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Pencil genus for normal surface singularities

Dedicated to Professor Kei-ichi Watanabe on his sixtieth birthday

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Abstract. Let (X, o) be a normal complex surface singularity. We define an invariant $p_e(X, o)$ for (X, o) in terms of pencils of compact complex curves. Similarly, for a pair of (X, o) and $h \in \mathfrak{m}_{X,o}$ (the maximal ideal of $\mathscr{O}_{X,o}$), we define an invariant $p_e(X, o, h)$. We call $p_e(X, o)$ (resp. $p_e(X, o, h)$) the *pencil genus of* (X, o) (resp. *a pair of* (X, o) *and* h). In this paper, we give a method to construct pencils of compact complex curves by gluing a resolution space of (X, o) and resolution spaces of some cyclic quotient singularities. Using this, we prove some formulae on $p_e(X, o, h)$ and estimate $p_e(X, o)$. We also characterize Kodaira singularities in terms of $p_e(X, o, h)$.

1. Introduction.

In the field of complex surface singularity theory, there have been several works until now about relations between singularities and pencils (one parameter families) of algebraic curves. With respect to their relation, V. Kulikov [**Ku**] showed that unimodal and bimodal singularities classified by V. I. Arnold [**Arn**] are obtained through a procedure (see Definition 3.7 of this paper) from Kodaira's list [**Ko**] of pencils of elliptic curves. In addition, M. Reid [**Re1**] pointed out relations between minimally elliptic singularities and pencils of elliptic curves. Further, generalizing the procedure of Kulikov, U. Karras [**Ka1**] introduced the notion of *Kodaira singularities* in terms of pencils of curves. He also applied it to the deformation theory of surface singularities. In [**St1**], J. Stevens studied a subclass of Kodaira singularities (called *Kulikov singularities*) and proved some relations between them and deformations of curve singularities. In this paper, we also study some relations between surface singularities and pencils of compact complex algebraic curves.

Before describing our main results, we review some facts and definitions. Let S be a non-singular complex surface and $\Delta \subset \boldsymbol{C}$ a small open disc around the origin. If $\Phi : S \to \Delta$ is a proper surjective holomorphic map with connected fibers and the generic fiber $S_t := \Phi^{-1}(t)$ $(t \neq 0)$ is a smooth curve of genus g, it is called a *pencil of curves of genus* g. If $\Phi : S \to \Delta$ is a pencil of curves, the intersection matrix of any connected one-dimensional analytic proper subset E in $\mathrm{supp}(S_o)$ is negative definite from Zariski's lemma ([**BPV**, p. 90]). Hence E is contracted to a normal surface singularity by Grauert's result ([**G**, p. 367]). In this paper, we consider the converse problem. Namely, we will construct pencils of curves from resolution spaces of normal surface singularities and holomorphic functions on them (Theorem 2.4).

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Definition 1.1. Let (X, o) be a normal surface singularity and $\pi : (\tilde{X}, E) \to (X, o)$ a resolution. Let $\Phi : S \to \Delta$ be a pencil of curves.

(i) If $(S, \mathrm{supp}(S_o)) \supset (\tilde{X}, E)$ (i.e., $S \supset \tilde{X}$ and $\mathrm{supp}(S_o) \supset E$), then Φ is called *a pencil of curves including* $(\tilde{X}, E)$, where $\mathrm{supp}(S_o)$ is the support of S_o.

(ii) If $h \in \mathfrak{m}_{X,o}$ satisfies $h \circ \pi = \Phi$, then Φ is called *a pencil of curves extending* $h \circ \pi$ *or an extension of* $h \circ \pi$. Namely it implies the following diagram:

$$\begin{array}{ccc} (X, o) & \xleftarrow{\ \pi\ } & (\tilde{X}, E) \subset S \\ & \searrow^{h} & \downarrow \Phi, \\ & & \Delta \end{array}$$

where $\mathrm{supp}(S_o) \supset E$ and $h \circ \pi = \Phi|_{\tilde{X}}$.

(iii) Under the situation of (ii), if there is no (-1) curve in $\mathrm{supp}(S_o) \backslash E$ which does not intersect E, then we call Φ a *pencil of curves minimally extending* $h \circ \pi$ or a *minimal extension of* $h \circ \pi$.

Definition 1.2.

(i) Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$. If h defines a reduced curve on X, then h is called a *reduced element*.

(ii) Let $(R, \mathfrak{m})$ be a commutative local ring and h a non-zero element of $\mathfrak{m}$. Then h is called a *perfect power element* if there is an element $g \in \mathfrak{m}$ satisfying $h = g^k$ for some positive integer $k \geqq 2$.

In this paper, we prove that if $h \in \mathfrak{m}_{X,o}$ is not a perfect power element, then there is a pencil of curves extending $h \circ \pi$ (see Theorem 2.4). Here we can state the following definition.

Definition 1.3. Let (X, o) be a normal surface singularity.

(i) We define a holomorphic invariant for (X, o) as follows:

$$p_e(X, o) = \min\{\text{the genus of a pencil of curves including a resolution of } (X, o)\}.$$

(ii) Let $h \in \mathfrak{m}_{X,o}$ be not a perfect power element. We also define a holomorphic invariant for a pair of (X, o) and h as follows:

$$p_e(X, o, h) = \min\{\text{the genus of a pencil of curves extending } h \circ \pi \text{ for a resolution } \pi \text{ of } (X, o)\}.$$

Then, $p_e(X, o)$ (resp. $p_e(X, o, h)$) is called the *pencil genus* of (X, o) (resp. a pair of (X, o) and h).

Remark 1.4. Let (X, o) be a normal isolated singularity. In general, if $h \in \mathfrak{m}_{X,o}$ is a reduced element, then it defines a one parameter smoothing of (X, o) ([**Mah1**, p. 150]). Using formal completion argument presented by M. Artin, E. Looijenga [**Lo**, p. 301] proved that if $h \in \mathfrak{m}_{X,o}$ is an element that gives a one parameter smoothing

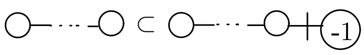
$h : (X, o) \to (\boldsymbol{C}^1, o)$, then there is a flat projective morphism $\psi : Z \to \boldsymbol{C}^1$ and an embedding $\phi : X \to Z$ satisfying $h = \psi \circ \phi$. In Section 2 of this paper, we prove a similar result (see Theorem 2.4) in the case of $\dim_{\boldsymbol{C}}(X, o) = 2$ and h is not a perfect power element by a different method. From this method we can compute $p_e(X, o, h)$ for (X, o) and h.

Remark 1.5.

(i) Let (X, o) be a normal isolated singularity. Let $(\tilde{X}, E) \xrightarrow{\pi} (X, o)$ be an arbitrary resolution. Then there is a pencil of curves $S \xrightarrow{\Phi} \Delta$ of genus $p_e(X, o)$ and including $(\tilde{X}, E)$. In fact, there is a pencil of curves $\hat{S} \xrightarrow{\hat{\Phi}} \Delta$ of genus $p_e(X, o)$ and including a resolution space $(\hat{X}, \hat{E})$ of (X, o). There is a birational transformation $\bar{\varphi}$ from $\tilde{X}$ to $\hat{X}$. Thus, there is a complex surface S and a birational transformation $S \xrightarrow{\varphi} \hat{S}$ such that $\varphi|_{\tilde{X}} = \bar{\varphi}$ and $\Phi := \hat{\Phi} \circ \varphi : S \to \Delta$ gives a pencil of curves. The genus of Φ is $p_e(X, o)$ and $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_o))$.

(ii) Let $(\tilde{X}_i, E(i)) \xrightarrow{\pi_i} (X_i, o)$ be a resolution of a normal surface singularity for $i = 1, 2$. If $(\tilde{X}_1, E(1)) \supset (\tilde{X}_2, E(2))$, then $p_e(X_1, o) \geqq p_e(X_2, o)$. In fact, there is a pencil of curves $S_1 \xrightarrow{\Phi_1} \Delta$ of genus $p_e(X_1, o)$ and including $(\tilde{X}_1, E(1))$ from (i). Then $(S_1, \mathrm{supp}(S_1)_o) \supset (\tilde{X}_2, E(2))$; hence $p_e(X_1, o) \geqq p_e(X_2, o)$. Further, if $h_i \in \mathfrak{m}_{X_i, o}$ for $i = 1, 2$ and $h_1 \circ \pi_1|_{\tilde{X}_2} = h_2 \circ \pi_2$, then $p_e(X_1, o, h_1) \geqq p_e(X_2, o, h_2)$. In fact, if $S_1 \xrightarrow{\Phi_1} \Delta$ is a pencil of curves of genus $p_e(X_1, o, h_1)$ which is an extension of $h_1 \circ \pi_1$, then $h_2 \circ \pi_2 = h_1 \circ \pi_1|_{\tilde{X}_2} = \Phi_1|_{\tilde{X}_2}$; hence $p_e(X_1, o, h_1) \geqq p_e(X_2, o, h_2)$.

Remark 1.6. For rational double points, $p_e(X, o) = 0$ for any singularity of type A_n $(n = 1, 2, \dots)$ or D_n $(n = 4, 5, \dots)$, as shown in the following configurations:



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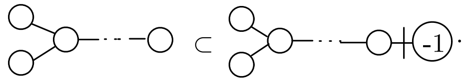


On the other hand, we prove $p_e(X, o) = 1$ for a singularity of type E_n for $n = 6, 7$ and 8 (see Proposition 3.12).

Let (X, o) be a normal surface singularity. Let $\pi : (\tilde{X}, E) \to (X, o)$ be a resolution and let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition. For an element $h \in \mathfrak{m}_{X,o}$, let $(h \circ \pi)_{\tilde{X}}$ be the divisor defined by $h \circ \pi$ on $\tilde{X}$. The *exceptional part* $E(h \circ \pi)$ of $(h \circ \pi)_{\tilde{X}}$ is defined by $E(h \circ \pi) = \sum_{i=1}^{r} v_{E_i}(h \circ \pi) E_i$, where $v_{E_i}(h \circ \pi)$ indicates the vanishing order of $h \circ \pi$ on E_i. Let $\Lambda(h \circ \pi)$ be the strict transform of the divisor $\{h = 0\}$ in X by π. If $\mathrm{supp}(\Lambda(h \circ \pi)) = \bigcup_{j=1}^{s} C_j$ is the irreducible decomposition, then $\Lambda(h \circ \pi) = \sum_{j=1}^{s} v_{C_j}(h \circ \pi) C_j$ and it is called the *non-exceptional part of* $(h \circ \pi)_{\tilde{X}}$. For an effective cycle $D = \sum_{i=1}^{r} d_i E_i$ on E, the arithmetic genus of D is defined as

$$p_a(D) := 1 + \frac{1}{2}\left(D^2 + K_{\tilde{X}} D\right), \tag{1.1}$$

where $K_{\tilde{X}}$ is the canonical bundle (or divisor) on $\tilde{X}$. For the (Artin's) *fundamental cycle* $Z_E := \min\{D = \sum_{i=1}^{r} a_i E_i | a_i > 0 \text{ and } DE_i \leqq 0 \text{ for } i\}$ ([**Art**]), the value of $p_a(Z_E)$ is independent of the choice of a resolution, and so we put it $p_f(X,o)$ in this paper. The positive cycle $M_E := \min\{E(h\circ\pi) | h \in \mathfrak{m}_{X,o}\}$ on E is called the *maximal ideal cycle* on E and we have $Z_E \leqq M_E$ ([**Y**]). In this paper, $\boldsymbol{M}_X$ represents the maximal ideal cycle on the minimal resolution.

From now on we explain our main results of this paper. In the following, let $h \in \mathfrak{m}_{X,o}$ be not a perfect power element; furthermore, assume that $\mathrm{red}(h\circ\pi)_{\tilde{X}}$ is a simple normal crossing divisor on $\tilde{X}$.

In Section 2, we construct a pencil of curves $\Phi : S \to \Delta$ of genus $p_e(X,o,h)$ such that Φ is a minimal extension of $h\circ\pi$ and all connected components of $\mathrm{supp}(S_o)\backslash E$ are minimal $\boldsymbol{P}^1$-chains started from E (Theorem 2.4). By the properties of the pencil of curves constructed from $h\circ\pi$ as in Theorem 2.4, we can characterize the numerical type (i.e., the weighted dual graph and the coefficient for any irreducible component of the singular fiber) of a pencil of curves of genus $p_e(X,o,h)$ which is an extension of $h\circ\pi$ (Theorem 2.9). Also, using the construction of Theorem 2.4, we prove the following equality (Corollary 2.12),

$$p_e(X,o,h) = \delta(h) - r(h) + 1,$$

for a reduced element $h \in \mathfrak{m}_{X,o}$, where $\delta(h)$ (resp. $r(h)$) is the conductor number (resp. the number of irreducible components) of the curve singularity $\{h=0\} \subset X$.

In Section 3, first we prove the following inequality (Theorem 3.5),

$$p_f(X,o) \leqq p_e(X,o) \leqq p_a(\boldsymbol{M}_X) + \mathrm{mult}(X,o) - 1,$$

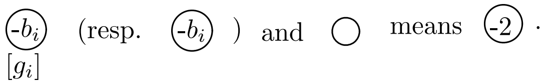
where $\mathrm{mult}(X,o)$ is the multiplicity. If $p_f(X,o) = p_e(X,o)$, then (X,o) is called *a weak Kodaira singularity* (Definition 3.6). From a result by U. Karras (cf. [**Ka2**, Lemma 3.4] and Proposition 3.10 (ii) of this paper), any Kodaira singularity is a weak Kodaira singularity. Second, in Theorem 3.11, we characterize Kodaira singularities and Kulikov singularities in terms of pencil genus. The statements are given by the existence of a good function $h \in \mathfrak{m}_{X,o}$ satisfying the equality $p_f(X,o) = p_e(X,o) = p_e(X,o,h)$. Third, we compute $p_e(X,o)$ for log-canonical singularities and rational triple points (Proposition 3.12 and 3.13).

In Section 4, we consider cyclic covers of normal surface singularities and pencil genus for them. Let (Y,o) be a normal complex surface singularity and let $h \in \mathfrak{m}_{Y,o}$ be a semi-reduced element (Definition 4.6). Let (X,o) be a normalization of the cyclic covering defined by $z^n = h$ over (Y,o). We prove that there is a positive integer $N_h(Y,o)$ such that (X,o) is a weak Kodaira singularity satisfying $p_f(X,o) = p_e(Y,o,h)$ for any positive integer $n \geqq N_h(Y,0)$ (Theorem 4.12). Furthermore, we prove that if h is a reduced element, then (X,o) is a Kulikov singularity (Theorem 4.14).

Let (X,o) be a normal hypersurface singularity defined by $z^n = f(x,y)$ and $\ell \in \mathfrak{n}\backslash\mathfrak{n}^2$, where $\mathfrak{n}$ is the maximal ideal $(x,y) \subset \boldsymbol{C}\{x,y\}$. Let $I_o(\ell,h)$ be the intersection multiplicity of $\{h=0\}$ and $\{\ell=0\}$ at $\{0\} \in \boldsymbol{C}^2$ ([**BK**, p. 47], [**Na**, p. 231]). In Section 5, a formula of $p_e(X,o,\ell)$ is proven in terms of n and $I_o(\ell,h)$ (Theorem 5.4).

NOTATIONS AND TERMINOLOGIES. Let M be a complex surface and let $D =$

$\sum_{i=1}^{r} d_i A_i$ be a divisor on M, where each A_i is a reduced and irreducible curve. In this paper, we put $\mathrm{supp}(D) = \bigcup_{j=1}^{r} A_j$ (the support of D), $red(D) = \sum_{j=1}^{r} A_j$ (the reduced divisor of D) and $\mathrm{Coeff}_{A_j} D = d_j$. Futher, if A is a reduced divisor with $\mathrm{supp}(A) \subset \mathrm{supp}(D)$, then we put $\mathrm{supp}(D)\backslash A := \mathrm{supp}(\mathrm{red}(D) - A)$. Assume that $B := \bigcup_{j=1}^{m} A_{i_j} \subset \mathrm{supp}(D)$, where $m < r$. Let $D|_B = \sum_{j=1}^{m} (\mathrm{Coeff}_{A_{i_j}} D) A_{i_j}$. Let $E = \bigcup_{j=1}^{r} E_j \subset M$ be the irreducible decomposition of a compact algebraic curve E. Suppose that $E_j^2 \leqq 0$ for any j and $E = \sum_{j=1}^{r} E_j$ is a simple normal crossing divisor on M. For (M, E), the *weighted dual graph* (= w.d.graph) Γ_E of E is a graph such that each vertex of Γ_E represents an irreducible component E_j weighted by E_j^2 and $g(E_j)$ (= genus), while each edge connecting to E_i and E_j, $i \neq j$, corresponds to point $E_i \cap E_j$. For example, if $E_i^2 = -b_i$ and $g(E_i) = g_i > 0$ (resp. $g_i = 0$), then E_i corresponds to a vertex that is configured as follows:

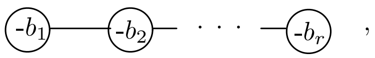


For fundamental notations and terminologies, please refer [**La3**], [**O**] and [**Re2**].

2. Pencils of curves constructed from normal surface singularities.

With respect to the construction of pencils of curves, there are several results after Kodaira's work [**Ko**] for pencils of elliptic curves. For a given curve (not necessarily irreducible) satisfying some condition, G. Winters [**Wi**] proved the existence of a pencil of curves whose singular fiber is equal to the given curve. In [**Ka1**] and [**Ka2**], U. Karras constructed pencils of curves by gluing resolution spaces of Kodaira singularities and open neighborhoods of some (-1) curves. Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ not a perfect power element. This paper, generalizing his method, presents a way to construct pencils of curves of genus $p_e(X, o, h)$ by gluing a resolution space of (X, o) and some cyclic quotient singularities.

Let us prepare some facts on cyclic quotient singularities. Let n and q be positive integers. Let $G_{n,q}$ be the cyclic group generated by (e_n, e_n^q) $\left(:= \begin{pmatrix} e_n & 0 \\ 0 & e_n^q \end{pmatrix} \in GL(2, \boldsymbol{C}) \right)$, where $e_n = exp\left(\frac{2\pi\sqrt{-1}}{n}\right)$. Then we obtain a cyclic quotient singularity $(\boldsymbol{C}^2/G_{n,q}, o)$. It is indicated by $C_{n,q}$. Also, we call it a cyclic quotient singularity of type $C_{n,q}$. However, for reasons of our argument, we do not assume that $n > q$ and $\gcd(n, q) = 1$. Hence, if $\gcd(n, q) = r$ and $n = rn_1$ and $q = rq_1$, then $C_{n,q} = C_{n_1,q_1}$. In this paper, a non-singular point is a cyclic quotient singularity; it is expressed by $C_{1,0}$. The cyclic quotient singularity $C_{n,q}$ has a good resolution whose w.d.graph is given as



(2.1)

where $\frac{n}{q} = [[b_1, \ldots, b_r]] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$ and $b_1 \geqq 1$ and $b_i \geqq 2$ for $i \geqq 2$. Such a good resolution is said to be the *standard resolution* of $C_{n,q}$. If $1 \leqq q < n$, then $b_1 \geqq 2$ and the standard resolution equals the minimal resolution of $C_{n,q}$; also $b_1 = 1$ if $q > n$. The standard resolution of a non-singular point $C_{1,0}$ is designated a neighborhood of a (-1) curve. For cyclic quotient singularities, please refer [**Fu**] and [**Ri**].

Every quotient singularity is a rational singularity. For rational singularities, the following is known ([**Li**, Theorem 12.1]). Here we give a simple proof for it in the category of complex geometry, which is indicated by M. Tomari.

Lemma 2.1. *Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution of a rational singularity. If D is a divisor on $\tilde{X}$ with $DE_i = 0$ for any irreducible component E_i of E, then D is linearly equivalent to the zero divisor (i.e., there is a meromorphic function f on $\tilde{X}$ with $D = \{f = 0\}$). Especially, if D is effective, then there is an element $h \in \mathfrak{m}_{X,o}$ such that $(h \circ \pi)_{\tilde{X}} = D$.*

Proof. From the exact sequence $0 \to \boldsymbol{Z}_{\tilde{X}} \to \mathscr{O}_{\tilde{X}} \to \mathscr{O}^*_{\tilde{X}} \to 0$, we have the following:

$$0 = H^1\big(\tilde{X}, \mathscr{O}_{\tilde{X}}\big) \to H^1\big(\tilde{X}, \mathscr{O}^*_{\tilde{X}}\big) \xrightarrow{\delta} H^2\big(\tilde{X}, \boldsymbol{Z}_{\tilde{X}}\big) \to H^2\big(\tilde{X}, \mathscr{O}_{\tilde{X}}\big) = 0.$$

We have $H^2(\tilde{X}, \boldsymbol{Z}_{\tilde{X}}) \simeq \bigoplus_{i=1}^r H^2(E_i, \boldsymbol{Z})$ since $\tilde{X}$ is contractible to E. Because the isomorphic map $H^1(\tilde{X}, \mathscr{O}^*_{\tilde{X}}) \simeq \bigoplus_{i=1}^r H^2(E_i, \boldsymbol{Z})$ is given by the restriction of the first Chern class of $\mathscr{O}_{\tilde{X}}(D)$ to E, it is given by $D \mapsto (DE_1, \ldots, DE_r)$. This yields the proof. □

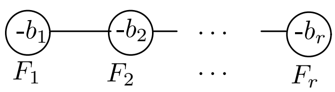
Definition 2.2. Let E be supp(S_o) for a pencil of curves $\Phi : S \to \Delta$ or the exceptional set of a resolution of a normal surface singularity. Let $F = \bigcup_{i=1}^r F_i$ be the irreducible decomposition of a one-dimensional analytic subset F of E.

(i) If $r \geqq 3$ and

$$F_iF_j = \begin{cases} 1 & \text{if } j = i+1 \text{ or } (i,j) = (1,r) \\ 0 & \text{otherwise} \end{cases}$$

for $i \neq j$, then F is called a *cyclic chain*. If $r = 2$ and $F_1F_2 = 2$ and F_1 and F_2 intersect at two different points, then F is also called *a cyclic chain*.

(ii) Assume that the w.d.graph of F is given as



If $(E \backslash F)F = E_1F_1 = 1$ for an irreducible component E_1 of $E \backslash F$, then F is said to be a $\boldsymbol{P}^1$*-chain* (*of type* $(b_1, \ldots, b_r)$) *started from* E_1. If $(E \backslash F)F = 2$ and $E_1F_1 = E_2F_r = 1$ for irreducible components E_1, E_2 of $E \backslash F$, then F is said to be a $\boldsymbol{P}^1$*-chain* (*of type* $(b_1, \ldots, b_r)$) *between* E_1 *and* E_2. For these cases, r is called the *length of* F, and F is said to be a *minimal* $\boldsymbol{P}^1$*-chain* if $b_i \leqq -2$ for any i.

Definition 2.3. Let S be a non-singular complex surface and $\Delta \subset \boldsymbol{C}$ a small open disc around the origin. If $\Phi : S \to \Delta$ is a proper surjective holomorphic map and the generic fiber $S_t := \Phi^{-1}(t)$ $(t \neq 0)$ is a smooth curve (but not necessarily connected), then it is called a *quasi-pencil of curves*.

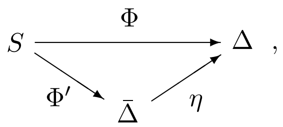
If Φ is a quasi-pencil of curves whose any fiber is connected, it is a pencil of curves. If $\Phi : S \to \Delta$ is a pencil of curves, then $\Phi^k : S \to \Delta$ $(k \geqq 2)$ is a quasi-pencil of curves, but not a pencil of curves, because the general fiber is not connected. Conversely, using Stein factorization, we can observe that if S_o is a singular fiber of a quasi-pencil of curves, then $S_o = k\bar{S}_o$ (as a formal sum of curves) for the singular fiber $\bar{S}_o$ of a pencil of curves and $k \geqq 1$.

