J. Math. Soc. Japan
Vol. 58, No. 4 (2006) pp. 1031–1036

Elementary divisors of Cartan matrices for symmetric groups

By Katsuhiro UNO and Hiro-Fumi YAMADA

(Received Oct. 28, 2005)

Abstract. In this paper, we give an easy description of the elementary divisors of the Cartan matrices for symmetric groups in terms of the lengths of p-regular partitions and their Glaisher correspondents. Moreover, for $p = 2$, it is done block wise. There we use certain kinds of cores and weights, which are similar to but different from the usual ones.

1. Introduction.

The purpose of this note is to give a simple expression of the elementary divisors of the Cartan matrices for the symmetric groups.

Let G be a finite group and k be an algebraically closed field of characteristic $p > 0$. The group algebra kG, which affords the left regular representation of G, is a direct sum of indecomposable representations. There is a natural one-to-one correspondence between the equivalence classes of indecomposable summands of kG and those of irreducible representations. Let $c_{\lambda\mu}$ be the multiplicity of the irreducible representation F_μ occurring as a composition factor of the indecomposable summand U_λ of kG. The Cartan matrix is, by definition $C = (c_{\lambda\mu})$, and is an $\ell \times \ell$ square matrix, where ℓ is the number of inequivalent irreducible representations. The following is known for the elementary divisors of C. Let $\{x_1, \ldots, x_\ell\}$ be a set of representatives of p-regular conjugacy classes of G. Then the elementary divisors of C are given by $\{|Z_G(x_i)|_p ; i = 1, \ldots, \ell\}$, where $Z_G(x)$ denotes the centralizer of x in G [**NT**]. In the case of the symmetric groups, explicit computations are made in [**O1**] (see also [**BO2**]), by using generating functions.

Here we give in this note a simple expression of the elementary divisors for the symmetric groups by utilizing the one-to-one correspondence between the p-regular partitions and p-class regular partitions. By our result, one can obtain them by looking at p-regular partitions only. The case $p = 2$ is of special interest. We can state the block version of our expression. As is well-known, the elementary divisors of the Cartan matrix depend only on the weight w of the block. For the case $p = 2$ we determine the elementary divisors of the given weight by looking at a certain abacus. It is interesting that the same abacus plays a role in the Q-function realization of the basic representation of the affine Lie algebra $A_1^{(1)}$ [**NY**].

2. The Glaisher map and elementary divisors.

In this section p always denotes a fixed prime number. A partition λ is said to be p-regular if it does not contain p equal parts. We denote the set of all p-regular partitions

2000 *Mathematics Subject Classification.* Primary 20C20; Secondly 20C30, 05E10.
Key Words and Phrases. Cartan matrix, symmetric groups.

of n by $P^{r(p)}(n)$. A partition λ is said to be p-class regular if p does not divide any part of λ. We denote the set of all p-class regular partitions of n by $P^{c(p)}(n)$. There is a bijection between $P^{r(p)}(n)$ and $P^{c(p)}(n)$, which is defined as follows. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a p-regular partition. Write each part as $\lambda_i = p^{a_i} q_i$ with $(p, q_i) = 1$. Let $\mu(i)$ be the rectangular partition of λ_i given by $\mu(i) = (q_i, \dots, q_i)$ with length p^{a_i}. Suppose that $q_{j_1} \geq \cdots \geq q_{j_\ell}$. Then let $\tilde{\lambda}$ be the vertical concatenation $(\mu(j_1), \dots, \mu(j_\ell))$, which is p-class regular. For example, if $p = 2$ and $\lambda = (5, 4, 3)$, then $\tilde{\lambda} = (5, 3, 1, 1, 1, 1)$. Then, the map $\gamma : P^{r(p)}(n) \to P^{c(p)}(n)$ sending any λ in $P^{r(p)}(n)$ into $\tilde{\lambda}$ is well defined, and it is easily seen that γ gives a bijection. This γ is sometimes called the Glaisher map. Note that $\ell(\tilde{\lambda}) \geq \ell(\lambda)$. The elementary divisors of the p-Cartan matrix can be described as follows.

