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The asymptotic cones of manifolds of roughly non-negative radial curvature

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Abstract. We prove that the asymptotic cone of every complete, connected, non-compact Riemannian manifold of roughly non-negative radial curvature exists, and it is isometric to the Euclidean cone over their Tits ideal boundaries.

0. Introduction.

Throughout this paper, let M be a complete, connected and non-compact Riemannian manifold with a base point $o \in M$. We say that *M has the asymptotic cone* if the pointed Gromov-Hausdorff limit of $((1/t)M, o)$ as $t \to \infty$ exists, and it is isometric to a Euclidean cone. The existence of asymptotic cones has been shown for manifolds with restricted sectional curvature, and then we see that the cone is generated by the Tits ideal boundary. In the case where M is a Hadamard manifold, Gromov [**5**] has shown that M has the asymptotic cone if its Tits ideal boundary is compact. If M is non-negatively curved, then M has the asymptotic cone, and its Tits ideal boundary is an Alexandrov space with curvature bounded below by 1. On the other hand, Gromov [**11**] and Abresch [**1**] have studied manifolds of asymptotically non-negative curvature, and their topologies, and Kasue [**17**] has introduced the ideal boundary of such a manifold and has given its compactification. However, Drees [**10**] pointed out a gap in the argument of [**17**] (cf. 1 and the end of 4 in [**10**]). Without smoothing the gap, Kasue's compactification is not yet completed. The main purpose of the present paper is to show the existence of asymptotic cones for a class of manifolds with restricted radial curvature. Our class includes the class of all manifolds of asymptotically non-negative curvature.

Some notions are needed for the statements of our results. A *model surface* $(\widetilde{M}, \tilde{o})$ *with radial curvature function* $K : [0,\infty) \to \boldsymbol{R}$ *at* $\tilde{o}$ is a surface of revolution with the metric

$$ds^2 = dt^2 + f(t)^2 d\theta^2 \tag{0.1}$$

in the geodesic polar coordinates $(t, \theta) \in (0,\infty) \times S^1$ centered at $\tilde{o} \in \widetilde{M}$. Here $f \colon [0,\infty) \to [0,\infty)$ denotes the unique solution of the following:

$$f''(t) + K(t) f(t) = 0, \quad f(0) = 0, \ \ f'(0) = 1, \ \ f > 0 \text{ on } (0,\infty). \tag{0.2}$$

Obviously, the Gaussian curvature $G(\tilde{p})$ at $\tilde{p} \in \widetilde{M}$ is equal to $K\big(d_{\widetilde{M}}(\tilde{o}, \tilde{p})\big)$. We say that the *radial sectional curvature at o of M is bounded below by K*: $[0,\infty) \to \boldsymbol{R}$ if along every minimizing geodesic $\gamma \colon [0,a) \to M$, $a \in [0,\infty)$, emanating from o, the sectional curvature K_M on M satisfies

$$K_M\big(\dot{\gamma}(t), v\big) \geq K(t)$$

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for any $t \in [0,a)$ and for any vector $v \in T_{\gamma(t)}M$ perpendicular to the tangent vector $\dot{\gamma}(t)$. We then assign to M a model surface $(\widetilde{M},\tilde{o})$ whose radial curvature at $\tilde{o}$ is K. If the function K is non-positive, we then say that (M,o) is dominated by the CH (Cartan-Hadamard)-model surface $(\widetilde{M},\tilde{o})$. The M is by definition a *manifold of roughly non-negative radial curvature* if (M,o) is dominated by a CH-model surface $(\widetilde{M},\tilde{o})$. Here $(\widetilde{M},\tilde{o})$ is assumed to admit the finite total curvature

$$c(\widetilde{M}) = \int_{\widetilde{M} \ni \tilde{p}} G(\tilde{p})\, d\widetilde{M} > -\infty. \tag{0.3}$$

With the notions above, we state our main theorem:

THEOREM 0.1. *Every manifold M of roughly non-negative radial curvature has the asymptotic cone.*

In Section 2, we introduce the ideal boundary M_∞ of M as equivalence classes on rays in M, and we then provide a way to equip the intrinsic distance into M_∞, called the *Tits distance*. The M_∞ equipped with the Tits distance is called the Tits ideal boundary of M, and the asymptotic cone in Theorem 0.1 is generated by the Tits ideal boundary M_∞. We also state in Section 2 that every connected component of M_∞ is a geodesic space whose Hausdorff dimension is not greater than $\dim M - 1$.

The existence of the asymptotic cone itself imposes restrictions on the topology of M. We prove in Proposition 2.8 that the distance function from the base point o is almost regular outside a bounded set, and hence the set of all critical points of the distance function from o is bounded. Consequently, M has the *finite topological type*, that is, there exists $R > 0$ such that $M \setminus B_o^M(R)$ is homeomorphic to $\partial B_o^M(R) \times [0,\infty)$, where $B_o^M(R)$ is the R-distance ball centered at o and $\partial B_o^M(R)$ is the boundary of $B_o^M(R)$.

COROLLARY 0.2. *Let M be a manifold of roughly non-negative radial curvature dominated by a CH-model surface $\widetilde{M}$. Then M has the finite topological type, and there exists a universal upper bound $N = N(c(\widetilde{M}), \dim M)$ for the number of ends of M.*

REMARK 0.3. Machigashira [**20**] has proved the same results as described in Corollary 0.2. In the present paper, we give an alternative proof.

We now mention manifolds of asymptotically non-negative curvature (see [**1**]). We say that M is a *manifold of asymptotically non-negative curvature* if there exists a monotone increasing and negative function $K\colon [0,\infty) \to (-\infty,0)$ with

$$\int_0^\infty tK(t)dt > -\infty \tag{0.4}$$

such that the sectional curvatures at every $p \in M$ are bounded below by $K\big(d_M(o,p)\big)$ for all plane sections at p. Then the condition (0.4) induces a curvature decay condition

$$\lim_{t \to \infty} t^2 K(t) = 0 \tag{0.5}$$

(see Remarks 1.2 in [**1**]). Furthermore, the integral condition (0.4) implies the condition (0.3) for the CH-model surface. To see this, consider the solution $f : [0,\infty) \to [0,\infty)$ of (0.2) for the function K above, and let $(\widetilde{M},\tilde{o})$ be the CH-model surface with the metric (0.1). According to

Zhu's result (see Lemma 2.1 in [**32**]), f satisfies $f(t) \leq e^{-b}t$ for $b := -\int_0^\infty tK(t)dt$, which means $c(\widetilde{M}) \geq 2\pi(-b)e^{-b} > -\infty$. Therefore, as a corollary to Theorem 0.1, we obtain the following:

COROLLARY 0.4. *Every manifold M of asymptotically non-negative curvature has the asymptotic cone.*

REMARK 0.5. The sectional curvature of M is said to be *faster than quadratic decay* if there exist a constant $C > 0$ and $\delta > 0$ such that the sectional curvatures at $p \in M$ are bounded below by $-C/d_M(o,p)^{2+\delta}$ for all plane sections at p (see [**28**]). Such a manifold is of asymptotically non-negative curvature. Petrunin and Tuschmann state in [**24**] without proof that every manifold with the sectional curvature of faster than quadratic decay has the asymptotic cone.

