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Laplace approximations for large deviations of diffusion processes on Euclidean spaces

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Abstract. Consider a class of uniformly elliptic diffusion processes $\{X_t\}_{t\geq 0}$ on Euclidean spaces $\boldsymbol{R}^d$. We give an estimate of $E^{P_x}\left[\exp(T\Phi(1/T\int_0^T \delta_{X_t}dt))\middle|X_T = y\right]$ as $T\to\infty$ up to the order $1+o(1)$, where $\delta_\cdot$ means the delta measure, and Φ is a function on the set of measures on $\boldsymbol{R}^d$. This is a generalization of the works by Bolthausen-Deuschel-Tamura [**3**] and Kusuoka-Liang [**10**], which studied the same problems for processes on compact state spaces.

1. Introduction.

Let $Y(t,x) = (Y_1(t,x),\cdots,Y_d(t,x))$ be the solution of the following stochastic differential equation:

$$\begin{cases} dY_i(t,x) = \displaystyle\sum_{j=1}^d \sigma_{ij}(Y(t,x))dB_j(t) + b_i(Y(t,x))dt, \qquad i = 1,\cdots,d, \\ Y(0,x) = x, \end{cases}$$

where $(B_1(t),\cdots,B_d(t))$ is a d-dimensional Brownian motion on $\boldsymbol{R}^d$, and let P_x be the distribution of $\{Y(t,x)\}_{t\geq 0}$ on $\Omega = C([0,\infty),\boldsymbol{R}^d)$. Note that $\{P_x\}_{x\in\boldsymbol{R}^d}$ is the solution of the L_0-martingale problem with $L_0 := \sum_{i,j=1}^d a_{ij}(\partial^2/(\partial x_i\partial x_j)) + b\cdot\nabla$, where $a_{ij} = \sum_{k=1}^d \sigma_{ik}\sigma_{jk}$ and $b = (b_1,\cdots,b_d)$. Let $X_t(\omega) = \omega(t), \omega\in\Omega, t\geq 0$, and let $L_t = (1/t)\int_0^t \delta_{X_s}ds$, where $\delta_\cdot$ means the delta measure.

Under some conditions, there exists a unique invariant probability measure π of $\{P_x\}_{x\in\boldsymbol{R}^d}$, and the ergodic theorem induces the convergence of L_t to π in law as $t\to\infty$ under $P_x\left(\cdot\middle|X_t = y\right)$ for any $x,y\in\boldsymbol{R}^d$. Hence for any closed set $A\subset\wp(\boldsymbol{R}^d)$ (the set of all probabilities on $\boldsymbol{R}^d$) that does not contain π, we have $P_x(L_t\in A)\to 0$ as $t\to\infty$. Large deviation principle (LDP) studies the order of this convergence in terms of the so-called entropy function determined by the generator L_0. Under some conditions, we have that $\{L_t\}_{t>0}$ under $P_x\left(\cdot\middle|X_t = y\right)$ satisfies the LDP, *i.e.*,

$$\begin{aligned} -\inf_{\nu\in A^0} I(\nu) &\leq \liminf_{T\to\infty}\frac{1}{T}\log P_x\left(L_T\in A\middle|X_T = y\right) \\ &\leq \limsup_{T\to\infty}\frac{1}{T}\log P_x\left(L_T\in A\middle|X_T = y\right) \leq -\inf_{\nu\in\bar{A}} I(\nu) \qquad (1.1) \end{aligned}$$

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for any $A \in \mathscr{B}(\wp(\boldsymbol{R}^d))$, where $I : \wp(\boldsymbol{R}^d) \to \boldsymbol{R} \cup \{\infty\}$ is the entropy function given by

$$I(\nu) = \sup\left\{ -\int_{\boldsymbol{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty(\boldsymbol{R}^d), u > 0, \frac{L_0 u}{u} \text{ is bounded} \right\}, \quad \nu \in \wp(\boldsymbol{R}^d). \tag{1.2}$$

Here $\mathscr{B}(\cdot)$ means the Borel σ-field, and A^0 and $\overline{A}$ mean the interior and the closure of A, respectively.

Let $V : \boldsymbol{R}^d \times \boldsymbol{R}^d \to \boldsymbol{R}$ be a symmetric bounded continuous function, and let $\nu_T^{x,y}$ be the probability on Ω with mean field potential V given by

$$\nu_T^{x,y}(d\omega) = (Z_T^{x,y})^{-1} e^{1/T \int_0^T \int_0^T V(X_t, X_s) dt ds} P_x\big(d\omega \big| X_T = y\big), \quad \omega \in \Omega, T > 0,$$

where $Z_T^{x,y}$ is the normalizing constant. One wants to know the asymptotic behavior of L_T under $\nu_T^{x,y}$ as $T \to \infty$. The LDP described above implies

$$\frac{1}{T} \log Z_T^{x,y} \to \sup_{\nu \in \wp(\boldsymbol{R}^d)} \left\{ \int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} V(x,y) \nu(dx) \nu(dy) - I(\nu) \right\}.$$

Unfortunately, this is not enough to tell us the asymptotic behavior of L_T under $\nu_T^{x,y}$. Indeed, it even does not give us whether L_T under $\nu_T^{x,y}$ weakly converges or not. To study whether L_T under $\nu_T^{x,y}$ weakly converges or not, we need to study more precise estimates of $Z_T^{x,y}$.

In this paper, we study the problem cited above in a general frame work: let Φ be a function on $\mathscr{M}(\boldsymbol{R}^d)$ that is "good" enough, ($\Phi(R) = \int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} V(x,y) R(dx) R(dy)$ in the example just mentioned), where $\mathscr{M}(\boldsymbol{R}^d)$ means the set of signed measures on $\boldsymbol{R}^d$ with finite total variations, and let $Z_T^{x,y} = E^{P_x}\big[e^{T\Phi(L_T)} \big| X_T = y\big]$. Then as before, by the LDP,

$$\lim_{T \to \infty} \frac{1}{T} \log Z_T^{x,y} = \lambda,$$

where $\lambda = \max_{\wp(\boldsymbol{R}^d)} \{\Phi - I\}$. We study the $1 + o(1)$ order estimate of $Z_T^{x,y}$ as $T \to \infty$. To be more precise, we show that under some conditions, $e^{-\lambda T} Z_T^{x,y}$ converges to a constant as $T \to \infty$. See Section 2 for the details.

The $1 + o(1)$-order precise estimating problem for the case of the sums of Banach space-valued *i.i.d.* random variables has been discussed by many authors, *e.g.*, Bolthausen [**1**], Kusuoka-Liang [**9**] and Liang [**13**], *etc*. As for the continuous time case, Bolthausen-Deuschel-Tamura [**3**] considered the same problem for Markov processes on compact state spaces under some conditions that derive the "Central Limit Theorem Assumption" as a result, Kusuoka-Liang [**10**] for diffusion processes without the "Central Limit Theorem Assumption", but on torus, still a compact space. The same problem for diffusion processes on non-compact state spaces is very less studied. One of the obvious and most vital difficulity is that, since the state space is not compact, many properties such as bounded, which were trivial for continuous functions in the compact case, become very difficult.

In this paper, we succeeded in dealing with diffusion processes on $\boldsymbol{R}^d$, a non-compact state space. The main idea of this paper is as follows. First, by using [**15**], we have (See Section 4 and

Lemma 6.3) that $|\nabla Gf(x)|$ can be dominated by $(1+|x|^2)^{\xi/2}\|f\|_\infty$, where G means some Green operator and ξ is some constant depending on the drift term of the generator not too large. The main point is that, although the state space is not compact, when the drift term is "good" enough, the probability that the process goes to infinity converges to 0 exponentially fast *uniformly* with respect to the starting point. This absorbs the extra factor $(1+|x|^2)^{\xi/2}$ above (See Lemma 3.2).

Let us explain shortly the outline of the proof: By using the measure changes discussed in Section 4, the considered quantity $e^{-\lambda T}Z_T^{x,y}$ is approximately equal to $E^{Q_x}\big[e^{1/T\int_0^T\int_0^T V(X_s,X_t)dsdt}\big|\,X_T=y\big]$, where $V(\cdot,*)$ is the symmetric translation of $\Phi^{(2)}(\nu_0;\cdot,*)$, and Q_x is the new diffusion measure which has invariant measure ν_0. So in order to get our assertion, we only need to show that this expectation converges as $T\to\infty$, *i.e.*, we want to show the L^1-convergence. This will be done if we can show the convergence in law and the L^p-bounded property with respect to $T\to\infty$ for some $p>1$. The convergence in law is not difficult by using the central limit for Hilbert space-valued random variables (see "Proof of Lemma 8.2" of Section 8 for the details).

The proof of the L^p-bounded property for some $p>1$ is the most difficult part of this study. By large deviation principle, there is no problem for the part of the integral on the set $\mathrm{dist}(L_T,\nu_0)>\varepsilon$ for any $\varepsilon>0$, where dist means the Prohorov metric. For the part on the set $\mathrm{dist}(L_T,\nu_0)\le\varepsilon$, we deal in the following way: We first prove (see Section 5) the Ito's formula with respect to the Green operator (this is easy if everything is smooth, but not so trivial in our case since the related quantities are not necessarily smooth). we apply this to the given diffusion process, and then to the time-inversed diffusion (see Section 7). In the way of doing so, we are faced to some estimates with respect to ∇G, where G stands for the Green operator. This uses the "main idea" we just explained, *i.e.*, we observe that, although $|\nabla Gf(x)|$ is not bounded, it can be dominated by $(1+|x|^2)^{\xi/2}\|f\|_\infty$ for some constant $\xi\ge 0$ not too large (see Sections 4, 6), then this extra factor $(1+|x|^2)^{\xi/2}$ can be absorbed by the uniform estimate given in Section 3.

The rest of the paper is organized as follows: We state in Section 2 the setting of the problem and the result. In Section 3 we give some uniform bounded property. In Section 4 we define measure changes and discuss their basic properties. In Section 5 we prove a generalization of Ito's formula for the Green operator. The proof of Theorem 2.1 is given in Sections 6, 7 and 8. Finally, in Section 9, we give some examples that satisfy our conditions.

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2. Set up and Main results.

Let $\mathscr{M}(\boldsymbol{R}^d)$ be the set of all signed measures on $\boldsymbol{R}^d$ with total variation norm $\|\cdot\|$. We also think of the weak*-topology in $\mathscr{M}(\boldsymbol{R}^d)$. Let $\mathscr{M}_0(\boldsymbol{R}^d)=\{\mu\in\mathscr{M}(\boldsymbol{R}^d);\mu(\boldsymbol{R}^d)=0\}$. Let $\wp(\boldsymbol{R}^d)$ be the set of all probabilities on $\boldsymbol{R}^d$, and let $\mathrm{dist}(\cdot,\cdot)$ denote the Prohorov metric on $\wp(\boldsymbol{R}^d)$. Note that the topology induced by the Prohorov metric and the weak*-topology coincide. The path space $\Omega=C([0,\infty);\boldsymbol{R}^d)$ is the set of continuous functions $\omega:[0,\infty)\to\boldsymbol{R}^d$. Let $X_t(\omega)=\omega(t), t\ge 0$, $\mathscr{F}_t=\sigma\{\omega(s);s\le t\}$ and $\mathscr{F}=\vee_t\mathscr{F}_t$ as before.

Let $L_0=(1/2)\sum_{i,j=1}^d a_{ij}\nabla_i\nabla_j+b\cdot\nabla$, where $\nabla_i=(\partial/\partial x_i)$, $i=1,\cdots,d$, and $\nabla=(\nabla_1,\cdots,\nabla_d)$. Also, denote $\sigma=a^{1/2}$. Before stating the assumptions, we prepare some notations.

Let $\{R_t\}_{t\geq 0}$ be any diffusion semi-group with the associated generator written as $L^R = \sum_{i,j=1}^d a_{ij}^R \nabla_i \nabla_j + b^R \cdot \nabla$. We define several conditions.
(**C1**): $a^R := (a_{ij}^R)_{i,j=1}^d \in C_b^\infty(\boldsymbol{R}^d; \boldsymbol{R}^{d^2})$ and is uniformly elliptic, *i.e.*, there exist $c_1, c_2 > 0$ such that

$$c_1 \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a_{ij}^R(x)\xi_i\xi_j \leq c_2 \sum_{i=1}^d \xi_i^2, \qquad \text{for all } x, \xi \in \boldsymbol{R}^d. \tag{2.1}$$

For $\kappa_1, \kappa_2 > 0$, we define the following conditions:
($\mathbf{C2}_{\boldsymbol{\kappa_1}}$): $b^R \in C(\boldsymbol{R}^d; \boldsymbol{R}^d)$ and there exist $c_3, c_4 > 0$ such that

$$x \cdot b^R(x) \leq c_3 - c_4 |x|^{\kappa_1 + 1}, \qquad x \in \boldsymbol{R}^d. \tag{2.2}$$

($\mathbf{C3}_{\boldsymbol{\kappa_2}}$): $b^R \in C^\infty(\boldsymbol{R}^d; \boldsymbol{R}^d)$ and there exist $c_5, c_6 > 0$ such that

$$\sum_{i,j=1}^d \xi_i \xi_j \nabla_i b^R{}_j(x) \leq c_5 \sum_{i=1}^d \xi_i^2,$$
$$|b^R(x)| + |\nabla b^R(x)| \leq c_6(1 + |x|^{\kappa_2}), \qquad x, \xi \in \boldsymbol{R}^d.$$

We use these conditions to define two sets:

$$H_1(\kappa_1) = \{\text{diffusion semi-group } \{R_t\}_{t\geq 0} \text{ satisfying } (C1) \text{ and } (C2_{\kappa_1})\},$$
$$H_2(\kappa_1, \kappa_2) = \{\text{diffusion semi-group } \{R_t\}_{t\geq 0} \text{ satisfying } (C1), (C2_{\kappa_1}) \text{ and } (C3_{\kappa_2})\}.$$

Now, we are ready to give our first assumption:

A1. *There exist* $\gamma_1 > 1$ *and* $\gamma_2 \in [\gamma_1, \gamma_1 + (1/2)(\gamma_1 - 1))$ *such that* $\{P_t\}_{t\geq 0} \in H_2(\gamma_1, \gamma_2)$.

Let $\{P_x\}_{x\in \boldsymbol{R}^d}$ be the family of probabilities on Ω which is the solution of the martingale problem L_0, *i.e.*,
(1) $f(\omega_t) - f(\omega_0) - \int_0^t L_0 f(\omega_s) ds$ is a $(\{\mathscr{F}_t\}, P_x)$-martingale for any $f \in C_b^\infty(\boldsymbol{R}^d; \boldsymbol{R})$,
(2) $P_x(\omega_0 = x) = 1$.
We denote by $\{P_t\}_{t\geq 0}$ the corresponding semi-group.

By [**11**], $\{P_x\}_{x\in \boldsymbol{R}^d}$ has an invariant probability π, $P_t(x, dy) = P_x(X_t \in dy)$ has density $p_t(x,y) \in C^\infty(\boldsymbol{R}^d \times \boldsymbol{R}^d; \boldsymbol{R}^+)$ with respect to π, and $\sup_{x\in \boldsymbol{R}^d, |y|\leq r} p_t(x,y) < \infty$ for any $r, t > 0$. Therefore, we can define the pinned probability $P_x\big(\cdot \,\big|X_t = y\big)$ for all $x, y \in \boldsymbol{R}^d$ and $t > 0$.