Theorem 1. *The elementary divisors of the p-Cartan matrix for the symmetric group S_n on n letters are given by*

$$\left\{ p^{\frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1}}; \lambda \in P^{r(p)}(n) \right\}.$$

Proof. Take $\tilde{\lambda} \in P^{c(p)}(n)$ and fix a representative $x_{\tilde{\lambda}}$ of the corresponding p-regular class in S_n. We will prove that

$$\left| Z_{S_n}(x_{\tilde{\lambda}}) \right|_p = p^{\frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1}}.$$

Putting m_k to be the multiplicity of the natural number k as the parts of $\tilde{\lambda}$, we write $\tilde{\lambda}$ exponentially as $(1^{m_1}, \dots, k^{m_k}, \dots)$. Let $m_k = \sum_{i \geq 0} b_i^{(k)} p^i$ be the p-adic expansion of m_k. $(0 \leq b_i^{(k)} < p)$. Since

$$\left| Z_{S_n}(x_{\tilde{\lambda}}) \right| = \prod_k k^{m_k} (m_k!),$$

we have

$$\left| Z_{S_n}(x_{\tilde{\lambda}}) \right|_p = \prod_k (m_k!)_p = p^{\Sigma_k d_p(m_k)},$$

where

$$d_p(m) = \sum_{i \geq 1} \left[\frac{m}{p^i} \right].$$

A simple computation shows that

$$d_p(m_k) = \frac{1}{p-1} \left(m_k - \sum_{i \geq 0} b_i^{(k)} \right)$$

and we have

$$\sum_k d_p(m_k) = \frac{1}{p-1}\left(\sum_k m_k - \sum_{k,i} b_i^{(k)}\right) = \frac{1}{p-1}\big(\ell(\tilde{\lambda}) - \ell(\lambda)\big).$$

Thus, the assertion holds. □

The invariant $(\ell(\tilde{\lambda}) - \ell(\lambda))/(p-1)$ can also be interpreted as follows. As in the paragraph preceding Theorem 1, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ we write $\lambda_i = p^{a_i} q_i$ with $(p, q_i) = 1$. We then have

$$\frac{1}{p-1}\big(\ell(\tilde{\lambda}) - \ell(\lambda)\big) = \sum_i \frac{p^{a_i} - 1}{p-1} = \sum_{i,\ a_i \neq 0} (1 + p + \cdots + p^{a_i - 1}).$$

Let $p\lambda$ denote the partition $(p\lambda_1, p\lambda_2, \ldots, p\lambda_\ell)$. Then, it is easy to see that

$$\frac{1}{p-1}\big(\ell(\tilde{p\lambda}) - \ell(p\lambda)\big) = \sum_i \frac{p^{a_i+1} - 1}{p-1} = \sum_i (1 + p + \cdots + p^{a_i}) = e(\lambda).$$

Here, the invariant $e(\lambda)$ is the one first introduced incorrectly on p. 54 in [**O1**] and then correctly on p. 299 in [**BO2**]. In these articles, the multiplicity of p^i as an elementary divisor of the Cartan matrix is given in terms of the number of p-regular partition μ of k with $e(\mu) = i$ and the invariant $m(w-k)$. (Proposition 3.19 of [**O1**]) Here w is the weight of the block. However, we know only the generating function of m and to obtain the multiplicity it is necessary to know the numbers of partitions μ of all k smaller than or equal to w. Thus it is quite interesting that from Theorem 1, in order to compute the elementary divisor corresponding to $\tilde{\lambda}$, it suffices to look at the partitions λ and $\tilde{\lambda}$ only. On the other hand, it should be noticed that the arguments in [**O1**] and [**BO2**] are given block wise, while ours is stated for the entire group. The block wise statement in our setting can be done only for $p = 2$ at present and is given in the next section.

3. A block version for $p = 2$.

In this section we restrict our attention to the special case $p = 2$. Then the Glaisher map turns out to be a bijection between the set of all strict partitions and that of all odd partitions. Theorem 1 tells us that an elementary divisor has the form $2^{\ell(\tilde{\lambda}) - \ell(\lambda)}$ for a strict partition λ.

We introduce the following "H-abacus" ([**NY**]). This notion is also given on p. 278–279 of [**BO1**], where it is called the $\bar{4}$-abacus. They represent strict partitions. For example, the H-abacus of $\lambda = (9, 5, 3, 2)$ is shown below.