For a normal surface singularity (X, o) and not a perfect power element $h \in \mathfrak{m}_{X,o}$, let $\pi : (\tilde{X}, E) \to (X, o)$ be a resolution such that $\operatorname{red}(h \circ \pi)_{\tilde{X}}$ is simple normal crossing. In the following, a pencil of curves extending $h \circ \pi$ is constructed by gluing $\tilde{X}$ and standard resolution spaces of cyclic quotient singularities.

Theorem 2.4. *Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$. Let $\pi : (\tilde{X}, E) \to (X, o)$ be a good resolution such that $red((h \circ \pi)_{\tilde{X}})$ is a simple normal crossing divisor on $\tilde{X}$. Then there exists a quasi-pencil of curves $\Phi : S \to \Delta$ such that $\Phi|_{\tilde{X}} = h \circ \pi$ and all connected components of* $\operatorname{supp}(S_o) \backslash E$ *are minimal $\boldsymbol{P}^1$-chains started from E. Further, if h is not a perfect power element, then $\Phi : S \to \Delta$ above is a pencil of curves extending $h \circ \pi$.*

Proof. Let $(h \circ \pi)_{\tilde{X}} = \sum_{i=1}^{r} a_i E_i + \sum_{j=1}^{s} \gamma_j C_j$, where $E = \bigcup_{i=1}^{r} E_i$ and $\operatorname{supp}(\Lambda(h \circ \pi)) = \bigcup_{j=1}^{s} C_j$ are the irreducible decompositions of the exceptional set and the non-exceptional part respectively. Let E_{i_j} be an irreducible component of E intersecting C_j for $j = 1, \dots, s$. Let $(V_j; z_{j,1}, z_{j,2})$ be a local coordinate neighborhood of $E_{i_j} \cap C_j$ in $\tilde{X}$ such that $E_{i_j} = \{z_{j,1} = 0\}$ and $C_j = \{z_{j,2} = 0\}$ and $h \circ \pi|_{V_j} = z_{j,1}^{a_{i_j}} z_{j,2}^{\gamma_j}$ on V_j. Consider a cyclic quotient singularity $C_{\bar{a}_j, \bar{r}_j}$ for $j = 1, \dots, s$, where $\bar{a}_j = a_{i_j} / \gcd(a_{i_j}, \gamma_j)$ and $\bar{\gamma}_j = \gamma_j / \gcd(a_{i_j}, \gamma_j)$. Let $(\tilde{Y}_j, F_j)$ be the standard resolution of $C_{\bar{a}_j, \bar{\gamma}_j}$ and $F_j = \bigcup_{k=1}^{\ell_j} F_{j,k}$ the irreducible decomposition. Thereby, $F_{j,k}^2 = -\delta_{j,k}$, where $\frac{\bar{a}_j}{\bar{\gamma}_j} = [[\delta_{j,1}, \dots, \delta_{j,\ell_j}]]$. From Lemma 2.1, there is a holomorphic function h_j on $\tilde{Y}_j$ such that $(h_j)_{\tilde{Y}_j} = a_{i_j} F_{j,0} + \gamma_j F_{j,1} + \sum_{k=2}^{\ell_j} \epsilon_{j,k} F_{j,k}$, where $F_{j,0}$ is a non-exceptional curve on $\tilde{Y}_j$ intersecting $F_{j,1}$ transversally and $\epsilon_{j,2}, \dots, \epsilon_{j,\ell_j}$ are positive integers determined by $(h_j)_{\tilde{Y}_j} F_{j,k} = 0$ for any k. Further, we choose a small open neighborhood W_j of $F_{j,0} \cap F_{j,1}$ in $\tilde{Y}_j$ and a local coordinate $(w_{j,1}, w_{j,2})$ on W_j such that $F_{j,0} = \{w_{j,1} = 0\}$ and $F_{j,1} = \{w_{j,2} = 0\}$ and $h_j|_{W_j} = w_{j,1}^{a_{i_j}} w_{j,2}^{\gamma_j}$. Gluing V_j and W_j by $z_{j,1} = w_{j,1}$ and $z_{j,2} = w_{j,2}$ for each j, we can obtain a surface $\bar{S} = \tilde{X} \cup \tilde{Y}_1 \cup \cdots \cup \tilde{Y}_s / \sim$ and a holomorphic function $\bar{\Phi} : \bar{S} \longrightarrow \boldsymbol{C}$ such that $\bar{\Phi}|_{\tilde{X}} = h \circ \pi$ and $\bar{\Phi}|_{\tilde{Y}_j} = h_j$ for $j = 1, \dots, s$. Then $A := \operatorname{supp}(\bar{\Phi}^{-1}(o)) = E \cup (\bigcup_{j=1}^{s} F_j)$ is a one-dimensional compact analytic subset of $\bar{S}$. From [**St**] (or [**Fi**, p. 56]), there are open neighborhoods S of A in $\bar{S}$ and Δ of $\{o\}$ in $\boldsymbol{C}$, respectively, such that $\Phi := \bar{\Phi}|_S : S \longrightarrow \Delta$ is a proper holomorphic map. From the construction we can easily see that $\Phi|_{\tilde{X}} = h \circ \pi$. For any point $P \in A$, we can choose a local coordinate (u, v) on an open neighborhood U of P such $\Phi = u^{\alpha} v^{\beta}$ on U for non-negative integers α and β. Then the fiber $S_t = \Phi^{-1}(t)$ is a non-singular curve for any $t \in \Delta - \{o\}$ if we take Δ is sufficiently small. Therefore $\Phi : S \longrightarrow \Delta$ is a desired quasi-pencil of curves.

Next, assume that h is not a perfect power element. Consider the Stein factorization of Φ as

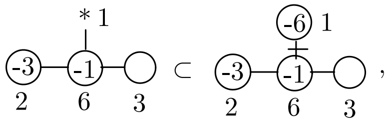


where Φ' is proper and the fiber is connected and η is a finite map. Then η is given by $t = \eta(v) = v^n \eta_1(v)$ $(n \geqq 1)$, where v is a coordinate on $\bar{\Delta}$ and $\eta_1(0) \neq 0$. Because $\Phi'|_{\tilde{X}}$ is a holomorphic map with $\Phi'|_{\tilde{X}}(E) = 0$, there exists $g \in \mathfrak{m}_{X,o}$ satisfying $g \circ \pi = \Phi'|_{\tilde{X}}$. Let $u \in \mathscr{O}_{X,o}$ be a unit determined by $u \circ \pi = \eta_1 \circ \Phi'|_{\tilde{X}}$. Since there is $\bar{u} \in \mathscr{O}_{X,o}$ with $\bar{u}^n = u$, we have $h = (g\bar{u})^n$ from $h \circ \pi = \Phi = \eta \circ \Phi' = \Phi'^n \cdot (\eta_1 \circ \Phi') = (g \circ \pi)^n (u \circ \pi) = (g^n u) \circ \pi = (g\bar{u})^n \circ \pi$ on $\tilde{X}$. Because h is not a perfect power element, we have $n = 1$; therefore, any fiber of Φ is connected. Hence Φ gives a pencil of curves. □

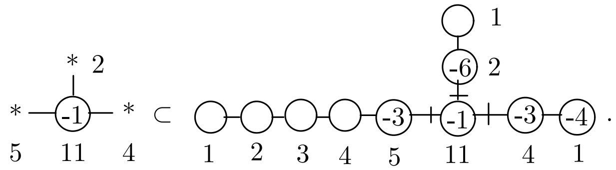
Here we give some examples to explain the procedure in Theorem 2.4.

Example 2.5.

(i) Let $(X, o) = (\boldsymbol{C}^2, o)$ and $h_1 = x^2 + y^3$ and $h_2 = x^5 y^4 (x + y)^2$. Let $\sigma_j : (\tilde{X}_j, E(j)) \longrightarrow (\boldsymbol{C}^2, o)$ be the minimal embedded resolution of a curve singularity $\{h_j = 0\}$ for $j = 1, 2$. Let $\Phi_j : S(j) \to \Delta_j$ be a pencil of curves constructed as in Theorem 2.4. Then the divisor $(h_1 \circ \sigma_1)_{\tilde{X}_1}$ and the singular fiber $S(1)_o$ is given as follows:



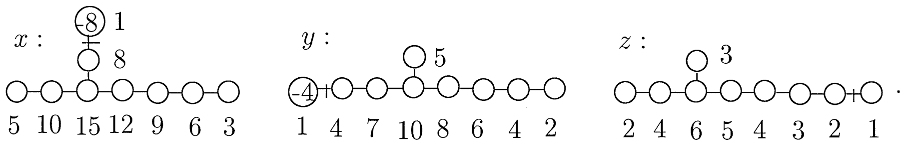
where $*$ indicates the strict transforms of irreducible components of $\{h_1 = 0\}$. Then we have $p_a(S(1)_o) = 1$ from (1.1) and the adjunction formula (i.e., $K_S E_i = -E_i^2 + 2g(E_i) - 2$). Also the divisor $(h_2 \circ \sigma_2)_{\tilde{X}_2}$ and the singular fiber $S(2)_o$ is given as



Then we have $p_a(S(2)_o) = 5$. In Theorem 2.9, we prove that the value of $p_a(S(j)_o)$ is equals to $p_e(X, o, h_j)$. Hence $p_e(\boldsymbol{C}^2, o, x^2 + y^3) = 1$ and $p_e(\boldsymbol{C}^2, o, x^5 y^4 (x + y)^2) = 5$.

(ii) Let $(X, o) = (\{x^2 + y^3 + z^5 = 0\}, o)$ (a rational double point of type E_8). (X, o) is a double covering over $\boldsymbol{C}^2$ branched along a plane curve $C := \{y^3 + z^5 = 0\}$. Let $V \xrightarrow{\sigma} \boldsymbol{C}^2$ the minimal embedded resolution of C. Taking the double coveing over

V branched along $\sigma^*(C)$ (total transform of C), we can obtain the minimal resolution $(\tilde{X}, E) \xrightarrow{\pi} (X, o)$ and the divisor $(x \circ \pi)_{\tilde{X}}$ (cf. [**Tt2**, Lemma 3.1] or [Lemma 4.3 in this paper]). Similarly, we can obtain the divisors $(y \circ \pi)_{\tilde{X}}$ and $(z \circ \pi)_{\tilde{X}}$. Applying Theorem 2.4 for them, the singular fibers of the pencils of curves constructed from x, y and z are given as



Then we can see that $p_e(X, o, x) = 4$, $p_e(X, o, y) = 2$ and $p_e(X, o, z) = 1$ from (1.1) and the adjunction formula and Theorem 2.9.

Remark 2.6. If (X, o) is a normal surface singularity with $\boldsymbol{C}^*$-action, then it is known to construct pencils of curves including a resolution space of (X, o) using a weighted projective space. For example, let (X, o) be the singularity of 2.5 (ii) and $\bar{X}_o := \{x^2 + y^3 + z^5 = 0\} \subset \boldsymbol{P}(15, 10, 6, 1)$. If we put $P = [0 : 0 : 0 : 1] \in \bar{X}_o$, then $(\bar{X}_o, P) \simeq (X, o)$. Let $\psi : \boldsymbol{P}(15, 10, 6, 1) \longrightarrow \boldsymbol{P}^1$ be a map defined by $[x : y : z : w] \longrightarrow [x : w^{15}]$. Let $\pi : S \longrightarrow \bar{X}_o$ be a resolution of $(\bar{X}_o, P)$ and let $\Phi : S \longrightarrow \boldsymbol{P}^1$ be a map $\psi \circ \pi$. Then Φ is isomorphic to the pencil of curves determined by x in 2.5 (ii) near the singular fiber $\Phi^{-1}([0 : 1])$. In [**Sa**], K. Saito considered similar constructions of pencils of curves in a slightly different situation.

All pencils of curves constructed as in Theorem 2.4 form a subclass of the class of all pencils of curves. However, there is not a large gap between them. For these classes, we prove the following.

Theorem 2.7. *Let $\Phi : S \to \Delta$ be a pencil of curves. Let take a suitable successive blowing-up $\tilde{S} \xrightarrow{\sigma} S$ and choose a sufficiently small disc Δ' ($\subset \Delta$). Let $\tilde{\Phi} := \Phi \circ \sigma$ and let consider a pencil of curves $\tilde{\Phi}' : S' := \tilde{\Phi}^{-1}(\Delta') \to \Delta'$. Then there is a normal surface singularity (X_1, o) and a cyclic quotient singularity (X_2, o) and resolutions $\pi_k : (\tilde{X}_k, E(k)) \to (X_k, o)$ ($k = 1, 2$) such that $\tilde{S}$ is a gluing of $\tilde{X}_1$ and $\tilde{X}_2$ and $\tilde{\Phi}$ is an extension of $h_1 \circ \pi_1$ and $h_2 \circ \pi_2$, where $h_1 \in \mathfrak{m}_{X_1,o}$ is not a perfect power element and $h_2 \in \mathfrak{m}_{X_2,o}$.*

Proof. By taking suitable successive blowing-up $\tilde{S} \xrightarrow{\sigma} S$, we may assume that $\mathrm{red}(\tilde{S}_o)$ is a simple normal crossing divisor and $\mathrm{supp}(\tilde{S}_o)$ contains a $\boldsymbol{P}^1$-chain F started from $E := \mathrm{supp}(\tilde{S}_o) \backslash F$ and $E \cap F$ is a non-singular point P of $\mathrm{red}(\tilde{S}_o)$. Let E_1 ($\subset E$) and F_1 ($\subset F$) be irreducible components with $E_1 \cap F_1 = P$. Let $(U; u, v)$ be a small open coordinate neighborhood in $\tilde{S}$ around P such that $E_1 = \{u = 0\}$ and $F_1 = \{v = 0\}$ in U. Let $\tilde{X}_1$ and $\tilde{X}_2$ be small open neighborhoods of E and F, respectively, such that $\tilde{X}_1 \cap \tilde{X}_2 = U$. Let $\pi_k : (\tilde{X}_k, E) \to (X_k, o)$ be the contraction to a normal surface singularity ($k = 1, 2$). Then (X_2, o) is a cyclic quotient singularity. There is an element $h_k \in \mathfrak{m}_{X_k,o} \subset \mathscr{O}_{X_k,o}$ with $\Phi|_{\tilde{X}_k} = h_k \circ \pi_k$ ($k = 1, 2$). Therefore, π_k gives a good resolution of (X_k, o) such that $\mathrm{red}((h_k \circ \pi_k)_{\tilde{X}_k})$ is simple normal crossing. By gluing $(\tilde{X}_1, h_1 \circ \pi_1)$

and $(\tilde{X}_2, h_2 \circ \pi_2)$ on U by identification on $\tilde{S}$, we obtain a pencil of curves $\tilde{\Phi}' : \tilde{S}' \to \Delta'$ satisfying $S' \subset S$ and $\Delta' \subset \Delta$ and $\Phi|_{S'} = \tilde{\Phi}'$.

If Φ is a non-multiple pencil of curves, then h_1 is obviously not a perfect power element; it completes the proof. Assume that Φ is a multiple pencil of curves. Suppose that h_1 is a perfect power element with $h_1 = h_3^\ell$ $(\ell \geqq 2)$ for $h_3 \in \mathfrak{m}_{X_1,o}$. If we put $S_o|_{\tilde{X}_2} = c_0 E_1|_U + \sum_{i=1}^r c_i F_i$, then $\ell | c_j$ and so put $\bar{c}_j = c_j/\ell$ for $i = 0, \dots, r$. From Lemma 2.1, there exists $h_4 \in \mathfrak{m}_{X_2,o}$ with $(h_4 \circ \pi_2)|_{\tilde{X}_2} = \bar{c}_0 E_1|_U + \sum_{i=1}^r \bar{c}_i \bar{F}_i$. If we put $\delta := h_2/h_4^\ell$, then $\delta \in \mathscr{O}_{X_2,o} \backslash \mathfrak{m}_{X_2,o}$; and so there is $\delta_1 \in \mathscr{O}_{X_2,o}$ with $\delta = \delta_1^\ell$. Hence $h_2 = (\delta_1 h_4)^\ell$. On the other hand, we have $h_k \circ \pi_k = u^{c_0} v^{c_1}$ $(k = 1, 2)$ on U; thereby, $h_3 \circ \pi_1 = (\delta_1 h_4) \circ \pi_2 = u^{\bar{c}_0} v^{\bar{c}_1}$ on U after choosing suitable branches h_3 and h_4. By gluing as above, there is a quasi-pencil of curves $\Phi_1 : S' \to \Delta'$ such that $\Phi_1|_{\tilde{X}_1} = h_3 \circ \pi_1$ and $\Phi_1|_{\tilde{X}_2} = (\delta_1 h_4) \circ \pi_2$. Hence $\tilde{\Phi}' = \Phi_1^\ell$ $(\ell \geqq 2)$ and then $\tilde{\Phi}'$ is a quasi-pencil of curves but not a pencil of curves. This is a contradiction; thus h_1 is not a perfect power element. □

From the proof above, we can see that if $\mathrm{red}(S_o)$ is normal crossing and $\mathrm{supp}(S_o)$ contains a $\boldsymbol{P}^1$-chain F and $F \cap (\mathrm{supp}(S_o) \backslash F)$ is a non-singular point of $\mathrm{red}(S_o)$, then we may assume that σ in Theorem 2.7 is the identity map (i.e., $\tilde{S} = S$)

In the following, we prove that the genus of a pencil of curves obtained as in Theorem 2.4 is equal to $p_e(X, o, h)$. We prepare the following lemma for it.

Lemma 2.8. *Let $(\tilde{X}, E) \to (X, o)$ be a good resolution of a normal surface singularity and let $D = \sum_{j=0}^r d_j E_j$ be a cycle on $\bigcup_{j=0}^r E_j$ $(\subset E)$, where $\bigcup_{j=1}^r E_j$ is a $\boldsymbol{P}^1$-chain started from E_0. Assume $DE_j = 0$ for $j = 1, \dots, r$. Then $K_{\tilde{X}}(\sum_{j=1}^r d_j E_j) = d_0 - d_1 - \gcd(d_0, d_1)$, where $K_{\tilde{X}}$ is the canonical bundle.*

Proof. From $DE_j = 0$ for $j = 1, \dots, r$, we have

$$d_j - d_{j+1} b_{j+1} + d_{j+2} = 0 \text{ for } j = 0, \dots, r-1, \tag{2.2}$$

where $d_{r+1} := 0$. Therefore,

$$\begin{aligned} 0 &= \sum_{j=0}^{r-1} (d_j - d_{j+1} b_{j+1} + d_{j+2}) = d_0 - d_1 - d_r - \sum_{j=1}^r d_j (b_j - 2) \\ &= d_0 - d_1 - d_r - K_{\tilde{X}} \left(\sum_{j=1}^r d_j E_j \right) \end{aligned}$$

by the adjunction formula. From (2.2), we can easily check that $d_r = \gcd(d_0, d_1)$; hence this completes the proof. □

Theorem 2.9. *Let (X, o) be a normal surface singularity and let $h \in \mathfrak{m}_{X,o}$ not a perfect power element. Let $\pi : (\tilde{X}, E) \to (X, o)$ be a good resolution such that red$((h \circ \pi)_{\tilde{X}})$ is simple normal crossing on $\tilde{X}$. Suppose that $\Phi : S \to \Delta$ is a pencil of curves of genus g extending $h \circ \pi$. If $g = p_e(X, o, h)$ and Φ is minimally extending $h \circ \pi$, then any connected component of* $\mathrm{supp}(S_o) \backslash E$ *is a $\boldsymbol{P}^1$-chain. Conversely, if any connected component of* $\mathrm{supp}(S_o) \backslash E$ *is a $\boldsymbol{P}^1$-chain, then $g = p_e(X, o, h)$. Therefore, any pencil of* *curves constructed from $h \circ \pi$ as in Theorem 2.4 is minimally extending $h \circ \pi$ and the genus is equal to $p_e(X,o,h)$.*

Proof. Assume that $g = p_e(X,o,h)$ and Φ is minimally extending $h \circ \pi$. We put $F = \mathrm{supp}(S_o) \backslash E$. By taking some blowing-ups on S, we may assume that $\mathrm{red}(S_o)$ is simple normal crossing.

From now on we show that there are no cyclic chains in $\mathrm{supp}(S_o)$ which contains at least one irreducible component of F. Suppose that there is a cyclic chain $\bigcup_{j=0}^{r} A_j \subset \mathrm{supp}(S_o)$ (i.e., $A_i A_{i+1} \neq 0$ for $i = 0, \ldots, r-1$ and $A_r A_0 \neq 0$) such that A_0 is an irreducible component of F. Let B be the connected component of $\mathrm{supp}(S_o) \backslash A_0$ which contains E. Let $\tilde{Y}$ be a small neighborhood of B and let $\varphi := \Phi|_{\tilde{Y}}$. From the assumption of the cyclic chain, we can choose A_0 with $\sharp(A_0 \cap B) \geqq 2$ and so put $A_0 \cap B = \{P_1, \ldots, P_s\}$ ($s \geqq 2$). Similarly to Theorem 2.4, construct a pencil of curves $\Phi'' : S'' \longrightarrow \Delta$ with $\tilde{Y} \subset S''$ and $\Phi''|_{\tilde{Y}} = \varphi$ by gluing $\tilde{Y}$ and resolution spaces of s cyclic quotient singularities on small neighborhoods of $P_1, \ldots, P_s$. Let g_o be the genus of Φ''. Let $b = -A_0^2$ and $d = \mathrm{Coeff}_{A_0} S_o$. Let $A_{i_1}, \ldots, A_{i_s}$ be irreducible components of B which intersect A_0 at $P_1, \ldots, P_s$, respectively, and let $a_j = \mathrm{Coeff}_{A_{i_j}} S_o$ for $j = 1, \ldots, s$. Then $db \geqq \sum_{j=1}^{s} a_j$. Let F_j ($j = 1, \ldots, s$) be a connected component of $\mathrm{supp}(S''_o) \backslash B$; thus it is a $\boldsymbol{P}^1$-chain started from A_{i_j}. Therefore, Lemma 2.8 implies that $K_{S''}(S''_o|_{F_j}) = a_j - d - \gcd(a_j, d) \leqq a_j - d - 1$ for any j. Since we may assume that there are no (-1) curves in $\mathrm{supp}(S_o) \backslash (B \cup A_0)$, we have

$$\begin{aligned} p_e(X,o,h) &= 1 + \frac{K_S S_o}{2} = 1 + \frac{K_S(S_o|_{B \cup A_0} + S_o|_{\mathrm{supp}(S_o) \backslash (B \cup A_0)})}{2} \\ &\geqq 1 + \frac{K_S(S_o|_{B \cup A_0})}{2} = 1 + \frac{K_S(S_o|_B) + K_S(dA_0)}{2}. \end{aligned}$$

Since

$$K_S(S_o|_B) = K_{S''}\big(S''_o|_B\big) \text{ and } db \geqq \sum_{j=1}^{s} a_j,$$

$$\begin{aligned} p_e(X,o,h) - g_o &= \frac{1}{2}\big(K_S S_o - K_{S''} S''_o\big) \geqq \frac{1}{2}\left\{K_S(dA_0) - \sum_{j=1}^{s} K_{S''}\big(S''_o|_{F_j}\big)\right\} \\ &\geqq \frac{1}{2}\left\{d(b-2) - \sum_{j=1}^{s}(a_j - d - 1)\right\} \geqq \frac{1}{2}\{(s-2)d + s\} > 0. \end{aligned}$$

Another hand, $\Phi'' : S'' \to \Delta$ is an extension of $h \circ \pi$ and so $g_o \geqq p_e(X,o,h)$ from the definition of $p_e(X,o,h)$. This is a contradiction. Consequently, there are no cyclic chains in $\mathrm{supp}(S_o)$ containing an irreducible component of F.

