \Sigma^\infty \Omega G \to \vee_{i=1}^\infty \Sigma^\infty (\Omega G)_{2i} \to \vee_{i=1}^N \Sigma^\infty (\Omega G)_{2i} \to \Sigma^\infty (\Omega G)_{2N}$$

satisfies the equality $h_* = p^m$ on $K_0(\Sigma^\infty \Omega G)_{2p^m} \cong K_0(\Omega G)_{2p^m}$, where $q : \vee_{i=1}^\infty (\Omega G)_{2i} \to$

$\vee_{i=1}^{N}(\Omega G)_{2i}$ collapses $\vee_{i=N+1}^{\infty}(\Omega G)_{2i}$ to the base point. We consider the stable Adams operation ψ_p in K-homology groups, that is, for an element $\eta \in K_0(\Sigma^\infty X) \cong \varinjlim K_0(\Sigma^{2n}X)$ we take a representative $\eta_n \in K_0(\Sigma^{2n}X)$ and define $\psi_p(\eta) = p^{-n}\psi^p(\eta_n)$, where $\psi^p : K_0(\Sigma^{2n}X) \mapsto K_0(\Sigma^{2n}X)$ is the unstable Adams operation. Since $\psi^p(\xi_2^{p^s}) = (\psi^p\xi_2)^{p^s} = p^{p^s}\xi_2^{p^s}$ in $K_0(\Omega G)$, we have $\psi_p h_*(\xi_2^{p^s}) = p^{p^s}h_*(\xi_2^{p^s})$ in $K_0(\Sigma^\infty(\Omega G)_{2N})$. Since eigenvalues of the linear map $\psi_p : K_0(\Sigma^\infty(\Omega G)_{2N}) \otimes \boldsymbol{Q} \mapsto K_0(\Sigma^\infty(\Omega G)_{2N}) \otimes \boldsymbol{Q}$ are bounded, there is an $s > \max\{N,\ m\}$ such that $h_*(\xi_2^{p^s}) = 0$. Here we claim

Lemma 2.2. *There are $\eta \in K_0(\Omega G)_{2p^m}$ and $\eta' \in K_0(\Omega G)_{2p^s}$ such that $\xi_2^{p^s} = \xi_2 + p\eta + p^{m+1}\eta'$.*

We postpone the proof of Lemma 2.2 and continue to prove Theorem 1.5. Applying h_* to the equality obtained in Lemma 2.2 we have

$$0 = h_*\big(\xi_2^{p^s}\big) = h_*(\xi_2) + h_*(p\eta) + p^{m+1}h_*(\eta') = p^m\xi_2 + p^{m+1}\big(\eta + h_*(\eta')\big)$$

since $h_* = p^m$ on $K_0(\Omega G)_{2p^m}$. The equality above implies that $\xi_2 = -p(\eta + h_*(\eta'))$ in $K_0(\Omega G)$. Clearly this is impossible and completes the proof. □

Proof of Lemma 2.2. We have

$$\xi_2^{p^s} = \big(\xi_2^p\big)^{p^{s-1}} = (p\xi_{2p} + \xi_2)^{p^{s-1}} = \sum_{i=0}^{p^{s-1}} \binom{p^{s-1}}{i} p^i \xi_{2p}^i \xi_2^{p^{s-1}-i}.$$

Since for $i = p^t j$, where $0 < i \le p^{s-1}$ and $(p, j) = 1$, we have

$$\binom{p^{s-1}}{i} = \binom{p^{s-1}}{p^t j} = \frac{p^{s-1}}{p^t j}\binom{p^{s-1}-1}{p^t j - 1},$$

we obtain

$$\nu_p\left(\binom{p^{s-1}}{i} p^i\right) \ge s - 1 - t + p^t j \ge s > m,$$

where $\nu_p(k)$ denotes the p-exponent of an integer k. We proved that $\xi_2^{p^s} \equiv \xi_2^{p^{s-1}}$ (mod $p^{m+1}K_0(\Omega G)_{2p^s}$) for $s > m$. Thus inductively we know that

$$\xi_2^{p^s} \equiv \xi_2^{p^m} \pmod{p^{m+1}K_0(\Omega G)_{2p^s}}.$$

Clearly $\xi_2^{p^m} \equiv \xi_2 \pmod{pK_0(\Omega G)_{2p^m}}$ and we complete the proof. □

3. ΩE_6 and ΩE_7 are stably indecomposable.

In this section we will prove that ΩE_6 and ΩE_7 are stably indecomposable assuming technical theorems. From now on $H_*(X)$ stands for $H_*(X; \boldsymbol{F}_2)$.