Let $L_t = (1/t)\int_0^t \delta_{X_s} ds$, where $\delta_\cdot$ denotes the delta measure. We have by [**14**] that $P_x\big(L_t \in \cdot \big| X_t = y\big)$ satisfies the LDP, *i.e.*, (1.1) holds for any $A \in \mathscr{B}(\wp(\boldsymbol{R}^d))$ with I given by (1.2). Let $\Phi : \mathscr{M}(\boldsymbol{R}^d) \to \boldsymbol{R}$ be a bounded function such that $\Phi\big|_{\wp(\boldsymbol{R}^d)}$ is continuous with respect to $\mathrm{dist}(\cdot,\cdot)$ and let

$$Z_T^{x,y} = E^{P_x}\big[\exp\big(T\Phi(L_T)\big)\big| X_T = y\big].$$

Then $(1/T)\log Z_T^{x,y} \to \lambda$ for every $x, y \in \boldsymbol{R}^d$, where $\lambda = \sup\{\Phi(\nu) - I(\nu); \nu \in \wp(\boldsymbol{R}^d)\}$.

The purpose of this paper is to give a precise estimate of $Z_T^{x,y}$ as $T \to \infty$, up to the order $1 + o(1)$. We need some more assumptions.

For any $T > 0$, $\{X_{T-t}(\omega)\}_{t\in[0,T]}$ under $P_\pi(d\omega)$ is a diffusion associated with the semi-group $\{P_t^{*\pi}\}_{t\geq 0}$, where $P_t^{*\pi}$ denotes the dual operator of P_t in $L_2(d\pi)$.

We need to prepare some notations first. Define $\psi(x) = \sqrt{1+x^2}, x \geq 0$. Also, for $x \in \boldsymbol{R}^d$, let $\psi(x) = \psi(|x|)$. For any $\alpha \in \boldsymbol{R}$, define B_α^0 as

$$B_\alpha^0 = \left\{ f \in C(\boldsymbol{R}^d; \mathbf{C}); \|f\|_{B_\alpha^0} := \sup_{x\in\boldsymbol{R}^d} \psi(x)^{-\alpha} |f(x)| < \infty \right\}.$$

For any $\{R_t\}_{t\geq 0} \in H_1(\gamma)$ with $\gamma > 1$ and any $\varphi \in B_\theta^0$ with $\theta \in [0, \gamma - 1)$, we can define a new semi-group of operators $\{Q(R)_t^\varphi\}_{t\geq 0}$ in the following way (see Section 4 for the details): Define

$$R_t^\varphi f(x) = E^{R_x}\left[\exp\left(\int_0^t \varphi(X_s)ds\right) f(X_t)\right], \qquad x \in \boldsymbol{R}^d, \tag{2.3}$$

where $\{R_x\}_{x\in\boldsymbol{R}^d}$ is the family of diffusion measures associated with $\{R_t\}_{t\geq 0}$. For any $\alpha > 0$, R_t^φ is a continuous linear operator on B_α^0 and $\Lambda^{R,\varphi} := \lim_{t\to\infty}(1/t)\log\|R_t^\varphi\|_{B_\alpha^0 \to B_\alpha^0}$ is well-defined, finite and not depending on $\alpha > 0$, (α could be $\alpha = 0$ if $\varphi = 0$), and there exists a unique (up to constant multiplications) $h^{R,\varphi} \in B_\alpha^0$ such that $R_t^\varphi h^{R,\varphi} = e^{\Lambda^{R,\varphi}t} h^{R,\varphi}$ for any $t > 0$. Let $\{Q(R)_t^\varphi\}_{t\geq 0}$ be the Markovian semi-group given by

$$Q(R)_t^\varphi f := e^{-\Lambda^{R,\varphi}t}(h^{R,\varphi})^{-1} R_t^\varphi (h^{R,\varphi} f), \tag{2.4}$$

and let $\{Q(R)_x^\varphi\}_{x\in\boldsymbol{R}^d}$ denote the corresponding diffusion measures.

REMARK 1. If $\{R_t\}_{t\geq 0} \in H_2(\gamma_1, \gamma_2)$ with $\gamma_1 > 1$ and $\gamma_2 \in [\gamma_1, \gamma_1 + (1/2)(\gamma_1 - 1))$, then $h^{R,\varphi}$ is differentiable and the generator of $\{Q(R)_t^\varphi\}_{t\geq 0}$ is $L^R + a^R(\nabla h^{R,\varphi}/h^{R,\varphi}) \cdot \nabla$, where L^R is the generator of $\{R_t\}_{t\geq 0}$ and a^R is the coefficient of the diffusion term of L^R. Also, $h^{R,\varphi} \in C^\infty(\boldsymbol{R}^d)$ if $\varphi \in C^\infty(\boldsymbol{R}^d)$, and $h^{R,\varphi} \in C_b(\boldsymbol{R}^d)$ if $\varphi \in C_b(\boldsymbol{R}^d)$.

Now, we are ready to give our second assumption.

A2. *There exist a $\{S_t\}_{t\geq 0} \in H_2(\gamma_1', \gamma_2')$ with $\gamma_1' > 1$ and $\gamma_2' \in [\gamma_1', \gamma_1' + (1/2)(\gamma_1' - 1))$, and a $\varphi_0 \in C^\infty(\boldsymbol{R}^d) \cap B_{\theta_0}^0$ with $\theta_0 \in [0, ((\gamma_1' - 1)/2) - (\gamma_2' - \gamma_1'))$ such that $\{P_t^{*\pi}\}_{t\geq 0} = \{Q(S)_t^{\varphi_0}\}_{t\geq 0}$.*

Note that, by Remark 1, the diffusion term of $\{S_t\}_{t\geq 0}$ in A2 is $a := (a_{ij})_{i,j=1}^d$.

Let $K = \{\nu \in \wp(\boldsymbol{R}^d) : \Phi(\nu) - I(\nu) = \lambda\}$. Then K is not empty and is compact in $\wp(\boldsymbol{R}^d)$. We assume the following as in [**10**]:

A3. *There exists only one element in K, say ν_0, i.e., $K = \{\nu_0\}$.*

A4. *$\Phi : \mathscr{M}(\boldsymbol{R}^d) \to \boldsymbol{R}$ is three times continuously Fréchet differentiable and satisfies the following: there exist $\Phi^{(k)} \in C(\wp(\boldsymbol{R}^d) \times (\boldsymbol{R}^d)^k; \boldsymbol{R})$ $(k = 1, 2, 3)$ such that*

$$D^k\Phi(\nu)(R_1,\cdots,R_k)=\int_{\boldsymbol{R}^d}\cdots\int_{\boldsymbol{R}^d}\Phi^{(k)}(\nu;x_1,\cdots,x_k)R_1(dx_1)\cdots R_k(dx_k),$$

for any $\nu\in\wp(\boldsymbol{R}^d)$ *and any* $R_1,R_2,R_3\in\mathscr{M}(\boldsymbol{R}^d)$, $k=1,2,3$.

We construct, as in [**2**] and [**10**], a family of diffusion probabilities $\{Q_x\}_{x\in\boldsymbol{R}^d}$ whose invariant measure is ν_0. Let $\phi^{\nu_0}\in C_b(\boldsymbol{R}^d)$ be given by

$$\phi^{\nu_0}(x)=D\Phi(\nu_0)(\delta_x-\nu_0)+\Phi(\nu_0),\qquad x\in\boldsymbol{R}^d.$$

Then $\lambda=\Lambda^{P,\phi^{\nu_0}}$ and $\{Q_x\}_{x\in\boldsymbol{R}^d}$ is given by $\{Q_x\}_{x\in\boldsymbol{R}^d}=\{Q(P)_x^{\phi^{\nu_0}}\}_{x\in\boldsymbol{R}^d}$. Note that by Remark 1, $h^{P,\phi^{\nu_0}}\in C_b(\boldsymbol{R}^d)\cap C^1(\boldsymbol{R}^d)$ and the generator of $\{Q_x\}_{x\in\boldsymbol{R}^d}$ is $L=L_0+a(\nabla h^{P,\phi^{\nu_0}}/h^{P,\phi^{\nu_0}})\cdot\nabla$. Let G denote the Green operator corresponding to $\{Q_x\}_{x\in\boldsymbol{R}^d}$ and let $\overline{G}=G+G^*$, where G^* means the dual operator of G in $L^2(d\nu_0)$ (See Section 6 for the details).

As in [**3**] and [**10**], we also assume the following:

A5. $I-D^2\Phi(\nu_0)$ *is non-degenerate.*

i.e., we assume that $I-D^2\Phi(\nu_0)\big|_{H\times H}$ is strictly positive-definite. Here H is the Hilbert space whose norm is essentially the second Fréchet differential of the entropy function I. H can be regarded as a dense subset of $\mathscr{M}_0(\boldsymbol{R}^d)$. As I is not smooth, this description is not mathematically precise. See Section 6 for the precise definition of H and statement of (A5).

Finally, we assume the following as in [**10**]:

A6. *For any* $\delta>0$, *there exist an* $\varepsilon_\delta>0$ *and a symmetric* $K_\delta\in C_b(\boldsymbol{R}^d\times\boldsymbol{R}^d;\boldsymbol{R})$ *such that the function* $\widetilde{K_\delta}$ *given by* $\widetilde{K_\delta}(R_1,R_2):=\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}K_\delta(x,y)R_1(dx)R_2(dy)$, $R_1,R_2\in\mathscr{M}_0(\boldsymbol{R}^d)$, *satisfies* $\|\widetilde{K_\delta}\big|_{H\times H}\|_{H.S.}\le\delta$ *and*

$$D^3\Phi(R)(\nu-\nu_0,\nu-\nu_0,\nu-\nu_0)\le\widetilde{K_\delta}(\nu-\nu_0,\nu-\nu_0)$$

for any $R,\nu\in\wp(\boldsymbol{R}^d)$ *with* $\mathrm{dist}(R,\nu_0)<\delta$ *and* $\mathrm{dist}(\nu,\nu_0)<\delta$.

THEOREM 2.1. *Assume* A1$\sim$ A6. *Then*

$$\lim_{T\to\infty}e^{-T\lambda}Z_T^{x,y}=\frac{h(x)}{h(y)}\cdot\exp\left\{\frac{1}{2}\int(\overline{G}\otimes I)\Phi^{(2)}(\nu_0;\cdot,\cdot)\big|_{(u,u)}\nu_0(du)\right\}$$
$$\times\det{}_2(I_H-D^2\Phi(\nu_0))^{-1/2},\qquad \textit{for any } x,y\in\boldsymbol{R}^d.$$

REMARK 2. It is not difficult to see, by checking the proofs in Sections 6 and 7 carefully, that the conditions with respect to $\gamma_1,\gamma_2,\gamma_1',\gamma_2'$ and θ_0 in the assumptions A1 and A2 can be relaxed as follows: $\gamma_1>1$, $\gamma_1'>1$, $\gamma_2<\gamma_1+((\gamma_1\vee\gamma_1')-1)/2$, $\gamma_2'<\gamma_1'+[(\gamma_1'-1)\wedge((\gamma_1\vee\gamma_1')-1)/2]$ and $\theta_0\in[0,\gamma_1'-1-(\gamma_2'-\gamma_1'))$.

3. Some bounded property.

Let $\gamma > 1$, and let $\{R_t\}_{t\geq 0} \in H_1(\gamma)$. Write the generator of $\{R_t\}_{t\geq 0}$ as $L^R = \sum_{i,j=1}^d a_{ij}^R \nabla_i \nabla_j + b^R \cdot \nabla$. Then $a^R = (a_{ij}^R)_{i,j=1}^d$ is uniformly elliptic, and by (2.2) with $\kappa_1 = \gamma$, there exist constants $c_3, c_4 > 0$ such that

$$x \cdot b^R \leq c_3 - c_4 |x|^{\gamma+1}. \tag{3.1}$$

Let $\{R_x\}_{x\in\boldsymbol{R}^d}$ denote the associated family of probabilities.

We first have the following by Ito's formula and a simple calculation:

LEMMA 3.1. *For any $\alpha \geq 2$ and $x \in \boldsymbol{R}^d$, $t > 0$, we have that*

$$\frac{d}{dt} E^{R_x}[|X_t|^\alpha]^{2/\alpha} \leq (2c_3 + c_2 d + (\alpha-2)c_2) - 2c_4 \big(E^{R_x}[|X_t|^\alpha]^{2/\alpha}\big)^{(\gamma+1)/2}.$$

PROOF. Let $\{X_t\}_{t\geq 0}$ be the diffusion corresponding to L^R, *i.e.*,

$$dX_t = \sigma^R(X_t) dB_t + b^R(X_t) dt, \tag{3.2}$$

where σ^R is the $d \times d$-matrix given by $\sigma^R = (a^R)^{1/2}$. Then by Ito's formula,

$$d|X_t|^2 = 2X_t \cdot \sigma^R(X_t) dB_t + 2X_t \cdot b^R(X_t) dt + \sum_{i=1}^d a_{ii}^R(X_t) dt.$$

So by Ito's formula again, we get

$$\begin{aligned} d|X_t|^\alpha &= d(|X_t|^2)^{\alpha/2} \\ &= \Bigg(\alpha |X_t|^{\alpha-2} X_t \cdot b^R(X_t) + \frac{\alpha}{2} |X_t|^{\alpha-2} \sum_{i=1}^d a_{ii}^R(X_t) \\ &\qquad + \frac{\alpha}{2}(\alpha-2)|X_t|^{\alpha-4} X_t \cdot a^R(X_t) X_t \Bigg) dt + \text{martingale}. \end{aligned} \tag{3.3}$$

By our condition with respect to a^R, we have that $\sum_{i=1}^d a_{ii}^R(X_t) \leq c_2 d$ and $X_t \cdot a^R(X_t) X_t \leq c_2 |X_t|^2$. Also, by (3.1),

$$X_t \cdot b^R(X_t) \leq c_3 - c_4 |X_t|^{r+1}.$$

Therefore, (3.3) gives us that

$$\begin{aligned} \frac{d}{dt} E^{R_x}[|X_t|^\alpha] &\leq \frac{\alpha}{2}(2c_3 + c_2 d + (\alpha-2)c_2) E^{R_x}[|X_t|^{\alpha-2}] - \alpha c_4 E^{R_x}[|X_t|^{\alpha+\gamma-1}] \\ &\leq \frac{\alpha}{2}(2c_3 + c_2 d + (\alpha-2)c_2) E^{R_x}[|X_t|^\alpha]^{(\alpha-2)/\alpha} - \alpha c_4 E^{R_x}[|X_t|^\alpha]^{(\alpha+\gamma-1)/\alpha}, \end{aligned}$$

where in the last inequality we used the fact that $\alpha \geq 2$ and $\gamma > 1$.

This gives us our asserion since $(d/dt)E^{R_x}[|X_t|^\alpha]^{2/\alpha} = (2/\alpha)E^{R_x}[|X_t|^\alpha]^{2/\alpha-1}\ (d/dt)E^{R_x}[|X_t|^\alpha]$. □

Also, we have by Stirling's formula

$$\sum_{n=0}^{\infty} \frac{C^n}{n!}(\alpha n)^{\alpha n/(\gamma+1)} < \infty, \qquad \text{for any } \alpha \in [0, \gamma+1), C > 0. \tag{3.4}$$

Now, we are ready to show the following

LEMMA 3.2. $\sup_{x\in \boldsymbol{R}^d, t\geq T} E^{R_x}\left[e^{C|X_t|^\beta}\right] < \infty$ *for any* $\beta \in [0, \gamma+1)$, $T > 0$ *and* $C > 0$.