Next we prove that any connected component of F is a $\boldsymbol{P}^1$-chain started from E. We may assume that $\mathrm{red}(S_o)$ is a simple normal crossing divisor and any irreducible component of F is $\boldsymbol{P}^1$. Assume that there is a connected component G of F which is not a $\boldsymbol{P}^1$-chain started from E. Since F does not contain any cyclic chain, G is a tree and it intersects E transversally at only one point. Further, we can easily see that G contains an irreducible component A_o such that $\mathrm{supp}(S_o)\backslash A_o$ has a connected component G_o containing E and at least two other different $\boldsymbol{P}^1$-chains started from A_o. Let $G_1,\dots,G_s$ be those $\boldsymbol{P}^1$-chains started from A_o ($s \geqq 2$). Let A_j (resp E_1) be the irreducible component of G_j (resp G_o) intersecting A_o for $j = 1,\dots,s$. Let $b = -A_o^2$, $d = \mathrm{Coeff}_{A_o} S_o$ and $a_j = \mathrm{Coeff}_{A_j} S_o$ for $j = 1,\dots,s$. Since Φ is minimally extending $h \circ \pi$, G_i does not contain any (-1) curve. Hence G_i is not a (-1) curve or a $\boldsymbol{P}^1$-chain of type $(1,2,\dots,2)$. Then we can easily see that d does not divide a_j for any j (so $\gcd(d,a_j) \leqq \frac{d}{2}$). Let $\tilde{Y}$ be a small neighborhood of G_o in S and let $\hat{\varphi} = \Phi|_{\tilde{Y}}$. Let U be a small neighborhood of a point $G_o \cap A_o$ in $\tilde{Y}$. Let $(\tilde{Z}, \bigcup_{j=1}^p H_j)$ be the standard resolution of a cyclic quotient singularity $C_{N,d}$ ($N = bd - \sum_{j=1}^s a_j = \mathrm{Coeff}_{E_1} S_o$). By gluing $\tilde{Y}$ and $\tilde{Z}$, we can construct a pencil $\hat{\Phi} : \hat{S} \longrightarrow \Delta$ satisfying $\hat{\Phi}|_{\tilde{Y}} = \hat{\varphi}$. Hence $\hat{\Phi}$ is an extension of $h \circ \pi$, and the genus $\hat{g}$ is greater than or equal to $p_e(X,o,h)$. Also $\bigcup_{j=1}^p H_j$ is the glued $\boldsymbol{P}^1$-chain started from G_o, and we can say the following:

$$\mathrm{supp}\left(\hat{S}_o\right) = G_o \cup \left(\bigcup_{j=1}^{p} H_j\right),\ S_o|_{G_o} = \hat{S}_o|_{G_o},\ \mathrm{Coeff}_{E_1} \hat{S}_o = N \text{ and } \mathrm{Coeff}_{H_1} \hat{S}_o = d,$$

where E_1 intersects H_1 in $\hat{S}$. Lemma 2.8 implies that

$$\begin{aligned} 0 \geqq p_e(X,o,h) - \hat{g} &= \frac{1}{2}\left\{K_S\left(\sum_{j=1}^{s} S_o|_{G_j}\right) + K_S(dA_o) - K_{\hat{S}}\left(\hat{S}_o|_{\bigcup_{k=1}^p H_k}\right)\right\} \\ &= \frac{1}{2}\left\{\sum_{j=1}^{s}(d - a_j - \gcd(d,a_j)) + d(b-2) - N + d + \gcd(N,d)\right\} \\ &= \frac{1}{2}\left\{sd - \sum_{j=1}^{s}\gcd(d,a_j) - d + \gcd(N,d)\right\}. \end{aligned}$$

Then we have $\sum_{j=1}^s \gcd(d,a_j) + d \geqq sd + \gcd(N,d) \geqq sd + 1$. Since $\gcd(d,a_j) \leqq \frac{d}{2}$, it yields a contradiction: $0 \geqq \frac{ds}{2} - d + 1 > 0$. Hence any connected component of $\mathrm{supp}(S_o)\backslash E$ is contracted to a $\boldsymbol{P}^1$-chain started from E.

Now consider the converse. Let $\Phi : S \longrightarrow \Delta$ be a pencil of curves of genus g such that Φ is a minimal extension of $h \circ \pi$ and also any connected component of $\mathrm{supp}(S_o)\backslash E$ is a $\boldsymbol{P}^1$-chain started from E. Let $\Phi' : S' \longrightarrow \Delta$ be a pencil of curves of genus $p_e(X,o,h)$ such that Φ' is also a minimal extension of $h \circ \pi$. From the proof above, any connected component of $\mathrm{supp}(S'_o)\backslash E$ is a $\boldsymbol{P}^1$-chain started from E. We may assume that any connected component in $\mathrm{supp}(S'_o)\backslash E$ and $\mathrm{supp}(S_o)\backslash E$ is a minimal $\boldsymbol{P}^1$-chain. Since $\Phi|_{\tilde{X}} = \Phi'|_{\tilde{X}} = h \circ \pi$, any $\boldsymbol{P}^1$-chain in $\mathrm{supp}(S_o)\backslash E$ is equal numerically to the corresponding one in $\mathrm{supp}(S'_o)\backslash E$ (i.e., same w.d.graph and same coefficients for any irreducible component). It turns out that S_o and S'_o are numerically equal; consequently, $p_e(X,o,h) = g$. □

Corollary 2.10. *Let (X,o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ not a perfect power element. Let $\pi : (\tilde{X},E) \to (X,o)$ be a good resolution and $\Phi : S \to \Delta$*

a pencil of curves of genus $p_e(X,o,h)$ which is a minimal extension of $h\circ\pi$. Then the numerical type of S_o (i.e., $\mathrm{Coeff}_{E_i} S_o$ *for any* $E_i \subset \mathrm{supp}(S_o)$ *and the w.d.graph of* $\mathrm{supp}(S_o)$*) is determined uniquely.*

In the following, we prove a formula of $p_e(X,o,h)$.

THEOREM 2.11. *Let (X,o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ not a perfect power element. Let $\pi : (\tilde{X}, E) \longrightarrow (X,o)$ be a resolution such that* $\mathrm{red}((h\circ\pi)_{\tilde{X}})$ *is a simple normal crossing divisor. Let $\Lambda(h\circ\pi) = \sum_{j=1}^{r(h)} \gamma_j C_j$ and put $C = \sum_{j=1}^{r(h)} C_j$, where C_j is an irreducible component for any j. Let $n_1, \ldots, n_{r(h)}$ be positive integers denoted by $n_j = v_{E_{i_j}}(h\circ\pi)$ if E_{i_j} intersects C_j. Then*

$$p_e(X,o,h) = p_a(E(h\circ\pi)) - E(h\circ\pi)^2 - \frac{1}{2}\Big\{(E(h\circ\pi)+E)(\Lambda(h\circ\pi)-C) + r(h) + \sum_{j=1}^{r(h)} \gcd(n_j,\gamma_j)\Big\}.$$

Further, if h is a reduced element, then

$$p_e(X,o,h) = p_a(E(h\circ\pi)) - E(h\circ\pi)^2 - r(h).$$

PROOF. Let $\Phi : S \longrightarrow \Delta$ be a pencil of curves of genus $p_e(X,o,h)$ which is a minimal extension of $h\circ\pi$. From Theorem 2.9, any connected component of $\mathrm{supp}(S_o)\backslash E$ is a minimal $\boldsymbol{P}^1$-chain. Let $F_1, \ldots, F_{r(h)}$ be such all connected components of $\mathrm{supp}(S_o)\backslash E$ and $F_j = \bigcup_{k=1}^{t_j} F_{j,k}$ such that $C_j \subset F_{j,1}$. Then $\mathrm{Coeff}_{F_{j,1}} S_o = \gamma_j$. From Lemma 2.8, $K_S(S_o|_{F_j}) = n_j - \gamma_j - \gcd(n_j,\gamma_j)$ for any j. Since

$$0 = E(h\circ\pi)S_o = E(h\circ\pi)\Big(E(h\circ\pi) + \sum_{j=1}^{r(h)}\gamma_j C_j\Big) = E(h\circ\pi)^2 + \sum_{j=1}^{r(h)} n_j\gamma_j,$$

we have

$$E(h\circ\pi)^2 = -\sum_{j=1}^{r(h)} n_j\gamma_j.$$

Since

$$S_o = E(h\circ\pi) + \sum_{j=1}^{r(h)} n_j(S_o|_{F_j})$$

and

$$p_e(X,o,h) = p_a(S_o),$$

we have

$$\begin{aligned} & p_e(X,o,h) - p_a(E(h\circ\pi)) + E(h\circ\pi)^2 \\ & = \frac{1}{2}\Big\{K_S\big(S_o|_{\bigcup_{j=1}^{r(h)} F_j}\big) + E(h\circ\pi)^2\Big\} \\ & = \frac{1}{2}\Big\{\sum_{j=1}^{r(h)}\big(n_j - \gamma_j - \gcd(n_j,\gamma_j)\big) - \sum_{j=1}^{r(h)} n_j\gamma_j\Big\} \\ & = -\frac{1}{2}\sum_{j=1}^{r(h)}\{(n_j+1)(\gamma_j-1) + 1 + \gcd(n_j,\gamma_j)\} \\ & = -\frac{1}{2}\Big\{(E(h\circ\pi)+E)(\Lambda(h\circ\pi)-C) + r(h) + \sum_{j=1}^{r(h)}\gcd(n_j,\gamma_j)\Big\}. \qquad \square \end{aligned}$$

Let (C,o) be a curve singularity and $\nu : \tilde{C} \longrightarrow C$ the normalization. The conductor number $\delta(C,o)$ is defined by $\dim_{\boldsymbol{C}}(\mathscr{O}_{\tilde{C}}/\nu^*\mathscr{O}_{C,o})$ (cf. [**BK**, p. 573], [**Na**, p. 120]).

Corollary 2.12.

(i) *Let (X,o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ a reduced element. Let $\delta(h)$ be the conductor number of a curve singularity $(X\cap\{h=0\},o)$. Then*

$$p_e(X,o,h) = \delta(h) - r(h) + 1.$$

(ii) *For a generic element $h \in \mathfrak{m}_{X,o}\backslash\mathfrak{m}_{X,o}^2$, we have*

$$p_e(X,o) \leqq \delta(h) - r(h) + 1.$$

(iii) *Let $(X,o) = \{z^n = h(x,y)\}$ be a normal hypersurface singularity. Then*

$$p_e(X,o,z) = p_e(\boldsymbol{C}^2,o,h) = \delta(h) - r(h) + 1 = \frac{\mu(h) - r(h) + 1}{2},$$

where $\mu(h)$ is the Milnor number of a plane curve singularity $(\{h=0\},o) \subset (\boldsymbol{C}^2,o)$.

Proof.

(i) From a result of M. Morales ([**Mo**, 2.1.2]), we have $\delta(h) = 1 + \frac{1}{2}E(h\circ\pi)(E(h\circ\pi) - K_{\tilde{X}})$. Then we have the equality of (i) from Theorem 2.11, and (ii) is obvious from (ii).

(iii) From (i) and $\mu(h) = 2\delta(C,o) - r(C,o) + 1$ ([**Mi**, Section 10]), we complete the proof. $\square$

The Milnor number is the number of vanishing cycles and so it is defined for smoothable singularities. However, in the case of curve singularities, Buchweitz and Greuel [**BG**, p. 244] generalized the definition of the Milnor number algebraically and showed that it is equal to $2\delta(C,o) - r(C,o) + 1$, where $\delta(C,o)$ (resp $r(C,o)$) is the conductor number (resp. the number of irreducible components) of a curve singularity (C,o). Hence, if we assume $\mu(h)$ is the Milnor number for a curve singularity $\{h=0\}$ on (X,o) in the sense of them, then we have also $p_e(X,o,h) = \frac{\mu(h)-r(h)+1}{2}$ in Corollary 2.12 (i). Here we remark that we can obtain the values of $p_e(X,o,h)$ of Example 2.5 using Corollary 2.12 (ii) and $\mu(\{x^a+y^b=0\},o) = \frac{(a-1)(b-1)}{2}$.

With respect to the equality of Corollary 2.12 (i), it is already known that $\delta(h) \geqq r(h)-1$ from Hironaka's formula ([**H**, Lemma 1.2.2], [**BG**, p. 246]). Therefore, $p_e(X,o,h)$ gives the difference between $\delta(h)$ and $r(h)-1$.

In [**St1**], J. Stevens proved that if $h : (X,o) \longrightarrow (\boldsymbol{C},o)$ is a semistable smoothing of a curve singularity $X \cap \{h=0\}$ and $\pi : (\tilde{X},E) \longrightarrow (X,o)$ is a resolution such that $(h\circ\pi)_{\tilde{X}}$ is a reduced divisor, then $p_a(E) = \delta(h) - r(h) + 1$. From his result and Theorem 4.5 in this paper, we can give another proof of Corollary 2.12.

In the definition of $p_e(X,o)$, we do not use elements of $\mathfrak{m}_{X,o}$. However, there is $h \in \mathfrak{m}_{X,o}$ not a perfect power element with $p_e(X,o,h) = p_e(X,o)$.

Theorem 2.13. *Let (X,o) be a normal surface singularity. Let $\Phi : S \to \Delta$ be a pencil of curves of genus $p_e(X,o)$ and including $(\tilde{X},E)$ for a resolution $\pi : (\tilde{X},E) \longrightarrow (X,o)$.*

(i) *If $h \in \mathfrak{m}_{X,o}$ satisfies $\Phi|_{\tilde{X}} = h\circ\pi$, then h is not a perfect power element. Hence we have the following*:

$$p_e(X,o) = \min\{p_e(X,o,h) | h \in \mathfrak{m}_{X,o} \textit{ is not a perfect power element}\}.$$

(ii) *If π is a good resolution, then any connected component of* $\mathrm{supp}(S_o)\backslash E$ *is a minimal $\boldsymbol{P}^1$-chain after suitable contractions of* (-1) *curves in* $\mathrm{supp}(S_o)\backslash E$.

Proof.

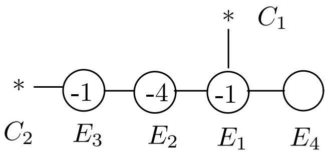
(i) After taking suitable blowing-ups $\tilde{S} \xrightarrow{\sigma} S$, we have a good resolution $\pi' := \pi\circ\sigma : (\tilde{X}',E') \longrightarrow (X,o)$ such that $(\tilde{S}, \mathrm{supp}(\tilde{S}_o)) \supset (\tilde{X}',E')$ for a pencil of curves $\tilde{\Phi} = \Phi\circ\sigma : \tilde{S} \longrightarrow \Delta$ and $(h\circ\pi')_{\tilde{X}'}$ is simple normal crossing. Suppose that h is a perfect power element. Let put $h = h_1^\ell$ ($\ell \geqq 2$), where h_1 is not a perfect power element. Let $\Phi' : S' \longrightarrow \Delta$ be a pencil of curves of genus $p_e(X,o,h_1)$ as in Theorem 2.4 which is a minimal extension of $h_1\circ\pi'$. Since $\ell S'_o|_E = S_o|_E$, $\ell S'_o$ and S_o are numerically equal from Theorem 2.9 (i.e., both w.d.graphs are equal and the both coefficients for any irreducible component are equal). Then we have $p_a(\ell S'_o) = p_a(S_o) = p_e(X,o)$ and

$$1 \leqq p_e(X,o,h_1) = p_a(S'_o) \leqq p_a(\ell S'_o) = p_e(X,o).$$

If $p_e(X,o,h_1) \geqq 2$, then $p_a(S'_o) < p_a(\ell S'_o)$; therefore $p_e(X,o,h_1) < p_e(X,o)$. This is a contradiction and so h is not a perfect power element. If $p_e(X,o,h_1) = 1$, then $p_e(X,o) = p_e(X,o,h_1)$ for not a perfect power element h_1 from $0 < p_e(X,o) \leqq p_e(X,o,h_1) = 1$

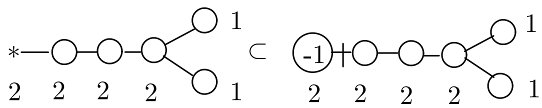
(ii) As in the proof of (i), there exists $h \in \mathfrak{m}_{X,o}$ which is not a perfect power element with $h\circ\pi = \Phi|_{\tilde{X}}$ and $p_e(X,o,h) = p_e(X,o)$. After contracting (-1) curves in $\mathrm{supp}(S_o)\backslash E$ suitably, Φ become a minimal extension of $h\circ\pi$. From Theorem 2.9, we complete our proof. □

Remark 2.14. It is not always true that there is a reduced element h with $p_e(X,o,h) = p_e(X,o)$. For example, let $(X,o) = (\{z^2 = y(x^2 + y^{n-2})\}, o)$ (i.e., a rational double point of type D_n ($n \geqq 4$)). Then y is a non-reduced element. Let $n = 5$ and consider a following embedded resolution $\sigma : V \longrightarrow \boldsymbol{C}^2$ of a plane curve singularity $(\{y(x^2+y^3) = 0\}, o) \subset (\boldsymbol{C}^2, o)$ as follows:



,

where C_1 (resp. C_2) indicates the strict transform of a curve $\{x^2 + y^3 = 0\}$ (resp. $\{y = 0\}$) in $\boldsymbol{C}^2$. We can easily see that $(y \circ \sigma)_V = 2E_1 + E_2 + 2E_3 + E_4 + C_2$. By taking a double covering of V given by $z^2 = y \circ \sigma$ ([**D**], [**Tt2**]), we obtain the minimal resolution $(\tilde{X}, E) \to (X, o)$. Therefore, we can construct a pencil of curves S as follows:



.

Then we have $p_a(S_o) = 0$. Hence $p_e(X,o,y) = 0$ from Theorem 2.9. However, we have $p_e(X,o,h) \geqq 1$ for any reduced element h. We explain this according to the suggestion by the referee. Assume $p_e(X,o,h) = 0$. From Corollary 2.12, we have $\delta(h) = r(h) - 1$. Then $(C,o) := (X \cap \{h = 0\}, o)$ is a Gorenstein singularity because it is a complete intersection. Also (C,o) is the ordinary n-tuple point of $n = 2$ or $n = 3$ ([**BG**, Lemma 1.2.4]). If (C,o) is the ordinary 2-tuple point (i.e., (C,o) is isomorphic to $(\{x^2 + y^2 = 0\}, 0) \subset \boldsymbol{C}^2$), then (X,o) is isomorphic to a rational double point of type A_n. Therefore, (C,o) is the ordinary 3-tuple point from the assumption on (X,o). From [**Sal**, Theorem 3.1], the Cohen-Macaulay type ($:= \dim_{\boldsymbol{C}} \omega_{C,o}/\mathfrak{m}_{C,o}\omega_{C,o}) = \mathrm{mult}(C,o) - 1 = 2$. Then (C,o) is not a Gorenstein singularity. This is a contradiction and so $p_e(X,o,h) > 0$.

We also remark that if there is a reduced element h with $p_e(X,o,h) = p_e(X,o) = p_f(X,o)$, then (X,o) is a Kulikov singularity (Definition 3.5 and Theorem 3.11 (ii)). Therefore, from this and $p_e(D_n) = 0$ (Section 1), we can also check that $p_e(X,o,h) \geqq 1$ for any reduced element h.

3. Some results on pencil genus for normal surface singularities.

In this section, first we estimate $p_e(X,o)$ for normal surface singularities (Theorem 3.5). Second we give a necessary and sufficient condition for normal surface singularities to be Kodaira (or Kulikov) singularities (Theorem 3.11). Further, we determine the values of log-canonical surface singularities and rational triple points (Propositions 3.12 and 3.13).

The following proposition was suggested by M. Tomari together with the proof. We use it for the proof of Theorem 3.5.

PROPOSITION 3.1. *Let (X, o) be a normal surface singularity. Then there exists a resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ and a reduced element $h \in \mathfrak{m}_{X,o}$ such that $\pi^*\mathfrak{m}_{X,o}$ is invertible and $(h \circ \pi)_{\tilde{X}}$ is simple normal crossing and $E(h \circ \pi)$ equals the maximal ideal cycle $M_{\tilde{X}}$.*

PROOF. Let us consider the following commutative diagram:

$$
\begin{array}{ccccccc}
(X,o) & \xleftarrow{\phi_1} & X' & \xleftarrow{\phi_2} & X'' & \xleftarrow{\phi_3} & \tilde{X}, \\
\psi \downarrow & & \psi' \downarrow & & & & \\
(\boldsymbol{C}^2, o) & \xleftarrow{\sigma} & V' & & & &
\end{array}
\tag{3.1}
$$

where ψ is a covering map with $deg(\psi) = \mathrm{mult}(X, o)$, σ is an embedded resolution of the branch locus B_ψ of ψ, X' is the fiber product $X \times_{\boldsymbol{C}^2} V'$, ϕ_2 is the normalization of X', and ϕ_3 is a good resolution such that $\phi^*\mathfrak{m}_{X,o}$ is invertible for $\phi := \phi_1 \circ \phi_2 \circ \phi_3$. From the definition, $mult(X, o) = e(\mathfrak{m}_{X,o}, \mathscr{O}_{X,o})$ ([**Mah2**]); thus $e(\mathfrak{m}_{X,o}, \mathscr{O}_{X,o}) = deg(\psi) = e((x, y), \mathscr{O}_{X,o})$, where x and y are coordinate functions of $\boldsymbol{C}^2$. Hence there is a positive integer r with $\mathfrak{m}_{X,o}^{r+1}\mathscr{O}_{\tilde{X}} = \mathfrak{m}_{X,o}^{r}(x, y)\mathscr{O}_{\tilde{X}}$ from Rees's result ([**Re**, Theorem 3.2]). Then we have $\mathscr{O}_{\tilde{X}}(M_{\tilde{X}}) = \mathfrak{m}_{X,o}\mathscr{O}_{\tilde{X}} = (x, y)\mathscr{O}_{\tilde{X}}$ because $\pi^*\mathfrak{m}_{X,o}$ is invertible. Therefore, $E(\alpha x + \beta y) = M_{\tilde{X}}$ for general elements $\alpha, \beta \in \boldsymbol{C}$. Then we may assume that $E(\alpha x + \beta y) = M_{\tilde{X}}$ and the line $L(\alpha, \beta)$ defined by $\{\alpha x + \beta y = 0\}$ in $\boldsymbol{C}^2$ is not contained in the tangent cone of B_ψ at $\{o\}$. Let $\boldsymbol{C}^2 \xleftarrow{\sigma_1} V_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_s} V_s = V'$ be a sequence of blowing-ups with $\sigma = \sigma_1 \circ \cdots \circ \sigma_s$ and let $E_1 := \sigma_1^{-1}(\{o\})$. Then the strict transform of $L(\alpha, \beta)$ by σ_1 intersects E_1 transversally and it does not intersect B_ψ on V_1. Hence the strict transform of $L(\alpha, \beta)$ by $\sigma \circ \psi' \circ \phi_2 \circ \phi_3$ intersects E transversally on $\tilde{X}$. If we put $h = (\alpha x + \beta y) \circ \psi$, then h is a reduced element and $(h \circ \pi)_{\tilde{X}}$ is simple normal crossing. □

Now let $S \longrightarrow \Delta$ be a pencil of curves and $E = \mathrm{supp}(S_o)$. The *fundamental cycle Z_E on E* is defined as the smallest positive cycle D satisfying $DE_i \leqq 0$ for any irreducible component E_i of E. We show the existence according to [**La2**]. Let D_λ ($\lambda = 1, 2$) be a positive cycle on E such that $D_\lambda E_i \leqq 0$ for any irreducible component E_i of E. Such D_λ always exists because $cS_oE_i = 0$ for any E_i and any $c \in \boldsymbol{N}$. Let $D = \sum_{i=1}^n r_iE_i$, where $r_i = \min\{\mathrm{Coeff}_{E_i} D_1, \mathrm{Coeff}_{E_i} D_2\}$ for $i = 1, \dots, n$. For any fixed $j \in \{1, \dots, n\}$, we have $DE_j = r_jE_j^2 + \sum_{i \neq j} r_iE_iE_j \leqq (\mathrm{Coeff}_{E_j} D_k)E_j^2 + \sum_{i \neq j}(\mathrm{Coeff}_{E_i} D_k)E_iE_j = D_kE_j \leqq 0$, where $k = 1$ if $\mathrm{Coeff}_{E_j} D_1 \leqq \mathrm{Coeff}_{E_j} D_2$ and otherwise $k = 2$. Therefore, there exists the minimal element Z_E of the set $\{D > 0 | DE_i \leqq 0 \text{ for } i = 1, \dots, n\}$.