First we recall the ring structure of $H_*(\Omega E_6)$, $H_*(\Omega E_7)$ and the action of the Steenrod algebra on them [**9**]:

$$H_*(\Omega E_6) = \Lambda(x_2) \otimes \boldsymbol{F}_2[x_4, x_8, x_{10}, x_{14}, x_{16}, x_{22}],$$
$$H_*(\Omega E_7) = \Lambda(x_2, x_4, x_8) \otimes \boldsymbol{F}_2[x_{10}, x_{14}, x_{16}, x_{18}, x_{22}, x_{26}, x_{34}],$$

$$\begin{array}{llll} Sq^2 x_4 = x_2, & Sq^2 x_8 = x_2 x_4, & Sq^4 x_8 = x_4, & Sq^2 x_{10} = x_4^2, \\ Sq^4 x_{14} = x_{10}, & Sq^2 x_{16} = x_{14} + x_2 x_4 x_8, & Sq^4 x_{16} = x_4 x_8, & Sq^8 x_{16} = x_8, \\ Sq^8 x_{18} = x_{10}, & Sq^2 x_{22} = x_{10}^2, & Sq^8 x_{22} = x_{14}, & Sq^4 x_{26} = x_{22}, \\ Sq^8 x_{26} = x_{18}, & Sq^2 x_{34} = x_{16}^2, & Sq^{16} x_{34} = x_{18}, & \end{array}$$

and $Sq^{2^i} x_{2j} = 0$ in all cases not explicitly recorded. Here the degree of x_{2j} is $2j$. Since we are working in homology theory, the Adem relations are given as follows: for $0 < a < 2b$ we have

$$Sq^b Sq^a = \sum \binom{b-1-t}{a-2t} Sq^t Sq^{a+b-t}.$$

Thus, for example, we have $Sq^6 x_{34} = (Sq^4 Sq^2 + Sq^1 Sq^5) x_{34} = Sq^4 x_{16}^2 = x_{14}^2$, $Sq^{12} x_{26} = (Sq^8 Sq^4 + Sq^1 Sq^{11} + Sq^2 Sq^{10}) x_{26} = Sq^8 x_{22} = x_{14}$, and so on.

We have to calculate the subrings of $H_*(\Omega E_6)$ and $H_*(\Omega E_7)$ which consist of those elements annihilated by Sq^i for all $i > 0$.

Theorem 3.1.

$$\{x \in H_*(\Omega E_6) | Sq^i x = 0 \textit{ for all } i > 0\} = \boldsymbol{F}_2[x_4^2, x_{20}, \bar{x}_{16}]\{1, x_2, x_2 x_4, x_2 x_{10} + x_4^3\},$$

where

$$x_{20} = x_4^3 x_8 + x_{10}^2 + x_2(x_4^2 x_{10} + x_4 x_{14} + x_8 x_{10}),$$
$$\bar{x}_{16} = x_4^5 x_{16} + x_4^3 x_{10} x_{14} + x_4^2 x_{14}^2 + x_4^2 x_8 x_{20} + x_8^2 x_{20} + x_2 x_4^2 (x_{10} x_{16} + x_4^3 x_{14} + x_4 x_{22}).$$

Theorem 3.2.

$$\begin{aligned} &\{x \in H_*(\Omega E_7) | Sq^i x = 0 \textit{ for all } i > 0\} \\ &= \Lambda(\bar{x}_4) \otimes \boldsymbol{F}_2[x_{10}, \bar{x}_{18}, x_{74}, x_{56}^2] \bar{x}_{26} \\ &\quad + \boldsymbol{F}_2[x_{10}, \bar{x}_{18}, x_{74}, x_{56}^2]\{1, x_2, x_2 x_4, x_2 x_4 x_8, x_2(x_4 x_{14} + x_8 x_{10}), \\ &\qquad \bar{x}_4, x_2 x_{56}, x_2 x_4 x_{56}, x_2 x_4 x_8 x_{56}, x_2(x_4 x_{14} + x_8 x_{10}) x_{56}\}, \end{aligned}$$

where

$$\bar{x}_{18} = x_{10}x_{18} + x_{14}^2,$$
$$\bar{x}_4 = x_4x_{10}^4 + x_2\bigl(x_{10}x_{16}^2 + x_{10}^2x_{22} + x_{14}^3\bigr),$$
$$x_{56} = x_{10}^3x_{26} + x_{10}^2x_{14}x_{22} + x_{10}x_{14}^2x_{18} + x_{10}^4x_{16} + x_{14}^4 + x_8x_{10}^2x_{14}^2 + x_4x_{10}^2x_{16}^2 + x_4x_8x_{10}^3x_{14},$$
$$x_{74} = x_{10}^6x_{14} + x_{10}x_{14}^2x_{18}^2 + x_{10}x_{16}^4 + x_{10}^2x_{18}^3 + x_{10}^3x_{22}^2 + x_{14}^4x_{18},$$
$$\bar{x}_{26} = x_2\bigl(x_{10}^4x_{34} + x_4x_8x_{10}^4x_{22} + x_{14}^4x_{18} + x_{10}x_{16}^4 + x_4x_{10}^4x_{14}x_{16} + x_8x_{10}^5x_{16}\bigr) + x_{10}^2x_{56}.$$

Here we remark that $Sq^2x_{56} \neq 0$ but $Sq^i(x_2x_{56}) = 0$ for all $i > 0$.