PROOF. Let $n \in \boldsymbol{N}$ be such that $n\beta \geq 2$ and fix it for a while. Let $u_x(t) = E^{R_x}[|X_t|^{n\beta}]^{2/(n\beta)}$ and $K_1 = 2c_3 + c_2 d + (n\beta - 2)c_2$. Then by Lemma 3.1,

$$\frac{d}{dt}u_x(t) \leq K_1 - 2c_4 u_x(t)^{(\gamma+1)/2}. \tag{3.5}$$

Let $\tau_x = \inf\{t \geq 0; K_1 - 2c_4 u_x(t)^{(\gamma+1)/2} \geq 0\}$. Then (3.5) implies

$$\int_{u_x(t)}^{u_x(0)} \frac{ds}{2c_4 s^{(\gamma+1)/2} - K_1} \geq t, \qquad t \in (0, \tau_x]. \tag{3.6}$$

On the other hand, let $K_2 = (K_1/c_4)^{2/(\gamma+1)}$ and $t_1 = 1/(c_4(\gamma-1))K_2^{-(\gamma-1)/2}$. Then since $\gamma > 1$, we have that for any $t \in (0, t_1]$, there exists an $a(t) > 0$ such that

$$\int_{a(t)}^{\infty} \frac{ds}{2c_4 s^{(\gamma+1)/2} - K_1} = t. \tag{3.7}$$

(3.6) and (3.7) imply that $u_x(t) \leq a(t)$ for any $t \in (0, \tau_x \wedge t_1]$. Also, it is trivial that $u_x(t) \leq (1/2)^{2/(\gamma+1)}K_2$ if $t \geq \tau_x$. Note that $a(t) \leq K_2 \vee (2/(c_4(\gamma-1)))^{2/(\gamma-1)} t^{-2/(\gamma-1)}$. Combining these, we get that

$$u_x(t) \leq \left(\frac{2}{c_4(\gamma-1)}\right)^{2/(\gamma-1)} t^{-2/(\gamma-1)}, \qquad t \in (0, t_1].$$

So by the semi-group property,

$$u_x(t) \leq \max\left\{2^{2/(\gamma-1)}K_2, \left(\frac{2}{c_4(\gamma-1)}\right)^{2/(\gamma-1)} t^{-2/(\gamma-1)}\right\}, \qquad t > 0, x \in \boldsymbol{R}^d.$$

This combined with Taylor expansion $E^{R_x}\left[e^{C|X_t|^\beta}\right] = \sum_{n=0}^{\infty}(C^n/n!)E^{R_x}[|X_t|^{n\beta}]$ and (3.4) completes the proof. □

LEMMA 3.3.

$$\limsup_{T\to\infty}\frac{1}{T-1}\log\sup_{x\in\boldsymbol{R}^d}E^{R_x}\left[\exp\left(C\int_1^T|X_t|^{2\xi}\,dt\right)\right]<\infty$$

for any $\xi\in[0,(\gamma+1)/2)$ and $C>0$.

PROOF. Let $\xi\in(1,(\gamma+1/2))$ and let $K_3=\sup_{x\in\boldsymbol{R}^d,t\geq 1}E^{R_x}[\exp(2C|X_t|^{2\xi})]$, which is finite by Lemma 3.2. Then by Schwartz's inequality and the Markovian property,

$$\begin{aligned}\sup_{x\in\boldsymbol{R}^d}E^{R_x}\left[e^{C\int_1^{2n+1}|X_t|^{2\xi}dt}\right]&\leq\sup_{x\in\boldsymbol{R}^d}\left(E^{R_x}\left[e^{2C\sum_{k=1}^n\int_{2k-1}^{2k}|X_t|^{2\xi}dt}\right]\right)^{1/2}\cdot\left(E^{R_x}\left[e^{2C\sum_{k=1}^n\int_{2k}^{2k+1}|X_t|^{2\xi}dt}\right]\right)^{1/2}\\&\leq\left(\sup_{x\in\boldsymbol{R}^d}E^{R_x}\left[e^{2C\int_1^2|X_t|^{2\xi}dt}\right]\right)^n\leq(K_3)^n,\qquad\text{for any } n\in\boldsymbol{N}.\end{aligned}$$

Our assertion is now easy. □

Let $\psi(x)=(1+|x|^2)^{1/2}$ as before. Also, for $\lambda>0$ and $\varphi\in B^0_\theta$ with $\theta\in[0,\gamma-1)$, let

$$g_{\lambda,\varphi}(x)=-\lambda\psi(x)^2+\varphi(x)\psi(x)^2+\alpha(c_3-c_4|x|^{\gamma+1})+\frac{\alpha}{2}c_2d+\frac{1}{2}\alpha(\alpha-2)\psi(x)^{-2}x\cdot a^R(x)x.$$

LEMMA 3.4. *Let $\varphi\in B^0_\theta$ with $\theta\in[0,\gamma-1)$. Then for any $\alpha>0$, there exists a $\lambda(\alpha,\varphi)>0$ such that*

$$E^{R_x}\left[e^{\int_0^t\varphi(X_s)ds}\psi(X_t)^\alpha\right]\leq e^{\lambda(\alpha,\varphi)t}\psi(x)^\alpha,\qquad x\in\boldsymbol{R}^d,t\geq 0.\tag{3.8}$$

Moreover, $\lambda(\alpha,\varphi)>0$ can be chosen to be continuous and monotone nondecreasing with respect to $\|\varphi\|_{B^0_\theta}$ and converges to 0 as $\|\varphi\|_{B^0_\theta}\to 0$.

PROOF. By assumption, $\gamma>1$ and $\varphi\in B^0_\theta$ with $\theta<\gamma-1$. So for any $\alpha>0$, there exists a constant $\lambda=\lambda(\alpha,\varphi)>0$ such that $g_{\lambda,\varphi}(x)\leq 0, x\in\boldsymbol{R}^d$. Let

$$Y_t^\lambda=e^{-\lambda t+\int_0^t\varphi(X_s)ds}\psi(X_t)^\alpha.$$

Then by (3.2), we have by Ito's formula

$$\begin{aligned}dY_t^\lambda=e^{-\lambda t+\int_0^t\varphi(X_s)ds}\psi(X_t)^{\alpha-2}\bigg(&-\lambda\psi(X_t)^2+\varphi(X_t)\psi(X_t)^2\\&+\alpha X_t\cdot b^R(X_t)+\frac{\alpha}{2}\sum_{i=1}^d a_{ii}^R(X_t)+\frac{\alpha}{2}(\alpha-2)\psi(X_t)^{-2}X_t\cdot a^R(X_t)X_t\bigg)dt\\&+\text{martingale}.\end{aligned}$$

Therefore, as in the proof of Lemma 3.1, by (3.1) and the assumption with respect to a^R, we have that

$$\frac{d}{dt}E^{R_x}[Y_t^\lambda] \le E^{R_x}\Big[e^{-\lambda t+\int_0^t \varphi(X_s)ds}\psi(X_t)^{\alpha-2}g_{\lambda,\varphi}(X_t)\Big]. \tag{3.9}$$

So by the choice of λ, we have $(d/dt)E^{R_x}[Y_t^\lambda] \le 0$. Therefore, $E^{R_x}[Y_t^\lambda] \le E^{R_x}[Y_0^\lambda] = \psi(x)^\alpha$. This gives us our first assertion. The others are now easy. □

We also have the following:

LEMMA 3.5. *Let φ be as in Lemma 3.4. Then for any $\alpha, \varepsilon > 0$, there exists a constant $\lambda(\varepsilon,\alpha,\varphi) > 0$ such that*

$$E^{R_x}\Big[e^{\int_0^t \varphi(X_s)ds}\Big] \le e^{\lambda(\varepsilon,\alpha,\varphi)t}(1+\varepsilon\psi(x)^\alpha), \qquad x \in \boldsymbol{R}^d, t \ge 0.$$

PROOF. By assumption, $\gamma > 1$ and $\varphi \in B_\theta^0$ with $\theta < \gamma - 1$. So for any $\alpha, \varepsilon > 0$, there exists a $\lambda = \lambda(\varepsilon,\alpha,\varphi) > 0$ such that

$$-\lambda + \varphi(x) + \varepsilon\psi(x)^{\alpha-2}g_{\lambda,\varphi}(x) \le 0.$$

Let

$$\widetilde{Y_t^\lambda} = e^{-\lambda t+\int_0^t \varphi(X_s)ds}(1+\varepsilon\psi(X_t)^\alpha).$$

Then

$$d\widetilde{Y_t^\lambda} = (-\lambda + \varphi(X_t))e^{-\lambda t+\int_0^t \varphi(X_s)ds}dt + \varepsilon dY_t^\lambda.$$

Therefore, by (3.9), we have

$$\frac{d}{dt}E^{R_x}[\widetilde{Y_t^\lambda}] \le E^{R_x}\Big[\Big(-\lambda + \varphi(X_t) + \varepsilon\psi(X_t)^{\alpha-2}g_{\lambda,\varphi}(X_t)\Big)e^{-\lambda t+\int_0^t \varphi(X_s)ds}\Big].$$

So by our choice of λ now, we get $(d/dt)E^{R_x}[\widetilde{Y_t^\lambda}] \le 0$. Therefore,

$$E^{R_x}\Big[e^{\int_0^t \varphi(X_s)ds}\Big] \le e^{\lambda t}E^{R_x}[\widetilde{Y_t^\lambda}] \le e^{\lambda t}E^{R_x}[\widetilde{Y_0^\lambda}] = e^{\lambda t}(1+\varepsilon\psi(x)^\alpha). \qquad \square$$

4. Measure Changes.

As in Section 3, let $\gamma > 1$, $\{R_t\}_{t\ge 0} \in H_1(\gamma)$, and $\varphi \in B_\theta^0$ with $\theta \in [0, \gamma-1)$. $\{R_t\}_{t\ge 0}$ has an invariant probability μ which has all moments finite, and $R_t(x,dy)$ has a continuous density $r_t(x,y)$ with respect to μ. The following is easy by [**15**, Lemma 4.3]:

LEMMA 4.1. *For any $\alpha \ge 0$ there exists a $C_\alpha > 0$ such that for any $\beta > 0$, $R_u : B_\alpha^0 \to B_\beta^0$*

is a compact operator satisfying

$$\|R_u\|_{B^0_\alpha \to B^0_\beta} \le C_\alpha u^{-(\alpha-\beta)/(\gamma-1)}, \qquad \text{for any } u \in (0,1].$$

Let R^φ_t be as in (2.3). Then it is easy to see that

$$R^\varphi_t = R_t + \int_0^t R_s \varphi R^\varphi_{t-s} ds, \qquad t > 0. \tag{4.1}$$

We use (4.1) and Lemma 4.1 to give the following:

LEMMA 4.2. *Let α, β be any positive constants satisfying $\alpha - \beta < \gamma - 1 - \theta$. Then $R^\varphi_t : B^0_\alpha \to B^0_\beta$ is a compact operator for any $t > 0$.*

PROOF. By (3.8), $R^c_t : B^0_\alpha \to B^0_\alpha$ is bounded for any $\alpha, t > 0$, so by semi-group property, we only need to show the lemma for $t \in (0,1]$. Also, it is trivial that we only need to show the lemma for any $\alpha \ge \beta > 0$ satisfying $\alpha - \beta < \gamma - 1 - \theta$.

Fix any such $\alpha, \beta > 0$. By (4.1) and Lemma 4.1, it is sufficient if $\int_0^t R_s \varphi R^\varphi_{t-s} ds$ is compact. We show this now. By Lemma 3.4, the definition of φ and Lemma 4.1, we have that for any $0 < s < t \le 1$, $R_s \varphi R^\varphi_{t-s} : B^0_\alpha \to B^0_\beta$ is a compact operator with norm

$$\|R_s \varphi R^\varphi_{t-s}\|_{B^0_\alpha \to B^0_\beta} \le e^{\lambda(\alpha,\varphi)} C_{\alpha+\theta} \|\varphi\|_{B^0_\theta} s^{-(\alpha-\beta+\theta)/(\gamma-1)}.$$

Since $(\alpha - \beta + \theta)/(\gamma - 1) < 1$ by assumption, this completes the proof. □

Let $\alpha > 0$ and fix it for a while. By Lemma 3.4, $\{R^\varphi_t\}_{t \ge 0}$ is a semi-group of continuous linear operators on B^0_α, so the logarithmic spectral radius $\Lambda^{R,\varphi} := \lim_{t\to\infty}(1/t)\log\|R^\varphi_t\|_{B^0_\alpha \to B^0_\alpha}$ of $R^\varphi_t : B^0_\alpha \to B^0_\alpha$ is well-defined.

The generator of $\{R_t\}_{t\ge 0}$ is L^R, so the 0-order term of the generator of the semi-group $\{e^{-\Lambda^{R,\varphi}t}\psi^{-\alpha}R^\varphi_t\psi^\alpha\}_{t\ge 0}$ on $C_b(\boldsymbol{R}^d)$ is $\psi^{-\alpha}L^R\psi^\alpha + \varphi - \Lambda^{R,\varphi}$, which goes to $-\infty$ as $|x| \to \infty$ by assumption. Also, it is easy to see that $\psi^{-\alpha}R^\varphi_t\psi^\alpha f \in C_\infty(\boldsymbol{R}^d)$ for $f \in C_b(\boldsymbol{R}^d)$ and $\alpha > 0$. Here $C_\infty(\boldsymbol{R}^d)$ denotes the set of continuous functions on $\boldsymbol{R}^d$ that converges to 0 at ∞.

With the preparations above, we are now ready to give the following:

LEMMA 4.3.

1. $\Lambda^{R,\varphi} := \lim_{t\to\infty}(1/t)\log\|R^\varphi_t\|_{B^0_\alpha \to B^0_\alpha}$ *is well-defined.*
2. *For any $t > 0$, $e^{-\Lambda^{R,\varphi}t}R^\varphi_t : B^0_\alpha \to B^0_\alpha$ is compact, the spectral radius 1 is a simple eigenvalue of it, and is the only eigenvalue with a positive eigenfunction. Also, the absolute value of any other eigenvalue is smaller than 1.*
3. *There exists a unique (up to constant multiplication) $h^{R,\varphi} \in B^0_\alpha$ such that $R^\varphi_t h^{R,\varphi} = e^{\Lambda^{R,\varphi}t}h^{R,\varphi}$ for any $t > 0$. Moreover, $(h^{R,\varphi})^{-1} \in B^0_\alpha$.*
4. *There exists a set of probabilities $\{Q(R)^\varphi_x\}_{x\in\boldsymbol{R}^d}$ on $(\Omega, \mathscr{F})$ such that*

$$Q(R)^\varphi_x(A) = \frac{e^{-\Lambda^{R,\varphi}t}}{h^{R,\varphi}(x)} E^{R_x}\left[1_A \exp\left(\int_0^t \varphi(X_s)ds\right) h^{R,\varphi}(X_t)\right]$$

for any $x \in \boldsymbol{R}^d$, $t \geq 0$ *and* $A \in \mathscr{F}_t$. *The corresponding semi-group of continuous linear operators on* B^0_α *is* $\{Q(R)^\varphi_t\}_{t\geq 0}$ *given by* (2.4).