In the following, we prepare some facts for effective cycles on the exceptional set of a resolution space or the support of the singular fiber of a pencil of curves. Let E be the exceptional set of a resolution space of a normal surface singularity or $E = \mathrm{supp}(S_o)$ for the singular fiber S_o of a pencil of curves.

DEFINITION 3.2. Let D_1, D_2 be effective cycles on E with $D_1 < D_2$. If there is a sequence $Z_0 = D_1, Z_1 = Z_0 + E_{i_1}, \dots, Z_s = Z_{s-1} + E_{i_s} = D_2$ which satisfies $Z_{j-1}E_{i_j} > 0$ for $j = 1, \dots, s$, then it is called a *computation sequence from D_1 to D_2*. In this case, a sequence $E_{i_1}, \dots, E_{i_s}$ is called the *component sequence* associated to the computation sequence.

Lemma 3.3. *Let D_1, D_2 be effective cycles on E with $D_1 < D_2$.*

(i) *If there is a computation sequence from D_1 to D_2 as follows*:

$$Z_0 = D,\ Z_1 = D + E_{i_1},\ \ldots,\ Z_s = Z_{s-1} + E_{i_s},$$

then $p_a(D_1) \leqq p_a(Z_1) \leqq \cdots \leqq p_a(Z_{s-1}) \leqq p_a(D_2)$.

(ii) *Under the condition* (i), *if $p_a(D_1) = p_a(D_2)$, then $E_{i_j} = \boldsymbol{P}^1$ and $Z_{j-1}E_{i_j} = 1$ for any j.*

(iii) *If D is an effective cycle on E with $D < Z_E$ for the fundamental cycle Z_E, then there is a computation sequence from D to Z_E.*

(iv) *Let D_1, D_2 be effective cycles on E with $D_1 < Z_E \leqq D_2$. If there is a computation sequence from D_1 to D_2, then $Z_E = D_2$.*

Proof.

(i) Since $p_a(Z_j) = p_a(Z_{j-1}) + p_a(E_{i_j}) + Z_{j-1}E_{i_j} - 1$ and $p_a(E_{i_j}) \geqq 0$, we have $p_a(Z_{j-1}) \leqq p_a(Z_j)$ for $j = 1, \ldots, s$.

(ii) Consider the adjunction formula $K_S E_{i_j} = -E_{i_j}^2 + 2g(E_{i_j}) - 2 + 2\delta(E_{i_j})$, where $\delta(E_{i_j})$ is the degree of the conductor of E_{i_j} (i.e., the sum of the conductor numbers of all singularities of E_{i_j}). We have $p_a(Z_{j-1}) = p_a(Z_j)$ from (i); then $Z_{j-1}E_{i_j} + g(E_{i_j}) - 1 + \delta(E_{i_j}) = 0$. Since $Z_{j-1}E_{i_j} > 0$ and $g(E_{i_j}) \geqq 0$ and $\delta(E_{i_j}) \geqq 0$, we have $Z_{j-1}E_{i_j} = 1$ and $g(E_{i_j}) = 0$ and $\delta(E_{i_j}) = 0$. Then E_{i_j} is a non-singular rational curve.

(iii) The existence of a computation sequence is proven as in [**La2**, Proposition 4.1]; therefore, it is omitted.

(iv) Let $Z_0 = D_1, Z_1, \ldots, Z_s = D_2$ be a computation sequence. If $Z_E < D_2$, then there is ℓ with $\ell < s$ such that $Z_{\ell-1} \leqq Z_E$ and $Z_\ell \not\leqq Z_E$. Then $\mathrm{Coeff}_{E_{i_\ell}} Z_\ell = \mathrm{Coeff}_{E_{i_\ell}} Z_E + 1$ and $\mathrm{Coeff}_{E_{i_\ell}} Z_E = \mathrm{Coeff}_{E_{i_\ell}} Z_{\ell-1}$; thus $Z_{\ell-1}E_{i_\ell} \leqq Z_E E_{i_\ell}$. This yields a contradiction: $1 \leqq Z_{\ell-1}E_{i_\ell} \leqq Z_E E_{i_\ell} \leqq 0$. Then $Z_E = D_2$. □

Proposition 3.4. *Let $S \longrightarrow \Delta$ be any pencil of curves of genus g and let $A = \mathrm{supp}(S_o)$.*

(i) *$S_o = cZ_A$ $(c \in \boldsymbol{N})$.*

(ii) *Let D be an effective cycle on A such that $D \not\leqq (m-1)Z_A$ and $D \leqq mZ_A$ for $m \in \boldsymbol{N}$. Then there is a computation sequence from D to mZ_A.*

(iii) *If D is an effective cycle on A with $D \leqq S_o$, then $p_a(D) \leqq g$.*

(iv) *Suppose that the pencil of curves is minimal* (*i.e., S does not contain any* (-1) *curve*). *If $g \geqq 2$ or it is a non-multiple elliptic pencil, then $p_a(D) < g$ for any effective cycle D with $D < S_o$.*

Proof.

(i) From the definition of Z_A, we have $S_o \geqq Z_A$ and then $c = 1$ if $S_o = Z_A$. Suppose $S_o > Z_A$ and put $D_1 = S_o - Z_A$. Hence $0 = S_o^2 = Z_A^2 + 2Z_A D_1 + D_1^2$. Then $D_1^2 = 0$ because Z_A^2, ZD_1 and D_1^2 are non-positive. If $\mathrm{supp}(D_1) \subsetneq A$, then $D_1 = 0$ because $\mathrm{supp}(D_1)$ is an exceptional set. This contradicts $D_1 > 0$; therefore, $\mathrm{supp}(D_1) = A$. Let $D_1 = \sum_{i=1}^r d_i A_i$. For any irreducible component $A_i \subset A$, we have $D_1 A_i \geqq 0$ because $0 = S_o A_i = D_1 A_i + Z_A A_i$. Also $0 = D_1^2 = \sum_{i=1}^r d_i D_1 A_i$ and so $D_1 A_i = 0$ for any i since $d_i > 0$ for any i. Then $Z_A \leqq D_1$ from the definition of the fundamental cycle. If

$D_1 = Z_A$, then $S_o = 2Z_A$. If $D_1 > Z_A$, then we put $D_2 = D_1 - Z_A$ and continue the process above. Because it stops after finite steps, we complete the proof.

(ii) We can make a computation sequence of cycles as follows: $Z_0 = D$, $Z_1 = Z_0 + A_{i_1}$, $Z_2 = Z_1 + A_{i_2}, \dots$ with $Z_{j-1}A_{i_j} > 0$ for any j. Since that $Z_j \leqq mZ_A$ for each j, the sequence stops at a cycle Z_ℓ satisfying $Z_\ell A_i \leqq 0$ for any $A_i \subset A$; therefore, $Z_A \leqq Z_\ell$. If $Z_A = Z_\ell$, then $m = 1$ and concludes the proof. If $Z_A < Z_\ell$, then $(Z_\ell - Z_A)A_i = Z_\ell A_i \leqq 0$; therefore, $Z_A \leqq Z_\ell - Z_A$. If $2Z_A = Z_\ell$, then $m = 2$ and concludes the proof. If $2Z_A < Z_\ell$, then $(Z_\ell - 2Z_A)A_i \leqq 0$. Hence, continuing this argument, we can say that $Z_\ell = mZ_A$. The above computation sequence connects D and mZ_A.

(iii) If D is an effective cycle on A with $D \leqq S_o$, then there is a positive integer m as (ii) and $mZ_A \leqq S_o$. Then there is a computation sequence:

$$Z_0 = D,\ Z_1 = D + A_{i_1},\ \dots,\ Z_\ell = mZ_A = Z_{\ell-1} + A_{i_\ell}.$$

From Lemma 3.3 (i), if $g \geqq 1$, then $p_a(D) \leqq p_a(mZ_A) \leqq p_a(S_o) = g$ from $mZ_A \leqq S_o = cZ_A$. If $g = 0$, then $c = 1$ becasue any rational pencil is non-multiple. Hence $m = 1$ and $p_a(D) \leqq p_a(Z_A) = p_a(S_o) = 0$. This completes the proof.

(iv) Assume that $0 < D < S_o$ and $p_a(D) = g$. Let m be a positive integer such that $D \nleqq (m-1)Z_A$ and $D \leqq mZ_A \leqq S_o$. From (ii), there is a computation sequence $Z_0 = D$, $Z_1 = Z_0 + A_{i_1}$, $Z_2 = Z_1 + A_{i_2}, \dots, Z_\ell = mZ_A = Z_{\ell-1} + A_{i_\ell}$. By Lemma 3.3 (i), we have $g = p_a(D) \leqq p_a(mZ_A) \leqq p_a(S_o) = g$. From Lemma 3.3 (ii), $0 = mZ_A A_{i_\ell} = (Z_{\ell-1} + A_{i_\ell})A_{i_\ell} = 1 + A_{i_\ell}^2$; thereby, A_{i_ℓ} is a (-1) curve, which contradicts the minimality of S_o. □

THEOREM 3.5. *Let (X, o) be a normal surface singularity and let h a reduced element of $\mathfrak{m}_{X,o}$ satisfying the properties of Proposition* 3.1. *Then*

$$p_f(X, o) \leqq p_e(X, o) \leqq p_a(\boldsymbol{M}_X) + \mathrm{mult}(X, o) - r(h).$$

Especially, if (X, o) is a rational singularity, then $0 \leqq p_e(X, o) \leqq \mathrm{mult}(X, o) - 1$. Also, if (X, o) is an elliptic singularity (i.e., $p_f(X, o) = 1$), then $1 \leqq p_e(X, o) \leqq \mathrm{mult}(X, o)$.

PROOF. Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution that satisfies the properties of Proposition 3.1. From Theorem 2.11, we have $p_e(X, o) \leqq p_a(E(h \circ \pi)) - E(h \circ \pi)^2 - r(h)$. Also, we have $-E(h \circ \pi)^2 = \mathrm{mult}(X, o)$ from [**Wa**, Theorem 2.7] because $\pi^*\mathfrak{m}_{X,o}$ is invertible. Also $p_a(E(h \circ \pi)) = p_a(M_{\tilde{X}}) \leqq p_a(\boldsymbol{M}_X)$ from [**Tm2**, Proposition 3.9 (i)]. Then we have the inequality of the right hand side. Consider the left hand side. Let $(\hat{X}, \hat{E})$ be a resolution of (X, o) and let $S \longrightarrow \Delta$ be any pencil of curves of genus g including $(\hat{X}, \hat{E})$. From the definition of the fundamental cycle $Z_{\hat{E}}$, we have $Z_{\hat{E}} \leqq S_o$. Since $\hat{E} \subsetneq \mathrm{supp}(S_o)$, we have $Z_{\hat{E}} < S_o$. Then there is a computation sequence from $Z_{\hat{E}}$ to S_o by Lemma 3.3 (i), and $p_f(X, o) = p_a(Z_{\hat{E}}) \leqq p_a(S_o) = g$; therefore, $p_f(X, o) \leqq p_e(X, o)$.

On the minimal resolution of every rational singularity, $\boldsymbol{M}_X$ equals the fundamental cycle ([**Y**]); thus $p_a(\boldsymbol{M}_X) = p_f(X, o) = 0$. For every elliptic singularity, we have $p_a(D) \leqq$

1 for any positive cycle D on the exceptional set of any resolution. Thus, $p_a(\boldsymbol{M}_X) \leqq 1$ ([**Wa**]). $\square$

In the following, we consider the singularities that satisfy a minimality condition $p_e(X,o) = p_f(X,o)$.

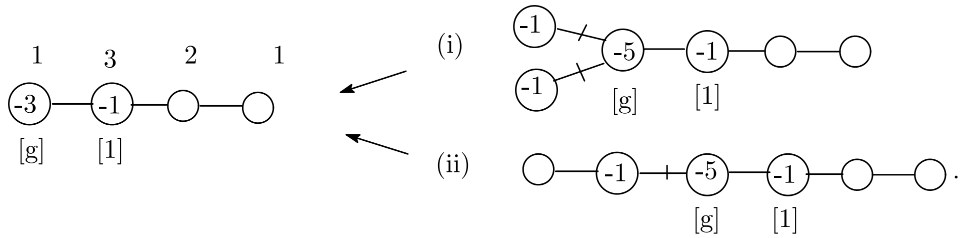
DEFINITION 3.6. If (X,o) satisfies $p_e(X,o) = p_f(X,o) = g$, then (X,o) is called *a weak Kodaira singularity of genus g*.

We remark that every Kodaira singularity in the sense of Karras [**Ka1**] is a weak Kodaira singularity (Proposition 3.10 (iii)). Here we recall the definition of Kodaira singularities and Kulikov singularities.

DEFINITION 3.7 ([**Ka1**], [**St1**], [**St2**]). Let $\Phi : S \to \Delta$ be a pencil of curves of genus g which has reduced components. Let $P_1, \ldots, P_r \in \mathrm{supp}(S_o)$ be non-singular points of S_o (i.e., they are contained in components whose coefficients of S_o equal one and also smooth points of $\mathrm{red}(S_o)$). Let $S' \xrightarrow{\sigma} S$ be a finite number of blowing-ups with centers $P_1, \ldots, P_r$. Let $\tilde{X}$ be an open neighborhood of the proper transform $E \subset S'$ of $\mathrm{supp}(S_o)$ by σ. By contracting E in $\tilde{X}$, we obtain a normal surface singularity (X,o). Then, the contraction map $\varphi : (\tilde{X}, E) \longrightarrow (X,o)$ is a resolution of (X,o). If a normal surface singularity is isomorphic to a singularity obtained in this way, then it is called a *Kodaira singularity* of genus g (or *Kodaira singularity* associated to Φ). Also, if σ is just one blowing-up at every center P_i $(i = 1, \ldots, r)$ in the above construction, then (X,o) is called a *Kulikov singularity* of genus g (or *Kulikov singularity* associated to Φ). Moreover, if $h \in \mathfrak{m}_{X,o}$ satisfies $h \circ \varphi = \Phi \circ \sigma|_{\tilde{X}}$, then h (or $h \circ \varphi$) is called a *projection function* of a Kodaira singularity (X,o).

Let Γ be the w.d.graph of the exceptional set of the minimal good resolution of a normal surface singularity. If there exists a Kodaira singularity whose w.d.graph for the minimal good resolution is equal to Γ, then Γ is called *a Kodaira graph*.

The definitions shows that every Kulikov singularity is a Kodaira singularity, but the converse is not true. Here, let us explain the procedure of the definition by drawing the following figure:



The figure of the left hand side is the w.d.graph of the singular fiber of a pencil of curves. The figure (i) (resp. (ii)) of the right hand side is the w.d.graph associated to a Kulikov (resp. Kodaira) singularity. Nevertheless, it depends on the choice of the center of the blowing-up of the procedure above whether the singularity associated to the figure (ii) is Kulikov or not.

In Theorem 3.11, we characterize Kodaira or Kulikov singularities in terms of $p_e(X,o,h)$. However, the author knows of no reference containing a concrete example that distinguishes both classes. Then we prove the following and give a such example as its application.

PROPOSITION 3.8. *Let (X,o) be a normal surface singularity obtained by the contraction of the zero-section of a negative line bundle L on a non-singular algebraic curve E.*

(i) *(X,o) is a Kodaira singularity if and only if $L \sim -\sum_{i=1}^{r} n_i P_i$ (linearly equivalent), where $n_i > 0$ for any i.*

(ii) *(X,o) is a Kulikov singularity if and only if $L \sim -\sum_{i=1}^{r} P_i$, where $P_1,\dots,P_d$ are r different points.*

(iii) *In the case of* (i) (*resp.* (ii)), *(X,o) is a Kodaira* (*resp. Kulikov*) *singularity associated to the trivial pencil; it is obtained by taking n_i blowing-up at $Q_i := (P_i, 0) \in E \times \boldsymbol{C}$ for $i = 1,\dots,r$, where $n_1 = \cdots = n_r = 1$ in the case of* (ii).

PROOF.

(i) We prove the "if" part. Assume that $P_1,\dots,P_r$ are r different points. Let $\Phi : E \times \boldsymbol{C} \to \boldsymbol{C}$ be the trivial pencil. Consider a $\boldsymbol{C}^*$-action on $E \times \boldsymbol{C}$ defined by $t \cdot (p,z) = (p,tz)$ for $t \in \boldsymbol{C}^*$. Then each point of $E \times \{0\}$ is a fixed point of this action. By taking n_i blowing-ups at $(P_i,0)$ in $E \times \boldsymbol{C}$ for each i, we have $\sigma : \tilde{S} \to E \times \boldsymbol{C}$. Let $\tilde{E}$ be the strict transform of E by σ. Then we can easily check that $\boldsymbol{C}^*$-action is extended onto $\tilde{S}$ and each point of $\tilde{E}$ is a fixed point for it (cf. [**OrW**, Section 1.2]). We can contract $\tilde{E}$ in $\tilde{S}$ and obtain a Kodaira singularity (Y,o) with $\boldsymbol{C}^*$-action. If $(\tilde{X},E)$ is the minimal resolution, the normal bundle of E (resp. $\tilde{E}$) in $\tilde{X}$ (resp. $\tilde{S}$) is L (resp. $-\sum_{i=1}^{r} n_i P_i$). Since $L \sim -\sum_{i=1}^{r} n_i P_i$, we have $(X,o) \simeq (Y,o)$ (i.e., holomorphically isomorphic). Then (X,o) is a Kodaira singularity.

Now we prove the "only if" part. There is a normal surface singularity (Y,o) and a pencil of curves $\Phi : S \to \Delta$ including a resolution $(\tilde{Y},F)$ of (Y,o) such that $(Y,o) \simeq (X,o)$ and $S_o|_{\tilde{Y}} = F + \sum_{i=1}^{r} n_i G_i$, where any G_i is a non-exceptional irreducible component in $\tilde{Y}$. Then there is a following diagram:

$$\begin{array}{ccccc} & \overset{\pi}{\swarrow} & (\tilde{Y},F)\subset(S,\mathrm{supp}(S_o)) & \overset{\Phi}{\searrow} & \\ (Y,o) & & \varphi\big\downarrow \qquad\qquad \varphi\big\downarrow & & \Delta\ , \\ & \underset{\bar{\pi}}{\nwarrow} & (\bar{Y},\bar{F})\subset(\bar{S},\mathrm{supp}(\bar{S}_o)) & \underset{\bar{\Phi}}{\nearrow} & \end{array} \tag{3.2}$$

where $\bar{\pi}$ is the minimal resolution. Therefore, $\bar{F}$ is an irreducible smooth curve which is holomorphically isomorphic to E and $\bar{S}_o|_{\bar{Y}} = \bar{F} + \sum_{i=1}^{r} n_i \varphi(G_i)$. Then

$$0 \sim \bar{S}_o \bar{F} = N_{\bar{F}/\bar{Y}}|_{\bar{F}} + \sum_{i=1}^{r} n_i \bar{P}_i \text{ for } \bar{P}_i := \varphi(G_i) \cap \bar{F} \quad (i = 1,\dots,r).$$

Let $(\tilde{X},E)$ be the minimal resolution of (X,o). Since there is a holomorphic isomorphism $\psi : (\bar{Y},\bar{F}) \xrightarrow{\simeq} (\tilde{X},E)$, we have $L = N_{E/\tilde{X}}|_E \simeq \psi_* N_{\bar{F}/\bar{Y}}|_{\bar{F}} \sim -\sum_{i=1}^{r} n_i P_i$ on E, where

$P_i := \psi(\bar{P}_i)$ for any i.

(ii) Assuming $n_i = 1$ for any i in the argument of (i), we can prove "if" part in the same manner. Thus we prove the "only if" part. From the definition of Kulikov singularities, there is a normal surface singularity (Y, o) and a pencil of curves $\Phi : S \to \Delta$ including a resolution $(\tilde{Y}, F)$ of (Y, o) such that $(Y, o) \simeq (X, o)$ and $S_o|_{\tilde{Y}} = F + \sum_{j=1}^{r} G_j$, where G_j is contained in a (-1)-curve $\tilde{G}_j$ with $\mathrm{Coeff}_{\tilde{G}_j} S_o = 1$ for $j = 1, \ldots, r$. We can consider the same diagram as (3.2). Let F_{i_j} be an irreducible component in F with $F_{i_j} \cap G_j \neq \varnothing$ $(j = 1, \ldots, r)$; therefore $\mathrm{Coeff}_{F_{i_j}} S_o = 1$. Let F_o be an irreducible component of F with $\bar{F} = \varphi(F_o) \simeq E$. Any connected component of $F \backslash (F_o \cup \bigcup_{j=1}^{r} G_j)$ is contracted to a non-singular point by φ. Therefore, if there is $j \in \{1, \ldots, r\}$ with $F_{i_j} \neq F_o$, then $\mathrm{Coeff}_{F_{i_j}} S_o \geqq 2$. This contradicts $\mathrm{Coeff}_{F_{i_j}} S_o = 1$; thus $F_{i_1} = \cdots = F_{i_r} = F_o$. Therefore, if we put $\bar{P}_j := \bar{F} \cap \varphi(G_j)$ $(j = 1, \ldots, r)$, each $\varphi(\tilde{G}_j)$ is a (-1)-curve which intersects $\bar{F}$ transversally at $\bar{P}_j$ and $\bar{P}_i \neq \bar{P}_j$ if $i \neq j$. Then $0 \sim \bar{S}_o \bar{F}_o = N_{\bar{F}/\bar{S}}|_{\bar{F}} + \sum_{i=1}^{r} \bar{P}_i$; thus (ii) is proven as in (i).

The proof of (iii) is contained in the proof of (i). □

Example 3.9. Let E be a non-hyperelliptic curve and let P be any point of E. Then there are no two points Q_1 and Q_2 in E with $2P \sim Q_1 + Q_2$. Let L be the negative line bundle associated to $-2P$, and let (X, o) be a normal surface singularity obtained by the blowing down of the zero-section of L. Then (X, o) is a Kodaira singularity, but not a Kulikov singularity from Proposition 3.8.

Proposition 3.10.

(i) ([**Ka1**, Proposition 2.7]) *Let E be the exceptional set of the minimal good resolution of a normal surface singularity and Z_E the fundamental cycle. Then the w.d.graph Γ_E is a Kodaira graph if and only if* $\mathrm{Coeff}_{E_i} Z_E = 1$ *for any E_i with $Z_E E_i < 0$.*

(ii) ([**Ka1**, Theorem 2.9]) *If (X, o) is a rational or minimally elliptic singularity with K-graph, then it is a Kodaira singularity.*

(iii) ([**Ka2**, Lemma 3.4]) *If (X, o) is a Kodaira singularity of genus g, then $p_f(X, o) = g$.*

Let (X, o) be a rational double point. If (X, o) is of type A_n, then (X, o) is a Kulikov singularity (see Section 1). If (X, o) is of type D_n, then $p_e(X, o) = 0$ as noted in Section 1. Then (X, o) is a weak Kodaira singularity, but not a Kodaira singularity from Proposition 3.10 (ii). If (X, o) is of type E_6, E_7 or E_8, then it is not a weak Kodaira singularity (Proposition 3.12).