Theorems 3.1 and 3.2 will be proved in sections 4 and 5, respectively. In this section by assuming Theorems 3.1 and 3.2 we prove Theorem 1.6.

PROOF OF THEOREM 1.6 FOR E_6. According to the remark after Theorem 1.6 we will show that the graph $G(\Omega E_6)$ is connected. To prove this it is sufficient to prove that for any non-zero element x of $H_*(\Omega E_6)$ with $|x| > 2$ there is a path connecting x and a lower dimensional vertex.

If $Sq^ix \neq 0$ for some $i > 0$, then the claim is clearly true.

We assume, therefore, that $Sq^ix = 0$ for all $i > 0$. By Theorem 3.1 x is in $\mathscr{A} = \boldsymbol{F}_2[x_4^2, x_{20}, \bar{x}_{16}]\{1, x_2, x_2x_4, x_2x_{10}+x_4^3\}$. As the list below shows, for a given multiplicative generator u of $\mathscr{A}$ there is an element v such that $Sq^2v = u$. For given x, therefore, there is also an element y such that $Sq^2y = x$.

v	$u = Sq^2v$	$\lvert u\rvert$
x_4	x_2	2
x_8	x_2x_4	6
x_{10}	x_4^2	8
x_4x_{10}	$x_2x_{10} + x_4^3$	12
$x_{22} + x_4x_8x_{10} + x_2x_4x_{16}$	x_{20}	20
x_{38}	$\bar{x}_{16}$	36

where

$$x_{38} = x_4^3x_{10}x_{16} + x_{10}x_{14}^2 + \bigl(x_4^2x_8 + x_8^2\bigr)(x_{22} + x_4x_8x_{10} + x_2x_4x_{16}) + x_2x_4^5x_{16}.$$

If $|x| = 8$ or $|x| \geq 12$, there is an element z such that $|z| = |y|$, $Sq^2z = 0$ and $Sq^iz \neq 0$ for some $i > 2$ as the following list shows.

$\lvert x\rvert$	$\lvert z\rvert$	z	Sq^iz
$8n+8$	$8n+10$	$x_2x_8x_4^{2n}$	$Sq^4\bigl(x_2x_8x_4^{2n}\bigr) = x_2x_4^{2n+1}$
$8n+18$	$8n+20$	$x_{10}^2x_4^{2n}$	$Sq^4\bigl(x_{10}^2x_4^{2n}\bigr) = x_4^{2n+4}$
$8n+12$	$8n+14$	$x_{14}x_4^{2n}$	$Sq^4\bigl(x_{14}x_4^{2n}\bigr) = x_{10}x_4^{2n}$
$8n+14$	$8n+16$	$x_8^2x_4^{2n}$	$Sq^8\bigl(x_8^2x_4^{2n}\bigr) = x_4^{2n+2}$

Then $Sq^2 y = Sq^2(y+z) = x$ and $Sq^i y \neq Sq^i(y+z)$ for some $i > 2$, that is, there is a path connecting x and a lower dimensional vertex $Sq^i y$ or $Sq^i(y+z)$.

If $|x| < 8$ or $|x| = 10$, then $x = x_2 x_4$ or $x = x_2 x_4^2$. If $x = x_2 x_4$, $x_2 x_4 \overset{Sq^2}{\hookleftarrow} x_8 \overset{Sq^4}{\mapsto} x_4$ is a path connecting x and a lower dimensional vertex. If $x = x_2 x_4^2$, $x_2 x_4^2 \overset{Sq^2}{\hookleftarrow} x_2 x_{10} \overset{Sq^6}{\hookleftarrow} x_2 x_{16} \overset{Sq^4 Sq^8}{\longmapsto} x_2 x_4$ is a path connecting x and a lower dimensional vertex. $\square$

Proof of Theorem 1.6 for E_7. Similarly to the argument for the case E_6, we only have to prove that there is a path connecting x and a lower dimensional vertex for any non-zero element x of $H_*(\Omega E_7)$ with degree greater than 2 and $Sq^i x = 0$ for all $i > 0$. We consider the following lists.