We discuss the structure of the asymptotic cones of manifolds of asymptotically non-negative curvature via the geometry of Alexandrov spaces with curvature bounded below (cf. [**2**], [**8**], [**7**]). We see that the number of connected components of M_∞ is finite since M_∞ is compact (see Section 2). Let $M_{\infty,0}$ be a connected component of M_∞, and denote by $\mathrm{Cone}\,M_{\infty,0}$ the Euclidean cone over $M_{\infty,0}$ with the vertex o^*. Since the sectional curvature $K_{(1/t)M}$ on $(1/t)M$ satisfies $K_{(1/t)M} = t^2K_M$ for all plane sections, the curvature decay condition (0.5) implies that $\mathrm{Cone}\,M_{\infty,0} \setminus \{o^*\}$ is an Alexandrov space with curvature bounded below by 0. In particular, $\dim_{\mathscr{H}} \mathrm{Cone}\,M_{\infty,0}$ is an integer not greater than $m = \dim M$, and $\dim_{\mathscr{H}} M_{\infty,0}$ is an integer not greater than $m-1$. Here $\dim_{\mathscr{H}}$ denotes the Hausdorff dimension.

COROLLARY 0.6. *Let M be as in Corollary 0.4, and $M_{\infty,0}$ be a connected component of the Tits ideal boundary of M. Then $\mathrm{Cone}\,M_{\infty,0} \setminus \{o^*\}$ is an Alexandrov space with curvature bounded below by 0, and $\dim_{\mathscr{H}} M_{\infty,0}$ is an integer not greater than $\dim M - 1$. More precisely, the following hold*:

(1) *If $\dim_{\mathscr{H}} M_{\infty,0} = 1$ and if the diameter of $M_{\infty,0}$ is not greater than π, then $\mathrm{Cone}\,M_{\infty,0}$ itself and $M_{\infty,0}$ are Alexandrov spaces with curvature bounded below by 0 and 1, respectively.*
(2) *If $\dim_{\mathscr{H}} M_{\infty,0} \geq 2$, then $\mathrm{Cone}\,M_{\infty,0}$ and $M_{\infty,0}$ are Alexandrov spaces with curvature bounded below by 0 and 1, respectively. In particular, the diameter of $M_{\infty,0}$ is not greater than π.*

We remark that there actually exists a complete, open surface M of asymptotically non-negative curvature such that its asymptotic cone is isometric to a circle whose circumference is greater than 2π, and in particular, the cone is not an Alexandrov space with curvature bounded below (see Example 1.6).

Furthermore, we observe the relation between the Hausdorff dimension of M_∞ and the volume growth of manifolds of asymptotically non-negative curvature:

COROLLARY 0.7. *Let M be an m-dimensional one as in Corollary 0.4. Then, $\dim_{\mathscr{H}} M_\infty = m-1$ if and only if $\lim_{t\to\infty} \mathrm{vol}\,B_o^M(t)/t^m$ is bounded away from 0. Here, $\mathrm{vol}\,B_o^M(t)$ denotes the volume of $B_o^M(t)$.*

In Section 3, we show Corollaries 0.6 and 0.7 stated above.

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1. Preliminaries and examples.

Throughout this paper, the following conventions are used in any geodesic space: (1) Every geodesic is parameterized by arc-length. (2) For two geodesics γ,σ emanating from the same point $x=\gamma(0)=\sigma(0)$, we denote by $\angle_x(\gamma,\sigma)$ the angle at x between γ and σ. (3) We mean by $\triangle(x_1,x_2,x_3)$ a geodesic triangle whose vertices are three points x_1,x_2,x_3 in the space, and by x_ix_{i+1} the edge of $\triangle(x_1,x_2,x_3)$ for i mod 3. All of the edges are minimizing geodesics unless otherwise stated. We then denote by $\angle(x_{i-1}x_ix_{i+1})$ the angle at x_i between the edges x_ix_{i+1} and x_ix_{i-1} for i mod 3.

1.1. Generalized Toponogov comparison theorem.

Let (M,o) be a manifold whose radial sectional curvature at o is bounded below by $K\colon [0,\infty)\to \boldsymbol{R}$, and $(\widetilde{M},\tilde{o})$ the model surface with the metric (0.1) determined by the solution f of (0.2) for the K. Itokawa, Machigashira and Shiohama established the following comparison theorem on such a manifold:

THEOREM 1.1 ([**15**], [**16**], [**20**]). *Let (M,o) and $(\widetilde{M},\tilde{o})$ be as above.*
(I) *For every geodesic triangle $\triangle(x,y,o)$ in M, there exists a geodesic triangle $\triangle(\hat{x},\hat{y},\tilde{o})$ in $\widetilde{M}$* (*the edge $\hat{x}\hat{y}$ is not necessarily a minimizing geodesic*) *such that*:

(1) $d_{\widetilde{M}}(\tilde{o},\hat{x})=d_M(o,x)$ *and* $d_{\widetilde{M}}(\tilde{o},\hat{y})=d_M(o,y)$.
(2) *The length of* $\hat{x}\hat{y}=d_M(\gamma_1(s),\gamma_2(t))$.
(3) $\angle(xyo)\geq\angle(\hat{x}\hat{y}\tilde{o})$ *and* $\angle(yxo)\geq\angle(\hat{y}\hat{x}\tilde{o})$.

(II) *Let $\gamma:[0,a]\to M$ and $\sigma:[0,b]\to M$ be two minimizing geodesics emanating from $o=\gamma(0)=\sigma(0)$. For any $(t,s)\in(0,a]\times(0,b]$, consider a geodesic triangle $\triangle(\gamma(t),\sigma(s),o)$ in M, and take the corresponding geodesic triangle $\triangle(\hat{\gamma}(t),\hat{\sigma}(s),\tilde{o})$ to $\triangle(\gamma(t),\sigma(s),o)$ satisfying* (1)–(3) *in* (I). *Then, if the edge $\hat{\gamma}(t)\hat{\sigma}(s)$ is a minimizing geodesic for every $(t,s)\in(0,a]\times(0,b]$, the angle $\angle(\hat{\gamma}(t)\tilde{o}\hat{\sigma}(s))$ is monotone non-increasing as t and s increase. In particular, we have $\angle(\gamma(t)o\sigma(s))\geq\angle(\hat{\gamma}(t)\tilde{o}\hat{\sigma}(s))$.*

1.2. On the geometry of Hadamard surfaces.

Throughout this subsection, let $(\widetilde{M},\tilde{o})$ be a Hadamard surface satisfying the condition (0.3). We here recall the ideal boundary and its properties: Let us denote by $\widetilde{M}_\infty$ the ideal boundary of $\widetilde{M}$, which is obtained as the asymptotic classes of rays in $\widetilde{M}$, equipped with the angle distance $\angle_\infty^{\widetilde{M}}$.