Under the situation of Definition 3.7, we can easily show that the fundamental cycle Z_E is equal to the strict transform $\sigma_*^{-1}(S_o)$. Hence, we have the equality $p_f(X, o) = g$ of Proposition 3.10 (iii). From the definitions and 3.10 (iii), any Kodaira singularity is a weak Kodaira singularity, although the converse is not true.

In the following, we consider necessary and sufficient conditions for a weak Kodaira singularity to be a Kodaira or a Kulikov singularity.

Theorem 3.11. *Let (X, o) be a normal surface singularity.*

(i) *(X, o) is a Kodaira singularity if and only if there exists $h \in \mathfrak{m}_{X,o}$ which is not a perfect power element satisfying $p_e(X, o, h) = p_f(X, o)$ and $E(h \circ \pi) = Z_E$, where*

$\pi : (\tilde{X}, E) \longrightarrow (X, o)$ *is a resolution such that* $\mathrm{red}(h \circ \pi)_{\tilde{X}}$ *is simple normal crossing.*

(ii) (X, o) *is a Kulikov singularity if and only if there exists a reduced element* $h \in \mathfrak{m}_{X,o}$ *with* $p_e(X, o, h) = p_f(X, o)$.

PROOF.

(i) Let consider the "only if part". Assume that (X, o) is a Kodaira singularity. Let $h \in \mathfrak{m}_{X,o}$ be a projection function of (X, o) (i.e., there exists a resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ and a pencil of curves $\Phi : S \to \Delta$ such that $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_o)))$ and $h \circ \pi = \Phi|_{\tilde{X}}$). Also, from the definition of Kodaira singularities, we have $E(h \circ \pi) = Z_E$ and $\mathrm{Coeff}_{E_i} E(h \circ \pi) = \mathrm{Coeff}_{E_i} Z_E = 1$ for an irreducible component E_i with $Z_E E_i < 0$. Hence $p_e(X, o, h) = p_f(X, o)$ and h is not a perfect power element.

Consider the "if part". Theorem 2.4 allows construction of a pencil of curves $\Phi : S \to \Delta$ of genus $p_f(X, o)$ such that $h \circ \pi = \Phi|_{\tilde{X}}$ and any connected component of $\mathrm{supp}(S_o) \backslash E$ is a minimal $\boldsymbol{P}^1$-chain. From the condition $E(h \circ \pi) = Z_E$ and Proposition 3.4 (i), Φ is non-multiple. Suppose there is an irreducible component E_{i_o} with $Z_E E_{i_o} < 0$ and $\mathrm{Coeff}_{E_{i_o}} Z_E \geqq 2$. From the construction of S, there is an irreducible component F_{i_1} of $\mathrm{supp}(S_o) \backslash E$ with $F_{i_1} E_{i_o} \neq 0$. From Proposition 3.4 (i), and Lemma 3.3 (iii), there is a computation sequence from Z_E to S_o as

$$Z_o = Z_E,\ Z_1 = Z_E + F_{i_1},\ \ldots,\ Z_\ell = S_o,$$

Since $p_a(Z_E) = p_f(X, o) = p_a(S_o)$, we have $Z_E F_{i_1} = 1$ from Lemma 3.3 (ii). However, $Z_E F_{i_1} \geqq 2$ from $\mathrm{Coeff}_{E_{i_o}} Z_E \geqq 2$. This is a contradiction. Then $\mathrm{Coeff}_{E_i} Z_E = 1$ if $Z_E E_i < 0$. We have $Z_E = E(h \circ \pi) = S_o|_E$ from the assumption and $h \circ \pi = \Phi|_{\tilde{X}}$, and so $\mathrm{Coeff}_{E_i} S_o = 1$ if $Z_E E_i < 0$. Let $F(1), \ldots, F(m)$ be connected components of $\mathrm{supp}(S_o) \backslash E$. If $F(i) := \bigcup_{j=1}^{r_i} F_{i,j}$ and $b_{i,j} = -F_{i,j}^2$ for any i, j, then $b_{i,1} \geqq 1$ and $b_{i,j} \geqq 2$ for $j = 2, \ldots, r_i$. Suppose that $b_{1,1} = \cdots = b_{s,1} = 1$ and $b_{s+1,1} \geqq 2, \ldots, b_{m,1} \geqq 2$ $(0 \leqq s \leqq m)$. Suppose $s < m$. We have $\mathrm{Coeff}_{F_{i,j}} S_o = 1$ for any $i > s$ and any j by considering a computation sequence from Z_E to S_o. This is a contradiction: $0 = S_o F_{i,r_i} = 1 - b_{i,r_i} < 0$ for $i \geqq s + 1$. Then $s = m$ (i.e., $b_{1,1} = \cdots = b_{m,1} = 1$). Suppose there is i such that $b_{i,2} = \cdots = b_{i,t_i} = 2$ and $b_{i,t_i+1} \geqq 3$ for $t_i < r_i$. From Proposition 3.4 (i), we have $S_o|_{F(i)} = Z_{\mathrm{supp}(S_o)}|_{F(i)} = \sum_{j=1}^{t_i} (t_i - j + 2) F_{i,j} + F_{i,t_i+1} + \cdots + F_{i,r_i}$ which yields a contradiction: $0 = S_o F_{i,r_i} = \epsilon - b_{i,r_i} < 0$, where $\epsilon = 2$ (resp. 1) if $r_i = t_i + 1$ (resp. $r_i > t_i + 1$). Then $b_{i,2} = \cdots = b_{i,r_i} = 2$ if $r_i \geqq 2$. Therefore, any connected component of $\mathrm{supp}(S_o) \backslash E$ is a minimal $\boldsymbol{P}^1$-chain of type (1) or $(1, 2, \ldots, 2)$. Then (X, o) is a Kodaira singularity.

(ii) Let consider the "only if part". Assume that (X, o) is a Kulikov singularity. Let $h \in \mathfrak{m}_{X,o}$ be a projection function of (X, o) (i.e., $h \circ \pi = \Phi|_{\tilde{X}}$, where $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ is a resolution satisfying the properties of (i)). From (i), we have $p_e(X, o, h) = p_f(X, o)$. From the definition of Kulikov singularities, the non-exceptional part $\Lambda(h \circ \pi)$ is a reduced divisor. Then h is a reduced element.

Next we prove the "if part". From Theorem 2.4, 2.9 and 2.13, there are a non-multiple pencil of curves $\Phi : S \longrightarrow \Delta$ of genus $p_f(X, o)$ and a good resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ such that $\mathrm{red}(S_o)$ is simple normal crossing and $\Phi|_{\tilde{X}} = h \circ \pi$ and any connected component of $\mathrm{supp}(S_o) \backslash E$ is a minimal $\boldsymbol{P}^1$-chain. Because h is a reduced element, the length of any $\boldsymbol{P}^1$-chain in $\mathrm{supp}(S_o) \backslash E$ is one. Then we can put $F =$

$\mathrm{supp}(S_o)\backslash E$ and $F = \bigcup_{i=1}^m F_i$, where F_i is a $\boldsymbol{P}^1$ with $\mathrm{Coeff}_{F_i} S_o = 1$ for any i and $F_i \cap F_j = \varnothing$ for $i \neq j$. Let $\varphi : S \longrightarrow \bar{S}$ be a successive contraction map of (-1) curves in E such that $\bar{E} = \varphi(E)$ is minimal (i.e., $\bar{E}$ contains no (-1) curve) and $\varphi|_{S\backslash E}$ is an isomorphism. We put $\bar{F}_i = \varphi(F_i)$ for $i = 1, \dots, m$. It suffices to prove $\bar{F}_i^2 = -1$ for any i. Assume $Z_{\bar{E}} + \sum_{i=1}^m \bar{F}_i < Z_{\bar{E}\cup\bar{F}} = \bar{S}_o$. From Lemma 3.3 (iii), we can make a computation sequence $Z_0 = Z_{\bar{E}} + \sum_{i=1}^m F_i$, $Z_1 = Z_0 + E'_{i_1}, \dots, Z_s = Z_{s-1} + E'_{i_s} = Z_{\bar{E}\cup\bar{F}}$, where E'_{i_j} is an irreducible component of $\bar{E} \cup \bar{F}$. From Lemma 3.3 (i),

$$\begin{aligned} p_f(X,o) &= p_a(Z_E) = p_a(Z_{\bar{E}}) \leqq p_a(Z_{s-1}) \\ &= p_a\big(Z_{\bar{E}\cup\bar{F}} - E'_{i_s}\big) \leqq p_a(Z_{\bar{E}\cup\bar{F}}) = p_e(X,o) = p_f(X,o). \end{aligned}$$

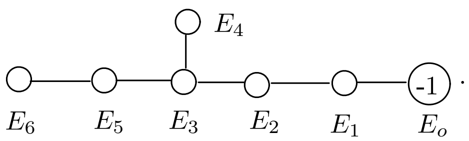
Therefore, $-{E'_{i_s}}^2 = (\bar{S}_o - E'_{i_s})E'_{i_s} = (Z_{\bar{E}\cup\bar{F}} - E'_{i_s})E'_{i_s} = 1$ from Lemma 3.3 (ii). Hence $E'_{i_s} \not\subset \bar{E}$ since $\bar{E}$ is minimal. Then $E'_{i_s} \subset \bar{F}$, thus yielding a contradiction: $1 = \mathrm{Coeff}_{E'_{i_s}} \bar{S}_o = \mathrm{Coeff}_{E'_{i_s}} Z_{\bar{E}\cup\bar{F}} \geqq 2$. Then $Z_{\bar{E}\cup\bar{F}} = Z_{\bar{E}} + \sum_{i=1}^m F_i$. By considering a computation sequence $Z_{\bar{E}}, Z_{\bar{E}} + F_1, \dots, Z_{\bar{E}} + \sum_{i=1}^m F_i = S_o$, we have $\bar{F}_1^2 = \cdots = \bar{F}_m^2 = -1$ from Lemma 3.3 (ii). Then (X,o) is a Kulikov singularity. □

The condition $E(h \circ \pi) = Z_E$ is necessary in Theorem 3.11 (ii). Because, in Remark 2.14 (i.e., $(X,o) = (\{z^2 = y(x^2 + y^{n-2})\}, o))$, $y \in \mathfrak{m}_{X,o}$ is not a perfect power element and $p_e(X,o,y) = p_f(X,o) = 0$, but (X,o) is not a Kodaira singularity.

In Remark 1.6, we explained that $p_e(X,o) = 0$ for any rational double point of type A_n and D_n. In the following, we obtain the value of the pencil genus for rational singularities with lower multiplicity.

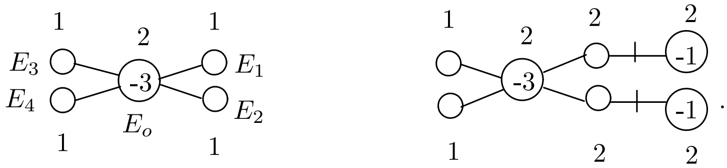
Proposition 3.12. *Rational double points of type E_6, E_7 and E_8 have $p_e(X,o) = 1$. Further, all other log-canonical surface singularities except for them are weak Kodaira singularities.*

Proof. Let (X,o) be a rational double point of E_6, E_7 or E_8. Every quotient singularity is a taut singularity ([**La1**]) (i.e., the analytic structure of (X,o) is determined by the weighted dual graph of the minimal good resolution). Then we can readily see that the minimal resolution space of (X,o) is included in the relatively minimal elliptic pencil ([**Ko**]); therefore, so we have $p_e(X,o) \leqq 1$. From Remark 1.5, $p_e(E_6) \leqq p_e(E_7) \leqq p_e(E_8) \leqq 1$. Hence we need only to prove $p_e(E_6) = 1$. Suppose $p_e(E_6) = 0$ and let $\Phi : S \to \Delta$ be a rational pencil with $(S, \mathrm{supp}(S_o) \supset (\tilde{X}, E)$, where $(\tilde{X}, E) \longrightarrow (X,o)$ is the minimal resolution. Because the relatively minimal pencil of genus 0 is isomorphic to the trivial pencil $\boldsymbol{P}^1 \times \Delta$, we may assume that $\mathrm{supp}(S_o)\backslash E$ contains a (-1) curve E_0 intersecting E. Let $F = E \cup E_0$ and let $Z_E, Z_E + E_0, \dots, Z_F$ be a computation sequence from Z_E to Z_F. From Lemma 3.3 (i) and Proposition 3.4 (ii), $0 = p_a(Z_E) \leqq p_a(Z_E + E_0) \leqq \cdots \leqq p_a(Z_F) = 0$. From Lemma 3.3 (ii), we have $Z_E E_0 = 1$. Therefore, E_0 intersects E as follows:



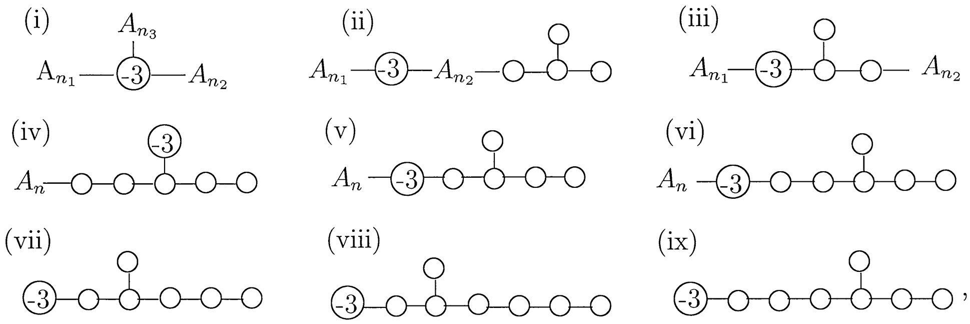
Then a proper subset $\bigcup_{j=0}^{5} E_j$ of F is an exceptional set by Zariski's Lemma ([**BPV**, p. 90]). However, we can easily see that its intersection matrix of $\bigcup_{j=0}^{5} E_j$ is not negative definite. This is a contradiction; consequently, $p_e(E_6) = 1$.

Because log-canonical elliptic singularities are simple elliptic or cusp singularities, they are minimally elliptic singularities. Hence they are Kodaira singularities from Proposition 3.10 (ii). If a log-canonical rational singularity has a Kodaira graph, then it is a Kodaira singularity from Proposition 3.10 (ii). It must be proven that all log-canonical rational singularities except for E_6, E_7 and E_8 are weak Kodaira singularities. We have the classification of two-dimensional log-canonical singularities (cf. [**Mak**, p. 237]). Hence, we need to check the above for only log-canonical rational singularities which have not Kodaira graphs. We show this for one case; other cases are treated similarly. Let (X, o) be a log-canonical rational singularity such that the w.d.graph associated to the minimal resolution $(\tilde{X}, E) \xrightarrow{\pi} (X, o)$ is given as the left side in the following:



Let C_j be a non-compact smooth complex curve $\tilde{X}$ such that it intersects E_j transversally $(j = 1, 2)$. If we put $D = 2E_0 + 2E_1 + 2E_2 + E_3 + E_4 + 2C_1 + 2C_2$, then $DE_j = 0$ for $j = 0, 1, \ldots, 4$. From Lemma 2.1, there exists an element $h \in \mathfrak{m}_{X,o}$ such that $(h \circ \pi)_{\tilde{X}} = D$. Using the method of Theorem 2.4, we can construct a rational pencil as above. Then (X, o) is a weak Kodaira singularity of $g = 0$. $\square$

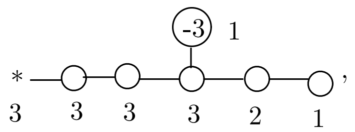
In [**Art**, p. 135], M. Artin exhausted the w.d.graphs associated to rational triple points as follows:



where A_n indicates $\bigcirc - \cdots - \bigcirc$ (i.e., the w.d.graph is of $\boldsymbol{P}^1$-chain of length $n \geqq 0$ and of type $(2, \ldots, 2)$). In the following, we can determine the value of $p_e(X, o)$ for rational triple points.

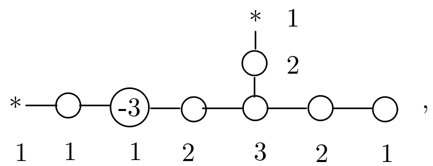
Proposition 3.13. *If (X, o) is a rational triple point whose w.d.graph is one of* (i)–(iv) (*resp.* (v)–(ix)), *then* $p_e(X, o) = 0$ (*resp.* $p_e(X, o) = 1$).

Proof. First we consider the cases of (i)–(iv). Let (X, o) be a singularity of type (iv) with $n = 0$. Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be the minimal resolution. Consider a following divisor D on $\tilde{X}$:

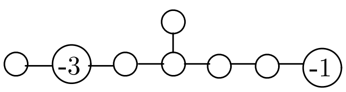


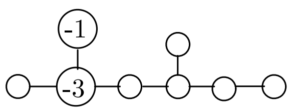
where $*$ is a non-exceptional curve on $\tilde{X}$ which intersects E transversally at one point. From Lemma 2.1, there is an element $h \in \mathfrak{m}_{X,o}$ such that $D = (h \circ \pi)_{\tilde{X}}$. By gluing suitably a neighborhood of (-1)-curve as in Theorem 2.4, we can construct a rational pencil $\Phi : S \to \Delta$ such that $\Phi|_{\tilde{X}} = h \circ \pi$. Then $p_e(X, o) = 0$. We can prove the case of $n \geqq 1$ similarly; in addition, we can prove cases of (ii)–(iv) similarly.

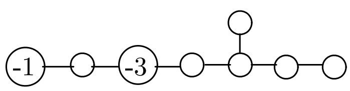
Second we consider cases of (v)–(ix). Let (X, o) be a singularity of type (v) with $n = 1$. Consider the following divisor D on $\tilde{X}$:



where $*$ is a non-exceptional curve on $\tilde{X}$. By gluing a suitable neighborhood of a (-1)-curve on a neighborhood of each $*$ as above, we obtain an elliptic pencil. Then $p_e(X, o) \leqq 1$. Suppose $p_e(X, o) = 0$. Then there is a rational pencil $\Phi : S \longrightarrow \Delta$ and a resolution $(\tilde{X}, E) \longrightarrow (X, o)$ satisfying $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_o))$. By the same way as in Proposition 3.12, we can easily show that there is a (-1) curve E_o in $\mathrm{supp}(S_o)$ which intersects E transversally and $Z_E E_o = 1$. Hence the configuration of $E \cup E_o$ is one of the following figures:

(v-1) 

(v-2) 

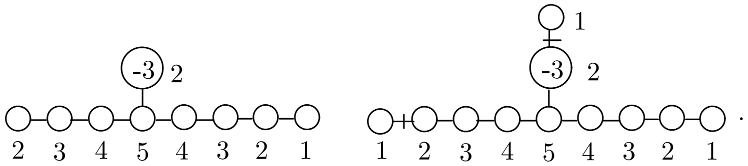
(v-3) 

For (v-1), we can easily see that the w.d.graph has a subgraph whose intersection matrix is not negative definite. For (v-2) or (v-3), the configuration contains the w.d.graph of rational double point of type E_6. Then $p_e(E_6) = 0$ from Remark 1.5 (ii), but this contradicts Proposition 3.12; thus $p_e(X, o) = 1$. We can check other cases (vi)–(ix) similarly. □

Here we propose the following problem:

Problem 3.14. Is there a finite upper bound of $p_e(X,o)$ for rational singularities?

Example 3.15. Let (X,o) be a rational singularity of multiplicity 4 whose fundamental cycle $\boldsymbol{Z_X}$ on the minimal resolution $(\tilde{X},E)$ is given by the left one of the following figures:



Because (X,o) is a rational singularity, there exists $h \in \mathfrak{m}_{X,o}$ with $E(h\circ\pi) = \boldsymbol{Z_X} = \boldsymbol{M_X}$. Therefore, as in Theorem 2.4, we can construct a pencil of curves of genus 2 whose singular fiber is figured by the right one above. Then $p_e(X,o) \leqq 2$. Suppose $p_e(X,o) \leqq 1$ and let $\Phi : S \longrightarrow \Delta$ be a pencil of curves of genus $p_e(X,o)$ and including a resolution of (X,o). From Kodaira's classification of elliptic pencils ([**Ko**]), we can see that there are no minimal pencils of curves of genus $g \leqq 1$ whose singular fiber contains E. Hence the pencil of curves is not minimal. After suitable contractions of (-1)-curves, we may assume that there is a (-1) curve E_0 in $\mathrm{supp}(S_o)$ intersecting E. If E_0 intersects a (-2)-curve of E, then it yields a contradiction. If E_0 intersects the (-3)-curve of E, then the w.d.graph of $E_0 \cup E$ is star-shaped and the intersection matrix is not negative definite ([**P**, p. 185]). This yields a contradiction. Therefore, $p_e(X,o) = 2$.

4. Pencil genus of cyclic coverings of normal surface singularities.

After fundamental results on surface singularities due to M. Artin, H. Laufer and P. Wagreich, S. S. T. Yau [**Y**] introduced the notion of elliptic sequences for elliptic surface singularities. He also considered a sequence $(X_n,o) = (\{z^{6n+c} = x^2 + y^3\},o)$ ($n = 1,2,3,\dots$ and $c = 3,5$) of elliptic hypersurface singularities associated to an elliptic sequence; thus $p_f(X_n,o) = 1$ for any n. Inspiring this, the author proved that if $(X,o) = (\{z^n = x^a + y^b\},o) \subset (\boldsymbol{C}^3,o)$ is normal and $n \geqq \mathrm{lcm}(a,b)$, then $p_f(X,o) = \frac{1}{2}\{(a-1)(b-1) - \gcd(a,b) + 1\}$ ([**Tt1**, Theorem 4.4]). Further, in Theorem 4.5 in [**Tt2**], he also proved that if $(X,o) = (\{z^n = h(x,y)\},o) \subset (\boldsymbol{C}^3,o)$ is normal and n is "sufficiently" large, then (X,o) is a Kodaira singularity of genus $\frac{\mu(h)-r(h)+1}{2}$.

Let (Y,o) be a normal complex surface singularity and let (X,o) be a normalization of the cyclic covering defined by $z^n = h$ over (Y,o), where $h \in \mathfrak{m}_{Y,o}$ is a semi-reduced element (Definition 4.6). In this section, we prove that (X,o) is a weak Kodaira singularity satisfying $p_f(X,o) = p_e(Y,o,h)$ for "sufficiently" large positive integer n (Theorem 4.12). Next, using the characterization of Kulikov singularities in Theorem 3.11, we prove that if h is a reduced element, then (X,o) is a Kulikov singularity for "sufficiently" large positive integer n (Theorem 4.14). This is a generalization of Theorem 4.5 in [**Tt2**].

Definition 4.1. Let $(Y,o) \subset (\boldsymbol{C}^N,o)$ be a normal singularity and I its defining ideal in $\boldsymbol{C}\{y_1,\dots,y_N\}$. Further, let $h \in \mathfrak{m}_{Y,o}$ be an element and let assume that $\tilde{h} \in \boldsymbol{C}\{y_1,\dots,y_N\}$ is correspondent to h. Let (X,o) $(\subset (\boldsymbol{C}^{N+1},o))$ be a singularity defined by the ideal generated by I and $z^n - \tilde{h}(y_1, \dots, y_N)$ in $\boldsymbol{C}\{y_1, \dots, y_N, z\}$. Then (X, o) is called the *n-fold cyclic covering of (Y, o) defined by $z^n = h$.*

In this paper, we usually assume that h is not a perfect power element. Then (X, o) of Definition 4.1 is irreducible ([**TW1**, Proposition 1.8]); also (X, o) is a normal singularity if and only if h is a reduced element in $\mathcal{O}_{Y,o}$ ([**TW2**, Theorem 3.2]).

