J. Math. Soc. Japan
Vol. 58, No. 4 (2006) pp. 1197–1202

An extension of Yamamoto's theorem on the eigenvalues and singular values of a matrix

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(Received Sep. 12, 2005)
(Revised Dec. 6, 2005)

Abstract. We extend, in the context of real semisimple Lie group, a result of T. Yamamoto which asserts that $\lim_{m\to\infty}[s_i(X^m)]^{1/m} = |\lambda_i(X)|$, $i = 1, \dots, n$, where $s_1(X) \geq \cdots \geq s_n(X)$ are the singular values, and $\lambda_1(X), \dots, \lambda_n(X)$ are the eigenvalues of the $n \times n$ matrix X, in which $|\lambda_1(X)| \geq \cdots \geq |\lambda_n(X)|$.

1. Introduction.

Let $X \in \boldsymbol{C}_{n\times n}$. It is well known [**9**, p. 70] that

$$\lim_{m\to\infty} \|X^m\|^{1/m} = r(X), \tag{1.1}$$

where $r(X)$ denotes the spectral radius of X and $\|X\|$ denotes the spectral norm of X. Since $\|X\|^m \geq \|X^m\| \geq r(X^m) = r^m(X)$,

$$\|X\| \geq \|X^m\|^{1/m} \geq r(X), \quad m = 1, 2, \dots \tag{1.2}$$

Yamamoto [**10**, p. 174] showed that when m_{i+1} is divisible by m_i, $i = 1, 2, \dots$, the sequence $\{\|X^{m_i}\|^{1/m_i}\}_{i\in\boldsymbol{N}}$ is monotonically decreasing, that is,

$$\|X\| \geq \|X^{m_i}\|^{1/m_i} \geq \|X^{m_{i+1}}\|^{1/m_{i+1}} \geq r(X). \tag{1.3}$$

Suppose that the singular values $s_1(X), \dots, s_n(X)$ of X and the eigenvalues $\lambda_1(X), \dots, \lambda_n(X)$ of X are arranged in nonincreasing order

$$s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X), \quad |\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)|. \tag{1.4}$$

Since $\|X\| = s_1(X)$ and $r(X) = |\lambda_1(X)|$, the following result of Yamamoto [**10**] is a direct generalization of (1.1):

$$\lim_{m\to\infty} [s_i(X^m)]^{1/m} = |\lambda_i(X)|, \quad i = 1, \dots, n. \tag{1.5}$$

2000 *Mathematics Subject Classification.* Primary 15A45, 22E46.
Key Words and Phrases. Yamamoto's theorem, Cartan decomposition, complete multiplicative Jordan decomposition.

Loesener [**6**] rediscovered (1.5). We remark that (1.1) remains true for Hilbert space operators [**1**, p. 45]. Also see [**3**], [**4**], [**8**] for some generalizations of Yamamoto's theorem.

Equation (1.5) relates the two important sets of scalars of X in (1.4) in a very nice asymptotic way. It may be interpreted as a relation between the singular value decomposition and the complete multiplicative Jordan decomposition of a nonsingular matrix. Since $GL_n(\boldsymbol{C})$ is dense in $\boldsymbol{C}_{n\times n}$ and the eigenvalues and singular values of a matrix are continuous functions of its entries [**9**, p. 44], it is sufficient to consider $X \in GL_n(\boldsymbol{C})$ or $SL_n(\boldsymbol{C})$ when we study (1.5). Let $A_+ \subset GL_n(\boldsymbol{C})$ denote the set of all positive diagonal matrices with diagonal entries in nonincreasing order. Recall that the singular value decomposition of $X \in GL_n(\boldsymbol{C})$ asserts [**9**, p. 129] that there exist unitary matrices U, V such that

$$X = Ua_+(X)V, \tag{1.6}$$

where $a_+(X) = \operatorname{diag}(s_1(X), \dots, s_n(X)) \in A_+$. Though U and V in the decomposition (1.6) are not unique, $a_+(X) \in A_+$ is uniquely defined. The complete multiplicative Jordan decomposition [**2**, p. 430–431] of $X \in GL_n(\boldsymbol{C})$ asserts that $X = ehu$ where e is diagonalizable with eigenvalues of modulus 1, h is diagonalizable over $\boldsymbol{R}$ with positive eigenvalues and $u = \exp \ell$ where ℓ is nilpotent [**2**]. The eigenvalues of h are the moduli of the eigenvalues of X, counting multiplicities. The elements e, h, u commute with each others and are uniquely defined. Moreover h is conjugate to a unique element in A_+, namely, $b(X) = \operatorname{diag}(|\lambda_1(X)|, \dots, |\lambda_n(X)|) \in A_+$. Thus (1.5) may be rewritten as

$$\lim_{m\to\infty} a_+(X^m)^{1/m} = b(X), \quad X \in GL_n(\boldsymbol{C}). \tag{1.7}$$

Our main result is to establish (1.7) in the context of real semisimple Lie groups.