	1	③
②		
4	⑤	7
6		
8	⑨	11

For a strict partition we put a set of beads on the assigned positions. Any two beads do not pile up. From the H-abacus of the given strict partition λ, we obtain the "H-core" λ^H by moving and removing the beads as follows,

(1) Move a bead one position up along the leftmost runner.
(2) Remove a bead at the position 2.
(3) Move a bead one position up along the runner of 1 or of 3.
(4) Remove the two beads at the positions 1 and 3 simultaneously.

The H-cores are the same as the $\overline{4}$-cores in [**BO1**] and characterized by the "stalemates", which constitute the set

$$HC = \{\phi, (4m+1, 4m-3, \ldots, 5, 1), (4m+3, 4m-1, \ldots, 7, 3) \quad (m \geq 0)\}.$$

For example, the H-core of the above $\lambda = (9, 5, 3, 2)$ is $\lambda^H = (1)$.

Here, notice that the number of nodes in every H-core is a triangular number, $m(m+1)/2$, and conversely, for any triangular number r, there is a unique H-core with r nodes. Recall that a similar thing is true for 2-cores since each 2-core has the form $\Delta_m = (m, m-1, \ldots, 2, 1)$. Hence, there is a unique bijection between HC and the set of 2-cores that preserves the number of nodes. In fact, the bijection can be obtained by applying "unfolding", which is defined as taking the hook lengths of the main diagonal in the Young diagram. Namely, we have

$$HC = \big\{\Delta_m^u; m \geq 0\big\},$$

where λ^u stands for the "unfolding" of λ. For example, $\Delta_4^u = (7, 3)$. For detail, see §4 of [**NY**]. See also p. 282 of [**BO1**].

In the computation of the elementary divisors block wise, the H-core λ^H of λ plays an essential role, instead of the 2-core λ^c. Let λ be a strict partition of n. To obtain the H-core λ^H, we perform the moving (or removing) of the beads on the H-abacus. Let each move (1) and (2) have H-weight 1, while (3) and (4) have H-weight 2. Thus λ has its own H-weight, which is, of course, equal to $w = (|\lambda| - |\lambda^H|)/2$. For example $\lambda = (9, 5, 3, 2)$ has H-weight $w = 9$.

Let $SP(n)_w$ (resp. $SP(n)^w$) be the set of those strict partitions of n with 2-weight w (resp. H-weight w). For example, $SP(7)_2 = \{(6, 1), (4, 3)\}$, while $SP(7)^2 = \{(7), (4, 3)\}$.

The number of strict partitions of n with a given H-core can be computed as follows. Let $p(n)$ be the number of partitions of n, and let $q(n)$ be the number of strict partitions of n. Recall that $q(n)$ is also the number of partitions of n with only odd parts. For a partition λ, let $\lambda^{(2')}$ and $\lambda^{(2)}$ denote the partition obtained from λ by picking up all the odd and even parts, respectively. Then, we have $\lambda^{(2)} = 2\mu = (2\mu_1, 2\mu_2, \ldots, 2\mu_\ell)$ for some partition μ. The partitions $\lambda^{(2')}$ and μ are determined uniquely by λ, and conversely, λ is determined by an odd partition ν and a partition μ with $\lambda^{(2')} = \nu$ and $\lambda^{(2)} = 2\mu$. This gives us

$$p(n) = \sum_{k=0}^{n} q(k) p\bigg(\frac{n-k}{2}\bigg),$$

where $p(x) = 0$ if x is not an integer.

Note that the H-core of λ depends only on $\lambda^{(2')}$. Moreover, by arguments similar to those in I.4 of [**O2**], partitions λ in $SP(n)^w$ is determined uniquely by a pair of a strict partition μ with $2\mu = \lambda^{(2)}$ and a partition ν obtained as the Frobenius symbol given by the $\overline{4}$-quotients, denoted by λ_1 and λ_3 in the notation in [**O2**], such that $|\mu| + 2|\nu| = w$. Though some arguments in I.4 in [**O2**] are given for odd numbers p rather than 4, the above conclusion can be proved equally well for $p = 4$. This shows that $|SP(n)^w| = \sum_{k=0}^{w} q(k)p(\frac{w-k}{2})$. Hence, we have $|SP(n)^w| = p(w)$.