Here we prepare some facts. Let (X, o) be the normalization of the cyclic covering $(\bar{X}, o)$ of normal surface singularity (Y, o) defined by $z^n = h$, where $h \in \mathfrak{m}_{Y,o}$ is not a perfect power element. Let $\sigma : (\tilde{Y}, E) \to (Y, o)$ be a resolution such that $(h \circ \sigma)_{\tilde{Y}}$ is simple normal crossing on $\tilde{Y}$. We can construct a good resolution of (X, o) from $(\tilde{Y}, E)$ and $(h \circ \sigma)_{\tilde{Y}}$. By taking the fiber product of σ and ψ, we have the following diagram:

$$\begin{array}{ccccc} (X,o) & \xleftarrow{\quad\phi_1\quad} & X' \xleftarrow{\phi_2} X'' & \xleftarrow{\phi_3} & (\tilde{X},\tilde{E}), \\ \psi\Big\downarrow & & \psi'\Big\downarrow & \swarrow\,\delta & \\ (Y,o) & \xleftarrow{\quad\sigma\quad} & (\tilde{Y},E) & & \end{array} \tag{4.1}$$

where ψ is the composition of the normalization map $(X, o) \longrightarrow (\bar{X}, o)$ and the restriction to $(\bar{X}, o) = \{z^n = h(x_1, \dots, x_N)\}$ of the projection map $\boldsymbol{C}^{N+1} \to \boldsymbol{C}^N$ $((x_1, \dots, x_N, z) \mapsto (x_1, \dots, x_N))$. Further, let X'' be the normalization of X' and then X'' has only cyclic quotient singularities. Let $\tilde{X}$ be the minimal resolution of all cyclic quotient singularities on X''. We put $E = \mathrm{supp}(E(h \circ \sigma))$ and $C = \mathrm{supp}(\Lambda(h \circ \sigma))$. If $\psi'(Q) \in E \cup C$ for a point $Q \in X'$, Q is included in the singular locus of X'. $\psi'(Q)$ is included in only one or two irreducible components of $E \cup C$. A local coordinate (u, v) can be chosen in a neighborhood of $\psi'(Q)$ such that X' is represented by $z^n = u^a$ or $z^n = u^a v^b$. The singularity $z^n = u^a$ is resolved by normalization ϕ_2. Hence we need only to resolve singularities of type $z^n = u^a v^b$ on X' by ϕ_3. It is well-known that the normalization of such singularities are cyclic quotient singularities ([**BPV**, p. 83]), and we can easily compute the type from a, b and n. Let us define three integers a_1, b_1 and n_1 as follows:

$$\begin{cases} a_1 = a/\gcd(a, \mathrm{lcm}(n, b)), \quad b_1 = b/\gcd(b, \mathrm{lcm}(n, a)), \\ n_1 = n/\gcd(n, \mathrm{lcm}(a, b)). \end{cases} \tag{4.2}$$

Lemma 4.2 ([**Tt2**, Lemma 2.5]). *Let $k = \gcd(n, a, b)$. The normalization of $\{z^n = u^a v^b\}$ is disjoint k cyclic quotient singularities of type $C_{n_1,\mu}$, where μ is an integer defined by $a_1\mu + b_1 \equiv 0(n_1)$ and $0 \leqq \mu < n_1$.*

Let $g \in \mathfrak{m}_{Y,o}$ and put $(g \circ \sigma)_{\tilde{Y}} = \sum_{i=1}^{r} v_{E_i}(g \circ \sigma)E_i + \sum_{j=1}^{s} v_{C_j}(g \circ \sigma)C_j$. Let $\tilde{E}_i$ and $\tilde{C}_j$ be the strict transform of E_i and C_j by δ respectively. They are not necessarily irreducible curves. With respect to the vanishing orders of z and $g \circ \sigma$ on $\tilde{E}_i$ and $\tilde{C}_j$, we have the following.

Lemma 4.3 ([**Tt2**, Lemma 3.1]). *Let F be an irreducible component E_i or C_j of* $\mathrm{supp}((g \circ \sigma)_{\tilde{Y}})$ *and $\tilde{F}$ the strict transform of F by δ. If we put $\phi = \phi_1 \circ \phi_2 \circ \phi_3$, then*

$v_{\tilde{F}}(z \circ \phi) = \frac{v_F(h\circ\sigma)}{\gcd(n, v_F(h\circ\sigma))}$ *and* $v_{\tilde{F}}(g \circ \sigma \circ \delta) = \frac{n v_F(g\circ\sigma)}{\gcd(n, v_F(h\circ\sigma))}$.

Definition 4.4. Let $\Phi : S \to \Delta$ be a pencil of curves of genus g_o and let $\eta : \Delta' \to \Delta$ be a map given by $t = \eta(s) = s^n$. Taking the fiber product of Φ and η yields the following diagram:

$$\begin{array}{ccccccc} S & \xleftarrow{\varphi_1} & S' & \xleftarrow{\varphi_2} & S'' & \xleftarrow{\varphi_3} & S^{(n)}, \\ \Phi\big\downarrow & & \Phi'\big\downarrow & & & \swarrow \Phi^{(n)} & \\ \Delta & \xleftarrow{\eta} & \Delta' & & & & \end{array} \tag{4.3}$$

where $S' = S \times_\Delta \Delta'$ and S'' is the normalization of S' and $S^{(n)}$ is the minimal resolution of S''. Then $\Phi^{(n)} : S^{(n)} \to \Delta'$ is a pencil of curves of genus g_o. It is called the *n-th root fibration of* Φ ([**BPV**, pp. 92–93]).

Theorem 4.5. *Let* (Y, o) *be a normal surface singularity and* $h \in \mathfrak{m}_{Y,o}$ *not a perfect power element. If* (X, o) *is the normalization of the n-fold cyclic covering of* (Y, o) *defined by* $z^n = h$ *for* $n > 1$, *then* $p_e(X, o, z) = p_e(Y, o, h)$, *and so* $p_e(X, o) \leqq p_e(Y, o, h)$.

Proof. Let $\sigma : (\tilde{Y}, E) \to (Y, o)$ be a resolution such that $\mathrm{red}((h \circ \sigma)_{\tilde{Y}})$ is simple normal crossing. Let $\Phi : S \to \Delta$ be a pencil of genus $p_e(Y, o, h)$ constructed as in Theorem 2.4 such that Φ is an extension of $h \circ \sigma$. Let $\Phi^{(n)} : S^{(n)} \to \Delta'$ be the n-th root fibration of Φ, and so its genus equals $p_e(Y, o, h)$. From the construction of $S^{(n)}$, $S^{(n)}$ contains a good resolution space $\tilde{X}$ of (X, o). If we put $\phi = \phi_1 \circ \phi_2 \circ \phi_3$ in (4.1) and $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$ in (4.3), then $(\Phi^{(n)}|_{\tilde{X}})^n = \Phi \circ \varphi|_{\tilde{X}} = h \circ \sigma \circ \delta|_{\tilde{X}} = (z \circ \phi)^n$; therefore, we may assume that $\Phi^{(n)}|_{\tilde{X}} = z \circ \phi$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} \boldsymbol{C} \supset \Delta' & \xleftarrow{z} & (X, o) & \xleftarrow{\phi} & (\tilde{X}, \tilde{E}) & \subset & \big(S^{(n)}, \mathrm{supp}(S_o^{(n)})\big) \\ \eta\big\downarrow & & \psi\big\downarrow & & \delta\big\downarrow & & \varphi\big\downarrow \\ \boldsymbol{C} \supset \Delta & \xleftarrow{h} & (Y, 0) & \xleftarrow{\sigma} & (\tilde{Y}, E) & \subset & \big(S, \mathrm{supp}(S_o)\big). \end{array}$$

Since all connected components of $\mathrm{supp}(S_o)\backslash E$ are $\boldsymbol{P}^1$-chains and they are lifted to $\boldsymbol{P}^1$-chains on $\mathrm{supp}(S_o^{(n)})$ by φ, all connected components of $\mathrm{supp}(S_o^{(n)})\backslash\tilde{E}$ are also $\boldsymbol{P}^1$-chains. Since $\Phi^{(n)}|_{\tilde{X}} = z \circ \phi$, the genus of $\Phi^{(n)}$ is equal to $p_e(X, o, z)$ from Theorem 2.9. Hence $p_e(X, o) \leqq p_e(X, o, z) = p_e(Y, o, h)$. □

In this section, we give a sufficient condition for cyclic coverings of normal surface singularities to be weak Kodaira singularities (Definition 3.6). Let (X, o) be a normal surface singularity and let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution such that $\mathrm{red}((h\circ\pi)_{\tilde{X}})$ is simple normal crossing. Let $E = \bigcup_{i=1}^r E_i$ and $\mathrm{supp}(\Lambda(h \circ \pi)_{\tilde{X}}) = \bigcup_{i=1}^s C_j$ be irreducible decompositions.

Definition 4.6. Under the situation above, put $a_i = v_{E_i}(h \circ \pi)$ for any i, $b_j =$

$v_{C_j}(h \circ \pi)$ for any j and $N_h(\pi) = \max\{\mathrm{lcm}(a_i, b_j) | E_i C_j \neq 0\}$. Define a positive integer $N_h(X, o)$ as follows:

(i) $N_h(X, o) = \min\{N_h(\pi) | \pi$ is a resolution such that $\mathrm{red}(h \circ \pi)_{\tilde{X}}$ is simple normal crossing$\}$.

(ii) If $\gcd(a_1, \ldots, a_r, b_1, \ldots, b_s) = 1$, then h is called a *semi-reduced element*.

The well-definedness of "semi-reduced" is obvious, because the value of $\gcd(a_1, \ldots, a_r,\ b_1, \ldots, b_s)$ is independent of the choice of π. In addition, any reduced element is semi-reduced and any semi-reduced element is not a perfect power. Also, if h is semi-reduced and Φ is an extension of $h \circ \pi$, then Φ is non-multiple.

In the following, we prepare some lemmas to prove the main result of this section (Theorem 4.12).

Lemma 4.7 ([**Tt3**, Lemma 1.3]). *For relatively prime positive integers n, q, let $\frac{n}{q} = [[b_1, \ldots, b_r]]$ be the continued fractional expansion. For a real number α, we have*

$$[[b_1, \ldots, b_{r-1}, b_r + \alpha]] = \frac{n + q'\alpha}{q + q''\alpha},$$

where q' and q'' are denoted by $qq' \equiv 1 (n)$, $0 < q' < n$ and $nq'' = qq' - 1$.

For relatively prime positive integers a, b, let $H(a, b)$ be the semi-group generated by them. If $n \geqq (a-1)(b-1)$, then $n \in H(a, b)$ ([**OnW**, Proposition 3]).

Lemma 4.8. *Let a, b be relatively prime positive integers and $n \in H(a, b)$. Then there exist integers p, q and r uniquely which satisfy the following condition*:

$$n = ap + bq + abr, \quad 0 \leqq p < b,\ 0 \leqq q < a \text{ and } 0 \leqq r.$$

Proof. Assume that (p, q, r) and (p', q', r') satisfy conditions above and $ap + bq + abr = ap' + bq' + abr'$. Suppose that $(p, q) \neq (p', q')$ and $p \neq p'$. Since $a(p - p') + b(q - q') + ab(r - r') = 0$, we have $b | p - p'$ and so $b < p$ or $b < p'$. This is a contradiction. Then $(p, q) = (p', q')$ and so $r = r'$. □

Suppose that n, a and b are relatively prime positive integers satisfying $n \geqq \mathrm{lcm}(a, b)$; thereby, $n \in H(a, b)$. Let μ be an integer defined by $a\mu + b \equiv 0 (n)$ with $0 < \mu < n$, and let p, q and r be integers given by Lemma 4.8. Let p' and q' be integers defined by $pp' \equiv 1 (b)$ and $qq' \equiv 1 (a)$ respectively, where $1 \leqq p' < b$ and $1 \leqq q' < a$.

Lemma 4.9. *Under the situation above,*

$$\frac{n}{\mu} = \begin{cases} [[d_1, \ldots, d_{s_1-1}, d_{s_1} + e_{s_2}, e_{s_2-1}, \ldots, e_1]], & \text{if } r = 0, \\ [[d_1, \ldots, d_{s_1-1}, d_{s_1} + 1, \overbrace{2, \ldots, 2}^{r-1}, e_{s_2} + 1, e_{s_2-1}, \ldots, e_1]] & \text{if } r > 0, \end{cases}$$

where $a/q' = [[d_1, \ldots, d_{s_1}]]$ and $b/p' = [[e_1, \ldots, e_{s_2}]]$.

Proof. Consider the case of $r > 0$. From $b/p = [[e_{s_2}, \dots, e_1]]$ ([**Ri**]) and Lemma 4.7,

$$\begin{aligned}
&[[d_1, \dots, d_{s_1-1}, d_{s_1} + 1, 2, \dots, 2, e_{s_2} + 1, e_{s_2-1}, \dots, e_1]] \\
&= \left[\left[d_1, \dots, d_{s_1-1}, d_{s_1} + 1, 2, \dots, 2, 1 + \frac{b}{p}\right]\right] = \left[\left[d_1, \dots, d_{s_1-1}, d_{s_1} + \frac{b}{rb+p}\right]\right] \\
&= \frac{ap + bq + abr}{q'p + bq'' + q'br} = \frac{n}{q'p + bq'' + q'br},
\end{aligned}$$

where $qq' = aq'' + 1$. Also

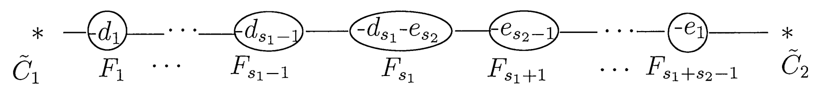
$$\begin{aligned}
a(q'p + bq'' + q'br) + b = q'(ap + bq + abr) = nq' \equiv 0(n) \\
\text{and } \ 0 < q'p + bq'' + q'br < ap + bq + abr = n \text{ from } q'' < q.
\end{aligned}$$

Since μ is determined uniquely by the relation $a\mu + b \equiv 0(n)$ and $0 < \mu < n$, we have $\mu = q'p + bq'' + q'br$. Then

$$[[d_1, \dots, d_{s_1-1}, d_{s_1} + 1, 2, \dots, 2, e_{s_2} + 1, e_{s_2-1}, \dots, e_1]] = \frac{n}{\mu}.$$

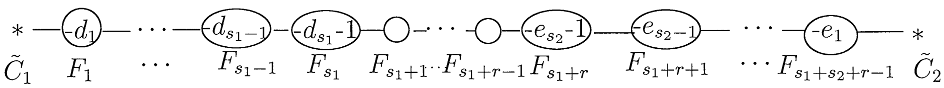
Because the case of $r = 0$ can be proven similarly, we omit the proof. □

Let $X = \{z^n = u^a v^b\} \subset \boldsymbol{C}^3$ and assume $n \geqq \gcd(a, b)$. Let a_1, b_1 and n_1 be positive integers determined by (4.2). Hence two elements of them are always relatively prime. By Lemma 4.2, the normalization of X is a cyclic quotient singularity $C_{n_1,\mu}$, where $a_1\mu + b_1 \equiv 0(n_1)$ and $0 < \mu < n_1$. Let $(\tilde{X}, E)$ be the minimal resolution of X. Let p, q and r be non-negative integers given by Lemma 4.8. Let p', q' be integers given by $pp' \equiv 1(b_1)$, $qq' \equiv 1(a_1)$, $1 \leqq p' < b_1$ and $1 \leqq q' < a_1$. Let $a_1/q' = [[d_1, \dots, d_{s_1}]]$ and $b_1/p' = [[e_1, \dots, e_{s_2}]]$. From Lemma 4.9, the configuration of $\mathrm{supp}(z)_{\tilde{X}}$ is given as



if $r_j = 0$, and

(4.4)



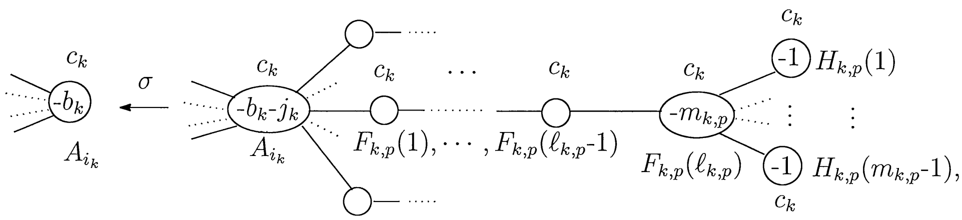
if $r_j > 0$,

where $\tilde{C}_1$ (resp. $\tilde{C}_2$) is the strict transform of $\{u = 0\}$ (resp. $\{v = 0\}$).

Lemma 4.10. *In the situation above, if we put* $\lambda = \gcd(a, b)/\gcd(a, b, n)$ *and put* $s_3 = s_1 + s_2 + r - 1$, *then* $\mathrm{Coeff}_{F_{S_3}}(z)_{\tilde{X}} = p'\lambda$, $\mathrm{Coeff}_{\tilde{C}_2}(z)_{\tilde{X}} = b_1\lambda$ *and* $v_{F_{s_1}}(z)_{\tilde{X}} = \cdots = v_{F_{s_1+r}}(z)_{\tilde{X}} = \lambda$ *in* (4.4).

Proof. From Lemma 4.3, $\mathrm{Coeff}_{\tilde{C}_1}(z)_{\tilde{X}} = a_1\lambda$ and $\mathrm{Coeff}_{\tilde{C}_2}(z)_{\tilde{X}} = b_1\lambda$. Let $f_0 = a_1\lambda$, $f_1 = q'\lambda$ and $f_2, \ldots, f_{s_1-1}$ be positive integers that are inductively defined by relations $f_i = d_{i-1}f_{i-1} - f_{i-2}$ $(i = 2, \ldots, s_1 - 1)$. Further, let $f_i = \lambda$ for $i = s_1, \ldots, s_1 + r$ and let $f_{s_3+1} = b_1\lambda$, $f_{s_3} = p'\lambda$ and $f_{s_3-1}, \ldots, f_{s_1+r+1}$ be positive integers defined by relations $f_{s_3-i} = e_i f_{s_3-i+1} - f_{s_3-i+2}$ inductively $(i = 1, \ldots, s_2 - 1)$. If we put $D = f_0\tilde{C}_1 + \sum_{i=1}^{s_3} f_i F_i + f_{s_3+1}\tilde{C}_2$, then $DF_j = 0$ for $j = 1, \ldots, s_3$ and $\mathrm{Coeff}_{\tilde{C}_k} D = \mathrm{Coeff}_{\tilde{C}_k}(z)_{\tilde{X}}$ for $k = 1, 2$. Therefore, $(z)_{\tilde{X}} = D$, which completes the proof. □

Let $\Phi : S \longrightarrow \Delta$ be a non-multiple pencil of genus g. Let $A = \mathrm{supp}(S_o)$ and let $A = \bigcup_{i=1}^{t} A_i$ be the irreducible decomposition of A. Let $P_{1,1}, \ldots, P_{1,j_1} \in A_{i_1}, \ldots, P_{s,1}, \ldots, P_{s,j_s} \in A_{i_s}$ be $\sum_{k=1}^{s} j_k$ different non-singular points of $\mathrm{red}(S_o)$. Let $\tilde{S} \xrightarrow{\sigma} S$ be a successive blowing-up started from those points such that it makes trees as follows:



where $m_{k,p} \geqq 2$ $(k = 1, \ldots, s$ and $p = 1, \ldots, j_k)$. Let $\tilde{A} = \mathrm{supp}(\tilde{S}_o)$ and let $\tilde{A}_i$ be the strict transform of A_i. Let $E = (\bigcup_{i=1}^{t} \tilde{A}_i) \cup (\bigcup_{k=1}^{s} \bigcup_{p=1}^{j_k} \bigcup_{q=1}^{\ell_{k,p}} F_{k,p}(q))$ and $\bar{E} = \bigcup_{i=1}^{t} \tilde{A}_i = \mathrm{supp}(\sigma_*^{-1} S_o)$, where $\sigma_*^{-1} S_o$ is the transform of S_o (i.e., $\sigma_*^{-1} S_o = \sum_{i=1}^{t} (\mathrm{Coeff}_{A_i} S_o)\tilde{A}_i$).

Lemma 4.11. *If $c_k = \mathrm{Coeff}_{A_{i_k}} S_o \leqq \ell_{k,p} + 1$ for any k and p, then the fundamental cycle Z_E is equal to*

$$D_0 := \sigma_*^{-1} S_o + \sum_{k=1}^{s} \sum_{p=1}^{j_k} \sum_{q=1}^{\ell_{k,p}} \min\{c_k, \ell_{k,p} - q + 1\} F_{k,p}(q)$$

and $p_a(Z_E) = g$.

Proof. Let $A(1), \ldots, A(N)$ be the component sequence associated to a computation sequence of S_o, where $A(k) \subset \{A_1, \ldots, A_s\}$ for any k. Let $\tilde{A}(k) = \sigma_*^{-1} A(k)$ for $k = 1, \ldots, N$ and let consider a sequence $\tilde{A}(1), \ldots, \tilde{A}(N)$. For a fixed k with $1 \leqq k \leqq s$, we assume that $\tilde{A}(\epsilon_1) = \cdots = \tilde{A}(\epsilon_{c_k}) = \tilde{A}_{i_k}$ and $\epsilon_1 < \cdots < \epsilon_{c_k}$, where $c_k = \mathrm{Coeff}_{A_{\epsilon_k}} S_o$. Inserting $F_{k,p}(1), \ldots, F_{k,p}(\ell_{k,p} - \delta + 1)$ between $\tilde{A}(\epsilon_\delta)$ and $\tilde{A}(\epsilon_\delta + 1)$ in the sequence $\tilde{A}(1), \ldots, \tilde{A}(N)$ for $p = 1, \ldots, j_k$ and $\delta = 1, \ldots, c_k$, we obtain a new sequence. Continuing this process for $k = 1, \ldots, s$, we obtain the component sequence associated to a computation sequence of D_0. The intersection number of D_0 and any irreducible component of E is non-positive. Hence $Z_E \leqq D_0$, and so

$$Z_E = D_0 := \sigma_*^{-1} S_o + \sum_{k=1}^{s} \sum_{p=1}^{j_k} \sum_{q=1}^{\ell_{k,p}} \min\{c_k, \ell_{k,p} - q + 1\} F_{k,p}(q)$$

from Lemma 3.3 (iv).