We denote by $\mathscr{R}_{\tilde{o}}$ the set of all rays emanating from $\tilde{o}$ (hence it consists of all geodesics emanating from $\tilde{o}$). Remark that there exists a natural bijection between $\widetilde{M}_\infty$ and $\mathscr{R}_{\tilde{o}}$. Fix any two rays $\tilde{\gamma}$, $\tilde{\sigma}\in\mathscr{R}_{\tilde{o}}$, and for $t,s\in(0,\infty)$ consider the geodesic triangle $\triangle(\tilde{\gamma}(t),\tilde{\sigma}(s),\tilde{o})$ whose vertices are $\tilde{\gamma}(t),\tilde{\sigma}(s)$ and $\tilde{o}$. We then take a corresponding Euclidean triangle $\triangle(\overline{\gamma}(t),\overline{\sigma}(s),\overline{o})$ to $\triangle(\tilde{\gamma}(t),\tilde{\sigma}(s),\tilde{o})$ such that $d_{\boldsymbol{R}^2}(\overline{\gamma}(t),\overline{\sigma}(s))=d_{\widetilde{M}}(\tilde{\gamma}(t),\tilde{\sigma}(s))$, $d_{\boldsymbol{R}^2}(\overline{\sigma}(s),\overline{o})=d_{\widetilde{M}}(\tilde{\sigma}(s),\tilde{o})$ and $d_{\boldsymbol{R}^2}(\overline{\gamma}(t),\overline{o})=d_{\widetilde{M}}(\tilde{\gamma}(t),\tilde{o})$. Then it follows from the Cartan-Alexandrov-Toponogov theorem that the angle $\angle(\overline{\gamma}(t)\overline{o}\overline{\sigma}(s))$ at $\overline{o}$ is monotone non-decreasing as t and s increase, and we have

$$\angle_\infty^{\widetilde{M}}(\tilde{\gamma},\tilde{\sigma})=\lim_{t,s\to\infty}\angle(\overline{\gamma}(t)\overline{o}\overline{\sigma}(s)). \tag{1.1}$$

For any $a,b>0$, by replacing t and s in (1.1) with at and bt, respectively, we obtain the following:

$$\lim_{t\to\infty} \frac{d_{\widetilde{M}}\big(\tilde{\gamma}(at),\tilde{\sigma}(bt)\big)}{t} = \big(a^2+b^2-2ab\cos\angle_\infty^{\widetilde{M}}(\tilde{\gamma},\tilde{\sigma})\big)^{1/2}. \tag{1.2}$$

To see (1.2), we observe that as $t\to\infty$ the $(1/t)$-scaled triangle of $\triangle(\overline{\gamma}(at),\overline{\sigma}(bt),\overline{o})$ converges to a Euclidean triangle whose two edges have lengths a,b and makes the angle $\angle_\infty^{\widetilde{M}}(\tilde{\gamma},\tilde{\sigma})$ at $\overline{o}$ (cf. 4.4 in [**5**], also II.4.4 in [**4**]).

The symbol $\angle_{\mathrm{Tits}}^{\widetilde{M}}$ denotes the *Tits distance* on $\widetilde{M}_\infty$, which is the intrinsic distance induced from $\angle_\infty^{\widetilde{M}}$. The ideal boundary $\widetilde{M}_\infty$ equipped with the distance $\angle_{\mathrm{Tits}}^{\widetilde{M}}$ is called the Tits ideal boundary of $\widetilde{M}$ (see [**5**]). Since $c(\widetilde{M})$ is finite, we can show the following proposition (cf. [**25**]):

PROPOSITION 1.2. *If a sequence $\{\tilde{\gamma}_i\}_i$ in $\mathscr{R}_{\tilde{o}}$ converges to $\tilde{\gamma}$ as $i\to\infty$ in the standard topology* (*the uniform convergence on bounded sets*)*, then we have* $\lim_{i\to\infty}\angle_\infty^{\widetilde{M}}(\tilde{\gamma}_i,\tilde{\gamma})=0$. *In particular, $\widetilde{M}_\infty$ is compact with respect to $\angle_\infty^{\widetilde{M}}$, and with respect to $\angle_{\mathrm{Tits}}^{\widetilde{M}}$.*

In our case, the Tits ideal boundary $(\widetilde{M}_\infty,\angle_{\mathrm{Tits}}^{\widetilde{M}})$ is isometric to a circle of length $2\pi-c(\widetilde{M})$ (cf. [**26**]). Once we establish the compactness of the Tits ideal boundary $\widetilde{M}_\infty$, it follows that $(\widetilde{M},\tilde{o})$ has the asymptotic cone over the Tits ideal boundary $(\widetilde{M}_\infty,\angle_{\mathrm{Tits}}^{\widetilde{M}})$ (cf. Proposition 2.2 in [**30**]). Therefore we summarize as follows:

PROPOSITION 1.3. *$\widetilde{M}$ has the asymptotic cone over a circle of circumference $2\pi-c(\widetilde{M})$.*

REMARK 1.4. In more general situation, Shioya [**25**] has investigated the ideal boundaries of open Riemannian surfaces admitting total curvatures.

Observing the cosine formula (1.2) combined with Propositions 1.2 and 1.3, we obtain the following:

PROPOSITION 1.5. *Let $\{\tilde{\sigma}_s\}_{s\geq 0}$ be any convergent sequence in $\mathscr{R}_{\tilde{o}}$ such that $\tilde{\sigma}_s\to\tilde{\sigma}_\infty$ as $s\to\infty$, and $\tilde{\gamma}$ any ray in $\mathscr{R}_{\tilde{o}}$. Then*

$$\lim_{t,s\to\infty} \frac{d_{\widetilde{M}}\big(\tilde{\gamma}(at),\tilde{\sigma}_s(bt)\big)}{t} = \big(a^2+b^2-2ab\cos\angle_\infty^{\widetilde{M}}(\tilde{\gamma},\tilde{\sigma}_\infty)\big)^{1/2}$$

holds for any $a,b>0$.

1.3. Examples.

We here provide an example of a manifold of roughly non-negative radial curvature but not of asymptotically non-negative curvature. This example also indicates that there exists a surface of asymptotically non-negative curvature whose asymptotic cone is not an Alexandrov space with curvature bounded below (recall Corollary 0.6).

EXAMPLE 1.6. Take a monotone increasing and negative function $K:[0,\infty)\to(-\infty,0)$, and consider the unique solution $f:[0,\infty)\to[0,\infty)$ of the Jacobi equation (0.2) for the K above. Here we assume

$$\int_0^\infty f(t)K(t)dt > -\infty. \tag{1.3}$$

We then obtain an n-dimensional manifold $(\widetilde{M}^n,\tilde{o})$ equipped with the metric

$$ds^2 = dt^2 + f(t)^2 d\Theta^2 \tag{1.4}$$

in the geodesic polar coordinates $(t,\Theta) \in (0,\infty) \times S^{n-1}$ centered at $\tilde{o}$. Here $d\Theta^2$ indicates the metric on the $(n-1)$-dimensional standard unit sphere S^{n-1}. We call the $\widetilde{M}^n$ an *n-dimensional model*.