2. Extension of Yamamoto's theorem.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$ and let G be any connected Lie group having $\mathfrak{g}$ as its Lie algebra. Let $K \subset G$ be the subgroup with Lie algebra $\mathfrak{k}$. Then K is connected and closed and that $\mathrm{Ad}_G(K)$ is compact [**2**, pp. 252–253]. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and set $A_+ := \exp \mathfrak{a}_+$. The Cartan decomposition [**5**, p. 434], [**2**, p. 402] asserts that

$$G = KA_+K.$$

Though $k_1, k_2 \in K$ are not unique in $g = k_1 a k_2$ ($g \in G$, $k_1, k_2 \in K$, $a \in A_+$), the element $a = a_+(g) \in A_+$ is unique.

An element $h \in G$ is called *hyperbolic* if $h = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\operatorname{ad} X \in \operatorname{End}(\mathfrak{g})$ is diagonalizable over $\boldsymbol{R}$. An element $u \in G$ is called *unipotent* if $u = \exp(N)$ where $N \in \mathfrak{g}$ is nilpotent, that is, $\operatorname{ad} N \in \operatorname{End}(\mathfrak{g})$ is nilpotent. An element $e \in G$ is *elliptic* if $\mathrm{Ad}(e) \in \operatorname{Aut}(\mathfrak{g})$ is diagonalizable over $\boldsymbol{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition [**5**, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

REMARK 2.1. By [**5**, Proposition 3.4] and its proof, if π is a representation of G, then $\pi(g) = \pi(e)\pi(h)\pi(u) \in SL(V_\pi)$ is the complete multiplicative Jordan decomposition of $\pi(g)$ in $SL(V_\pi)$, where V_π is the representation space.

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a unique element $b(h) \in A_+$ [**5**, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

Since $\exp : \mathfrak{a} \to A$ is bijective, $\log b(g)$ is well-defined. We denote

$$A(g) := \exp\big(\operatorname{conv}(W \log b(g))\big), \tag{2.1}$$

where $\operatorname{conv} Wx$ denotes the convex hull of the orbit of $x \in \mathfrak{a}$ under the action of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. Notice that $A(g) = A(b(g))$ is compact in G since the Weyl group W is finite.

Given any $g \in G$, we consider the two sequences $\{(a_+(g^m))^{1/m}\}_{m\in\boldsymbol{N}}$ and $\{(b(g^m))^{1/m}\}_{m\in\boldsymbol{N}}$. The latter is simply a constant sequence since if $g = ehu$, then $h(g^m) = h(g)^m$ (because e, h, u commute) so that $b(g^m) = b(g)^m$ and thus $(b(g^m))^{1/m} = b(g)$ for all $m \in \boldsymbol{N}$. The following is an extension of Yamamoto's theorem (1.5).

THEOREM 2.2. *Given $g \in G$, let $b(g) \in A_+$ be the unique element in A_+ conjugate to the hyperbolic part $h(g)$ of g. Then*

$$\lim_{m\to\infty} [a_+(g^m)]^{1/m} = b(g).$$

When m_{i+1} is divisible by m_i, $i = 1, 2, \ldots,$ the sequence of compact sets $\{A([a_+(g^{m_i})]^{1/m_i})\}_{i\in\boldsymbol{N}}$ is monotonically decreasing and converges to $A(b(g))$ with respect to set inclusion.

PROOF. Denote by $I(G)$ the set of irreducible representations of G, V_π the representation space of $\pi \in I(G)$, $r(X)$ the spectral radius of the endomorphism X. For each $\pi \in I(G)$, there is an inner product structure [**5**, p. 435] such that

(1) $\pi(k)$ is unitary for all $k \in K$,
(2) $\pi(a)$ is positive definite for all $a \in A_+$.

We will assume that V_π is endowed with this inner product. Thus for any $g \in G$, if $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$, then

$$\|\pi(g)\| = \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1)\pi(a_+(g))\pi(k_2)\|$$
$$= \|\pi(a_+(g))\| = r(\pi(a_+(g))), \tag{2.2}$$

since the spectral norm $\|\cdot\|$ is invariant under unitary equivalence, and $\|X\| = r(X)$ for every positive definite matrix X.