5. *There exists a unique* $\{Q(R)^\varphi_t\}$*-invariant probability* μ^φ. (*Now, we can determine the* $h^{R,\varphi}$ *above uniquely by requiring* $(h^{R,\varphi})^{-1}\mu^\varphi$ *to be a probability.*)
6. *For any* $t > 0$ *and* $x \in \boldsymbol{R}^d$, $Q(R)^\varphi_t(x, dy)$ *is absolutely continuous with respect to* μ^φ *with density* $q^{R,\varphi}_t(x,y) \in C(\boldsymbol{R}^d \times \boldsymbol{R}^d)$.
7. 1 *is a simple eigenvalue of* $Q(R)^\varphi_t$ *with eigenfunction* 1, *is the only eigenvalue of it with a positive eigenfunction, and the absolute value of any other eigenvalue is smaller than* 1. *Therefore, there exist constants* $C_\varphi, \varepsilon_\varphi > 0$ *such that*

$$\left\| Q(R)^\varphi_t f - \int_{\boldsymbol{R}^d} f\, d\mu^\varphi \right\|_{B^0_\alpha} \leq C_\varphi e^{-\varepsilon_\varphi t} \|f\|_{B^0_\alpha}, \qquad \text{for any } t \geq 1 \text{ and } f \in B^0_\alpha.$$

PROOF. By applying the same argument as in [**14**, Section 3] to $e^{-\Lambda^{R,\varphi} t}\psi^{-\alpha} R^\varphi_t \psi^\alpha$, we get all of our assertions except the fact $(h^{R,\varphi})^{-1} \in B^0_\alpha$. We show this in the following. As a corollary of Lemma 3.2, $\sup_{x\in\boldsymbol{R}^d} R_x(|X_1| \geq r) \to 0$ as $r \to \infty$, so there exists an $r > 0$ such that

$$\inf_{x\in\boldsymbol{R}^d} R_x(|X_1| \leq r) \geq \frac{1}{2}. \tag{4.2}$$

Also, note that

$$E^{R_x}\left[e^{\int_0^1 \varphi(X_s)ds} 1_A\right] \geq \frac{R_x(A)^2}{E^{R_x}\left[e^{\int_0^1 |\varphi(X_s)|ds}\right]}, \qquad x \in \boldsymbol{R}^d, A \in \mathscr{F}. \tag{4.3}$$

By Lemma 3.4

$$E^{R_x}\left[e^{\int_0^t |\varphi(X_s)|ds}\right] \leq e^{\lambda(\alpha,|\varphi|)t}\psi(x)^\alpha, \qquad \text{for any } t > 0, x \in \boldsymbol{R}^d. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) implies

$$e^{\Lambda^{R,\varphi}} h^{R,\varphi}(x) = E^{R_x}\left[e^{\int_0^1 \varphi(X_s)ds} h^{R,\varphi}(X_1)\right] \geq \frac{1}{4} e^{-\lambda(\alpha,|\varphi|)} \left(\inf_{B_r} h^{R,\varphi}\right) \psi(x)^{-\alpha}.$$

This gives us that $(h^{R,\varphi})^{-1} \in B^0_\alpha$. □

REMARK 3. It is easy to see by Perron-Frobnius argument that, if $\theta = 0$, then the same result of Lemma 4.3 holds with $\alpha = 0$.

REMARK 4. The uniqueness of the positive eigenfunction of R^φ_t on B^0_α for any $\alpha > 0$ gives us that $\Lambda^{R,\varphi}$ and $h^{R,\varphi}$ do not depend on $\alpha > 0$.

For any $\delta > 0$, we say that a semi-group $\{U_t\}_{t\geq 0}$ satisfies (B_δ) if for any $\alpha \geq 0$ and $\beta > 0$, there exist $C_{\alpha,\beta} > 0$ and $d_\beta \in (0,1)$ such that

$$|\nabla U_t f(x)| \le \psi(x)^{\delta+\beta+\alpha} \frac{C_{\alpha,\beta}}{t^{d_\beta}} \|f\|_{B^0_\alpha}, \qquad x \in \boldsymbol{R}^d, t \in (0,1], f \in B^0_\alpha. \tag{4.5}$$

We have the following by [**15**, Theorem 1.1]:

LEMMA 4.4. *Assume that there exist* $\delta_1 > 1$ *and* $\delta_2 \in [\delta_1, \delta_1 + (1/2)(\delta_1 - 1))$ *such that* $\{R_t\}_{t\ge 0} \in H_2(\delta_1, \delta_2)$. *Then* $\{R_t\}_{t\ge 0}$ *satisfies* (B_δ) *with* $\delta := \delta_2 - \delta_1 + 1$.

LEMMA 4.5. *Assume that* $\{R_t\}_{t\ge 0}$ *satisfies* (B_δ). *Then* $\{R^\varphi_t\}_{t\ge 0}$ *satisfies* $(B_{\delta+\theta})$.

PROOF. First note that by (4.1),

$$\nabla R^\varphi_t = \nabla R_t + \nabla \int_0^t R_s \varphi R^\varphi_{t-s} ds. \tag{4.6}$$

Also, by assumption,

$$\|\nabla R_t\|_{B^0_\alpha \to B^0_{\delta+\beta+\alpha+\theta}} \le \|\nabla R_t\|_{B^0_{\alpha+\theta} \to B^0_{\delta+\beta+\alpha+\theta}} \le \frac{C_{\alpha+\theta,\beta}}{t^{d_\beta}}, \qquad t \in (0,1].$$

Next, we estimate the second term on the right hand side of (4.6). By Lemma 3.4, the definition of φ and (4.5), we have $\|R^\varphi_u\|_{B^0_\alpha \to B^0_\alpha} \le e^{\lambda(\alpha,\varphi)}$, $\|\varphi\|_{B^0_\alpha \to B^0_{\alpha+\theta}} \le \|\varphi\|_{B^0_\theta}$ and $\|\nabla R_u\|_{B^0_{\alpha+\theta} \to B^0_{\delta+\beta+\alpha+\theta}} \le C_{\alpha+\theta,\beta} u^{-d_\beta}$ for $u \in (0,1)$. Therefore,

$$\begin{aligned}\left\| \nabla \int_0^t R_s \varphi R^\varphi_{t-s} ds \right\|_{B^0_\alpha \to B^0_{\delta+\beta+\alpha+\theta}} &\le \int_0^t e^{\lambda(\alpha,\varphi)} \|\varphi\|_{B^0_\theta} C_{\alpha+\theta,\beta} s^{-d_\beta} ds \\ &= C_{\alpha+\theta,\beta} e^{\lambda(\alpha,\varphi)} \|\varphi\|_{B^0_\theta} \frac{1}{1-d_\beta} t^{1-d_\beta}, \qquad t \in (0,1].\end{aligned}$$

These complete the proof. □

LEMMA 4.6. *Assume that* $\{R_t\}_{t\ge 0}$ *satisfies* (B_δ). *Then* $\{Q(R)^\varphi_t\}_{t\ge 0}$ *satisfies* $(B_{\delta+\theta})$.

PROOF. By the definition of $Q(R)^\varphi_t$,

$$\nabla Q(R)^\varphi_t f(x) = e^{-\Lambda^{R,\varphi} t} \frac{\nabla R^\varphi_t (h^{R,\varphi} f)}{h^{R,\varphi}} + e^{-\Lambda^{R,\varphi} t} \frac{\nabla h^{R,\varphi}}{(h^{R,\varphi})^2} R^\varphi_t (h^{R,\varphi} f), \quad f \in B^0_\alpha.$$

Note that $h^{R,\varphi} = e^{-\Lambda^{R,\varphi}} R^\varphi_1 h^{R,\varphi}$ and $h^{R,\varphi} \in B^0_{\alpha_0}$ for any $\alpha_0 > 0$ by the definition of $h^{R,\varphi}$. So $\nabla h^{R,\varphi} \in B^0_{\delta+\beta+\alpha_0+\theta}$ for any $\beta > 0$ by Lemma 4.5. Also, $(h^{R,\varphi})^{-1} \in B^0_{\alpha_0}$ by Lemma 4.3. These combined with Lemma 4.5 complete the proof. □

By Lemma 4.3, for any $\alpha > 0$, we can define the Green operator $\widetilde{G^{R,\varphi}}$ on B^0_α given by

$$\widetilde{G^{R,\varphi}} f = \int_0^\infty \left(Q(R)^\varphi_t f - \int_{\boldsymbol{R}^d} f d\mu^\varphi \right) dt. \tag{4.7}$$

$\widetilde{G^{R,\varphi}}$ maps B^0_α into B^0_α, and it is a continuous linear operator. Now, we are ready to give the following:

LEMMA 4.7. *Assume that $\{R_t\}_{t\geq 0}$ satisfies (B_δ). Then $\widetilde{G^{R,\varphi}} f \in C^1(\boldsymbol{R}^d)$ for any $f \in C_b(\boldsymbol{R}^d)$. Moreover, for any $\beta > 0$, there exists a $c_7 > 0$ (which may depend on β, $\{R\}_{t\geq 0}$ and φ) such that*

$$|\nabla \widetilde{G^{R,\varphi}} f(x)| \leq c_7 \psi(x)^{\delta+\theta+\beta} \|f\|_\infty, \qquad \text{for any } f \in C_b(\boldsymbol{R}^d), x \in \boldsymbol{R}^d.$$

PROOF. Let $\widetilde{f} = f - \int_{\boldsymbol{R}^d} f \, d\mu^\varphi$. Then $\|\widetilde{f}\|_\infty \leq 2\|f\|_\infty$ and $\widetilde{G^{R,\varphi}} f = \widetilde{G^{R,\varphi}} \widetilde{f}$. Therefore, without loss of generality, we may and do assume that $\int_{\boldsymbol{R}^d} f \, d\mu^\varphi = 0$.

Since

$$\widetilde{G^{R,\varphi}} f = \int_0^1 Q(R)^\varphi_t f \, dt + Q(R)^\varphi_1 \widetilde{G^{R,\varphi}} f,$$

we get from Lemma 4.6 for any $x \in \boldsymbol{R}^d$

$$\begin{aligned} |\nabla \widetilde{G^{R,\varphi}} f(x)| &\leq \int_0^1 |\nabla Q(R)^\varphi_t f(x)| \, dt + |\nabla Q(R)^\varphi_1 \widetilde{G^{R,\varphi}} f(x)| \\ &\leq \psi(x)^{\delta+\theta+\beta} \left(C_{0,\beta} \int_0^1 t^{-d_\beta} \, dt + C_{\beta/2,\beta/2} \|\widetilde{G^{R,\varphi}}\|_{B^0_{\beta/2} \to B^0_{\beta/2}} \right) \|f\|_\infty. \end{aligned}$$

Taking $c_7 = C_{0,\beta} \int_0^1 t^{-d_\beta} \, dt + C_{\beta/2,\beta/2} \|\widetilde{G^{R,\varphi}}\|_{B^0_{\beta/2} \to B^0_{\beta/2}}$ completes the proof. □

5. Ito's formula.

Let $\{R_t\}_{t\geq 0} \in H_1(\gamma)$ with $\gamma > 1$, and assume that it satisfies (B_δ) with $\delta \geq 1$. Denote the generator of $\{R_t\}_{t\geq 0}$ as $L^R = \sum_{i,j=1}^d a^R_{ij} \nabla_i \nabla_j + b^R \cdot \nabla$, let $a^R = (a^R_{ij})^d_{i,j=1}$ and $\sigma^R = (a^R)^{1/2}$. Also, let $\varphi \in B^0_\theta$ with $\theta \in [0, \gamma - 1)$.

Our main results of this section are the following two lemmas.

PROPOSITION 5.1. *$h^{R,\varphi} \in C^1(\boldsymbol{R}^d)$ and the generator of $\{Q(R)^\varphi_t\}_{t>0}$ is*

$$L^{R,\varphi} = L^R + a^R \frac{\nabla h^{R,\varphi}}{h^{R,\varphi}} \cdot \nabla.$$

By Proposition 5.1, we see that the diffusion corresponding to $\{Q(R)^\varphi_x\}$ is a semimartingale with respect to the canonical filtration.

PROPOSITION 5.2. *For any $f \in C_b(\boldsymbol{R}^d)$ with $\int_{\boldsymbol{R}^d} f \, d\mu^\varphi = 0$, let $g = -\widetilde{G^{R,\varphi}} f$ (where $\widetilde{G^{R,\varphi}}$ is as defined in (4.7)). Then $g \in C^1(\boldsymbol{R}^d)$. Also, let $\{X_t\}$ be the diffusion process corresponding to $\{Q(R)^\varphi_x\}$, and let*

$$B_t = \int_0^t (\sigma^R)^{-1}(X_s) \, dX_s - \int_0^t (\sigma^R)^{-1}(X_s) \left(b^R + a^R \frac{\nabla h^{R,\varphi}}{h^{R,\varphi}} \right)(X_s) \, ds.$$

Then $\{B_t\}_{t\geq 0}$ is a Brownian motion and

$$g(X_t)=g(X_0)+\int_0^t \nabla g(X_s)\cdot \sigma^R(X_s)dB_s+\int_0^t f(X_s)ds.$$

We make some preparation before giving the proofs of them. For any $\varphi\in B_\theta^0$, the operator $G^{R,\varphi}$ given by

$$G^{R,\varphi}f=\int_0^\infty\left(e^{-\Lambda^{R,\varphi}t}R_t^\varphi f-h^{R,\varphi}\int_{\boldsymbol{R}^d}\frac{f}{h^{R,\varphi}}d\mu^\varphi\right)dt,\qquad f\in B_\alpha^0,$$

is well-defined by Lemma 4.3. Note that by the same way as in the proof of Lemma 4.7, we get by Lemma 4.5 that $G^{R,\varphi}f\in C^1(\boldsymbol{R}^d)$. We will give a kind of Ito's formula for $G^{R,\varphi}$ first. Before doing so, we need to show some continuity. Let $B_n=\{x\in\boldsymbol{R}^d;|x|\leq n\}$, $n\in\boldsymbol{N}$.

Let us define two conditions.

(I) $\varphi,\{\varphi_n\}_{n\in\boldsymbol{N}}\subset B_\theta^0$ are bounded and $\varphi_n\to\varphi$ in $C_b(B_m)$ as $n\to\infty$ for any $m\in\boldsymbol{N}$,

(II) $f,\{f_n\}_{n\in\boldsymbol{N}}\subset B_\alpha^0$ are bounded and $f_n\to\varphi$ in $C_b(B_m)$ as $n\to\infty$ for any $m\in\boldsymbol{N}$.