Let $\tilde{p}$ be a point with $d_{\widetilde{M}}(\tilde{o},\tilde{p}) = t$, and take any plane section at $\tilde{p}$ with an orthonormal basis $x+v$ and w. Here x is a horizontal (radial) direction and v, w are vertical directions (hence $\|x\|^2 + \|v\|^2 = 1$ holds for the norm $\|\cdot\|$ determined by the metric (1.4)). Then by the Bishop-O'Neill formula [**6**], the sectional curvature $K_{\widetilde{M}}(x+v,w)$ of the plane section is calculated as

$$K_{\widetilde{M}}(x+v,w) = K(t)\|x\|^2 + \frac{1-f'(t)^2}{f(t)^2}\|v\|^2.$$

Since $\int_0^\infty f(t)K(t)dt = 1 - \lim_{t\to\infty} f'(t) > -\infty$, we have $f'(\infty) := \lim_{t\to\infty} f'(t) < \infty$. Furthermore it holds that $1 - f'(\infty) < 0$ and

$$\lim_{t\to\infty} t^2 K_{\widetilde{M}}(v,w) = \lim_{t\to\infty} t^2 \left(\frac{1-f'(t)^2}{f(t)^2}\right) = \frac{1-f'(\infty)^2}{f'(\infty)^2} < 0.$$

Hence the condition (0.5) in the Introduction is not satisfied, and $\widetilde{M}^n$ is not a manifold of asymptotically non-negative curvature if $n \geq 3$. It is obvious that the radial sectional curvature at $\tilde{o}$ is the function K. Thus the n-dimensional model $\widetilde{M}^n$ is a manifold of roughly non-negative radial curvature dominated by the 2-dimensional model $(\widetilde{M}^2,\tilde{o})$. Note that the total curvature $c(\widetilde{M}^2)$ is finite since $c(\widetilde{M}^2) = 2\pi \int_0^\infty f(t)K(t)dt > -\infty$. We see that the Tits ideal boundary of $\widetilde{M}^n$ is isometric to the $(n-1)$-dimensional round sphere each of whose great circle has the length $2\pi - c(\widetilde{M}^2)$ (cf. Proposition 1.3).

On the other hand, the 2-dimensional model $\widetilde{M}^2$ is a surface of asymptotically non-negative curvature. Indeed, the comparison theorem of solutions of (0.2) implies $f(t) \geq t$ for any $t \in [0,\infty)$, and hence $\int_0^\infty tK(t)dt > -\infty$ holds. Furthermore, the Tits ideal boundary of $\widetilde{M}^2$ is a circle of length $2\pi - c(\widetilde{M}^2) > 2\pi$.

2. The asymptotic cones of manifolds of roughly non-negative radial curvature.

Throughout this section let (M,o) be a manifold of roughly non-negative curvature dominated by a CH-model surface $(\widetilde{M},\tilde{o})$.

Hereafter, we use the following conventions: (1) Let (X,d_X) be a metric space. We denote by $B_x^X(R)$ the metric ball centered at $x \in X$ with radius $R > 0$, and denote by $S_x^X(R)$ the R-distance sphere $\{y \in X \mid d_X(x,y) = R\}$. (2) The symbol $\vartheta_\alpha(\varepsilon)$ indicates a positive function in ε such that $\lim_{\varepsilon\to 0} \vartheta_\alpha(\varepsilon) = 0$ for a fixed α.

2.1. The ideal boundary M_∞.

Let $\mathscr{R}_o$ denote the set of all rays in M emanating from o. For any two rays $\gamma, \sigma \in \mathscr{R}_o$, by using Theorem 1.1, we define the angle distance $\angle_\infty(\gamma,\sigma)$. Fix a ray $\tilde{\gamma} \in \mathscr{R}_{\tilde{o}}$ in the CH-model surface $(\widetilde{M},\tilde{o})$. We observe from Theorem 1.1 that, for any $t, s \in (0,\infty)$, there exists a ray $\tilde{\sigma}_{t,s}$ in $\mathscr{R}_{\tilde{o}}$ such that:

(i) $d_{\widetilde{M}}(\tilde{\gamma}(t),\tilde{\sigma}_{t,s}(s)) = d_M(\gamma(t),\sigma(s))$.

(ii) $\angle_{\tilde{o}}(\tilde{\gamma}, \tilde{\sigma}_{t,s})$ is monotone non-increasing as $t,s \to \infty$, and $\tilde{\sigma}_{t,s}$ converges to a ray $\tilde{\sigma}_\infty \in \mathscr{R}_{\tilde{o}}$ as $t,s \to \infty$.

We then define the *angle distance* $\angle_\infty(\gamma,\sigma)$ as

$$\angle_\infty(\gamma,\sigma) := \angle_\infty^{\widetilde{M}}(\tilde{\gamma}, \tilde{\sigma}_\infty). \tag{2.1}$$

The value in (2.1) does not depend on the choice of $\tilde{\gamma}$, $\tilde{\sigma}_{t,s}$ and $\tilde{\sigma}_\infty$ since $\widetilde{M}$ is rotationally symmetric around $\tilde{o}$.

The following is an immediate consequence of Proposition 1.5:

PROPOSITION 2.1. *For any $\gamma,\sigma \in \mathscr{R}_o$ and for any positive numbers $a,b > 0$, we obtain the following*:

$$\lim_{t\to\infty} \frac{d_M\big(\gamma(at),\sigma(bt)\big)}{t} = \big(a^2+b^2-2ab\cos\angle_\infty(\gamma,\sigma)\big)^{1/2}. \tag{2.2}$$

In particular, $\angle_\infty$ is a pseudo-distance on $\mathscr{R}_o$.

REMARK 2.2. (2.2) also implies that the value in (2.1) does not depend on the choice of the CH-model surface $\widetilde{M}$.

We thus define the *ideal boundary M_∞ of M* as the metric space $M_\infty := \mathscr{R}_o/_{\angle_\infty=0}$ equipped with the distance $\angle_\infty$. The symbol $[\gamma] \in M_\infty$ denotes the equivalence class containing $\gamma \in \mathscr{R}_o$.

PROPOSITION 2.3. M_∞ is compact with respect to $\angle_\infty$.