Recall also that for a given triangular number r, there are unique 2-core Δ and H-core Δ^u with r nodes. Thus the set $SP(n)_w$ and $SP(n)^w$ consist of strict partitions λ of n with $\lambda^c = \Delta$ and $\lambda^H = \Delta^u$, respectively, with $n - 2w = r$. It is known that the number of strict partitions of n with a given 2-core depends only on the weight w (Theorem 1 of [**O**]) and in fact we have $|SP(n)_w| = p(w)$. Thus, the following holds.

PROPOSITION 2. *We have* $|SP(n)_w| = |SP(n)^w| = p(w)$.

The above assertion is contained in §3 of [**BO1**]. However, we give a sketch of the proof above, since we need to refine its argument in order to prove our main theorem. In fact, the multiplicity of 2^i as an elementary divisor of the Cartan matrix can be computed as follows.

First note that $\ell(\tilde{\lambda}) - \ell(\lambda) = \ell(\widetilde{\lambda^{(2)}}) - \ell(\lambda^{(2)})$. In particular, $\ell(\tilde{\lambda}) - \ell(\lambda)$ depends only on $\lambda^{(2)}$. In other words, using the notation preceding the proposition, $\ell(\tilde{\lambda}) - \ell(\lambda)$ is determined only by μ and not by ν. Hence the number of strict partitions λ of n with H-weight w and $\ell(\tilde{\lambda}) - \ell(\lambda) = i$ is

$$\sum_{k=0}^{w} p_0^i(k) p\left(\frac{w-k}{2}\right),$$

where $p_0^i(k)$ is the number of $\mu \in P^{r(2)}(k)$ with $e(\mu) = \ell(\tilde{2\mu}) - \ell(2\mu) = i$. Since the function $m(n)$ defined on p. 53 in [**O1**] satisfies

$$m(n) = p\left(\frac{n}{2}\right),$$

(see (3.12) in [**O1**]), the formula $\sum_{k=0}^{w} p_0^i(k) m(w-k)$ in Proposition 3.19 in [**O1**] implies the following.

THEOREM 3. *Let B be the unique 2-block of the symmetric group on n letters corresponding to the 2-core Δ_m. Then, the elementary divisors of the Cartan matrix for B are given by*

$$\left\{2^{\ell(\tilde{\lambda})-\ell(\lambda)} \mid \lambda \in P^{r(2)}(n),\ \lambda^H = \Delta_m^u\right\}.$$

In particular, the multiplicity of 2^i as an elementary divisor of the Cartan matrix depends only on the H-weight, which is equal to the 2-weight of B.

The above theorem shows that H-cores and H-weights give a combinatorial meaning for the formula in Proposition 3.19 in [**O1**] for $p = 2$. For odd primes, we do not know such an explanation.

Example. For $n = 7$, there are five strict partitions described below.

λ	$\tilde{\lambda}$	$2^{\ell(\tilde{\lambda})-\ell(\lambda)}$	λ^c	λ^H
(7)	(7)	2^0	$\Delta_1 = (1)$	$\Delta_2^u = (3)$
$(6,1)$	$(3,3,1)$	2^1	$\Delta_2 = (2,1)$	$\Delta_1^u = (1)$
$(5,2)$	$(5,1,1)$	2^1	$\Delta_1 = (1)$	$\Delta_1^u = (1)$
$(4,3)$	$(3,1,1,1,1)$	2^3	$\Delta_2 = (2,1)$	$\Delta_2^u = (3)$
$(4,2,1)$	$(1,1,1,1,1,1,1)$	2^4	$\Delta_1 = (1)$	$\Delta_1^u = (1)$

Thus, the elementary divisors of the Cartan matrix of the principal 2-block corresponding to Δ_1 is $\{2^1, 2^1, 2^4\}$, and for the 2-block corresponding to Δ_2 we have $\{2^0, 2^3\}$. However, irreducible 2-modular representations of the principal 2-block are labeled by $\{(7), (5,2), (4,2,1)\}$, and those of the non-principal 2-block by $\{(6,1), (4,3)\}$. See [**JK**].

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Katsuhiro Uno
Division of Mathematical Sciences
Osaka Kyoiku University
Kashiwara, Osaka, 582-8582, Japan
E-mail: uno@cc.osaka-kyoiku.ac.jp

Hiro-Fumi Yamada
Department of Mathematics
Okayama University
Okayama, 700-8530, Japan
E-mail: yamada@math.okayama-u.ac.jp