LEMMA 5.3. *Assume* (I). *Then $\|R_t^{\varphi_n}-R_t^\varphi\|_{B_\alpha^0\to B_\alpha^0}\to 0$ as $n\to\infty$ for any $t,\alpha>0$.*

PROOF. First note that by Lemmas 3.2 and 3.4, there exists a $c_8>0$ such that

$$\begin{aligned}&\left|R_t^{\varphi_n}f(x)-R_t^\varphi f(x)\right|\\&\quad\leq E^{R_x}\left[\left|e^{\int_0^t(\varphi_n-\varphi)(X_s)ds}-1\right|^2\right]^{1/2}\cdot E^{R_x}\left[e^{4\int_0^t\varphi(X_s)ds}\right]^{1/4}E^{R_x}\left[\psi(X_t)^{4\alpha}\right]^{1/4}\|f\|_{B_\alpha^0}\\&\quad\leq c_8\psi(x)^{\alpha/2}\left(E^{R_x}\left[e^{2\int_0^t|\varphi_n-\varphi|(X_s)ds}\right]-1\right)^{1/2}\|f\|_{B_\alpha^0},\quad\text{for any } f\in B_\alpha^0, x\in\boldsymbol{R}^d.\end{aligned}$$

Therefore,

$$\|R_t^{\varphi_n}-R_t^\varphi\|_{B_\alpha^0\to B_\alpha^0}\leq c_8\left\{\sup_{x\in\boldsymbol{R}^d}\psi(x)^{-\alpha}\left(E^{R_x}\left[e^{2\int_0^t|\varphi_n-\varphi|(X_s)}\right]-1\right)\right\}^{1/2}.\tag{5.1}$$

We estimate the right hand side of (5.1) from now on. Let $K_2=\max\{\|\varphi_n\|_{B_\theta^0};n\in\boldsymbol{N}\}\vee\|\varphi\|_{B_\theta^0}<\infty$. Then by Lemma 3.5 and Hölder's inequality

$$\begin{aligned}&\sup_{x\in\boldsymbol{R}^d}\psi(x)^{-\alpha}\left(E^{R_x}\left[e^{2\int_0^t|\varphi_n-\varphi|(X_s)}\right]-1\right)\\&\quad\leq\left(e^{(1/2)\lambda(\varepsilon,\alpha,8K_2\psi^\theta)t_0}\sup_{x\in\boldsymbol{R}^d}E^{R_x}\left[e^{4\int_{t_0}^t|\varphi_n-\varphi|(X_s)}\right]^{1/2}-1\right)\\&\qquad+\varepsilon e^{(1/2)\lambda(\varepsilon,\alpha,8K_2\psi^\theta)t_0}\sup_{x\in\boldsymbol{R}^d}E^{R_x}\left[e^{4\int_{t_0}^t|\varphi_n-\varphi|(X_s)}\right]^{1/2},\quad\text{for any }\varepsilon,t_0>0.\end{aligned}\tag{5.2}$$

For any $t_0>0$, we have by Lemma 3.2

$$\begin{aligned}
&\sup_{x\in\boldsymbol{R}^d}\left(E^{R_x}\left[e^{4\int_{t_0}^t|\varphi_n-\varphi|(X_s)ds}\right]-1\right)\\
&\quad\le\left(e^{4(t-t_0)\|\varphi_n-\varphi\|_{C_b(B_m)}}-1\right)\\
&\qquad+\sup_{s\ge t_0,x\in\boldsymbol{R}^d}E^{R_x}\left[\exp(16tK\psi^\theta(X_s))\right]^{1/2}\cdot\sup_{s\ge t_0,x\in\boldsymbol{R}^d}R_x(|X_s|\ge m)^{1/2}\\
&\quad\to 0,\qquad \text{as } m\to\infty.
\end{aligned}$$

Here we used the fact $\lim_{m\to\infty}\sup_{s\ge t_0,x\in\boldsymbol{R}^d}R_x(|X_s|\ge m)=0$ which comes easily from Lemma 3.2. This combined with (5.2) implies

$$\sup_{x\in\boldsymbol{R}^d}\psi(x)^{-\alpha}\left(E^{R_x}\left[e^{2\int_0^t|\varphi_n-\varphi|(X_s)}\right]-1\right)\to 0,\qquad \text{as } n\to\infty,$$

which combined with (5.1) completes the proof. □

LEMMA 5.4. *Assume* (I). *Then for any* $\alpha>0$ *we have the following*:
(1) $\lim_{n\to\infty}\Lambda^{R,\varphi_n}=\Lambda^{R,\varphi}$,
(2) $\lim_{n\to\infty}h^{R,\varphi_n}=h^{R,\varphi}$ *in* B^0_α, *and* $\lim_{n\to\infty}\mu^{\varphi_n}=\mu^\varphi$ *in* $\wp(\boldsymbol{R}^d)$,
(3) $\lim_{n\to\infty}G^{R,\varphi_n}=G^{R,\varphi}$ *as operators on* B^0_α.

PROOF. (1) follows from Lemma 5.3, Lemma 4.3 and [**5**, Lemma VII.6.3].

As for (2), we have $e^{-\Lambda^{R,\varphi_n}t}R_t^{\varphi_n}$ and $e^{-\Lambda^{R,\varphi}t}R_t^{\varphi}$ are compact operators on B^0_α, and 1 is a simple eigenvalue of both of them. We have $\lim_{n\to\infty}e^{-\Lambda^{R,\varphi_n}t}R_t^{\varphi_n}=e^{-\Lambda^{R,\varphi}t}R_t^{\varphi}$ as operators on B^0_α. Hence by [**5**, Lemma VII.6.5] with respect to the convergence of projection operators, we have $\lim_{n\to\infty}h^{R,\varphi_n}\langle\cdot,(h^{R,\varphi_n})^{-1}d\mu^{\varphi_n}\rangle=h^{R,\varphi}\langle\cdot,(h^{R,\varphi})^{-1}d\mu^{\varphi}\rangle$ as operators on B^0_α. So $\lim_{n\to\infty}h^{R,\varphi_n}=h^{R,\varphi}$ in B^0_α and hence $\lim_{n\to\infty}\mu^{\varphi_n}=\mu^\varphi$ in $\wp(\boldsymbol{R}^d)$.

(3) follows easily from Lemma 4.3, [**5**, Lemma VII.6.5] and the dominated convergence theorem. □

LEMMA 5.5. *Fix any* $\alpha>0$. *Assume* (I) *and* (II). *Then* $G^{R,\varphi_n}f_n\to G^{R,\varphi}f$ *in* B^0_α *and* $\nabla G^{R,\varphi_n}f_n\to\nabla G^{R,\varphi}f$ *in* $C_b(B_m)$ *for any* $m\in\boldsymbol{N}$.

PROOF. The first assertion easily follows from Lemma 5.4 and the decomposition

$$\begin{aligned}
&\|G^{R,\varphi_n}f_n-G^{R,\varphi}f\|_{B^0_\alpha}\\
&\quad\le\|G^{R,\varphi_n}-G^{R,\varphi}\|_{B^0_\alpha\to B^0_\alpha}\cdot\sup_{n\in\boldsymbol{N}}\|f_n\|_{B^0_\alpha}+\|G^{R,\varphi}1_{B_m^C}\psi^\alpha\|_{B^0_\alpha}\left(\sup_{n\in\boldsymbol{N}}\|f_n\|_{B^0_\alpha}+\|f\|_{B^0_\alpha}\right)\\
&\qquad+\|G^{R,\varphi}((f_n-f)1_{B_m})\|_{B^0_\alpha}.
\end{aligned}$$

In the same way, $R_t^{\varphi_n}f_n\to R_t^{\varphi}f$ in B^0_α as $n\to\infty$. We show the second one.

CLAIM 1. *For any* $\alpha'\ge 0$ *and* $g_n,g\in B^0_{\alpha'}$ *bounded with* $g_n\to g$ *in* $C_b(B_m)$ *for any* $m\in\boldsymbol{N}$, *we have* $\lim_{n\to\infty}\nabla R_tg_n=\nabla R_tg$ *in* $C_b(B_m)$ *for any* $t>0$ *and* $m\in\boldsymbol{N}$.

PROOF OF CLAIM 1. Since $\{R_t\}_{t\ge0}$ satisfies (B_δ), we have for any $\alpha'\ge0$ and $\beta,\varepsilon>0$

$$
\begin{aligned}
&|\nabla R_t g_n(x) - \nabla R_t g(x)| \\
&\quad \le \psi(x)^{\delta+\beta+\alpha'+\varepsilon} C_{\alpha'+\varepsilon,\beta} t^{-d_\beta} (\sup_{n\in\boldsymbol{N}} \|g_n\|_{B^0_{\alpha'}} + \|g\|_{B^0_{\alpha'}}) \psi(m)^{-\varepsilon} \\
&\qquad + \psi(x)^{\delta+\beta+\alpha'} C_{\alpha'+\varepsilon,\beta} t^{-d_\beta} \|(g_n - g) 1_{B_m}\|_{B^0_{\alpha'}}, \quad m \in \boldsymbol{N}.
\end{aligned}
$$

The first term on the right hand side above converges to 0 as $m \to \infty$ uniformly in $n \in \boldsymbol{N}$ and $x \in B_r$ for any $r > 0$, and the second term converges to 0 as $n \to \infty$ uniformly in $x \in B_r$ for any $m \in \boldsymbol{N}$ and $r > 0$. This gives us Claim 1. (Proof of Claim 1) □

CLAIM 2. *For any $g_n, g \in B^0_\alpha$ bounded with $g_n \to g$ in $C_b(B_m)$ for any $m \in \boldsymbol{N}$, we have $\nabla R^{\varphi_n}_t g_n \to \nabla R^\varphi_t g$ in $C_b(B_m)$ as $n \to \infty$ for any $t \in (0,1]$ and $m \in \boldsymbol{N}$.*

PROOF OF CLAIM 2. Recall (4.6). Since $g_n \to g$ and $\varphi_n R^{\varphi_n}_{t-s} g_n \to \varphi R^\varphi_{t-s} g$ in $C_b(B_m)$ as $n \to \infty$ for any $s \in (0,t)$ and $m \in \boldsymbol{N}$, we get by Claim 1 that $\nabla R_t g_n \to \nabla R_t g$ and $\nabla R_s \varphi_n R^{\varphi_n}_{t-s} g_n \to \nabla R_s \varphi R^\varphi_{t-s} g$ in $C_b(B_m)$ as $n \to \infty$ for any $s \in (0,t)$ and $m \in \boldsymbol{N}$. This combined with (B_δ) and dominated convergence theorem gives us Claim 2. (Proof of Claim 2) □

Claim 2 combined with Lemma 5.4 and the definition of $h^{R,\cdot}$ gives us $\nabla h^{R,\varphi_n} \to \nabla h^{R,\varphi}$ and $\nabla R^{\varphi_n}_1 G^{R,\varphi_n} f_n \to \nabla R^\varphi_1 G^{R,\varphi} f$ in $C_b(B_m)$ as $n \to \infty$ for any $m \in \boldsymbol{N}$. Also,

$$
\begin{aligned}
\left|\nabla G^{R,\varphi_n} f_n - \nabla G^{R,\varphi} f\right| \le &\int_0^1 \left|e^{-\Lambda^{R,\varphi_n} t} \nabla R^{\varphi_n}_t f_n - e^{-\Lambda^{R,\varphi} t} \nabla R^\varphi_t f\right| dt \\
&+ \int_0^1 \left|\nabla h^{R,\varphi_n} \int \frac{f_n}{h^{R,\varphi_n}} d\mu^{\varphi_n} - \nabla h^{R,\varphi} \int \frac{f}{h^{R,\varphi}} d\mu^\varphi\right| dt \\
&+ \left|e^{-\Lambda^{R,\varphi_n}} \nabla R^{\varphi_n}_1 G^{R,\varphi_n} f_n - e^{-\Lambda^{R,\varphi}} \nabla R^\varphi_1 G^{R,\varphi} f\right|.
\end{aligned}
$$

These combined with Lemma 5.4 complete the proof. □

LEMMA 5.6. *Let $\eta \in B^0_\alpha$ and $\{X_t\}_{t\ge0}$ the diffusion corresponding to $L^R + \eta \cdot \nabla$. Let $B_t = \int_0^t (\sigma^R)^{-1}(X_s) dX_s - \int_0^t (\sigma^R)^{-1}(X_s)(b^R + \eta)(X_s) ds$. Then $\{B_t\}_{t\ge0}$ is a Brownian motion and*

$$
\begin{aligned}
G^{R,\varphi} f(X_t) = &G^{R,\varphi} f(X_0) + \int_0^t \nabla G^{R,\varphi} f(X_s) \cdot \sigma^R(X_s) dB_s \\
&- \int_0^t \left(f + (\varphi - \Lambda^{R,\varphi}) G^{R,\varphi} f - \eta \nabla G^{R,\varphi} f - h^{R,\varphi} \int_{\boldsymbol{R}^d} \frac{f}{h^{R,\varphi}} d\mu^\varphi\right)(X_s) ds.
\end{aligned}
$$

PROOF. The fact that $\{B_t\}_{t\ge0}$ is a Brownian motion is easy (*c.f.* Ikeda-Watanabe [**7**, Chapter 2]).

Since $\varphi \in B^0_\theta$ and $f \in B^0_\alpha$, there exist sequences $\{\varphi_n\}_{n\in\boldsymbol{N}} \in C^\infty(\boldsymbol{R}^d) \cap B^0_\theta$ and $\{f_n\}_{n\in\boldsymbol{N}} \in C^\infty(\boldsymbol{R}^d) \cap B^0_\alpha$ that satisfy the conditions (I) and (II) previous to Lemma 5.3. So by Lemma 5.5, $G^{R,\varphi_n} f_n \to G^{R,\varphi} f$ in B^0_α and $\nabla G^{R,\varphi_n} f_n \to \nabla G^{R,\varphi} f$ in $C_b(B_m)$ as $n \to \infty$ for any $m \in \boldsymbol{N}$.