PROOF. For any sequence $\{[\gamma_i]\}_i$ in M_∞, by taking a suitable subsequence if necessary, there exists a ray $\gamma \in \mathscr{R}_o$ such that $\angle_o(\gamma_i,\gamma) \to 0$ as $i \to \infty$. Recall the definition of the angle distance $\angle_\infty(\gamma_i,\gamma)$: For a fixed ray $\tilde{\gamma} \in \mathscr{R}_{\tilde{o}}$ and for any $t,s \in (0,\infty)$, let $\tilde{\gamma}_{t,s}$ be a ray in $\mathscr{R}_{\tilde{o}}$ such that:

(i) $d_{\widetilde{M}}(\tilde{\gamma}(t), \tilde{\gamma}_{t,s}(s)) = d_M(\gamma(t),\gamma_i(s))$.
(ii) $\angle_{\tilde{o}}(\tilde{\gamma}, \tilde{\gamma}_{t,s})$ is monotone non-increasing as $t,s \to \infty$, and $\tilde{\gamma}_{t,s}$ converges to a ray $\tilde{\gamma}_{i,\infty}$ in $\mathscr{R}_{\tilde{o}}$ as $t,s \to \infty$.

Then $\angle_\infty(\gamma_i,\gamma) = \angle_\infty^{\widetilde{M}}(\tilde{\gamma}_{i,\infty},\tilde{\gamma})$ holds for each i. Theorem 1.1 implies $\angle_{\tilde{o}}(\tilde{\gamma}_{i,\infty},\tilde{\gamma}) \le \angle_o(\gamma_i,\gamma)$, and we have $\angle_{\tilde{o}}(\tilde{\gamma}_{i,\infty},\tilde{\gamma}) \to 0$ as $i \to \infty$. Hence Proposition 1.2 implies that $\angle_\infty(\gamma_i,\gamma) = \angle_\infty^{\widetilde{M}}(\tilde{\gamma}_{i,\infty},\tilde{\gamma}) \to 0$ as $i \to \infty$. $\square$

2.2. Convergence to the asymptotic cone.

In this subsection, we prove Theorem 0.1. For simplicity, let C denote the Euclidean cone over $(M_\infty,\angle_\infty)$ with the vertex o^*. We first prove that the pointed Gromov-Hausdorff limit of $((1/t)M,o)$ as $t \to \infty$ is isometric to (C,o^*), and at the end of this subsection, we introduce the Tits distance on M_∞.

A pointed metric space (Y,o_Y) is by definition *a pointed Gromov-Hausdorff limit of pointed metric spaces (X_t,o_t) as $t\to\infty$* if for any fixed $R>0$ and for given $\varepsilon>0$, there exists $\vartheta_R(\varepsilon)$-*Hausdorff approximation* $h_t : B^{X_t}_{o_t}(R) \to B^{Y}_{o_Y}(R)$ (not necessary a continuous map) satisfying the following (1)–(3) for any sufficiently large $t>0$ (cf. [**12**], [**13**]):

(1) $\big|d_Y\big(h_t(x_1),h_t(x_2)\big) - d_X(x_1,x_2)\big| < \vartheta_R(\varepsilon)$ for every $x_1,x_2 \in B^{X_t}_{o_t}(R)$.

(2) $B^Y_{o_Y}(R)$ is contained in the $\vartheta_R(\varepsilon)$-neighborhood of $h_t(B^{X_t}_{o_t}(R))$. In other words, $h_t(B^{X_t}_{o_t}(R))$ is $\vartheta_R(\varepsilon)$-dense in $B^Y_{o_Y}(R)$.
(3) $h_t(o_t) = o_Y$.

Let us begin the proof of Theorem 0.1. Fix any $\varepsilon > 0$. Then from Proposition 2.3, we can take a finite ε-dense set $\{[\gamma_l]\}_{l=1,2,\ldots,L}$ in $(M_\infty, \angle_\infty)$. Define

$$\mathscr{N}_{\varepsilon,t} := \{\gamma_l(k\varepsilon t) \in (1/t)M \mid k = 0,1,2,\ldots,[R/\varepsilon],\ 1 \le l \le L\}.$$

We first prove the following:

CLAIM 2.4. *$\mathscr{N}_{\varepsilon,t}$ is a $\vartheta_R(\varepsilon)$-dense set in $B^{(1/t)M}_o(R)$ for any sufficiently large $t > 0$.*

Assume the validity of the claim. We then obtain a map

$$h_t : B^{(1/t)M}_o(R) \to B^C_{o^*}(R)$$

defined by $h_t(q) := (k\varepsilon, [\gamma_l]) \in C = [0,\infty) \times M_\infty/_{\{0\}\times M_\infty}$, where the pair of numbers (k,l) corresponds to the point $\gamma_l(k\varepsilon t) \in \mathscr{N}_{\varepsilon,t}$ which is a nearest point to q. It follows from Proposition 2.1 that the map h_t satisfies properties (1)–(3) of $\vartheta_R(\varepsilon)$-Hausdorff approximations.

PROOF OF CLAIM 2.4. Suppose that the claim is not true for $\vartheta_R(\varepsilon) = 4\varepsilon + 4R\sin 2\varepsilon + 4R\sin\big[(\varepsilon/4\pi)(2\pi - c(\widetilde{M}))\big]$. Then there are a sequence $\{x_j\} \subset M$ and a monotone divergent sequence $\{t_j\}$ such that

(1) $x_j \in B^{(1/t_j)M}_o(R)$,
(2) $d_{(1/t_j)M}(x_j, \gamma_l(k\varepsilon t_j)) \ge \vartheta_R(\varepsilon)$ for any $\gamma_l(k\varepsilon t_j) \in \mathscr{N}_{\varepsilon,t_j}$, and
(3) $d_{(1/t_j)M}(o, x_j) \in [k\varepsilon, (k+1)\varepsilon)$ for some $k \ge 1$.

Let σ_j be a minimizing geodesic joining o and x_j. From (3) above, we may assume that σ_j converges to a ray $\sigma \in \mathscr{R}_o$ in M as $j \to \infty$. Define $s_j := k\varepsilon t_j$, $y_j := \sigma_j(s_j)$ and $z_j := \sigma(s_j)$ for simplicity. Obviously, $d_{(1/t_j)M}(x_j, y_j) < \varepsilon$. Since $\{[\gamma_l]\}_l$ is an ε-dense set in M_∞, we have $\angle_\infty(\gamma_l, \sigma) < \varepsilon$ for some γ_l in $\{[\gamma_l]\}_{l=1,2,\ldots,L}$. This together with (2.2) yields that $d_{(1/t_j)M}(\gamma_l(s_j), z_j) \le 2R\sin 2\varepsilon$ for all sufficiently large j. By showing

$$d_{(1/t_j)M}(y_j, z_j) \le 2R\sin\left[\frac{\varepsilon}{4\pi}(2\pi - c(\widetilde{M}))\right], \tag{2.3}$$

we obtain a contradiction to (2):

$$d_{(1/t_j)M}(x_j, \gamma_l(s_j)) \le 2\varepsilon + 2R\sin 2\varepsilon + 2R\sin\left[\frac{\varepsilon}{4\pi}(2\pi - c(\widetilde{M}))\right].$$