Next we compute $p_a(Z_E)$. Let $A' := \mathrm{supp}(\sum_{i=1}^{t} A_i - \sum_{k=1}^{s} A_{i_k})$, and let $\tilde{A}'$ be the strict transform of A' by σ. If E_i is an irreducible component of A', then $Z_E E_i = 0$. Hence we have the following:

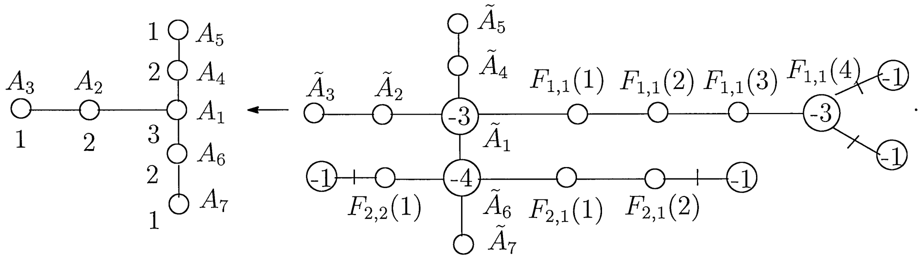
$$K_S S_o = K_S(S_o|_{A'}) + \sum_{k=1}^{s} c_k(b_k - 2) = K_{\tilde{X}}(\tilde{S}_o|_{\tilde{A}'}) + \sum_{k=1}^{s} c_k(b_k - 2),$$

$$K_{\tilde{X}} Z_E = K_{\tilde{X}}(\tilde{S}_o|_{\tilde{A}'}) + \sum_{k=1}^{s} c_k(b_k + j_k - 2) + \sum_{k=1}^{s}\sum_{p=1}^{j_k}(m_{k,p} - 2),$$

$$Z_E^2 = -\sum_{k=1}^{s}\sum_{p=1}^{j_k}(c_k + m_{k,p} - 2).$$

Therefore, we have $g - p_a(Z_E) = \frac{1}{2}(K_S S_o - K_{\tilde{X}} Z_E - Z_E^2) = 0$. □

We explain the procedure above to compute Z_E through the following example:



Then $A_1, A_2, \ldots, A_7, A_1, A_2, A_4, A_6, A_1$ is the component sequence (Definition 3.2) associated to a computation sequence of S_o; thereby, $\tilde{A}(1) = A(8) = A(12) = \tilde{A}_1$ and $\tilde{A}(6) = A(11) = \tilde{A}_6$. Let insert $F_{i,j}$'s into a sequence $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_7, \tilde{A}_1, \tilde{A}_2, \tilde{A}_4, \tilde{A}_6, \tilde{A}_1$ as follows:

$$\tilde{A}_1, F_{1,1}(1), \ldots, F_{1,1}(4), \tilde{A}_2, \ldots, \tilde{A}_6, F_{2,1}(1), F_{2,1}(2), F_{2,2}(1), \tilde{A}_7, \tilde{A}_1,$$

$$F_{1,1}(1), F_{1,1}(2), F_{1,1}(3), \tilde{A}_2, \tilde{A}_4, \tilde{A}_6, F_{2,1}(1), \tilde{A}_1, F_{1,1}(1), F_{1,1}(2).$$

This gives the component sequence associated to a computation sequence of Z_E. A similar lemma was proven in a slightly different situation ([**Tt4**, Lemma 2.4]).

Now we prove the main result of this section.

THEOREM 4.12. *Let $(Y, o) \subset (\boldsymbol{C}^N, o)$ be a normal surface singularity and $h \in \mathfrak{m}_{Y,o}$ a semi-reduced element* (Definition 4.6). *Let (X, o) be the normalization of the n-fold cyclic covering of (Y, o) defined by $z^n = h$. If $n \geqq N_h(Y, o)$, then (X, o) is a weak Kodaira singularity of genus $p_e(Y, o, h)$.*

PROOF. Let $\sigma : (\tilde{Y}, E) \longrightarrow (Y, o)$ be a resolution such that $\mathrm{red}((h\circ\sigma)_{\tilde{Y}})$ is a simple normal crossing divisor and $N_h(\sigma) = N_h(Y, o)$. Let $(\tilde{X}, \tilde{E})$ be a good resolution of (X, o) constructed as in (4.1); then consider diagram (4.1) and use the notations there. Let $\bigcup_{j=1}^{m} C_j$ be the irreducible decomposition of the strict transform C of $\mathrm{red}\{h = 0\}(\subset Y)$ by σ. For any C_j, let E_{i_j} be the irreducible component of E with $E_{i_j} \cap C_j \neq \varnothing$ for $j = 1, \ldots, m$. If we prove the following,

$$p_a(Z_{\tilde{E}}) = p_e(X, o, z), \tag{4.5}$$

then $p_f(X, o) = p_a(Z_{\tilde{E}}) = p_e(X, o, z) = p_e(Y, o, h)$ by Theorem 4.5 and (X, o) is a weak Kodaira singularity of genus $p_e(Y, o, h)$. This completes the proof; thereby, we prove (4.5) in the following.

Let U_j be a small open neighborhood of $E_{i_j} \cap C_j$ in $\tilde{Y}$, and let $a_j = v_{E_{i_j}}(h \circ \sigma)$, $b_j = v_{C_j}(h \circ \sigma)$. Hence $N_h(Y, o) = \max\{\mathrm{lcm}(a_j, b_j) \mid j = 1, \ldots, m\}$. Let (u_j, v_j) be a local coordinate of U_j such that $E_{i_j} = \{u_j = 0\}$ and $C_j = \{v_j = 0\}$ in U_j. Then $\psi'^{-1}(U_j) = \{z^n = u_j^{a_j} v^{b_j}\}$. Let $n_j, \bar{a}_j$ and $\bar{b}_j$ be integers given by $n_j = n/\gcd(n, \mathrm{lcm}(a_j, b_j))$, $\bar{a}_j = a_j/\gcd(a_j, \mathrm{lcm}(n, b_j))$ and $\bar{b}_j = b_j/\gcd(b_j, \mathrm{lcm}(n, a_j))$ as in (4.2). Then two elements of them are always relatively prime. Also, let μ_j be an integer determined by

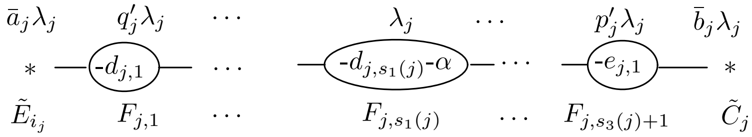
$$\bar{a}_j \mu_j + \bar{b}_j \equiv 0(n_j) \text{ and } 0 \leqq \mu_j < n_j.$$

Further we put $\ell_j = \gcd(n, a_j, b_j)$ and $\lambda_j = \gcd(a_j, b_j)/\ell_j$. Then $(\psi' \circ \phi_2)^{-1}(U_j)$ is the disjoint union of ℓ_j cyclic quotient singularities of type C_{n_j, μ_j} from Lemma 4.2.

From now on we assume $n \geqq N_h(Y, o)$. Then we have $n_j \geqq \bar{a}_j \bar{b}_j \lambda_j$. If $n_j = 1$, then any connected component of $(\psi' \circ \phi_2)^{-1}(U_j)$ is non-singular. Hence, we assume $n_j > 1$ in the following. From Lemma 4.8, there exist integers p_j, q_j and r_j uniquely which satisfy the conditions: $n_j = \bar{a}_j p_j + \bar{b}_j q_j + \bar{a}_j \bar{b}_j r_j$, $0 \leqq p_j < \bar{b}_j$, $0 \leqq q_j < \bar{a}_j$ and $0 \leqq r_j$. Hence we have

$$\lambda_j \leqq r_j + 1. \tag{4.6}$$

Let p_j' and q_j' be integers given by $p_j' p_j \equiv 1(\bar{b}_j)$, $q_j' q_j \equiv 1(\bar{a}_j)$, $1 \leqq p_j' < \bar{b}_j$ and $1 \leqq q_j' < \bar{a}_j$. Let $\bar{a}_j/q_j' = [[d_{j,1}, \ldots, d_{j,s_1(j)}]]$, $\bar{b}_j/p_j' = [[e_{j,1}, \ldots, e_{j,s_2(j)}]]$ and $s_3(j) = s_1(j) + s_2(j) + r_j - 1$. From (4.4), the divisor $(z \circ \phi)_{\tilde{X}}$ restricted to the $\boldsymbol{P}^1$-chain associated to the minimal resolution of C_{n_j, μ_j} is given as



, (4.7)

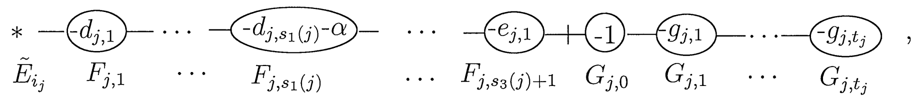
where $\tilde{C}_j$ (resp. $\tilde{E}_{i_j}$) is an irreducible component of $\delta^{-1}(C_j)$ (resp. $\delta^{-1}(E_{i_j})$) and $\alpha = 1$ if $r_j > 0$, and $\alpha = e_{j,s_2(j)}$ otherwise. Gluing $\tilde{X}$ and resolution spaces of $\sum_{j=1}^{m} \gcd(n, a_j, b_j)$ cyclic quotient singularities as in Theorem 2.4, we can construct a pencil of curves $\Phi : S \longrightarrow \Delta$ that satisfies the following commutative diagram:

$$\begin{array}{ccc} (\tilde{X},\tilde{E}) & \hookrightarrow & (S,\operatorname{supp}(S_o))\,. \\ & \overset{z\circ\phi}{\searrow} & \downarrow \Phi \\ & & \Delta \end{array}$$

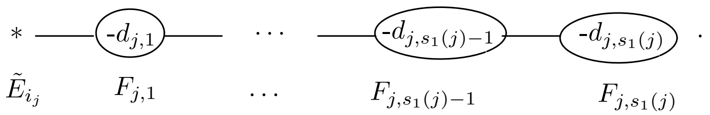
Because h is semi-reduced, $\Phi : S \longrightarrow \Delta$ is a non-multiple pencil of curves of genus $p_e(X,o,z)$ from Lemma 4.3 and Theorem 2.9. Therefore,

$$p_a\big(Z_{\tilde{E}}\big) \leqq p_e(X,o,z). \tag{4.8}$$

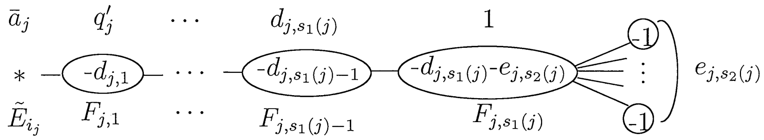
Since $p'_j\lambda_j < \bar{b}_j\lambda_j$, the configuration of (4.7) is embedded to a $\boldsymbol{P}^1$-chain started from $\tilde{E}_{i_j}$ in $\operatorname{supp}(S_o)$ as



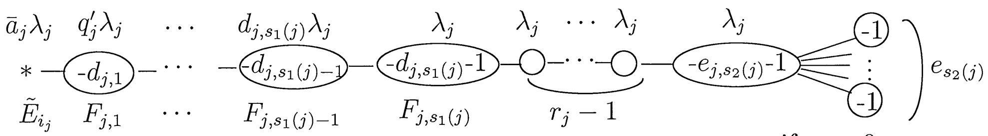
where $\tilde{C}_j \subset G_{j,0}$, $\frac{p'_j}{\bar{b}_j} = [[1,g_1,\ldots,g_{t_j}]]$ and $g_{j,k} \geqq 2$ if $k>0$. Then $(\bigcup_{k=1}^{s_3(j)+1}F_{j,k}) \cup (\bigcup_{\ell=0}^{t_j}G_{j,\ell})$ is a $\boldsymbol{P}^1$-chain started from $\tilde{E}_{i_j}$ which is a connected component of $\operatorname{supp}(S_o)\backslash\tilde{E}$. From Lemma 4.7, we can easily check $[[e_{j,s_2(j)}+1, e_{j,s_2(j)-1},\ldots,e_{j,1},1,g_{j,1},\ldots,g_{j,t_j}]] = 1$. Therefore, after successive contractions of (-1) curves of $\operatorname{supp}(S_o)\backslash\tilde{E}$, we can get a pencil $\Phi' : S' \longrightarrow \Delta$ of genus $p_e(Y,o,h)$ such that the $\boldsymbol{P}^1$-chain started from $\tilde{E}_{i_j}$ is given



After suitable blowing-ups $S'' \xrightarrow{\tau} S'$, we can construct a non-multiple pencil $\Phi'' = \Phi'\circ\tau : S'' \longrightarrow \Delta$ of genus $p_e(X,o,z)$ such that $\operatorname{supp}(S''_o)$ is given as follows:

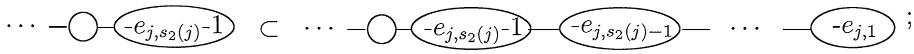


if $r_j = 0$, and



if $r_j > 0$.

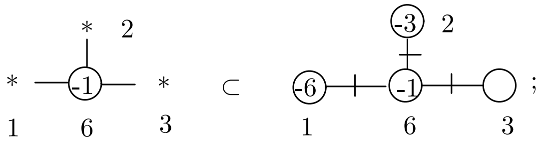
Let H_j be the union of all (-1) curves of the right hand side in the figure above; also, let $H = \bigcup_{k=1}^{m} H_j$. If we put $E' = \operatorname{supp}(S''_o)\backslash H$, then $p_a(Z_{E'}) = p_e(X,o,z)$ from (4.6) and Lemma 4.11. The w.d.graph of E' is embedded to the one of $\tilde{E}$ as



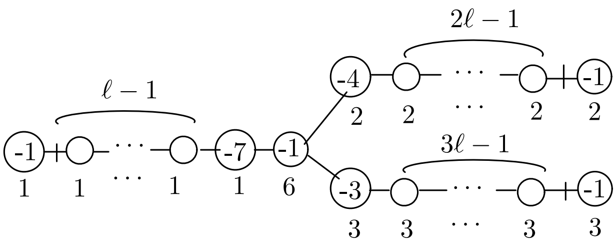
consequently, $p_a(Z_{E'}) \leqq p_a(Z_{\tilde{E}})$. Hence $p_e(X,o,z) = p_a(Z_{E'}) \leqq p_a(Z_{\tilde{E}}) \leqq p_e(X,o,z)$ from (4.8), thereby proving (4.5) and completing the proof. □

Example 4.13.

(i) Let $h = (x+y)x^2y^3$ and let $V \xrightarrow{\sigma} \boldsymbol{C}^2$ be a blowing-up at $\{0\} \in \boldsymbol{C}^2$. Then

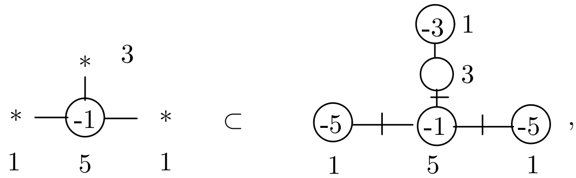


and so $p_e(\boldsymbol{C}^2, o, h) = 1$ and $N_h(\boldsymbol{C}^2, o) = 6$. Let (X,o) be the normalization of a non-normal singularity $\{z^n = h\} \subset (\boldsymbol{C}^3, o)$. From Theorem 4.12, (X,o) is a weak Kodaira singularity for any $n \geqq 6$. Assume $n = 6\ell + 1$ for $\ell \geqq 1$. Let $(Y,o) = (\boldsymbol{C}^2, o)$ and consider the diagram of (4.1). Then there are three cyclic quotient singularities of each type C_{n,μ_j} $(j = 1,2,3)$, where $\mu_1 = \ell$, $\mu_2 = 2\ell$ and $\mu_3 = 3\ell$ from Lemma 4.2. Since $\frac{6\ell+1}{\ell} = [[7,2,\dots,2]]$, $\frac{6\ell+1}{2\ell} = [[4,2,\dots,2]]$ and $\frac{6\ell+1}{3\ell} = [[3,2,\dots,2]]$, the associated elliptic pencil of curves which includes the minimal good resolution of (X,o) is given as

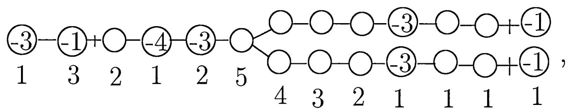


Therefore, (X,o) is a weak Kodaira elliptic singularity (but not a Kodaira singularity). From Nemethi's result ([**Ne**]), (X,o) is a maximally elliptic singularity with $-\boldsymbol{Z}_X^2 = 6$ $(=\mathrm{mult}(X,o) = \mathrm{emb.dim.}(X,o))$ and $p_g(X,o) = \ell$.

(ii) Let $h = xy(x+y)^3$ and let $V \xrightarrow{\sigma} \boldsymbol{C}^2$ be a blowing-up at $\{0\} \in \boldsymbol{C}^2$. Then we have



and so we have $p_e(\boldsymbol{C}^2, o, h) = 2$ and $N_h(\boldsymbol{C}^2, o) = 15$. Let (X,o) be the normalization of $\{z^n = h\} \subset (\boldsymbol{C}^3, o)$. From Theorem 4.12, (X,o) is a weak Kodaira singularity of genus 2 for any $n \geqq 15$. Assume $n = 15\ell + 19$ for $\ell \geqq 0$. Then we must consider three cyclic quotient singularities of each type C_{n,μ_j} $(j = 1, 2, 3)$, where $\mu_1 = \mu_2 = 12\ell + 15$ and $\mu_3 = 6\ell + 7$ from Lemma 4.2. Since $\frac{15\ell+19}{12\ell+15} = [[2, 2, 2, 3, 2, \dots, 2]]$ and $\frac{12\ell+19}{6\ell+7} = [[3, 3, 2, \dots, 2, 3, 2]]$, the associated pencil of curves of genus 2 which includes the minimal resolution of (X, o) is given as

$\ell = 0$: 

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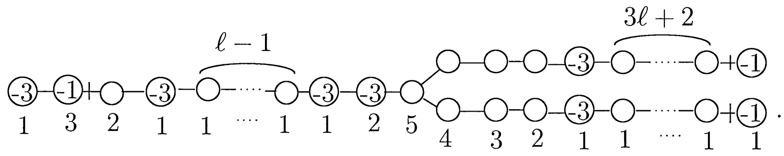
$\ell \geq 1$: 

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THEOREM 4.14. *Let (Y, o) be a normal surface singularity and $h \in \mathfrak{m}_{Y,o}$ a reduced element. Let (X, o) be the n-fold cyclic covering of (Y, o) defined by $z^n = h$. If $n \geqq N_h(Y, o)$, then (X, o) is a Kulikov singularity of genus $\delta(h) - r(h) + 1$ and z is the projection function* (Definition 3.7) *for an associated pencil of curves and $\boldsymbol{Z}_X^2 = -r(h)$.*

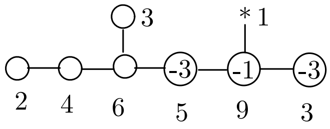
PROOF. Because h is reduced, (X, o) is normal ([**TW2**, Theorem 3.2]). Also, (X, o) is a weak Kodaira singularity of genus $\delta(h) - r(h) + 1$ from Corollary 2.12 and Theorems 4.5 and 4.12. Let $\sigma : (\tilde{Y}, E) \to (Y, o)$ be a resolution such that $\mathrm{red}((h \circ \sigma)_{\tilde{Y}})$ is a simple normal crossing divisor and $N_h(Y, o) = N_h(\sigma)$. Consider diagram (4.1); then $\phi := \phi_3 \circ \phi_2 \circ \phi_1 : (\tilde{X}, \tilde{E}) \to (X, o)$ is a good resolution such that $(z \circ \phi)_{\tilde{X}}$ is simple normal crossing. From Lemma 4.3, z is a reduced element of $\mathcal{O}_{X,o}$. By Theorem 3.11, (X, o) is a Kulikov singularity which has a projection function z. For non-exceptional part $\Lambda(z \circ \phi)$, $\mathrm{supp}(\Lambda(z \circ \phi))$ has $r(h)$ connected components and $\Lambda(z \circ \phi)$ is a reduced divisor. Since $\tilde{E}(z \circ \phi) = Z_{\tilde{E}}$, we have $0 \sim (z \circ \phi)_{\tilde{X}} = Z_{\tilde{E}} + \Lambda(z \circ \phi)$; therefore, $\boldsymbol{Z}_X^2 = Z_{\tilde{E}}^2 = -Z_{\tilde{E}}\Lambda(z \circ \phi) = -r(h)$. □

COROLLARY 4.15. *Let $(Y, o) = \{h(x, y, z) = 0\} \subset \boldsymbol{C}^3$ be a normal hypersurface singularity. If x is a reduced element of $\mathcal{O}_{Y,o}$, then a hypersurface singularity $(X, o) = \{h(x^n, y, z) = 0\}$ is a Kulikov singularity of genus $\frac{\mu(f)-r(f)+1}{2}$ and $\boldsymbol{Z}_X^2 = -r(f)$ if $n \geqq N_x(Y, o)$, where $f := h(0, y, z)$ and $\mu(f)$ and $r(f)$ are defined for a plane curve singularity $(\{f = 0\}, o)$.*

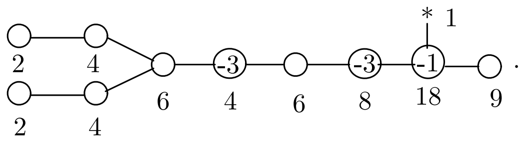
COROLLARY 4.16 ([**Tt2**, Theorem 4.5]). *Let $(X, o) = \{z^n = h(x, y)\}$ be a normal hypersurface singularity with $n > 1$, where $h \in \boldsymbol{C}\{x, y\}$. If $n \geqq N_h(\boldsymbol{C}^2, o)$, then (X, o) is a Kulikov singularity of genus $\frac{\mu(h)-r(h)+1}{2}$ and $\boldsymbol{Z}_X^2 = -r(h)$.*

EXAMPLE 4.17. Let $(Y, o) = \{z^3 = (x + y^2)(x + y^3)\} \subset (\boldsymbol{C}^3, o)$. This is a rational double point of type E_6. Because $p_f(Y, o) = 0$ and $p_e(Y, o) = 1$ from Proposition 3.12, (Y, o) is not a weak Kodaira singularity. Consider two resolutions of (Y, o) such that divisors determined by two reduced elements x and $x^3 + y^2$ are given as

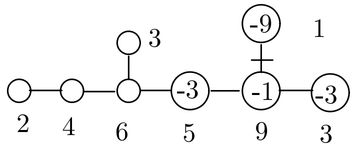
(x) :

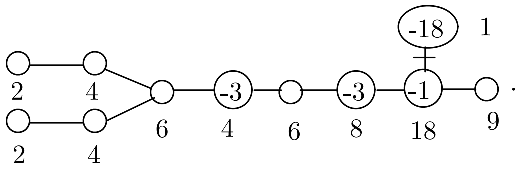


(x^3+y^2) :



As in Theorem 2.4, we can construct two pencils of curves from them. Their singular fibers are given as

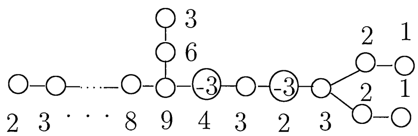




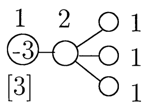
Then $N_x(Y,o)=9$ and $N_{x^3+y^2}(Y,o)=18$, and we have $p_e(Y,o,x)=4$ and $p_e(Y,o,x^3+y^2)=6$ from (1.1) and the adjunction formula and Theorem 2.9.

(i) Let $(X,o)=\{z^3=(x^n+y^2)(x^n+y^3)\}\subset(\boldsymbol{C}^3,o)$ for $n\geqq 2$. For $n=8,9$ and 10, the cycles defined by (x) on the minimal good resolutions of (X,o) are given as

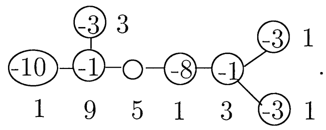
$n=8$:



$n=9$:

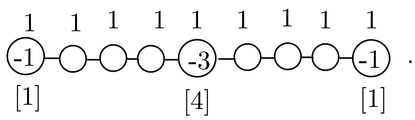


$n=10$:



A cycle above gives the fundamental cycle in each case. Though the w.d.graph is not a Kodaira graph for $n=8$, it is a Kodaira graph for $n=9$ or 10. From Corollary 4.15, (X,o) is a Kulikov singularity of genus 4 if $n\geqq N_x(Y,o)=9$.