v	$u = Sq^4 v$	$\lvert u\rvert$
?	x_2	2
$x_2 x_8$	$x_2 x_4$	6
x_{14}	x_{10}	10
$x_2 x_{16}$	$x_2 x_4 x_8$	14
$x_2 x_8 x_{14}$	$x_2(x_4 x_{14} + x_8 x_{10})$	20
$x_{14} x_{18} + x_{16}^2$	$\bar{x}_{18}$	28
$x_2\left(x_{10}^2 x_{26} + x_{14} x_{16}^2\right) + x_4 x_{10}^3 x_{14}$	$\bar{x}_4$	44
y_{60}	x_{56}	56
y_{78}	x_{74}	74
y_{80}	$\bar{x}_{26}$	76
y_{116}	x_{56}^2	112

where

$$
\begin{aligned}
y_{60} &= x_{10}^2 x_{14} x_{26} + x_{10} x_{16}^2 x_{18} + x_{10}^3 x_{14} x_{16} + x_{14}^2 x_{16}^2 + x_8 x_{10}^2 x_{16}^2,\\
y_{78} &= x_{10} x_{14} x_{18}^3 + x_{10}^2 x_{14} x_{22}^2 + x_{14} x_{16}^4 + x_{14}^2 x_{16}^2 x_{18} + x_{14}^3 x_{18}^2,\\
y_{80} &= x_2\left(x_{22}^2 x_{34} + x_4 x_8 x_{22}^3 + x_{14}^2 x_{16}^2 x_{18} + x_{14} x_{16}^4 + x_8 x_{10}^4 x_{14} x_{16}\right) + x_{10}^2 y_{60},\\
y_{116} &= x_{10}^2 x_{22}^2 x_{26}^2 + x_{10}^4 x_{16}^2 x_{22}^2 + x_{10}^2 x_{14}^2 x_{16}^2 x_{18}^2 + x_{14}^6 x_{16}^2.
\end{aligned}
$$

$\lvert x\rvert$	$\lvert z\rvert$	z	$Sq^8 z$
$10n+14$	$10n+18$	$x_{18} x_{10}^n$	$Sq^8\left(x_{18} x_{10}^n\right) = x_{10}^{n+1}$
$10n+16$	$10n+20$	$x_2 x_{18} x_{10}^n$	$Sq^8\left(x_2 x_{18} x_{10}^n\right) = x_2 x_{10}^{n+1}$
$10n+18$	$10n+22$	$x_{22} x_{10}^n$	$Sq^8\left(x_{22} x_{10}^n\right) = x_{14} x_{10}^n$
$10n+20$	$10n+24$	$x_2 x_{22} x_{10}^n$	$Sq^8\left(x_2 x_{22} x_{10}^n\right) = x_2 x_{14} x_{10}^n$
$10n+22$	$10n+26$	$z_{26} x_{10}^n$	$Sq^8\left(z_{26} x_{10}^n\right) = x_4 x_{14} x_{10}^n$

where $z_{26} = x_4 x_{22} + x_2 x_{10} x_{14}$.

The first list above shows that there is an element y such that $Sq^4 y = x$. The second list above shows that, if $|x| \geq 14$, there is a path connecting x and a lower dimensional vertex just as in the proof for E_6.

If $|x| < 14$, then $x = x_2x_4$, x_{10} or x_2x_{10}.

If $x = x_2x_4$, then $Sq^2x_8 = x$ and $Sq^4y = x_4 \neq 0$. If $x = x_{10}x'$, where $x' = 1$ or x_2, then $Sq^6(x_{16}x') = x$ and $Sq^8(x_{16}x') = x_8x' \neq 0$.

Thus we complete the proof of Theorem 1.6 for E_7. □

4. Proof of Theorem 3.1.

In this section we prove Theorem 3.1. We put

$$A = \{y \in H_*(\Omega E_6)[x_4^{-1}] \mid Sq^iy = 0 \text{ for all } i > 0\},$$
$$B = \boldsymbol{F}_2[x_4^2, x_{20}, \bar{x}_{16}][x_4^{-1}]\{1, x_2, x_2x_4, x_4^3 + x_2x_{10}\} = \Lambda(x_2) \otimes \boldsymbol{F}_2[\bar{x}_4, x_{20}, \bar{x}_{16}][x_4^{-1}],$$

where $\bar{x}_4 = x_4^3 + x_2x_{10}$. To prove the theorem it is sufficient to prove that $A = B$. Since we have the following isomorphisms as modules

$$\begin{aligned} H_*(\Omega E_6)[x_4^{-1}] &\cong \Lambda(x_2) \otimes \boldsymbol{F}_2[x_4, x_8, x_{10}, x_{14}, x_{16}, x_{22}][x_4^{-1}] \\ &\cong \Lambda(x_2) \otimes \boldsymbol{F}_2[x_4, x_8, x_{10}, x_{14}, \bar{x}_{16}, x_{22}][x_4^{-1}] \\ &\cong \Lambda(x_2) \otimes \boldsymbol{F}_2[\bar{x}_4, x_{20}, \bar{x}_{16}][x_4^{-1}] \otimes \Lambda(x_{10}) \otimes \boldsymbol{F}_2[x_8, x_{14}, x_{22}], \end{aligned}$$