We next verify (2.3). Let $\tilde{\sigma}$ be a fixed ray in the CH-model surface $(\widetilde{M}, \tilde{o})$ starting from $\tilde{o}$. Consider the comparison triangle $\triangle(\tilde{y}_j, \tilde{z}_j, \tilde{o}) \subset \widetilde{M}$ such that $\tilde{z}_j = \tilde{\sigma}(s_j)$, $d_{\widetilde{M}}(\tilde{y}_j, \tilde{z}_j) = d_M(y_j, z_j)$ and $\tilde{y}_j = \tilde{\sigma}_j(s_j)$ for some $\tilde{\sigma}_j \in \mathscr{R}_{\tilde{o}}$. Then Theorem 1.1 implies $\angle_{\tilde{o}}(\tilde{\sigma}_j, \tilde{\sigma}) \le \angle_o(\sigma_j, \sigma) < \varepsilon$ for all sufficiently large j. Hence we have $\angle^{\widetilde{M}}_\infty(\tilde{\sigma}_j, \tilde{\sigma}) \le (\varepsilon/2\pi)(2\pi - c(\widetilde{M}))$ since $\widetilde{M}$ is rotationally symmetric (cf. Proposition 1.3). Therefore it follows from the Cartan-Alexandrov-Toponogov theorem that $d_{(1/t_j)M}(y_j, z_j) = d_{(1/t_j)\widetilde{M}}(\tilde{y}_j, \tilde{z}_j) \le 2R\sin\big[(\varepsilon/4\pi)(2\pi - c(\widetilde{M}))\big]$. This completes the proof. $\square$

We have shown that the pointed Gromov-Hausdorff limit of $((1/t)M,o)$ as $t\to\infty$ is isometric to (C,o^*). We denote by $\angle_{\mathrm{Tits}}$ the intrinsic distance, called the *Tits distance*, induced from $\angle_\infty$. For our last goal, we discuss a way to introduce the Tits distance into M_∞: Since the Gromov-Hausdorff limit of geodesic spaces is again a geodesic space, the cone (C,o^*) is a geodesic space, and so is any connected component of M_∞. More precisely, for any $[\gamma],[\sigma]\in M_\infty$ with $\angle_\infty([\gamma],[\sigma])<\pi$, there exists a $\angle_\infty$-minimizing geodesic in M_∞ joining $[\gamma]$ and $[\sigma]$, and we then have $\angle_{\mathrm{Tits}}([\gamma],[\sigma])=\angle_\infty([\gamma],[\sigma])$. In this way, we have completed the proof of Theorem 0.1. $\square$

We here summarize the properties of the Tits ideal boundary M_∞:

THEOREM 2.5. *The Tits ideal boundary $(M_\infty,\angle_{\mathrm{Tits}})$ of M as in Theorem* 0.1 *satisfies the following*:

(1) *$(M_\infty,\angle_{\mathrm{Tits}})$ is compact.*
(2) *Every connected component of M_∞ is a geodesic space. In particular, if $\angle_{\mathrm{Tits}}([\gamma],[\sigma])\le\pi$ for $[\gamma],[\sigma]\in M_\infty$, then we have $\angle_{\mathrm{Tits}}([\gamma],[\sigma])=\angle_\infty([\gamma],[\sigma])$.*

2.3. Proof of Corollary 0.2.

In this subsection, we use the n-dimensional model $(\widetilde{M}^n,\tilde{o})$ with the metric (1.4). Note then that the function f in (1.4) is the solution of (0.2) for the radial curvature function K of M.

THEOREM 2.6. *Let M be an m-dimensional one. Then there exists an expanding map*

$$\Phi:(M_\infty,\angle_\infty)\to(\widetilde{M}^m_\infty,\angle_\infty^{\widetilde{M}}).$$

In particular, we have $\mathscr{H}^{m-1}(M_\infty)\le\mathscr{H}^{m-1}(\widetilde{M}^m_\infty)$ and $\dim_{\mathscr{H}}M_\infty\le\dim_{\mathscr{H}}\widetilde{M}^m_\infty=m-1$, where $\mathscr{H}^k(\cdot)$ denotes the k-dimensional Hausdorff measure.

REMARK 2.7. The authors do not know whether or not the connected component of M_∞ admits a locally uniform dimension (compare to Corollary 0.6).

PROOF OF THEOREM 2.6. We here give an outline of the proof. Consider a composition map

$$\Phi_t:=\exp_{\tilde{o}}\circ I\circ\log_o:(1/t)M\to(1/t)\widetilde{M},$$

where $\exp_{\tilde{o}}$ denotes the exponential map on the $(1/t)$-scaled space $(1/t)\widetilde{M}$, $I:T_oM\to T_{\tilde{o}}\widetilde{M}$ is a linear isometry identifying T_oM with $T_{\tilde{o}}\widetilde{M}$, and $\log_o:(1/t)M\to T_oM$ is the map satisfying for any $p,q\in(1/t)M$

(i) $\exp_o(\log_o(p))=p$,
(ii) the norm $\|\log_o(p)\|$ measured in $(1/t)M$ equals to $d_{(1/t)M}(o,p)$, and
(iii) $\angle(\log_o(p),\log_o(q))=\angle(poq)$ for some minimizing geodesics op and oq.

Then it follows from Theorem 1.1 (II) (the hinge comparison) that Φ_t is an expanding map. Denote by $(\widetilde{C},\tilde{o}^*)$ the Euclidean cone over $(\widetilde{M}_\infty,\angle_\infty^{\widetilde{M}})$. We here denote by $\tilde{o}^*$ the vertex of $\widetilde{C}$. We now know that $((1/t)M,o)$ and $((1/t)\widetilde{M},\tilde{o})$ converge as $t\to\infty$ to the cones (C,o^*) and $(\widetilde{C},\tilde{o}^*)$, respectively. Hence Φ_t converges as $t\to\infty$ to an expanding map $\Phi_\infty:B^C_{o^*}(2)\to B^{\widetilde{C}}_{\tilde{o}^*}(2)$ such that $\Phi_\infty(o^*)=\tilde{o}^*$ and $d_{\widetilde{C}}(\Phi_\infty(o^*),\Phi_\infty(u))=d_C(o^*,u)$ for every $u\in B^C_{o^*}(2)$. Thus the map

$$\Phi_\infty|_{S^C_{o^*}(1)} : S^C_{o^*}(1) \to S^{\widetilde{C}}_{\tilde{o}^*}(1)$$

restricted on the 1-distance sphere $S^C_{o^*}(1)$ gives the desired expanding map Φ if we identify $S^C_{o^*}(1)$ with M_∞, and $S^{\widetilde{C}}_{\tilde{o}^*}(1)$ with $\widetilde{M}^m_\infty$.