Since φ_n, f_n are smooth, we have by Gilbarg-Trudinger [**6**, Theorem 8.13] that $G^{R,\varphi_n} f_n \in C^\infty(\boldsymbol{R}^d)$. Let $\tau_m = \inf\{t \ge 0; X_t \notin B_m\}$. By Ito's formula,

$$G^{R,\varphi_n}(f_n)(X_{t\wedge\tau_m}) = G^{R,\varphi_n}(f_n)(X_0) + \int_0^{t\wedge\tau_m} \nabla G^{R,\varphi_n} f_n(X_s)\cdot\sigma^R(X_s)dB_s + \int_0^{t\wedge\tau_m}(L^R+\eta\nabla)\left(G^{R,\varphi_n} f_n\right)(X_s)ds, \quad m,n\in\boldsymbol{N}. \tag{5.3}$$

$G^{R,\varphi_n} f_n \to G^{R,\varphi} f$ in $C_b^1(B_m)$ as $n\to\infty$, and

$$L^R G^{R,\varphi_n} f_n = -f_n - (\varphi_n - \Lambda^{R,\varphi_n})G^{R,\varphi_n}(f_n) + h^{R,\varphi_n}\int_{\boldsymbol{R}^d}\frac{f_n}{h^{R,\varphi_n}}d\mu^{\varphi_n}$$
$$\to -f - (\varphi - \Lambda^{R,\varphi})G^{R,\varphi}(f) + h^{R,\varphi}\int_{\boldsymbol{R}^d}\frac{f}{h^{R,\varphi}}d\mu^{\varphi}$$

in $C_b(B_m)$. Take first $n\to\infty$ then $m\to\infty$. Since $\tau_m\to\infty$ as $m\to\infty$ almost surely, this completes the proof. □

PROOF OF PROPOSITION 5.1. By definition, $\lim_{t\to 0}(R_t h^{R,\varphi} - h^{R,\varphi})/t = (\Lambda^{R,\varphi}-\varphi)h^{R,\varphi}$. So

$$\int_{\boldsymbol{R}^d}(\varphi h^{R,\varphi} - \Lambda^{R,\varphi}h^{R,\varphi})d\mu = 0 \tag{5.4}$$

and

$$h^{R,\varphi} - \int_{\boldsymbol{R}^d} h^{R,\varphi}d\mu = G^{R,0}(\varphi h^{R,\varphi} - \Lambda^{R,\varphi}h^{R,\varphi}). \tag{5.5}$$

Let $\{X_t\}_{t\ge 0}$ be the diffusion corresponding to L^R. Then by Lemma 5.6,

$$h^{R,\varphi}(X_t) = h^{R,\varphi}(X_0) + \int_0^t \nabla h^{R,\varphi}(X_s)\cdot\sigma^R(X_s)dB_s + \int_0^t(\Lambda^{R,\varphi}h^{R,\varphi} - \varphi h^{R,\varphi})(X_s)ds.$$

So by Ito's formula

$$e^{-\Lambda^{R,\varphi}t}\frac{h^{R,\varphi}(X_t)}{h^{R,\varphi}(X_0)}\exp\left(\int_0^t\varphi(X_s)ds\right)$$
$$= \exp\left(\int_0^t h^{R,\varphi}(X_s)^{-1}\nabla h^{R,\varphi}(X_s)\cdot\sigma^R(X_s)dB_s - \frac{1}{2}\int_0^t (h^{R,\varphi}(X_s))^{-2}\nabla h^{R,\varphi}(X_s)\cdot a^R(X_s)\nabla h^{R,\varphi}(X_s)ds\right).$$

The left hand side above is nothing but $(dQ(R)^{\varphi}_{X_0}/dR_{X_0})(\omega)\big|_{\mathscr{F}_t}$. This gives us our assertion. □

PROOF OF PROPOSITION 5.2. The fact that $\{B_t\}_{t\ge 0}$ is a Brownian motion is easy (*c.f.* Ikeda-Watanabe [**7**, Chapter 2]). By Lemma 5.6, Proposition 5.1 and the assumption $\int_{\boldsymbol{R}^d} f d\mu^{\varphi} = 0$,

$$G^{R,\varphi}(h^{R,\varphi}f)(X_t) = G^{R,\varphi}(h^{R,\varphi}f)(X_0) + \int_0^t \nabla G^{R,\varphi}(h^{R,\varphi}f)(X_s)\cdot\sigma^R(X_s)dB_s$$
$$- \int_0^t \big(h^{R,\varphi}f + (\varphi - \Lambda^{R,\varphi})G^{R,\varphi}(h^{R,\varphi}f)$$
$$- (h^{R,\varphi})^{-1}\nabla h^{R,\varphi}\cdot a^R\nabla G^{R,\varphi}(h^{R,\varphi}f)\big)(X_s)ds.$$

Also, (5.4), (5.5) and Proposition 5.1 give us that

$$h^{R,\varphi}(X_t) = h^{R,\varphi}(X_0) + \int_0^t \nabla h^{R,\varphi}(X_s)\sigma^R(X_s)dB_s$$
$$- \int_0^t \big((\varphi - \Lambda^{R,\varphi})h^{R,\varphi} - (h^{R,\varphi})^{-1}\nabla h^{R,\varphi}\cdot a^R\nabla h^{R,\varphi}\big)(X_s)ds.$$

Since $\widetilde{G^{R,\varphi}} = (h^{R,\varphi})^{-1}G^{R,\varphi}h^{R,\varphi}$, the above and Ito's formula give us Proposition 5.2. □

As an immediate result of Lemma 4.3, Lemma 4.6 and Proposition 5.1, we have the following:

REMARK 5. *Suppose that* $\{R_t\}_{t\geq 0} \in H_1(\gamma)$ *and satisfies* (B_δ) *with* $\gamma > 1$ *and* $\delta \geq 1$. *Let* $\varphi \in B^0_\theta$ *with* $\theta \in [0, \gamma - \delta)$. *Then* $\{Q(R)^\varphi_t\}_{t\geq 0} \in H_1(\gamma)$ *and satisfies* $(B_{\delta+\theta})$.

6. Preparations and basic estimates.

Let us go back to the situation described in Section 2. From now on, we will omit the superscript R when $R = P$, if there is not risk of confusion, *i.e.*, we write $\Lambda^{P,\varphi}$ as Λ^φ, $h^{P,\varphi}$ as h^φ, *etc*.

Recall that $\{Q_t\}_{t\geq 0} = \{Q(P)^{\phi^{\nu_0}}\}_{t\geq 0}$ and $\{\widetilde{P^{*\pi}_t}\}_{t\geq 0} = \{Q(S)^{\varphi_0}_t\}_{t\geq 0}$. Let G be the Green operator corresponding to $\{Q_t\}_{t\geq 0}$, *i.e.*, $G = \widetilde{G^{P,\phi^{\nu_0}}}$. Then it is easy to see that the dual operator $\widetilde{Q^*_t \text{ of } Q_t}$ (resp., G^* of G) in $L^2(d\nu_0)$ is $Q^*_t = Q(P^{*\pi})^{\phi^{\nu_0}}_t = Q(S)^{\phi^{\nu_0}+\varphi_0}_t$, (resp., $G^* = \widetilde{G^{P^{*\pi},\phi^{\nu_0}}} = G^{S,\phi^{\nu_0}+\varphi_0}$).

We first have the following:

LEMMA 6.1. ν_0 *is the* (*unique*) *invariant probability measure of* $\{Q_x\}_{x\in\boldsymbol{R}^d}$.

PROOF. Let $I^{\phi^{\nu_0}}$ be the rate function corresponding to $\{Q^\phi_t\}_{t\geq 0}$, *i.e.*,

$$I^{\phi^{\nu_0}}(\nu) = \sup\left\{\int_{\boldsymbol{R}^d}\varphi d\nu - \lim_{t\to\infty}\frac{1}{t}\log\|(Q(P)^{\phi^{\nu_0}})^\varphi_t\|_{L^\infty\to L^\infty}; \varphi \in C_b(\boldsymbol{R}^d)\right\}, \qquad \nu \in \wp(\boldsymbol{R}^d).$$

Then it is easy that

$$I^{\phi^{\nu_0}}(\nu) = I(\nu) - \int_{\boldsymbol{R}^d}\phi^{\nu_0}d\nu + \Lambda^{\phi^{\nu_0}}, \qquad \text{for any } \nu \in \wp(\boldsymbol{R}^d).$$

ν minimize $I^{\phi^{\nu_0}}$ if and only if ν is the invariant probability measure of $\{Q_x\}_{x\in\boldsymbol{R}^d}$.

Since ν_0 maximize $\Phi - I$ and I is convex, we have for any $t \in (0,1)$ and $\nu \in \wp(\boldsymbol{R}^d)$

$$\begin{aligned}\Phi(\nu_0) - I(\nu_0) &\geq \Phi(t\nu + (1-t)\nu_0) - I(t\nu + (1-t)\nu_0)\\ &\geq \Phi(t\nu + (1-t)\nu_0) - tI(\nu) - (1-t)I(\nu_0).\end{aligned}$$

Hence

$$\frac{\Phi(t\nu + (1-t)\nu_0) - \Phi(\nu_0)}{t} \leq I(\nu) - I(\nu_0).$$

The left hand side above converges to $D\Phi(\nu_0)(\nu - \nu_0) = \int_{\boldsymbol{R}^d} \phi^{\nu_0} d\nu - \int_{\boldsymbol{R}^d} \phi^{\nu_0} d\nu_0$ as $t \to 0$. So

$$I(\nu_0) - \int_{\boldsymbol{R}^d} \phi^{\nu_0} d\nu_0 \leq I(\nu) - \int_{\boldsymbol{R}^d} \phi^{\nu_0} d\nu, \qquad \text{for any } \nu \in \wp(\boldsymbol{R}^d).$$

Therefore, ν_0 minimize $I^{\phi^{\nu_0}}$, which implies that ν_0 is the (unique) invariant probability measure of $\{Q_x\}_{x\in\boldsymbol{R}^d}$. □

LEMMA 6.2. *For any* $\beta < (\gamma_1 \vee \gamma_1') + 1$ *and* $C > 0$,

$$\int_{\boldsymbol{R}^d} e^{C|x|^\beta} \nu_0(dx) < \infty, \tag{6.1}$$

$$\limsup_{T\to\infty} \frac{1}{T} \log E^{Q_{\nu_0}}\left[\exp\left(C\int_0^T |X_t|^\beta dt\right)\right] < \infty. \tag{6.2}$$

PROOF. By Remark 5, we have $\{Q_t\}_{t\geq 0} = \{Q(P)_t^{\phi^{\nu_0}}\}_{t\geq 0} \in H_1(\gamma_1)$ and $\{Q_t^*\}_{t\geq 0} = \{Q(S)_t^{\phi^{\nu_0}+\varphi_0}\}_{t\geq 0} \in H_1(\gamma_1')$. Since ν_0 is $\{Q_t\}_{t\geq 0}$-invariant, by Lemma 3.2, we have (6.2) holds for any $\beta < (\gamma_1 \vee \gamma_1') + 1$ and $C > 0$. The second one is now easy by Schwartz inequality and Markovian property, as in the proof of Lemma 3.3. □

Also, we have the following two Lemmas by Lemma 4.7 and Proposition 5.2:

LEMMA 6.3. *For any* $\beta > 0$, *there exists a* $c_9 > 0$ (*depending on* β) *such that*

$$\begin{aligned}&|\nabla G f(x)| \leq c_9 \psi(x)^{\gamma_2 - \gamma_1 + 1 + \beta} \|f\|_\infty,\\ &|\nabla G^* f(x)| \leq c_9 \psi(x)^{\gamma_2' - \gamma_1' + 1 + \theta_0 + \beta} \|f\|_\infty, \qquad x \in \boldsymbol{R}^d, f \in C_b(\boldsymbol{R}^d).\end{aligned}$$

LEMMA 6.4. (1) *For any* $f \in C_b(\boldsymbol{R}^d)$ *with* $\int_{\boldsymbol{R}^d} f d\nu_0 = 0$, *let* $g_1 = -Gf$ *and* $g_2 = -G^* f$. *Then* $g_1, g_2 \in C^1(\boldsymbol{R}^d)$.
(2) *Let* $\{X_t\}_{t\geq 0}$ *be the diffusion corresponding to* $\{Q_t\}_{t\geq 0}$. *Then there exists a Brownian motion* $\{B_t\}_{t\geq 0}$ *such that*

$$g_1(X_t) = g_1(X_0) + \int_0^t \nabla g_1(X_s)\sigma(X_s) dB_s + \int_0^t f(X_s) ds,$$

and for any $T > 0$*, there exists a Brownian motion (with respect to the canonical backward filtraction)* $\{\hat{B}_t^T\}_{t\in[0,T]}$ *such that*

$$g_2(X_{T-t}) = g_2(X_T) + \int_0^t \nabla g_2(X_{T-s})\sigma(X_{T-s})d\hat{B}_s^T + \int_0^t f(X_{T-s})ds.$$

We remark that $\{X_{T-t}\}_{t\in[0,T]}$ is a diffusion with respect to the canonical backward filtraction associated with semi-group $\{Q_t^*\}_{t\in[0,T]}$ for any $T > 0$.

Let $\overline{G} = G + G^*$, and let $\Gamma(f,g) = \int_{\boldsymbol{R}^d} f\overline{G}g d\nu_0$, $f,g \in C_b(\boldsymbol{R}^d)$. Then we have the following:

LEMMA 6.5. *$\Gamma(f,f) = \int_{\boldsymbol{R}^d} \nabla Gf \cdot a\nabla Gf d\nu_0$ for any $f \in C_b(\boldsymbol{R}^d)$. In particular, $\Gamma(f,f) \geq 0$, and $\Gamma(f,f) = 0$ if and only if f is constant.*

PROOF. Let $\{X_t\}$ be the diffusion corresponding to $\{Q_x\}_{x\in\boldsymbol{R}^d}$. Then for any $f \in C_b(\boldsymbol{R}^d)$ with $\int_{\boldsymbol{R}^d} f d\nu_0 = 0$,

$$\Gamma(f,f) = \int_{\boldsymbol{R}^d} f\overline{G}f d\nu_0 = \lim_{T\to\infty} \frac{1}{T} E^{Q_{\nu_0}}\left[\left(\int_0^T f(X_s)ds\right)^2\right].$$

By Proposition 5.2,

$$\int_0^T f(X_s)ds = Gf(X_0) - Gf(X_T) + \int_0^T \nabla Gf(X_s)\cdot\sigma(X_s)dB_s, \tag{6.3}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion. Let $M_t = \int_0^t \nabla Gf \cdot \sigma(X_s)dB_s$, $t \geq 0$. Then $\{M_t\}_{t\geq 0}$ is a continuous local martingale. Since $\{Q_t\}_{t\geq 0}$ is ν_0-invariant and $\int_{\boldsymbol{R}^d} \psi^\alpha d\nu_0 < \infty$ for any $\alpha > 0$, we have by Lemma 4.7 and A1 that $E^{Q_{\nu_0}}\{\langle M,M\rangle_T\} = T\int_{\boldsymbol{R}^d} \nabla Gf \cdot a\nabla Gf d\nu_0 < \infty$ for all $T \geq 0$. So $\{M_t\}_{t\geq 0}$ is a martingale. Therefore, $(1/T)E^{Q_{\nu_0}}\{M_T^2\} = (1/T)E^{Q_{\nu_0}}\{\langle M,M\rangle_T\} = \int_{\boldsymbol{R}^d} \nabla Gf \cdot a\nabla Gf d\nu_0$. Also, Gf is bounded, so it is easy to see by (6.3) that

$$\frac{1}{T}E^{Q_{\nu_0}}\left[\left(\int_0^T f(X_s)ds\right)^2\right] \to \int_{\boldsymbol{R}^d} \nabla Gf\cdot a\nabla Gf d\nu_0 \qquad \text{as } T\to\infty.$$

These give us that $\Gamma(f,f) = \int_{\boldsymbol{R}^d} \nabla Gf \cdot a\nabla Gf d\nu_0$, for any $f \in C_b(\boldsymbol{R}^d)$. Now, the facts that $\Gamma(f,f) \geq 0$ and that $\Gamma(f,f) = 0$ if and only if f is constant are easy since the matrix a is strictly positive definite. □

Let $\sim$ be the equivalent relation in $C_b(\boldsymbol{R}^d)$ given by $f \sim g$ if and only if $f - g$ is equal to constant. Let $\widetilde{C_b}(\boldsymbol{R}^d) = C_b(\boldsymbol{R}^d)/\sim$. Then Γ is an inner product on $\widetilde{C_b}(\boldsymbol{R}^d)$. Let $H = \left(\overline{\widetilde{C_b}(\boldsymbol{R}^d)}^{\Gamma}\right)^*$. Then H is a Hilbert space, and can be regarded as a dense subspace of $\mathscr{M}_0(\boldsymbol{R}^d)$.