any element y of $H_*(\Omega E_6)[x_4^{-1}]$ is written uniquely as

$$y = \sum_{a,b,d \geq 0,\ c=0,1} x_{22}^a x_{14}^b x_{10}^c x_8^d P_{a,b,c,d},$$

where $P_{a,b,c,d} \in \Lambda(x_2) \otimes \boldsymbol{F}_2[\bar{x}_4, x_{20}, \bar{x}_{16}][x_4^{-1}]$. We define the second degree $|y|_2$ of y by

$$|y|_2 = \max\{|x_{22}^a x_{14}^b x_{10}^c x_8^d| \mid P_{a,b,c,d} \neq 0\}.$$

By A_i (resp. B_i) we denote the submodule of A (resp. B) which consists of elements with the second degree i.

It is easy to see that $B \subset A$ and $A_0 = B_0$ by definition. By induction on the second degree we prove that $A = B$.

Let M be a positive integer and assume that $A = B$ up to degree $2M - 2$. We prove that the equality holds in degree $2M$.

For a positive integer a we put

$$P_a^{22} = \sum_{b,d \geq 0,\ c=0,1} x_{14}^b x_{10}^c x_8^d P_{a,b,c,d},$$

then we have

Lemma 4.1. *$P_a^{22} = 0$ unless a is a power of 2. Moreover, P_a^{22} is an element of B.*

PROOF. We prove this lemma by downward induction on a. For sufficiently large a the assertion is trivially true. Assume that, for $a \geq 2^{n+1}$, $P_a^{22} = 0$ unless a is a power of 2 and P_a^{22} are elements of B. Put $Q = \sum_{2^n \leq a < 2^{n+1}} x_{22}^{a-2^n} P_a^{22}$, then

$$y = \sum_{i>n} x_{22}^{2^i} P_{2^i}^{22} + x_{22}^{2^n} Q + \sum_{a<2^n} x_{22}^a P_a^{22} + \text{terms without } x_{22}.$$

Since for any $\ell > 0$ we have

$$0 = Sq^\ell y = \sum_{i>n} Sq^\ell\bigl(x_{22}^{2^i}\bigr) P_{2^i}^{22} + \sum_{k>0} Sq^k\bigl(x_{22}^{2^n}\bigr) Sq^{\ell-k} Q$$
$$+ (Sq^\ell Q) x_{22}^{2^n} + \sum_{a<2^n} Sq^\ell\bigl(x_{22}^a P_a^{22}\bigr) + \text{terms without } x_{22}$$

and $Sq^k(x_{22}^{2^i}) \in \boldsymbol{F}_2[x_{10}, x_{14}]$ for $k > 0$, the coefficient of $x_{22}^{2^n}$, $Sq^\ell Q$, must be 0. Thus, by induction, we have $Q \in A_{2M-22\cdot 2^n} = B_{2M-22\cdot 2^n}$. This implies that $P_a^{22} = 0$ for $2^n < a < 2^{n+1}$ and that $P_{2^n}^{22} \in A_{2M-22\cdot 2^n} = B_{2M-22\cdot 2^n}$. □

By Lemma 4.1 y is written as

$$y = \sum_{a\geq 0} x_{22}^{2^a} P_{2^a}^{22} + \sum_{b,d\geq 0,\ c=0,1} x_{14}^b x_{10}^c x_8^d P_{0,b,c,d},$$

where $P_{2^a}^{22}$ and $P_{0,b,c,d}$ are elements of B.

As $Sq^i x_{22}, Sq^i x_{14}, Sq^i x_{10}$ are in $\boldsymbol{F}_2[x_4, x_{10}, x_{14}]$ for $i > 0$, if for a positive integer d we put

$$P_d^8 = \sum_{b\geq 0,\ c=0,1} x_{14}^b x_{10}^c P_{0,b,c,d},$$

then similarly we have

LEMMA 4.2. *$P_d^8 = 0$ unless d is a power of 2. Moreover, P_d^8 is an element of B.*

Thus we proved that y is written as

$$y = \sum_{a\geq 0} x_{22}^{2^a} P_{2^a}^{22} + \sum_{b\geq 0,\ c=0,1} x_{14}^b x_{10}^c P_{b,c} + \sum_{d\geq 0} x_8^{2^d} P_{2^d}^8, \tag{4.1}$$

where $P_{2^a}^{22},\ P_{b,c} = P_{0,b,c,0},\ P_{2^d}^8 \in B$.