We should note the following: We first construct an expanding map from a finite subset in $B^C_{o^*}(2)$ to $B^{\widetilde{C}}_{\tilde{o}^*}(2)$ in the same manner as above. Next, we make the finite subsets denser and denser in $B^C_{o^*}(2)$, and as a result, we obtain an expanding map from a countable and dense subset in $B^C_{o^*}(2)$ to $B^{\widetilde{C}}_{\tilde{o}^*}(2)$ (we then use the diagonal argument). The map Φ_∞ is then obtained as its extension to the whole $B^C_{o^*}(2)$. □

We denote by ρ_o the distance function from the base point o. A point $p \in M$ is, by definition, a *critical point of* ρ_o if for each unit vector $v \in T_pM$ there is a minimizing geodesic $\gamma : [0, \rho_o(p)] \to M$ joining $p = \gamma(0)$ and $o = \gamma(\rho_o(p))$ such that $\angle(v, \dot\gamma(0)) \le \pi/2$. Let $d\rho_o$ be the directional derivative of the ρ_o. For $p \in M$, we denote by $\nabla\rho_o(p)$ the set of all unit vectors $v \in T_pM$ such that $d\rho_o(v) = \max d\rho_o$, where max is taken over all unit vectors in T_pM. For $\varepsilon > 0$, the function ρ_o is, by definition, *ε-almost regular at $p \in M$* if

$$\angle(v, \dot\gamma(0)) \ge \pi - \varepsilon$$

holds for any $v \in \nabla\rho_o(p)$ and for any minimizing geodesic $\gamma : [0, \rho_o(p)] \to M$ joining $p = \gamma(0)$ and $o = \gamma(\rho_o(p))$. We say that the ρ_o is *ε-almost regular on a subset A in M* if ρ_o is ε-almost regular at each point $p \in A$. With these notions, we state:

PROPOSITION 2.8. *For given $\varepsilon > 0$, there exists $R > 0$ such that ρ_o is ε-almost regular outside $B^M_o(R)$.*

PROOF. Let $t > 0$ be sufficiently large against $\varepsilon > 0$, and choose any $p \in S^M_o(t)$. Then by Theorem 0.1, we can find a point $q \in S^M_o(2t)$ such that

$$\angle(\overline{opq}) \ge \pi - \varepsilon,$$

where $\angle(\overline{opq})$ denote the angle at $\bar p$ of a corresponding Euclidean triangle $\triangle(\bar o, \bar p, \bar q)$ to a geodesic triangle $\triangle(o, p, q)$ satisfying $d_{\boldsymbol{R}^2}(\bar o, \bar p) = d_M(o, p)$, $d_{\boldsymbol{R}^2}(\bar o, \bar q) = d_M(o, q)$ and $d_{\boldsymbol{R}^2}(\bar p, \bar q) = d_M(p, q)$. For the $\triangle(o, p, q)$, we take the corresponding geodesic triangle $\triangle(\tilde o, \hat p, \hat q)$ in $\widetilde{M}^2$ satisfying the conditions in Theorem 1.1 (I). Since $\widetilde{M}^2$ is a Hadamard surface, we have $\angle(\hat p \tilde o \hat q) \le \angle(\overline{poq}) \le \varepsilon$. This means that $\triangle(\tilde o, \hat p, \hat q)$ is contained entirely in a sector S, which is bounded by two rays in $\mathscr{R}_{\tilde o}$ and makes an angle ε at $\tilde o$. We then have $c(\triangle(\tilde o, \hat p, \hat q)) \ge c(S) = (\varepsilon/2\pi)c(\widetilde{M}^2)$, where $c(\triangle(\tilde o, \hat p, \hat q))$ and $c(S)$ are the total curvatures of $\triangle(\tilde o, \hat p, \hat q)$ and S, respectively. Moreover, by a comparison theorem due to Alexandrov and Zalgaller (see Theorem 6 in Chapter II of [**3**]), we have

$$\angle(opq) \ge \angle(\tilde o \hat p \hat q) \ge \angle(\overline{opq}) + c(\triangle(\tilde o, \hat p, \hat q)) \ge \pi - \frac{\varepsilon}{2\pi}\big(2\pi - c(\widetilde{M}^2)\big).$$

Hence this together with the first variation formula implies the almost-regularity of ρ_o. □

PROOF OF COROLLARY 0.2. If $\varepsilon > 0$ is sufficiently small in Proposition 2.8, there exists no critical point of ρ_o outside the ball $B^M_o(R)$. Hence by the isotopy lemma in [**11**], we conclude that M has the finite topological type. Thus, the remainder of the proof is the estimate of the number of ends. Suppose that there exist two rays $\gamma, \sigma \in \mathscr{R}_o$ tending to the distinct ends. Then we have $\angle_\infty([\gamma],[\sigma]) = \pi$, and hence by Theorem 2.6, we have $\angle_\infty^{\widetilde{M}}(\Phi([\gamma]),\Phi([\sigma])) = \pi$ and

$$\angle_{\mathrm{Tits}}^{\widetilde{M}}(\Phi([\gamma]),\Phi([\sigma])) \geq \pi.$$

Consequently, the number of ends of M is not greater than the largest possible number of pairwise disjoint $(\pi/2)$-balls in $\widetilde{M}_\infty^m$. □

3. The structure of the asymptotic cones of manifolds of asymptotically non-negative curvature.

Throughout this section, let M be a manifold of asymptotically non-negative curvature.

3.1. Alexandrov spaces with curvature bounded below.

We first recall the definition of Alexandrov spaces with curvature bounded below. Refer the basic tools and facts of Alexandrov spaces with curvature bounded below to [**2**], [**8**], [**7**].

Let N_κ^2 be the simply connected, complete model surface with constant Gaussian curvature κ. We say that a metric space N is an *Alexandrov space with curvature bounded below by κ* if the following condition (A) holds (cf. Definition 2.5 in [**8**]):
(A): Every point $p \in N$ has neighborhoods $U_p \supset V_p$ such that:

(1) Any two points in V_p are joined by a minimizing geodesic in U_p.
(2) For any $\triangle(x,y,z)$ with vertices in V_p, every w on the edge yz satisfies the inequality $d_N(x,w) \leq d_{N_\kappa^2}(\widetilde{x},\widetilde{w})$. Here $\widetilde{w}$ is the point on the edge $\widetilde{y}\,\widetilde{z}$ of the corresponding geodesic triangle $\triangle(\tilde{x},\tilde{y},\tilde{z}) \subset N_\kappa^2$ to $\triangle(x,y,z)$ such that $d_N(y,w) = d_{N_\kappa^2}(\tilde{y},\tilde{w})$. If $\kappa > 0$, we need an additional assumption: the perimeter of $\triangle(x,y,z)$ is $< 2\pi/\sqrt{\kappa}$.

If N is a 1-dimensional manifold and $\kappa > 0$, then we assume in addition that the diameter of N is not greater than $\pi/\sqrt{\kappa}$.

For example, complete Riemannian manifolds whose sectional curvatures are not smaller than κ are Alexandrov spaces with curvature bounded below by κ.

3.2. Proofs of the corollaries.

Let $M_{\infty,0}$ be any connected component of M_∞ with respect to the Tits distance $\angle_{\mathrm{Tits}}$, and define $C_0 := \mathrm{Cone}\, M_{\infty,0}$. As is stated in the Introduction, $C_0 \setminus \{o^*\}$ is an Alexandrov space with curvature bounded below by 0. In particular, $\dim_{\mathscr{H}} C_0 \setminus \{o^*\}$ is an integer not greater than $m = \dim M$, and $\dim_{\mathscr{H}} M_{\infty,0}$ is an integer not greater than $m-1$ (see §6, §8 in [**8**]).