Since ν_0 maximize $\Phi - I = \Phi - I^{\phi^{\nu_0}} + \langle\phi^{\nu_0},\cdot\rangle - \Lambda^{P,\phi^{\nu_0}}$, and is the invariant measure of $\{Q_t\}_{t\geq 0} = \{Q(P)_t^{\phi^{\nu_0}}\}_{t\geq 0}$, we have the following by the same method as in [**16**, Section 2]:

LEMMA 6.6.

$$D^2\Phi(\nu_0)(\overline{G}fd\nu_0,\overline{G}fd\nu_0)\le(f,\overline{G}f)_{L^2(d\nu_0)},\qquad \textit{for any } f\in C_b(\boldsymbol{R}^d).$$

By Lemma 6.6, all of the eigenvalues of $D^2\Phi(\nu_0)\big|_{H\times H}$ are not greater than 1. Now, we are ready to give a precise formulation of the assumption A5:

A5' *All of the eigenvalues of* $D^2\Phi(\nu_0)\big|_{H\times H}$ *are less than* 1, *i.e.,*

$$D^2\Phi(\nu_0)(\overline{G}fd\nu_0,\overline{G}fd\nu_0)<(f,\overline{G}f)_{L^2(d\nu_0)},\qquad \text{for any } f\in C_b(\boldsymbol{R}^d).$$

Let G_x and G_y be the continuous linear extensions of $G\otimes I$ and $I\otimes G$ on $C_b(\boldsymbol{R}^d)\times C_b(\boldsymbol{R}^d)$, respectively. G_y^*, $\overline{G}_x$, *etc*, are defined in the same way. Also, for any symmetric $V\in C_b(\boldsymbol{R}^d\times\boldsymbol{R}^d;\boldsymbol{R})$, define $A_V:\mathscr{M}_0(\boldsymbol{R}^d)\times\mathscr{M}_0(\boldsymbol{R}^d)\to\boldsymbol{R}$ by $A_V(R_1,R_2)=\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}V(x,y)R_1(dx)R_2(dy)$. Then A_V is symmetric, bilinear and continuous. The following is easy, and the proof is omitted.

LEMMA 6.7. *For any symmetric* $V\in C_b(\boldsymbol{R}^d\times\boldsymbol{R}^d;\boldsymbol{R})$, $\nabla_x\nabla_yG_xG_y^*V(x,y)$ *is well-defined and is in* $C(\boldsymbol{R}^d\times\boldsymbol{R}^d)$. *Moreover,* $A_V\big|_{H\times H}$ *is a Hilbert-Schmidt function and*

$$\begin{aligned}\Big\|A_V\big|_{H\times H}\Big\|_{H.S.}^2&=\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}V(x,y)\overline{G}_x\overline{G}_yV(x,y)\nu_0(dx)\nu_0(dy)\\&=\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}\sum_{i,j,k,l=1}^d\nabla_{x_k}\nabla_{y_i}G_xG_y^*V(x,y)a_{ij}(y)a_{kl}(x)\nabla_{x_l}\nabla_{y_j}G_xG_y^*V(x,y)\nu_0(dx)\nu_0(dy).\end{aligned}$$

7. Estimate for L^p-bounded.

Our main result of this section is the following:

PROPOSITION 7.1. *Let* $V\in C_b(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *be symmetric and satisfies* $\int_{\boldsymbol{R}^d}V(x,y)\nu_0(dy)=0$ *for any* $x\in\boldsymbol{R}^d$. *Also, suppose that all of the eigenvalues of* $A_V\big|_{H\times H}$ *are smaller than* 1. *Then there exists an* $\varepsilon_0>0$ *such that*

$$\sup_{T>0}E^{Q_x}\left[e^{(1/2T)\int_0^T\int_0^TV(X_t,X_s)dtds},A_{T,\varepsilon}\middle|X_T=y\right]<\infty \tag{7.1}$$

for any $x,y\in\boldsymbol{R}^d$ *and* $\varepsilon\le\varepsilon_0$. *Here* $A_{T,\varepsilon}=\{\mathrm{dist}(L_T,\nu_0)<\varepsilon\}$.

We first prepare several notations. For $0\le t\le T$, let $L_{t,T}=(1/(T-t))\int_t^T\delta_{X_s}ds$ and

$$\begin{aligned}A_{t,T,\varepsilon}&=\{\mathrm{dist}(L_{t,T},\nu_0)<\varepsilon\},\\A_{t,T,\varepsilon}^f&=\left\{\left|\int_{(\boldsymbol{R}^d)^k}fdL_{t,T}^{\otimes k}-\int_{(\boldsymbol{R}^d)^k}fd\nu_0^{\otimes k}\right|<\varepsilon\right\}\end{aligned}$$

for $f\in C((\boldsymbol{R}^d)^k)$, $k=1,2$. We write $A_{T,\varepsilon}^f=A_{0,T,\varepsilon}^f$.

We have the following Harnack inequality by Krylov-Safonov [**8**]:

LEMMA 7.2. *Let $\{R_t\}_{t\geq 0} \in H_2(\kappa_1,\kappa_2)$ with $\kappa_1 > 1$ and $\kappa_2 \in [\kappa_1, \kappa_1 + (1/2)(\kappa_1 - 1))$. Then for any $x \in \boldsymbol{R}^d$, there exists a $c_x > 0$ such that*

$$R_1 f(x) \leq c_x \int_{\boldsymbol{R}^d} f d\mu, \qquad \text{for any } f \in B_b(\boldsymbol{R}^d, \boldsymbol{R}^+).$$

Note that for any bounded $\sigma\{X_s; 1 \leq s \leq T-1\}$-measurable function f, we have

$$\int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} E^{Q_{\nu_0}}\left[f \middle| X_1 = x, X_{T-1} = y \right] \nu_0(dx)\nu_0(dy) = E^{Q_{\nu_0}}[f],$$

and that $q(1,\cdot,y)$ is bounded on $\boldsymbol{R}^d$. So we have by applying Lemma 7.2 to $\{Q_x\}_{x\in\boldsymbol{R}^d}$

LEMMA 7.3. *For any $x,y \in \boldsymbol{R}^d$, there exists a $c_{x,y} > 0$ such that*

$$E^{Q_x}\left[f \middle| X_T = y \right] \leq c_{x,y} E^{Q_{\nu_0}}[f]$$

for any $T > 2$ and $\sigma\{X_s; 1 \leq s \leq T-1\}$-measurable positive bounded function f.

For any $\varepsilon > 0$ and $T \geq 2 \vee 8/\varepsilon$, we have $A_{T,\varepsilon/2} \subset A_{1,T-1,\varepsilon} \subset A_{T,(3/2)\varepsilon}$. So by Lemma 7.2

$$\begin{aligned} &E^{Q_x}\left[e^{1/(2T)\int_0^T\int_0^T V(X_t,X_s)dtds}, L_{T,\varepsilon/2} \middle| X_T = y \right] \\ &\qquad \leq e^{4\|V\|_\infty} c_{x,y} E^{Q_{\nu_0}}\left[e^{1/(2T)\int_0^T\int_0^T V(X_t,X_s)dtds}, A_{T,2\varepsilon} \right]. \end{aligned}$$

Therefore, to show Proposition 7.1, it only remains to prove that there exists an $\varepsilon > 0$ such that

$$\sup_{T>1} E^{Q_{\nu_0}}\left[e^{1/(2T)\int_0^T\int_0^T V(X_t,X_s)dtds}, A_{T,\varepsilon} \right] < \infty. \tag{7.2}$$

We divide the proof of (7.2) into several steps. Let

$$\begin{aligned} U_1(x,y) &= -(G_x V)(x,y), \\ U(x,y) &= -(G_y^* U_1)(x,y), \\ W(x,y) &= \sum_{i,j,k,l=1}^d \nabla_{x_k}\nabla_{y_i} U(x,y) a_{kl}(x) a_{ij}(y) \nabla_{x_l}\nabla_{y_j} U(x,y), \quad x,y \in \boldsymbol{R}^d. \end{aligned}$$

Then we have the following by the continuity of G and Lemma 6.3:

LEMMA 7.4. *For any $\beta > 0$, there exists a $c_{10} > 0$ (depending on β) such that*

$$\begin{aligned} &|\nabla_x U(x,y)| \leq c_{10}\|V\|_\infty \psi(x)^{\gamma_2-\gamma_1+1+\beta}, \\ &|\nabla_x\nabla_y U(x,y)| \leq {c_{10}}^2 \|V\|_\infty \psi(x)^{\gamma_2-\gamma_1+1+\beta}\psi(y)^{\gamma_2'-\gamma_1'+1+\theta_0+\beta}, \quad x,y \in \boldsymbol{R}^d. \end{aligned}$$

Note that by Lemma 6.2, for any $\alpha < (\gamma_1 \vee \gamma_1') + 1$, there exists a $K_\alpha \in \boldsymbol{R}$ such that $E^{Q_{\nu_0}}\left[e^{\int_1^T \psi(X_t)^\alpha dt}\right] \le e^{K_\alpha(T-1)}$ for any $T > 1$ and $x \in \boldsymbol{R}^d$. Therefore,

$$Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T \psi(X_t)^\alpha dt \ge r\right) \le e^{-r(T-1)} e^{K_\alpha(T-1)}, \qquad \text{for any } r > 0. \tag{7.3}$$

Combining this with $1_{\{|X_t| \ge R\}} \le \psi(X_t)^\alpha / \psi(R)^\alpha$, $R > 0$, implies

$$Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T 1_{\{|X_t|\ge R\}} dt \ge r\right) \le e^{-r\psi(R)^\alpha(T-1)} e^{K_\alpha(T-1)}, \qquad r \in \boldsymbol{R}. \tag{7.4}$$

LEMMA 7.5. *Assume $\alpha < (\gamma_1 \vee \gamma_1') + 1$. Then for any $\varepsilon, C > 0$, there exists an $R > 0$ such that*

$$Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T \psi(X_t)^\alpha 1_{\{|X_t|\ge R\}} dt \ge \varepsilon\right) \le 2e^{-C(T-1)}, \qquad \textit{for any } T > 1.$$

PROOF. By assumption there exists a p such that $1 < p < \{(\gamma_1 \vee \gamma_1') + 1\}/\alpha$. Let $q > 1$ be the Hölder conjugate of p. Then by Hölder inequality,

$$\begin{aligned} &\frac{1}{T-1}\int_1^T \psi(X_t)^\alpha 1_{\{|X_t|\ge R\}} dt \\ &\quad \le \left(\frac{1}{T-1}\int_1^T \psi(X_t)^{p\alpha} dt\right)^{1/p} \cdot \left(\frac{1}{T-1}\int_1^T 1_{\{|X_t|\ge R\}} dt\right)^{1/q}. \end{aligned}$$

Therefore, by (7.4) with α substituted by $p\alpha$, we have

$$\begin{aligned} &Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T \psi(X_t)^\alpha 1_{\{|X_t|\ge R\}} dt \ge \varepsilon\right) \\ &\quad \le Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T \psi(X_t)^{p\alpha} dt \ge r\right) + Q_{\nu_0}\left(\frac{1}{T-1}\int_1^T 1_{\{|X_t|\ge R\}} dt \ge \left(\frac{\varepsilon}{r^{1/p}}\right)^q\right) \\ &\quad \le e^{-r(T-1)} e^{K_{p\alpha}(T-1)} + e^{-(\varepsilon r^{-1/p})^q \psi(R)^{p\alpha}(T-1)} e^{K_{p\alpha}(T-1)}, \qquad \text{for any } r > 0. \end{aligned}$$

This completes the proof. $\square$

LEMMA 7.6. *For any $\alpha < (\gamma_1 \vee \gamma_1') + 1$ and $\varepsilon, C > 0$, there exists a $R > 0$ such that*

$$Q_{\nu_0}\left(\left(\int_{\boldsymbol{R}^d} \psi(x)^\alpha L_{1,T}(dx)\right) \cdot \left(\int_{B_R^c} \psi(y)^\alpha L_{1,T}(dy)\right) > \varepsilon\right) \le 3e^{-C(T-1)}, \quad \textit{for any } T > 1,$$

where $B_R = \{x \in \boldsymbol{R}^d; |x| \le R\}$, $R > 0$, as before.

PROOF. Just notice that by (7.3),

$$Q_{\nu_0}\left(\left(\int_{\boldsymbol{R}^d}\psi(x)^\alpha L_{1,T}(dx)\right)\cdot\left(\int_{B_R^c}\psi(y)^\alpha L_{1,T}(dy)\right)>\varepsilon\right)$$
$$\leq e^{-r(T-1)}e^{K_\alpha(T-1)}+Q_{\nu_0}\left(\int_{B_R^c}\psi(x)^\alpha L_{1,T}(dx)>\frac{\varepsilon}{r}\right),\qquad \text{for any } r>0.$$

This combined with Lemma 7.5 completes the proof. □

LEMMA 7.7. *Let* $f\in C_b(\boldsymbol{R}^d\times\boldsymbol{R}^d)$. *Also, assume one of the following*:
(1) $g\in C(\boldsymbol{R}^d)$ *and there exist* $C_g>0$ *and* $\alpha<(\gamma_1\vee\gamma_1')+1$ *such that* $|g(x)|\leq C_g\psi(x)^\alpha$ *for any* $x\in\boldsymbol{R}^d$,
(2) $g\in C(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *and there exist* $C_g>0$ *and* $\alpha<(\gamma_1\vee\gamma_1')+1$ *such that* $|g(x,y)|\leq C_g\psi(x)^\alpha\psi(y)^\alpha$ *for any* $x,y\in\boldsymbol{R}^d$.
Then for any $\varepsilon>0$ *and* $C>0$, *there exists an* $\varepsilon_0>0$ *such that for any* $\varepsilon_1\leq\varepsilon_0$,

$$\lim_{T\to\infty}\frac{1}{T}\log E^{Q_{\nu_0}}\left[e^{(1/T)\int_0^T\int_0^T f(X_s,X_t)\,ds\,dt},A_{T,\varepsilon_1}\cap(A^g_{T,\varepsilon})^c\right]<-C. \tag{7.5}$$

PROOF. We only give the proof of (2) because the proof of (1) is similar. By (6.2) we have $\int_{\boldsymbol{R}^d}\psi^\beta d\nu_0<\infty$ for any $\beta>0$. So for any $\varepsilon>0$, there exists an $R_1>0$ such that

$$\iint_{(B_{R_1}\times B_{R_1})^c}|g(x,y)|\nu_0(dx)\nu_0(dy)<\frac{\varepsilon}{3}.$$

By Lemma 7.6, for any $C>0$ there exists an $R_2>0$ such that

$$Q_{\nu_0}\left(\iint_{(B_{R_2}\times B_{R_2})^c}\psi(x)^\alpha\psi(y)^\alpha L_{1,T}(dx)L_{1,T}(dy)>\frac{\varepsilon}{3C_g}\right)<6e^{-(C+\|f\|_\infty)(T-1)}$$

for any $T>0$. Let $R=R_1\vee R_2>0$. Then g is bounded on $B_R\times B_R$. So there exists an $\varepsilon_1>0$ such that, if $\omega\in A_{1,T,\varepsilon_1}$, then

$$\left|\int_{B_R}\int_{B_R}|g(x,y)|L_{1,T}(dx)L_{1,T}(dy)-\int_{B_R}\int_{B_R}|g(x,y)|\nu_0(dx)\nu_0(dy)\right|<\frac{\varepsilon}{3}.$$

Therefore,

$$Q_{\nu_0}(A_{1,T,\varepsilon_1}\cap(A^g_{1,T,\varepsilon})^c)\leq Q_{\nu_0}\left(\iint_{(B_{R_2}\times B_{R_2})^c}\psi(x)^\alpha\psi(y)^\alpha L_{1,T}(dx)L_{1,T}(dy)>\frac{\varepsilon}{3C_g}\right)$$
$$<6e^{-(C+\|f\|_\infty)(T-1)}.$$

This implies (7.5) easily. □

By condition, there exists a $\beta>0$ such that $\gamma_2-\gamma_1+1+\beta<((\gamma_1\vee\gamma_1')+1)/2$ and $\gamma_2'-\gamma_1'+1+\theta_0+\beta<((\gamma_1\vee\gamma_1')+1)/2$. So by Lemma 7.4 and A1, we get the following as a corollary of Lemma 7.7:

COROLLARY 7.8. *Let V be as in Proposition 7.1. Then for any $\varepsilon > 0$, there exists an $\varepsilon_0 > 0$ such that*

$$\sup_{T>0} E^{Q_{\nu_0}}\left[e^{1/(2T)\int_0^T\int_0^T V(X_t,X_s)dtds}, A_{T,\varepsilon_1}\cap\left(A^W_{T,\varepsilon}\right)^c\right] < \infty, \quad \textit{for any } \varepsilon_1 \le \varepsilon_0.$$

Corollary 7.8 completes the estimate about the integrals on $A_{T,\varepsilon_1}\cap(A^W_{T,\varepsilon})^c$. Therefore, in order to prove (7.2), we only need to deal with the integrals on $A^W_{T,\varepsilon}$.