By applying Sq^2 to the equality (4.1) we have

$$0 = Sq^2 y = x_{10}^2 P_1^{22} + \sum_{b\geq 0} x_{14}^b x_4^2 P_{b,1} + x_2 x_4 P_1^8$$
$$= \bigl(x_4^3 x_8 + x_{20} + x_2\bigl(x_4^2 x_{10} + x_4 x_{14} + x_8 x_{10}\bigr)\bigr) P_1^{22} + \sum_{b\geq 0} x_{14}^b x_4^2 P_{b,1} + x_2 x_4 P_1^8,$$

which implies that

$$P_1^{22} = 0, \quad P_{b,1} = 0 \quad \text{for } b > 0, \quad x_4^2 P_{0,1} = x_2 x_4 P_1^8.$$

Then y is written as

$$y = \sum_{a \geq 1} x_{22}^{2^a} P_{2^a}^{22} + \sum_{b \geq 0} x_{14}^b P_{b,0} + x_{10} P_{0,1} + \sum_{d \geq 0} x_8^d P_{2^d}^8. \tag{4.2}$$

By applying Sq^4 to the equality (4.2) we have

$$\begin{aligned} 0 = Sq^4 y &= x_{10}^4 P_2^{22} + \sum x_{14}^{2b} x_{10} P_{2b+1,0} + x_4 P_1^8 \\ &= \left(x_4^3 x_8 + x_{20}\right)^2 P_2^{22} + \sum x_{14}^{2b} x_{10} P_{2b+1,0} + x_4 P_1^8, \end{aligned}$$

which implies that

$$P_2^{22} = 0, \quad P_{2b+1,0} = 0, \quad P_1^8 = 0.$$

By the last equality we have $P_{0,1} = x_2 x_4^{-1} P_1^8 = 0$. Thus y is written as

$$y = \sum_{a \geq 2} x_{22}^{2^a} P_{2^a}^{22} + \sum_{b \geq 0} x_{14}^{2b} P_{2b,0} + \sum_{d \geq 1} x_8^d P_{2^d}^8.$$

Now it is easy to show, by induction on n, that y is written as

$$y = \sum_{a \geq n+1} x_{22}^{2^a} P_{2^a}^{22} + \sum_{b \geq 0} x_{14}^{2^n b} P_{2^n b,0} + \sum_{d \geq n} x_8^d P_{2^d}^8.$$

Therefore $y = P_{0,0} \in B$ as desired. □

5. Proof of Theorem 3.2.

As in the proof of Theorem 3.1 we proceed the calculation in the ring $H_*(\Omega E_7)[x_{10}^{-1}]$. Since

$$\begin{aligned} x_4 &= \bar{x}_4 x_{10}^{-4} + x_2 x_{10}^{-4}\left(x_{10} x_{16}^2 + x_{10}^2 x_{22} + x_{14}^3\right), \\ x_{18} &= x_{10}^{-1} \bar{x}_{18} + x_{10}^{-1} x_{14}^2, \\ x_{26} &= x_{10}^{-5} \bar{x}_{26} + x_2 x_{10}^{-5}\left(x_{10}^4 x_{34} + x_4 x_8 x_{10}^4 x_{22} + x_{14}^4 x_{18} + x_{10} x_{16}^4 \right. \\ &\quad \left. + x_4 x_{10}^4 x_{14} x_{16} + x_8 x_{10}^5 x_{16}\right) + x_{10}^{-3}\left(x_{10}^2 x_{14} x_{22} + x_{10} x_{14}^2 x_{18} \right. \\ &\quad \left. + x_{10}^4 x_{16} + x_{14}^4 + x_8 x_{10}^2 x_{14}^2 + x_4 x_{10}^2 x_{16}^2 + x_4 x_8 x_{10}^3 x_{14}\right), \\ x_{22}^2 &= x_{10}^{-3} x_{74} + x_{10}^{-2} x_{16}^4 + x_{10}^{-4} x_{14}^6 + x_{10}^3 x_{14} + x_{10}^{-4} \bar{x}_{18}^3, \end{aligned}$$

we have the following isomorphisms of modules:

$$\begin{aligned} H_*(\Omega E_7)\left[x_{10}^{-1}\right] &\cong \Lambda(x_2,x_4,x_8)\otimes \boldsymbol{F}_2[x_{10},x_{14},x_{16},x_{18},x_{22},x_{26},x_{34}]\left[x_{10}^{-1}\right] \\ &\cong \Lambda(x_2,\bar{x}_4)\otimes \boldsymbol{F}_2[x_{10},\bar{x}_{18},x_{74},\bar{x}_{26}]\left[x_{10}^{-1}\right] \\ &\quad \otimes \Lambda(x_8,x_{22})\otimes \boldsymbol{F}_2[x_{14},x_{16},x_{34}]. \end{aligned}$$