PROOF OF COROLLARY 0.6. First, note that $M_{\infty,0}$ satisfies the condition (A) if its cone itself is not a ray. This is obtained by the same argument as the one that appeared in the proof of Proposition 4.2.3 in [**8**] since $C_0 \setminus \{o^*\}$ is an Alexandrov space with curvature bounded below by 0.

If $\dim_{\mathscr{H}} C_0 \setminus \{o^*\} = 2$, then $C_0 \setminus \{o^*\}$ is a 2-dimensional topological manifold possibly with boundary, and $M_{\infty,0}$ is a 1-dimensional topological manifold possibly with boundary (12.9.3 in [**8**]). Hence if the diameter of $M_{\infty,0}$ is not greater than π, then $M_{\infty,0}$ is an Alexandrov space with curvature bounded below by 1.

If $\dim_{\mathscr{H}} C_0 \setminus \{o^*\} \geq 3$, then $M_{\infty,0}$ is not a 1-dimensional manifold. Furthermore, $M_{\infty,0}$ is an Alexandrov space with curvature bounded below by 1. In particular, from Theorem 3.6 and

Proposition 4.2.3 in [**8**], we see that the diameter of $M_{\infty,0}$ is not greater than π, and that C_0 is an Alexandrov space with curvature bounded below by 0. $\square$

Next, we show Corollary 0.7. We explain some notations needed in the proof: Let (X, d_X) be a metric space. (1) For $x \in X$, $R_1 > R_2 > 0$, we define the metric annulus $A_x^X(R_1, R_2) := B_x^X(R_1) \setminus B_x^X(R_2)$. (2) Let A be a bounded set in X. The *rough dimension* $\dim_r A$ *of* A is defined as

$$\dim_r A := \inf\{\alpha \mid \limsup_{\varepsilon\to 0} \varepsilon^\alpha \beta_A(\varepsilon) = 0\} = \sup\{\alpha \mid \limsup_{\varepsilon\to 0} \varepsilon^\alpha \beta_A(\varepsilon) = \infty\},$$

where $\beta_A(\varepsilon)$ is the largest possible number of points $x_i \in A$ satisfying $d_X(x_i, x_j) \geq \varepsilon$ for $i \neq j$.

PROOF OF COROLLARY 0.7. We here assume that M_∞ has the unique connected component. For the more general case where M_∞ does not necessarily have the unique connected component, we can similarly prove the corollary.

Let us take sufficiently large numbers $0 < R \ll T$ so that $T^2K(T) \geq T^2K(R) \geq -1$. Then, $A_o^{(1/T)M}(1, R/T)$ converges to $B_{o^*}^{C_0}(1)$ as $T \to \infty$ with respect to the Gromov-Hausdorff distance preserving the lower curvature bound -1. We also note that

$$\lim_{T\to\infty} \mathrm{vol}_T B_o^{(1/T)M}(R/T) = \lim_{T\to\infty} \frac{\mathrm{vol} B_o^M(R)}{T^m} = 0, \tag{3.1}$$

where vol_T denotes the volume measured in $(1/T)M$.

First, we assume that $\dim_{\mathscr{H}} M_\infty = m-1$, or equivalently, $\dim_{\mathscr{H}} C_0 = m$. Then, it is independently known by [**8**], [**27**], and [**29**] that

$$\lim_{T\to\infty} \mathrm{vol}_T A_o^{(1/T)M}(1, R/T) = \mathscr{H}^m\big(B_{o^*}^{C_0}(1)\big) > 0.$$

This together with (3.1) implies

$$\lim_{T\to\infty} \frac{\mathrm{vol} B_o^M(T)}{T^m} = \lim_{T\to\infty} \mathrm{vol}_T B_o^{(1/T)M}(1) = \mathscr{H}^m\big(B_{o^*}^{C_0}(1)\big) > 0.$$

We next assume $\dim_{\mathscr{H}} C_0 < m$ and $\lim_{T\to\infty}(\mathrm{vol} B_o^M(T)/T^m) > 0$. We lead a contradiction from the estimate of the rough dimension $\dim_r A_o^{(1/T)M}(1, R/T)$.

For simplicity, define $A_o := A_o^{(1/T)M}(1, R/T)$. Note then that the T and R in A_o vary satisfying the conditions stated at the beginning of the proof. From (3.1), we find a positive constant $V > 0$ such that $\mathrm{vol}_T A_o \geq V$. For a given sufficiently small $\varepsilon > 0$, let

$$\{p_i\}_{i=1}^{\beta_{A_o}(\varepsilon)} \subset A_o$$

be a maximal set of points in A_o satisfying $d_{(1/T)M}(p_i, p_j) \geq \varepsilon$ for $i \neq j$. Note that the points $\{p_i\}$ is a (2ε)-net of A_o, that is, $A_o \subset \bigcup_{i=1}^{\beta_{A_o}(\varepsilon)} B_{p_i}^{(1/T)M}(2\varepsilon)$. Then, by the Bishop volume comparison theorem, we have

$$\begin{aligned} V \leq \mathrm{vol}_T A_o &\leq \sum_{i=1}^{\beta_{A_o}(\varepsilon)} \mathrm{vol}_T B_{p_i}^{(1/T)M}(2\varepsilon) \\ &\leq \sum_{i=1}^{\beta_{A_o}(\varepsilon)} \omega_{-1}^m(2\varepsilon) \leq \mathrm{const}_m \cdot \varepsilon^m \beta_{A_o}(\varepsilon), \end{aligned}$$

where $\omega_{-1}^m(2\varepsilon)$ denotes the volume of a ball of radius 2ε in the m-dimensional model space of constant curvature -1, and const_m is a positive constant depending only on m. Let $h_{\varepsilon/10}\colon B_o^{(1/T)M}(1) \to B_{o^*}^{C_0}(1)$ be an $(\varepsilon/10)$-Hausdorff approximation. Then,

$$\{h_{\varepsilon/10}(p_i)\}_{i=1}^{\beta_{A_o}(\varepsilon)}$$

is an ε-discrete set in $2B_{o^*}^{C_0}(1) = B_{o^*}^{C_0}(2)$. Hence, we have

$$V \le \mathrm{const}_m \cdot \varepsilon^m \beta_{B_{o^*}^{C_0}(2)}(\varepsilon).$$

Since $C_0 \setminus \{o^*\}$ is an Alexandrov space with curvature bounded below, we have $\dim_r B_{o^*}^{C_0}(2) = \dim_{\mathscr{H}} B_{o^*}^{C_0}(2) = \dim_{\mathscr{H}} C_0 < m$ (see 6.4 in [**8**]). Hence, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^m \beta_{B_{o^*}^{C_0}(2)}(\varepsilon) = 0,$$

which yields a contradiction.

In this way, we conclude Corollary 0.7. □

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