Now the sequence $\{\|\pi(g^m)\|^{1/m}\}_{m\in\boldsymbol{N}} = \{\|\pi(g)^m\|^{1/m}\}_{m\in\boldsymbol{N}}$ converges to $r(\pi(g))$ by (1.1). According to (2.2),

$$\|\pi(g^m)\|^{1/m} = r(\pi[a_+(g^m)])^{1/m} = r([\pi(a_+(g^m))]^{1/m}) = r(\pi[a_+(g^m)^{1/m}]). \tag{2.3}$$

So $\{r(\pi[a_+(g^m)^{1/m}])\}_{m\in\boldsymbol{N}}$ converges to $r(\pi(g))$, and by (1.2) and (2.2)

$$r(\pi[a_+(g)]) \geq r(\pi[a_+(g^m)^{1/m}]) \geq r(\pi(g)), \quad m = 1, 2, \ldots. \tag{2.4}$$

A result of Kostant [**5**, Theorem 3.1] asserts that if $f, g \in G$, then $A(f) \supset A(g)$ if and only if $r(\pi(f)) \geq r(\pi(g))$ for all $\pi \in I(G)$, where $A(g)$ is defined in (2.1). Thus by (2.4),

$$A(a_+(g)) \supset A(a_+(g^m)^{1/m}) \supset A(g), \quad m = 1, 2, \ldots. \tag{2.5}$$

Thus the sequence $\{a_+(g^m)^{1/m}\}_{m\in\boldsymbol{N}}$ lies in the compact set $A(a_+(g))$ by (2.5). Let $\ell \in A(a_+(g)) \cap A_+$ be any limit point of the sequence $\{a_+(g^m)^{1/m}\}_{m\in\boldsymbol{N}} \subset A(a_+(g)) \cap A_+$, that is,

$$\lim_{i\to\infty} a_+(g^{p_i})^{1/p_i} = \ell,$$

for some natural number sequence $p_1 < p_2 < \cdots$. Since r and π are continuous,

$$r(\pi(a_+(g^{p_i})^{1/p_i})) \to r(\pi(\ell)).$$

So $r(\pi(\ell)) = r(\pi(g))$ for all $\pi \in I(G)$, which implies that $A(\ell) = A(g) = A(b(g))$ by the result of Kostant [**5**, Theorem 3.1] again. Both ℓ and $b(g)$ are in A_+. Thus $\ell = b(g)$ and

$$\lim_{m\to\infty} a_+(g^m)^{1/m} = b(g).$$

If m_{i+1} is divisible by m_i, $i = 1, 2, \ldots$, by (1.3) and the argument above,

$$A(a_+(g)) \supset A(a_+(g^{m_i})^{1/m_i}) \supset A(a_+(g^{m_{i+1}})^{1/m_{i+1}}) \supset A(g), \quad m = 1, 2, \ldots.$$

So the sequence of compact sets $\{A((a_+(g^{m_i}))^{1/m_i})\}_{i\in\boldsymbol{N}}$ is monotonically decreasing and converges to $A(b(g))$ with respect to set inclusion. □

REMARK 2.3. When $G = SL_n(\boldsymbol{C})$, Theorem 2.2 is reduced to Yamamoto's theorem (1.5) whose proof in [**10**] uses compound matrices which are corresponding to the fundamental representations on the exterior spaces. Also see [**7**] for an elementary proof of (1.5).

EXAMPLE 2.4. The following example exhibits that a subgroup $G \subset G'$ may not have the same $a_+(g)$ component in the KA_+K decomposition when $g \in G$ is viewed as an element of G'. But the complete multiplicative Jordan decomposition remains the same (See Remark 2.1). Let $G = SO_{2n}(\boldsymbol{C}) := \{g \in SL_{2n}(\boldsymbol{C}) : g^T g = 1\}$ be a connected group [**2**, p. 449] whose Lie algebra is

$$\mathfrak{g} := \mathfrak{so}_{2n}(\boldsymbol{C}) = \big\{X \in \boldsymbol{C}_{2n\times 2n} : X^T = -X\big\}.$$

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{k} = \{X \in \boldsymbol{R}_{2n\times 2n} : X^T = -X\}, \qquad \mathfrak{p} = i\mathfrak{k},$$

that is, the corresponding Cartan involution is $\theta(Y) = -Y^*$, $Y \in \mathfrak{g}$. So $K = SO(2n)$. Pick

$$\mathfrak{a} = \left\{\begin{pmatrix} 0 & it_1 \\ -it_1 & 0\end{pmatrix} \oplus \begin{pmatrix} 0 & it_2 \\ -it_2 & 0\end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & it_n \\ -it_n & 0\end{pmatrix} : t_1, \ldots, t_n \in \boldsymbol{R}\right\},$$

and

$$\mathfrak{a}_+ = \left\{\begin{pmatrix} 0 & it_1 \\ -it_1 & 0\end{pmatrix} \oplus \begin{pmatrix} 0 & it_2 \\ -it_2 & 0\end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & it_n \\ -it_n & 0\end{pmatrix} : t_1 \geq \cdots \geq t_{n-1} \geq |t_n|\right\}.$$