Therefore, any element y of $H_*(\Omega E_7)[x_{10}^{-1}]$ is written uniquely as

$$y=\sum_{a,c,d\geq 0,\ b,e=0,1} x_{34}^a x_{22}^b x_{16}^c x_{14}^d x_8^e P_{a,b,c,d,e},$$

where $P_{a,b,c,d,e}\in \Lambda(x_2,\bar{x}_4)\otimes \boldsymbol{F}_2[x_{10},\bar{x}_{18},x_{74},\bar{x}_{26}][x_{10}^{-1}]$. We define the second degree $|y|_2$ of y by

$$|y|_2=\max\left\{\left|x_{34}^a x_{22}^b x_{16}^c x_{14}^d x_8^e\right| \mid P_{a,b,c,d,e}\neq 0\right\}.$$

We put

$$\begin{aligned} A &= \left\{y\in H_*(\Omega E_7)\left[x_{10}^{-1}\right] \mid Sq^i y=0 \text{ for all } i>0\right\}, \\ B &= \Lambda(\bar{x}_4)\otimes \boldsymbol{F}_2\left[x_{10},\bar{x}_{18},x_{74},x_{56}^2\right]\left[x_{10}^{-1}\right]\bar{x}_{26} \\ &\quad + \boldsymbol{F}_2\left[x_{10},\bar{x}_{18},x_{74},x_{56}^2\right]\left[x_{10}^{-1}\right]\left\{1,x_2,x_2x_4,x_2x_4x_8,x_2(x_4x_{14}+x_8x_{10}),\right. \\ &\qquad \left.\bar{x}_4,x_2x_{56},x_2x_4x_{56},x_2x_4x_8x_{56},x_2x_{56}(x_4x_{14}+x_8x_{10})\right\}. \end{aligned}$$

Since $x_{56}^2=x_{10}^{-4}\bar{x}_{26}^2$, $x_2x_{56}=x_2\bar{x}_{26}x_{10}^{-2}$, $x_2x_4=x_2\bar{x}_4x_{10}^{-4}$,

$$B=\Lambda(x_2,\bar{x}_4)\otimes \boldsymbol{F}_2[x_{10},\bar{x}_{18},x_{74},\bar{x}_{26}]\left[x_{10}^{-1}\right]\{1,x_2(x_4x_{14}+x_8x_{10})\}.$$

Then it is easy to see that $B\subset A$ and $A_0=B_0$. By induction on the second degree we will prove that $A=B$. Let M be a positive integer and assume that $A=B$ up to degree $2M-2$. We prove that the equality holds in degree $2M$.

Let y be an element of A_{2M}. We recall that

$$\begin{array}{llll} Sq^2x_{34}=x_{16}^2, & Sq^{16}x_{34}=x_{18}=x_{10}^{-1}\left(\bar{x}_{18}+x_{14}^2\right), & & Sq^2x_{22}=x_{10}^2, \\ Sq^8x_{22}=x_{14}, & Sq^2x_{16}=x_{14}+x_2x_4x_8, & & Sq^4x_{16}=x_4x_8, \\ Sq^8x_{16}=x_8, & Sq^4x_{14}=x_{10}, & Sq^2x_8=x_2x_4, & Sq^4x_8=x_4, \end{array}$$

and $Sq^{2^i}x_{2j}=0$ in all cases not explicitly recorded.

Similarly to the case E_6 we have the following lemma.

LEMMA 5.1. *y is written as*

$$y = \sum_{a \geq 0} x_{34}^{2^a} P_{2^a}^{34} + x_{22} P^{22} + x_{16} P_1^{16} + \sum x_{16}^{2c} x_{14}^{d} x_8^{e} P_{2c,d,e},$$

where $P_{2^a}^{34}$, P^{22} *and* P_1^{16} *are in* B *and*

$$P_{2c,d,e} \in \Lambda(x_2, \bar{x}_4) \otimes \boldsymbol{F}_2[x_{10}, \bar{x}_{18}, x_{74}, \bar{x}_{26}]\left[x_{10}^{-1}\right].$$

By applying Sq^2 to y we have

$$0 = Sq^2 y = x_{16}^2 P_1^{34} + x_{10}^2 P^{22} + (x_{14} + x_2 x_4 x_8) P_1^{16} + \sum x_{16}^{2c} x_{14}^{d} x_2 x_4 P_{2c,d,1}.$$

As $P_1^{34}, P^{22}, P_1^{16}, P_{2c,d,1} \in B$, by comparing the coefficient of $x_{16}^{2c} x_{14}^{d}$ in the equality above we have

$$P_1^{34} = x_2 x_4 P_{2,0,1}, \quad P_1^{16} = x_2 x_4 P_{0,1,1}, \quad x_{10}^2 P^{22} = x_2 x_4 P_{0,0,1}. \tag{5.1}$$