(ii) Let (X,o) be a complete intersection singularity $\{z^3=(x+y^2)(x+y^3),u^n=x^3+y^2\}\subset(\boldsymbol{C}^4,o)$. From Theorem 4.14 and the configuration above for the divisor of x^3+y^2, (X,o) is a Kulikov singularity of genus 6 if $n\geqq N_{x^3+y^2}(Y,o)=18$. For example, assume $n=18$. Applying Lemmas 4.2 and 4.3 to the figure of the divisor (x^3+y^2) above on a resolution space of (Y,o), we can obtain the cycle $E(u\circ\pi)$ on the minimal resolution $(\tilde{X},E)\xrightarrow{\pi}(X,o)$ is given as



5. Pencil genus of hypersurface singularities defined by $z^n=f(x,y)$.

In this section we consider hypersurface singularities defined by $z^n=f(x,y)$. Though they construct a special class of normal surface singularities, they have been studied by many authors since Zariski's work [**Z**]. In addition, all normal double points (i.e., normal surface singularities of multiplicity 2) have $z^2 = f(x,y)$ as the defining equations, and there are many papers on them ([**D**], [**Tm1**] and their references). Hence, it seems worth while to consider the pencil genus for singularities defined by $z^n = f(x,y)$. We have already given a formula of $p_e(X,o,h)$ in Section 2 (Theorem 2.11). However we must to compute a resolution $(\tilde{X},E)$ and a divisor $E(h\circ\pi)$ for obtaining the value of $p_e(X,o,h)$. Therefore, it is not so easy to get the value of $p_e(X,o,h)$ even if (X,o) is a hypersurface singularity. In this section, we give a formula (Theorem 5.4) of $p_e(X,o,\ell)$ for hypersurface singularities of type $z^n = f(x,y)$ and $\ell \in \mathfrak{n}\backslash\mathfrak{n}^2$, where $\mathfrak{n}$ is the maximal ideal $(x,y) \subset \boldsymbol{C}\{x,y\}$. The author proved the following ([**Tt2**, Theorem 4.1] and [**Tt4**, Corollary 3.8]).

THEOREM 5.1. *Let (X,o) be a hypersurface singularity of type $z^n = f(x,y)$. If $n|ord(f)$, then (X,o) is a Kulikov singularity of genus $\frac{(n-1)(ord(f)-2)}{2}$, where $ord(f)$ is the order of f at the origin.*

In the situation above, for general elements α and β of $\boldsymbol{C}$, $\alpha x+\beta y$ is a projection function for (X,o). Since $p_e(X,o,\alpha x+\beta y) = \frac{(n-1)(ord(f)-2)}{2}$ from Proposition 3.10 (iii), Theorem 5.4 is a generalization of this formula (Corollary 5.5). In the following, let us prepare two lemmas to prove Theorem 5.4.

LEMMA 5.2. *Let $(\tilde{X},E) \longrightarrow (X,o)$ be a resolution of a Gorenstein surface singularity and $K_{\tilde{X}}$ the canonical divisor of $\tilde{X}$. Let $\bigcup_{j=1}^r E_j$ $(\subset E)$ be a $\boldsymbol{P}^1$-chain started from an irreducible component E_o of E, and let $b_j = -E_j^2$ $(1 \leqq j \leqq r)$ and $k_j = \mathrm{Coeff}_{E_j} K_{\tilde{X}}$ $(0 \leqq j \leqq r)$. Assume that $b_1 \geqq 1$ and $b_j \geqq 2$ $(2 \leqq j \leqq r)$. Let $\frac{n}{q} = [[b_1,\dots,b_r]]$ and $\gcd(n,q) = 1$. If $\bar{q}$ is an integer with $q\bar{q}+1 \equiv 0(n)$ and $0 \geqq \bar{q} < n$, then $k_r = \frac{k_o-\bar{q}+1}{n}$.*

PROOF. Let us prove this by induction on r. If $r = 1,2$, then the assertion is easily checked; hence, we assume $r \geqq 3$. Let $\delta_0 = n, \delta_1 = q$ and let $\delta_2,\dots,\delta_{r+1}$ be positive integers that are inductively determined by relations: $\delta_i = b_{i+1}\delta_{i+1} - \delta_{i+2}$ $(i = 0,1,\dots,r-1)$, and let $\delta_r = 1$, $\delta_{r+1} = 0$. Hence $\frac{\delta_{i-1}}{\delta_i} = [[b_i,\dots,b_r]]$ for $i = 1,2,\dots,r$ and $\bigcup_{j=i}^r E_j$ is the exceptional set of a resolution (not necessarily minimal) for a cyclic quotient singularity $C_{\delta_{i-1},\delta_i}$. Further, let $\epsilon_{r+1},\dots,\epsilon_1$ be non-negative integers those are inductively determined by the following relations:

$$\epsilon_{r+1} = 1, \quad \epsilon_r = b_r - 1, \quad \epsilon_{i-1} = b_{i-1}\epsilon_i - \epsilon_{i+1} \ (i = r, r-1,\dots,2).$$

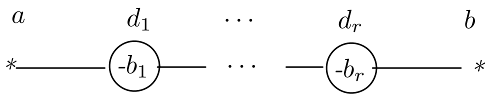
Then $\frac{\epsilon_1}{\epsilon_2} = [[b_1,\dots,b_{r-1},b_r-1]]$ and $\gcd(\epsilon_1,\epsilon_2) = 1$. Assume that $\epsilon_1 \geqq 2$, and let c be an integer with $0 < c < \epsilon_1$ and $\epsilon_2 c \equiv 1(\epsilon_1)$. Then $\frac{\epsilon_1}{c} = [[b_r-1,b_{r-1},\dots,b_1]]$ and $\gcd(c,\epsilon_1) = 1$. Then $\frac{\epsilon_1+c}{c} = \frac{\epsilon_1}{c}+1 = [[b_r,\dots,b_1]] = \frac{n}{q'}$, where q' is an integer defined by $qq' \equiv 1(n)$ and $0 < q' < n$. Hence $c = q'$ and $\epsilon_1+c = n$. Then $q\epsilon_1+1 = q(n-q')+1 \equiv 0(n)$ and so $\bar{q} = \epsilon_1$. Therefore, $\delta_1\epsilon_1+1 \equiv 0(\delta_o)$. Similarly, we obtain $\delta_{i+1}\epsilon_{i+1}+1 \equiv 0(\delta_i)$ for $i = 0,\dots,r-1$. From the assumption of length $r-1$ and $r-2$, we have $k_r = \frac{k_1-\epsilon_2+1}{\delta_1}$ and $k_r = \frac{k_2-\epsilon_3+1}{\delta_2}$. Since $n = b_1\delta_1-\delta_2$ and $\epsilon_1 = b_1\epsilon_2-\epsilon_3$,

$$
\begin{aligned}
b_1 - 2 = K_{\tilde{X}} E_1 &= k_o - b_1 k_1 + k_2 \\
&= k_o - b_1(k_r\delta_1 + \epsilon_2 - 1) + k_r\delta_2 + \epsilon_3 - 1 \\
&= k_o - (b_1\delta_1 - \delta_2)k_r - \epsilon_1 + b_1 - 1 = k_o - nk_r - \epsilon_1 + b_1 - 1;
\end{aligned}
$$

therefore, we have $k_r = \frac{k_o - \bar{q} + 1}{n}$.

Next, assume that $0 \leqq \epsilon_1 \leqq 1$. Then we can easily see that $n = \epsilon_1 + 1$, $[[b_1, \dots, b_r]] = [[1, 2, \dots, 2, \epsilon_1 + 2]]$ and $\bar{q} = \epsilon_1$. Hence $k_0 - k_1 + k_2 = -1$, $k_j - 2k_{j+1} + k_{j+2} = 0$ ($j = 1, \dots, r-2$) and $k_{r-1} - (\epsilon_1 + 2)k_r = \epsilon_1$. Taking their sum, we have $k_0 - (\epsilon_1 + 1)k_r = \epsilon_1 - 1$; hence, $k_r = \frac{k_0 - \epsilon_1 + 1}{\epsilon_1 + 1} = \frac{k_0 - \bar{q} + 1}{n}$. □

LEMMA 5.3. *Let $(\tilde{X}, E) \longrightarrow (X, o)$ be a resolution of a normal surface singularity and let $f \in \mathfrak{m}_{X,o}$. Let $E' = \bigcup_{j=1}^{r} E_{i_j}$ ($\subset E$) be a $\boldsymbol{P}^1$-chain such that the divisor $(f \circ \pi)_{\tilde{X}}|_{E'}$ is given as follows:*



, *where $E_{i_j}^2 = -b_j$ for $j = 1, \dots, r$.*

Then $d_r = \frac{a + bq'}{n}$, where $\frac{n}{q} = [[b_1, \dots, b_r]]$ and $qq' \equiv 1(n)$ and $0 < q' < n$.

PROOF. We have a linear equation on $d_1, \dots, d_r$ as follows:

$$
\begin{cases}
-b_1 d_1 + d_2 = -a \\
d_j - b_{j+1} d_{j+1} + d_{j+2} = 0 \qquad (j = 1, \dots, r-2) \\
d_{r-1} - b_r d_r = -b,
\end{cases}
$$

The determinant of the coefficient matrix of the linear equation is equal to $\pm n$. From Cramer's formula, it completes the proof. □

Now we prove the main result of this section in the following. Let h_1, h_2 be elements of the maximal ideal $\mathfrak{n}$ of $\boldsymbol{C}\{x, y\}$. Let $I_o(h_1, h_2)$ be the intersection multiplicity of $\{h_1 = 0\}$ and $\{h_2 = 0\}$ at $\{0\} \in \boldsymbol{C}^2$ ([**BK**, p. 47], [**Na**, p. 231]).

THEOREM 5.4. *Let (X, o) be a normal hypersurface singularity defined by $z^n = f(x, y)$. Let $\ell \in \mathfrak{n} \backslash \mathfrak{n}^2$ and*

$$
I_o := \begin{cases}
I_o(\ell, f_1) & \text{if } \ell \mid f \text{ and } f_1 = f/\ell, \\
I_o(\ell, f) & \text{if } \ell \nmid f.
\end{cases}
$$

Then

$$
p_e(X, o, \ell) = \frac{(n-1)(I_o - 1) + 1 - \gcd(n, I_o)}{2}.
$$

Proof. We prove the formula in the case of $\ell \mid f$. Let $f = \ell g_1 \dots g_\lambda$ be the irreducible decomposition and let $\bar{B}_j = \{g_j = 0\}$, $\bar{L}_o = \{\ell = 0\}$ and $s = \max_{1 \leqq j \leqq \lambda} \left\lceil \frac{I_o(\ell, g_j)}{ord(g_j)} \right\rceil$, where, for any real number $a \in \boldsymbol{R}$, $\lceil a \rceil$ is the least integer greater than or equal to a. Let $Y_o = \boldsymbol{C}^2 \xleftarrow{\sigma_o} Y_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{s-1}} Y_s$ be an iteration of blowing-ups such that the center of σ_o is the origin $P_o = \{0\} \in \boldsymbol{C}^2$ and the center of σ_k $(k \geqq 1)$ is $P_k := \bar{E}_k \cap \bar{L}_k$ $(k = 1, \dots, s-1)$, where $\bar{E}_k = \sigma_{k-1}^{-1}(P_{k-1})$ and $\bar{L}_k$ is the strict transform of $\bar{L}_o$ by $\sigma_o \circ \dots \circ \sigma_{k-1}$ $(k = 1, \dots, s)$. Let $\bar{B}_{j,k}$ be the strict transform of $\bar{B}_j$ by $\sigma_o \circ \dots \circ \sigma_{k-1}$. From the definition of s, $\bar{L}_s$ intersects $\bar{E}_s$ transversally and $\bar{L}_s$ does not intersect $\bar{B}_{j,s}$. If necessary, by taking suitable blowing-ups $Y_s \xleftarrow{\sigma_s} \cdots \xleftarrow{\sigma_{t-1}} Y_t$, an embedded resolution of the curve singularity $(\{f = 0\}, o) \subset (\boldsymbol{C}^2, o)$ is obtainable in which we assume that the center of σ_k $(k = s, \dots, t-1)$ is not P_s. Let $\bar{\sigma} = \sigma_0 \circ \dots \circ \sigma_{s-1}$ and $\sigma = \sigma_0 \circ \dots \circ \sigma_{t-1}$. Also, we put

$$m_{j,k} = \begin{cases} \operatorname{mult}_{P_k} \bar{B}_{j,k} & \text{if } P_k \in \bar{L}_k, \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, 1, \dots, s$. By M. Noether's theorem ([**BK**, p. 518]), we have $v_{\bar{E}_s}(g_j \circ \sigma) = \sum_{k=0}^{s-1} m_{j,k} = I_o(\ell, g_j)$. Hence $v_{\bar{E}_s}(f \circ \bar{\sigma}) = \sum_{j=1}^{\lambda} v_{\bar{E}_s}(g_j \circ \bar{\sigma}) + s = \sum_{j=1}^{\lambda} I_o(\ell, g_j) + s = I_o + s$ from $v_{\bar{E}_s}(\ell \circ \bar{\sigma}) = s$. Also $v_{\bar{L}_s}(f \circ \bar{\sigma}) = v_{\bar{L}_s}(\ell \circ \bar{\sigma}) = 1$. Let $\bar{E}_t$ (resp. $\bar{L}_t$) be the strict transform of $\bar{E}_s$ (resp. $\bar{L}_s$) by $\sigma_s \circ \dots \circ \sigma_{t-1}$. Since $\sigma_s \circ \dots \circ \sigma_{t-1}$ is an isomorphism on an open neighborhood of $\bar{L}_t$,

$$v_{\bar{E}_t}(f \circ \sigma) = I_o + s, \quad v_{\bar{E}_t}(\ell \circ \sigma) = s \text{ and } v_{\bar{L}_t}(f \circ \sigma) = v_{\bar{L}_t}(\ell \circ \sigma) = 1. \tag{5.1}$$

Now, consider the following diagram as in (4.1):

$$\begin{array}{ccccccc} & & \tilde{Y} \times \boldsymbol{C}^1 & & & & \\ & & \cup & & & & \\ (x,y,z) \in \boldsymbol{C}^3 \supset X & \xleftarrow{\ \phi_1\ } & X' & \xleftarrow{\phi_2} & X'' & \xleftarrow{\phi_3} & \tilde{X} \subset S, \\ \downarrow \quad \downarrow \psi & & \downarrow \psi' & & \swarrow \delta & & \downarrow \Phi \\ (x,y) \in \boldsymbol{C}^2 = Y_o & \xleftarrow{\ \sigma\ } & \tilde{Y} = Y_t & & & & \Delta \end{array} \tag{5.2}$$

where $\phi := \phi_1 \circ \phi_2 \circ \phi_3$ gives a good resolution of (X, o) such that $(f \circ \phi)_{\tilde{X}}$ is a simple normal crossing divisor. Also $\Phi : S \longrightarrow \Delta$ is a pencil of curves of genus $p_e(X, o, \ell)$ and satisfying $\Phi|_{\tilde{X}} f \circ \sigma \circ \delta$, which is constructed as in Theorem 2.4.

Let L_t (resp. E_t) $\subset \tilde{X}$ be the strict transform of $\bar{L}_t$ (resp. $\bar{E}_t$) by δ. Then L_t is an irreducible curve because $\ell \mid f$. Let $(U; u, v)$ be a local coordinate neighborhood of $\bar{E}_t \cap \bar{L}_t$ such that $\bar{E}_t = \{v = 0\}$ and $\bar{L}_t = \{u = 0\}$. By the definition of s, we have $(x, y) = \sigma(u, v) = (u^{s-1} v^s, uv)$ on U because $\sigma_s \circ \dots \circ \sigma_{t-1}$ is an isomorphism on U. Let put $\alpha = \gcd(n, I_o + s), n_1 = \frac{n}{\alpha}$ and $\epsilon = \frac{I_o + s}{\alpha}$. Since X' is represented by $z^n = uv^{I_o + s}$ on $\delta^{-1}(U)$ from (5.1), its normalization is a cyclic quotient singularity $C_{n_1, n_1 - \epsilon_1}$ from Lemma 4.2, where ϵ_1 is an integer with $\epsilon_1 \equiv \epsilon\,(n_1)$ and $0 < \epsilon_1 < n_1$. ϕ_3 gives the minimal resolution of $C_{n_1, n_1 - \epsilon_1}$; thereby, we have a $\boldsymbol{P}^1$-chain $\bigcup_{j=1}^{r} F_j$ between E_t and

L_t (Definition 2.2) of type $(b_1, \ldots, b_r)$ and so $\frac{n_1}{n_1 - \epsilon_1} = [[b_1, \ldots, b_r]]$.

Since (X, o) is a hypersurface singularity, the canonical divisor $K_{\tilde{X}}$ is given by the meromorphic 2-form $\omega = \frac{dx \wedge dy}{z^{n-1}}$ and $\mathrm{supp}(K_{\tilde{X}}) = E$ (cf. [**Re2**]). From $v_{\bar{E}_t}(f \circ \sigma) = I_o + s$, X' is represented by an equation $z^n = v^{I_o+s}$ around the general point of $\psi'^{-1}(\bar{E}_t)$ in $\tilde{Y} \times \boldsymbol{C}$. Then X' has a local parameterization $\eta : (u, \theta) \mapsto (u, v, z) = (u, \theta^{n_1}, \theta^{\epsilon})$. Therefore,

$$(\sigma \circ \delta)^*(\omega) = \frac{-u^{s-1} v^s du \wedge dv}{z^{n-1}} = \frac{-n_1 u^{s-1} du \wedge d\theta}{\theta^{n\epsilon - \epsilon - (s+1)n_1 + 1}}.$$

If we put $k_o = \mathrm{Coeff}_{E_t} K_{\tilde{X}}$ and $k_j = \mathrm{Coeff}_{F_j} K_{\tilde{X}}$ for $j = 1, \ldots, r$, then

$$\begin{aligned} k_o &= ord_{E_t}(\sigma \circ \delta)^*(\omega) = -n\epsilon + \epsilon + (s+1)n_1 - 1 \\ &= -\frac{(n-1)(I_o - 1) - 1 - s + \alpha}{\alpha}. \end{aligned} \tag{5.3}$$

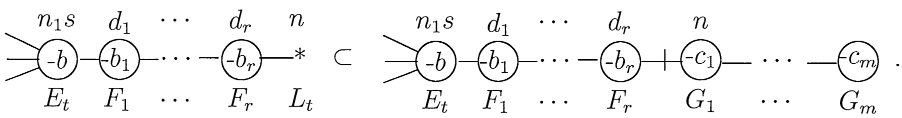
From Lemma 5.1,

$$k_r = \frac{k_o - \epsilon_2 + 1}{n_1}, \quad \text{where } \epsilon_1 \epsilon_2 \equiv 1(n_1) \text{ and } 0 < \epsilon_2 < n_1. \tag{5.4}$$

Let $d_j = v_{F_j}(\ell \circ \sigma \circ \delta)$ for $j = 1, \ldots, r$. From (5.1) and Lemma 4.3, $v_{E_t}(\ell \circ \sigma \circ \delta) = n_1 s$ and $v_{L_t}(\ell \circ \sigma \circ \delta) = n$. From $(\ell \circ \sigma \circ \delta)_{\tilde{X}} F_j = 0$ for $j = 1, \ldots, r$ and Lemma 5.3, we have

$$d_r = n + s - \alpha \epsilon_2. \tag{5.5}$$

Let $\bar{d}_r = \frac{d_r}{\gcd(n, d_r)}$, $n_2 = \frac{n}{\gcd(n, d_r)}$ and $\frac{\bar{d}_r}{n_2} = [[c_1, \ldots, c_m]]$, where $c_1 \geqq 1$ and $c_i \geqq 2$ for $i = 2, \ldots, m$. Using the method of Theorem 2.4, a pencil of curves $\Phi : S \longrightarrow \Delta$ is constructed by gluing $\tilde{X}$ and the standard resolution space $(W, G := \bigcup_{i=1}^m G_i)$ of a cyclic quotient singularity $C_{\bar{d}_r, n_2}$, where $G_i^2 = -c_i$ for any i and $L_t \subset G_1$ and $\Phi|_{\tilde{X}} = \ell \circ \sigma \circ \delta$. For divisors $(\ell \circ \sigma \circ \delta)_{\tilde{X}}|_F$ and $S_o|_{F \cup G} = \Phi^{-1}(o)|_{F \cup G}$, we have the following figure:



From Lemma 2.8,

$$K_S(S_o|_G) = d_r - n - \gcd(n, d_r). \tag{5.6}$$

Since $0 = S_o F_r = (S_o|_{E \cup F} + S_o|_G) F_r = (S_o|_{E \cup F}) F_r + n$, we have $(S_o|_{E \cup F}) F_r = -n$. Because the intersection number of $S_o|_{E \cup F}$ and any irreducible component of $E \cup F$ except for F_r is zero, we have

$$\begin{aligned} K_S(S_o|_{E\cup F}) &= K_{\tilde{X}}(S_o|_{E\cup F}) = (\mathrm{Coeff}_{F_r} K_{\tilde{X}})F_r(S_o|_{E\cup F}) \\ &= -k_r n = -\alpha k_o + \alpha\epsilon_2 - \alpha = (n-1)(I_o - 1) - 1 + n - d_r \end{aligned}$$

from (5.3), (5.4) and (5.5). Hence $I_o + d_r \equiv 0(n)$. Therefore,

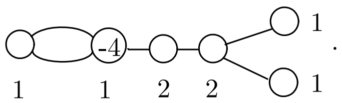
$$\begin{aligned} p_e(X, o, \ell) &= 1 + \frac{1}{2}K_S S_o = 1 + \frac{1}{2}\{K_S(S_o|_{E\cup F}) + K_S(S_o|_G)\} \\ &= 1 + \frac{1}{2}\{(n-1)(I_o - 1) - 1 - \gcd(n, d_r)\} \\ &= \frac{(n-1)(I_o - 1) + 1 - \gcd(n, I_o)}{2}. \end{aligned}$$

For the case of $\ell \nmid f$, we can prove the formula more easily than the case of $\ell \mid f$. We describe the outline of the proof. First we make an embedded resolution space of a curve singularity $\{f = 0\}$ and take an n-fold cyclic covering $\psi' : X' \longrightarrow \tilde{Y}$ as above. However, $\bar{L}_t$ is not contained in the ramification locus of ψ'. Then $L = (\psi')^{-1}(\bar{L}_t)$ has λ irreducible components $L_1, \ldots, L_\lambda$, where $\lambda = \gcd(n, I_o)$. Furthermore, X'' is non-singular near L and ϕ_2 is an isomorphism near L. By gluing $\tilde{X}$ and resolution spaces of λ cyclic quotient singularities, we can construct a pencil $\Phi : S \longrightarrow \Delta$. As above, we obtain the formula of $p_e(X, o, \ell)$. □

Corollary 5.5. *Let $(X, o) = \{z^n = f(x, y)\}$ be a normal surface singularity. For general elements $\alpha, \beta \in \boldsymbol{C}$,*

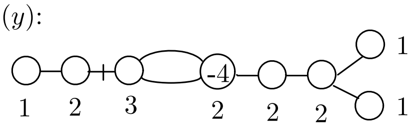
$$p_e(X, o, \alpha x + \beta y) = \frac{(n-1)(ord(f) - 1) + 1 - \gcd(n, ord(f))}{2}.$$

Example 5.6. Let $f(x, y) = (x^2 + y^3)y(y + x^2)(y + x^3)$ and let $(X, o) = \{z^2 = f(x, y)\} \subset \boldsymbol{C}^3$. Then the fundamental cycle on the minimal resolution is given as

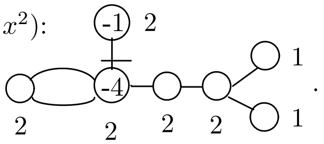


Consequently, (X, o) is an elliptic singularity. If we put $f_1 = f/y$ and $f_2 = f/(y + x^2)$, then $I_o(y, f_1) = 7$ and $I_o(y + x^2, f_2) = 6$. Therefore, $p_e(X, o, \alpha x + \beta y) = 2$ from Corollary 5.5, and $p_e(X, o, y) = 3$ and $p_e(X, o, y + x^2) = 2$ from Theorem 5.4. Two cycles determined by y and $y + x^2$ on the minimal resolution and their associated pencils are given as

(y):



$(y + x^2)$:



Kodaira's classification of elliptic pencils confirms that no elliptic pencil contains the exceptional set above. Then $p_e(X, o) \geqq 2$ and so (X, o) is an elliptic singularity of $p_e(X, o) = 2$.

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