Thus

$$A_+ = \left\{\begin{pmatrix} \cosh t_1 & i\sinh t_1 \\ -i\sinh t_1 & \cosh t_1\end{pmatrix} \oplus \begin{pmatrix} \cosh t_2 & i\sinh t_2 \\ -i\sinh t_2 & \cosh t_2\end{pmatrix} \oplus \cdots \oplus \right.$$
$$\left. \begin{pmatrix} \cosh t_n & i\sinh t_n \\ -i\sinh t_n & \cosh t_n\end{pmatrix} : t_1 \geq \cdots \geq t_{n-1} \geq |t_n|\right\}.$$

According to the Cartan decomposition, each $g \in SO_{2n}(\boldsymbol{C})$ may be written as $g = k_1 a k_2$ where $a \in A_+$ and $k_1, k_2 \in SO(2n)$ (Notice that it is different from the singular value decomposition of g in $SL_{2n}(\boldsymbol{C})$). By Remark 2.1 we may view $g \in SO_{2n}(\boldsymbol{C}) \subset SL_{2n}(\boldsymbol{C})$ as an element in $SL_{2n}(\boldsymbol{C})$ while computing its complete multiplicative Jordan decomposition $g = ehu$. Of course h is conjugate to $a_+(h) \in A_+$ via some element in $SO_{2n}(\boldsymbol{C})$. Notice that if $t_2 \neq 0$, then

$$h_1 = \begin{pmatrix} \cosh t_1 & i\sinh t_1 \\ -i\sinh t_1 & \cosh t_1\end{pmatrix} \oplus \begin{pmatrix} \cosh t_2 & i\sinh t_2 \\ -i\sinh t_2 & \cosh t_2\end{pmatrix}$$

and

$$h_2 = \begin{pmatrix} \cosh t_1 & i \sinh t_1 \\ -i \sinh t_1 & \cosh t_1 \end{pmatrix} \oplus \begin{pmatrix} \cosh t_2 & -i \sinh t_2 \\ i \sinh t_2 & \cosh t_2 \end{pmatrix}$$

are not in the same coset in $SO(4)\backslash SO_4(\boldsymbol{C})/SO(4)$. If $g_1 := yh_1y^{-1}$ and $g_2 := yh_2y^{-1}$, where $y \in SO_{2n}(\boldsymbol{C})$, then

$$\lim_{m\to\infty} \big(a_+(g_1^m)\big)^{1/m} = h_1, \quad \lim_{m\to\infty} \big(a_+(g_2^m)\big)^{1/m} = h_2.$$

But in $SL_4(\boldsymbol{C})$, if we pick $A_+ \subset SL_4(\boldsymbol{C})$ to be the group of positive diagonal matrices and $K = SU(4)$, then

$$\lim_{m\to\infty} \big(a_+(g_1^m)\big)^{1/m} = \lim_{m\to\infty} \big(a_+(g_2^m)\big)^{1/m} = \operatorname{diag}(e^{t_1}, e^{t_2}, e^{-t_2}, e^{-t_1}),$$

if $t_1 \geq t_2 \geq 0$, since g_1, g_2 have the same set of singular values.

ACKNOWLEDGEMENT. The authors are thankful to the referee for pointing out a misquote from [**10**] in the first version, that is, the sequence $\{\|X^m\|^{1/m}\}_{m\in\boldsymbol{N}}$ is not monotonically decreasing in general: $X = \left(\begin{smallmatrix} 0 & 2 \\ -1/2 & 0 \end{smallmatrix}\right)$.

References

[1] P. R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, New York, 1974.

[2] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[3] C. R. Johnson and P. Nylen, Yamamoto's theorem for generalized singular values, Linear Algebra Appl., **128** (1990), 147–158.

[4] C. R. Johnson and P. Nylen, Erratum: Yamamoto's theorem for generalized singular values [Linear Algebra Appl., **128** (1990), 147–158]. Linear Algebra Appl., **180** (1993), 4.

[5] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. (4), **6** (1973), 413–460.

[6] C. R. Loesener, Sur la recherche des valeurs propres de matrices mal conditionées, Thèse, Universitdés sciences et techniques du Languedoc, Montpellier, 1976.

[7] R. Mathias, Two theorems on singular values and eigenvalues, Amer. Math. Monthly, **97** (1990), 47–50.

[8] P. Nylen and L. Rodman, Approximation numbers and Yamamoto's theorem in Banach algebras, Integral Equations Operator Theory, **13** (1990), 728–749.

[9] D. Serre, Matrices: Theory and Applications, Springer, New York, 2002.

[10] T. Yamamoto, On the extreme values of the roots of matrices, J. Math. Soc. Japan, **19** (1967), 173–178.

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