Since

$$\begin{aligned} 0 &= x_4 Sq^2 Sq^4 y \\ &= x_4 \Big(\sum x_{16}^{4c} x_{14}^{d+2} x_2 x_4 P_{4c+2,d,1} + \sum x_{16}^{2c} x_{14}^{2d} x_{10} x_2 x_4 P_{2c,2d+1,1} + \sum x_{16}^{2c} x_{14}^{d} x_2 P_{2c,d,1} \Big) \\ &= \sum x_{16}^{2c} x_{14}^{d} x_2 x_4 P_{2c,d,1} \end{aligned}$$

and $P_{2c,d,e} \in \Lambda(x_2, \bar{x}_4) \otimes \boldsymbol{F}_2[x_{10}, \bar{x}_{18}, x_{74}, \bar{x}_{26}][x_{10}^{-1}]$, we have $x_2 x_4 P_{2c,d,1} = 0$. Then the equality $0 = Sq^2 Sq^4 y = \sum x_{16}^{2c} x_{14}^{d} x_2 P_{2c,d,1}$ implies that

$$x_2 P_{2c,d,1} = 0. \tag{5.2}$$

By the equalities (5.1) and (5.2) we have

$$P_1^{34} = 0, \quad P^{22} = 0, \quad P_1^{16} = 0.$$

Then y is written as

$$y = \sum_{a \geq 1} x_{34}^{2^a} P_{2^a}^{34} + \sum x_{16}^{2c} x_{14}^{d} x_8^{e} P_{2c,d,e}.$$

If we put

$$P_1^{14} = \sum x_{16}^{4c} x_{14}^{2d} x_8^{e} P_{4c,2d+1,e},$$

then y is written as

$$y = \sum_{a\geq 1} x_{34}^{2^a} P_{2^a}^{34} + x_{14}P_1^{14} + \sum x_{16}^{2c}x_{14}^{2d}x_8^e P_{2c,2d,e}$$

and the fact that $P_1^{16} = 0$ implies that $P_1^{14} \in B$ by the same argument as in the proof of Lemma 4.1.

By applying Sq^4 to the equality above we have

$$0 = Sq^4 y = x_{16}^4 P_2^{34} + x_{10}P_1^{14} + \sum x_{16}^{4c}x_{14}^{2d+2}x_8^e P_{4c+2,2d,e} + \sum x_{16}^{2c}x_{14}^{2d}x_4 P_{2c,2d,1},$$

which implies that $x_{10}P^{14} = x_4 P_{0,0,1}$.

Thus y is written as

$$y = \sum_{a\geq 1} x_{34}^{2^a} P_{2^a}^{34} + \sum_{(c,d)\neq(0,0)} x_{16}^{2c}x_{14}^{2d}x_8^e P_{2c,2d,e} + P_0,$$

where $P_0 = P_{0,0,0} + x_{10}^{-1}(x_4x_{14} + x_8x_{10})P_{0,0,1}$. Since $x_2P_{0,0,1} = 0$ by (5.2), $P_0 \in B$. Now it is easy to show that, by induction on n, y is written as

$$y = \sum_{a\geq n} x_{34}^{2^a} P_{2^a}^{34} + \sum_{(c,d)\neq(0,0)} x_{16}^{2^n c}x_{14}^{2^n d}x_8^e P_{2^n c,2^n d,e} + P_0.$$

Therefore $y = P_0 \in B$ as desired. □

References

[1] M. C. Crabb, On stable splitting of $U(n)$ and $\Omega U(n)$, Springer Lecture Notes in Math., **1298** (1986), 35–53.

[2] P. W. Duckworth, The K-theory Pontrjagin rings for the loop spaces on the exceptional Lie groups, Quart. J. Math. Oxford (2), **35** (1984), 253–262.

[3] B. Gray and C. A. McGibbon, Universal phantom maps, Topology, **32** (1993), 371–394.

[4] J. Grodal, The transcendence degree of the mod p cohomology of finite Postnikov systems, Fields Inst. Comm., **19** (1998), 111–130.

[5] M. J. Hopkins, Stable decompositions of certain loop spaces, Ph. D. thesis, Evanston, 1984.

[6] J. R. Hubbuck, Some stably indecomposable loop spaces, Springer Lecture Notes in Math., **1418** (1990), 70–77.

[7] K. Iriye, Universal phantom maps out of loop spaces, Proc. R. Soc. Edinburgh, **130A** (2000), 313–333.

[8] A. Kono and K. Kozima, The space of loops on a symplectic group, Japan. J. Math., **4** (1978), 461–486.

[9] A. Kono and K. Kozima, The mod 2 homology of the space of loops on the exceptional Lie group, Proc. R. Soc. Edinburgh, **112A** (1989), 187–202.

[10] J. P. May, The homology of E_∞-space, Springer Lecture Notes in Math., **533** (1976), 1–68.

[11] C. A. McGibbon, Phantom maps, Chapter 25 in The Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995.

[12] C. A. McGibbon, Some problems about phantom maps, Fields Inst. Comm., **19** (1998), 241–250.

[13] R. J. Wellington, The unstable Adams spectral sequence for free iterated loop spaces, Memoirs of Amer. Math. Soc., **258** (1982).

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