We first have the following result about multiple integral by [**10**, Lemma 3.1]:

LEMMA 7.9. *Let $\{\bar{B}_t\}_{t\ge 0}$ be a Brownian motion. Then for any $T > 0$ and symmetric $h(\cdot,\cdot) : [0,T]\times[0,T] \to \boldsymbol{R}$ satisfying $\int_0^T\int_0^T h(s,t)^2 dsdt < 1/4$, we have*

$$E\left[e^{\int_0^T\int_0^T h(s,t)d\bar{B}_s d\bar{B}_t}\right] \le e^{\int_0^T\int_0^T h(s,t)^2 dsdt}.$$

Since $V\big|_{H\times H}$ is a Hilbert-Schmidt function by Lemma 6.7, it can be written as the summation of a finite sum of bilinear terms and a term with Hilbert-Schmidt norm small enough. Lemma 7.10 and Lemma 7.11 deal with the term with Hilbert-Schmidt norm small enough, and Lemma 7.12 deals with the bilinear terms.

LEMMA 7.10. *Let V be as in Proposition 7.1. Also, suppose that V satisfies*

$$\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} V(x,y)\overline{G}_x\overline{G}_y V(x,y)\nu_0(dx)\nu_0(dy) < \frac{c_2^2}{256c_1^2}. \tag{7.6}$$

Then there exists an $\varepsilon_0 > 0$ such that

$$\sup_{T>1} E^{Q_{\nu_0}}\left[e^{1/(2T)\int_0^T\int_0^T V(X_t,X_s)dtds}, A^W_{T,\varepsilon}\right] < \infty, \qquad \textit{for any } \varepsilon \le \varepsilon_0.$$

PROOF. From the definition of U_1 and Lemma 6.4,

$$U_1(X_T,X_t) = U_1(X_t,X_t) + \int_t^T \nabla_x U_1(X_s,X_t)\cdot\sigma(X_s)dB_s + \int_t^T V(X_s,X_t)ds$$

for any $T,t > 0$ with $T > t$. Therefore, by the symmetry of V,

$$\begin{aligned}\int_0^T\int_0^T V(X_s,X_t)dsdt = {} & 2\left(\int_0^T (U_1(X_T,X_t) - U_1(X_t,X_t))dt\right)\\ & - 2\int_0^T dt\left(\int_t^T \nabla_x U_1(X_s,X_t)\cdot\sigma(X_s)dB_s\right).\end{aligned}$$

We have by Remark 3 that $\|U_1\|_\infty < \infty$. As for the second term on the right hand side above, it is equal to

$$M_T := -2\int_0^T \left(\int_0^s \nabla_x U_1(X_s,X_t)\cdot\sigma(X_s)dt\right)dB_s$$

by stochastic Fubini's theorem (*c.f.* Ikeda-Watanabe [**7**, Lemma 3.4.1]). Also, $\{M_t\}_{t\geq 0}$ is a local martingale with Q_{v_0}-intergable quadratic variation for any $T>0$, hence a continuous Q_{v_0}-martingale. So

$$E^{Q_{v_0}}\left[e^{(1/T)\int_0^T\int_0^T V(X_t,X_s)dsdt}, A^W_{T,\varepsilon}\right]$$
$$\leq \exp(4\|U_1\|_\infty)\cdot E^{Q_{v_0}}\left[e^{2\int_0^T|(2/T)\int_0^s \nabla_x U_1(X_s,X_t)\cdot\sigma(X_s)dt|^2 ds}, A^W_{T,\varepsilon}\right]^{1/2}.$$

By (2.1), it is sufficient to show there exists an $\varepsilon>0$ such that

$$\sup_{T>0} E^{Q_{v_0}}\left[e^{(8c_2)/T^2\int_0^T ds|\int_0^s \nabla_x U_1(X_s,X_t)dt|^2}, A^W_{T,\varepsilon}\right]<\infty. \tag{7.7}$$

Let $g(s,t)=\nabla_y\nabla_x U(X_{T-s},X_{T-t})$. Then by Lemma 6.4

$$\nabla_x U(X_{T-s},X_0)=\nabla_x U(X_{T-s},X_{T-s})+\int_s^T g(s,t)\cdot\sigma(X_{T-t})d\hat{B}^T_t$$
$$+\int_s^T \nabla_x U_1(X_{T-s},X_{T-t})dt, \qquad \text{for any } s\in(0,T)$$

where $\{\hat{B}^T_t\}_{t\in[0,T]}$ is a Brownian motion. Note that by assumption, there exists a $\beta>0$ such that $\gamma_2-\gamma_1+1+\beta<(\gamma_1+1)/2$, Also, v_0 is $\{Q_x\}_{x\in\boldsymbol{R}^d}$-invariant. So by Lemma 7.4 and Lemma 6.2,

$$\sup_{T>1} E^{Q_{v_0}}\left[e^{4/T^2\int_0^T|\nabla_x U(X_{T-s},X_0)-\nabla_x U(X_{T-s},X_{T-s})|^2 ds'}\right]<\infty.$$

Therefore, by Hölder's inequality, it only remains to show there exists an $\varepsilon>0$ such that

$$\sup_{T>0} E^{Q_{v_0}}\left[e^{(32c_2)/T^2\int_0^T|\int_s^T g(s,t)\cdot\sigma(X_{T-t})d\hat{B}^T_t|^2 ds}, A^W_{T,\varepsilon}\right]<\infty.$$

Let $\{\bar{B}_t\}_{t\geq 0}$ be a d-dimensional Brownian motion which is independent to $\{X_t\}_{t\in[0,\infty)}$. Then by a simple calculation with the help of Hölder's inequality, stochastic Fubini's theorem and A1, we have

$$E^{Q_{v_0}}\left[e^{(32c_2)/T^2\int_0^T|\int_t^T g(t,s)\cdot\sigma(X_{T-s})d\hat{B}^T_s|^2 dt}, A^W_{T,\varepsilon}\right]$$
$$\leq E^{Q_{v_0}}\left[E^{\bar{B}}\left[e^{(128c_2^2)/T^2\int_0^T|\int_0^s g(t,s)d\bar{B}_t|^2 ds}\right], A^W_{T,\varepsilon}\right]^{1/2}. \tag{7.8}$$

Note that

$$
\begin{aligned}
&\int_0^T \left| \int_0^s g(t,s)\, d\bar{B}_t \right|^2 ds \\
&\quad = \int_0^T \left(\int_t^T |g(t,s)|^2 ds \right) dt + \int_0^T \int_0^T \left(\int_{t\vee u}^T g(t,s) \otimes g(u,s)\, ds \right) d\bar{B}_t\, d\bar{B}_u.
\end{aligned}
\tag{7.9}
$$

For the first term on the right hand side of (7.9), we have by Lemma 6.7 and (7.6) there exists an $\varepsilon > 0$ such that

$$
\int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} W(x,y) L_T(dx) L_T(dy) < \frac{c_2^2}{256 c_1^2} \qquad \text{on } A_{T,\varepsilon}^W.
$$

For the second term on the right hand side of (7.9), we have

$$
\begin{aligned}
&\frac{(128c_2^2)^2}{T^4} \int_0^T \int_0^T dt\,du \left\| \int_{t\vee u}^T g(t,s) \otimes g(u,s)\, ds \right\|^2 \\
&\quad \le \frac{(128c_2^2)^2}{T^4} \int_0^T dt \int_0^T du \left(\int_t^T \|g(t,s)\|^2 ds \right) \left(\int_u^T \|g(u,s)\|^2 ds \right) \\
&\quad = (128c_2^2)^2 \left\{ \frac{1}{T^2} \int_0^T dt \left(\int_t^T \|g(t,s)\|^2 ds \right) \right\}^2 \\
&\quad \le \left\{ \frac{128c_2^2}{T^2} \int_0^T \int_0^T \|g(t,s)\|^2 dt\, ds \right\}^2 \\
&\quad = \left\{ 128c_2^2 \int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} \|\nabla_x \nabla_y U(x,y)\|^2 L_T(dx) L_T(dy) \right\}^2 \\
&\quad \le \left\{ \frac{128c_2^2}{c_1^2} \int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} (\nabla_x \nabla_y U(x,y))^t (a \otimes a)(x,y) \nabla_x \nabla_y U(x,y) L_T(dx) L_T(dy) \right\}^2 \\
&\quad < \frac{128c_2^2}{c_1^2} \cdot \left(\frac{c_2^2}{256 c_1^2} \right)^2 = \frac{1}{4} \qquad \text{on } A_{T,\varepsilon}^W.
\end{aligned}
\tag{7.10}
$$

So by Lemma 7.9,

$$
\begin{aligned}
&E^{Q_{\nu_0}} \left[E^{\bar{B}} \left[e^{(128c_2^2)/T^2 \int_0^T \int_0^T (\int_{t\vee u}^T g(t,s)\otimes g(u,s) ds) d\bar{B}_t d\bar{B}_u} \right], A_{T,\varepsilon}^W \right] \\
&\quad \le E^{Q_{\nu_0}} \left[e^{(128c_2^2)^2/T^4 \int_0^T \int_0^T dt\,du |\int_{t\vee u}^T g(t,s)\otimes g(u,s) ds|^2}, A_{T,\varepsilon}^W \right] \\
&\quad < e^{1/4}.
\end{aligned}
$$

This combined with (7.8) and (7.9) completes the proof. □

Now, we get the following as a direct result of Corollary 7.8 and Lemma 7.10:

LEMMA 7.11. *Let V be as in Lemma* 7.10. *Then there exists an $\varepsilon_0 > 0$ such that* (7.2) *holds for any $\varepsilon \le \varepsilon_0$.*

Similarly, we have the following:

LEMMA 7.12. *For any $e \in C_b(\boldsymbol{R}^d)$ with $\int_{\boldsymbol{R}^d} e d\nu_0 = 0$ and $\Gamma(e,e) = 1$, and any $c < 1$, there exists an $\varepsilon_0 > 0$ such that*

$$\sup_{T>0} E^{Q_{\nu_0}}\left[e^{c/(2T)(\int_0^T e(X_t)dt)^2}, A_{T,\varepsilon}\right] < \infty, \qquad \textit{for any } \varepsilon \le \varepsilon_0. \tag{7.11}$$

PROOF. Let $v = Ge$. Then $v \in C^1(\boldsymbol{R}^d)$ by Lemma 6.3. Let $u = \nabla v \cdot a \nabla v$. Then by Lemma 6.3, for any $\beta > 0$ there exists a $c_9 > 0$ (which may depend on β) such that $|u(x)| \le c_2 {c_9}^2 \|e\|_\infty^2 \psi(x)^{2(\gamma_2 - \gamma_1 + 1 + \beta)}$ for any $x \in \boldsymbol{R}^d$. So by Lemma 7.7, for any $\varepsilon > 0$, there exists an $\varepsilon_0 > 0$ such that

$$\sup_{T>1} E^{Q_{\nu_0}}\left[e^{(a/(2T))(\int_0^T e(X_t)dt)^2}, A_{T,\varepsilon_1} \cap \left(A^u_{T,\varepsilon}\right)^c\right] < \infty, \qquad \text{for any } \varepsilon_1 \le \varepsilon_0.$$

Next, we show that there exists an $\varepsilon > 0$ such that

$$\sup_{T>1} E^{Q_{\nu_0}}\left[e^{(a/(2T))(\int_0^T e(X_t)dt)^2}, A^u_{T,\varepsilon}\right] < \infty. \tag{7.12}$$

By Lemma 6.3,

$$v(X_T) - v(X_0) = \int_0^T \nabla v(X_t)\sigma(X_t)dB_t + \int_0^T e(X_t)dt.$$

v is bounded, so to prove (7.12), it is sufficient if there exists an $\varepsilon > 0$ such that

$$\sup_{T>0} E^{Q_{\nu_0}}\left[e^{(c/2)\cdot(1/T)(\int_0^T \nabla v(X_t)\cdot\sigma(X_t)dB_t)^2}, A^u_{T,\varepsilon}\right] < \infty.$$

Choose and fix a $\delta \in (0, (1/c) - 1)$. Since $\int_{\boldsymbol{R}^d} u(x)\nu_0(dx) = \|e\|^2_{H^*} = 1$, there exists an $\varepsilon_0 > 0$ such that $\int_{\boldsymbol{R}^d} u(x) L_T(dx) \le 1 + \delta$ on $A^u_{T,\varepsilon}$ for any $\varepsilon \le \varepsilon_0$. By Ikeda-Watanabe [**7**, Theorem II.7.2], there exists a Brownian motion $\widetilde{B}$ such that

$$\begin{aligned}\left(\int_0^T \nabla v(X_t)\cdot\sigma(X_t)dB_t\right)^2 &= \widetilde{B}\left(\int_0^T \nabla v(X_t)\cdot a(X_t)\nabla v(X_t)dt\right)^2 \\ &\le \sup_{0\le t\le(1+\delta)T} |\widetilde{B}(t)|^2 \qquad \text{on } A^u_{T,\varepsilon}.\end{aligned}$$

So by reflection principle,

$$\begin{aligned}&\sup_{T>0} E^{Q_{\nu_0}}\left[e^{(c/2)\cdot(1/T)(\int_0^T \nabla v(X_t)\cdot\sigma(X_t)dB_t)^2}, A^u_{T,\varepsilon}\right] \\ &\quad \le \sup_{T>0} E\left[e^{(c/2)\cdot(1/T)\sup_{0\le t\le(1+\delta)T}|\widetilde{B}(t)|^2}\right] \le \frac{2}{\sqrt{1-c(1+\delta)}} - 1 < \infty.\end{aligned}$$

This completes the proof. □

Now, we can prove Proposition 7.1 in the same way as in [**10**].

PROOF OF PROPOSITION 7.1. As mentioned before, it is sufficient to prove (7.2).

By Lemma 6.7, $A_V\big|_{H\times H}$ is a Hilbert-Schmidt type function. So by condition, the maximum eigenvalues $\overline{a}$ is smaller than 1. Write the eigenvalues of $A_V\big|_{H\times H}$ as $\{a_n\}_{n\in\boldsymbol{N}}$ with $|a_1|\geq|a_2|\geq|a_3|\geq\cdots$, and the corresponding eigenvectors as $\{\overline{G}e_m d\nu_0\}_{m=1}^{\infty}$ with $\int_{\boldsymbol{R}^d} e_m(x)\overline{G}e_n(x)\nu_0(dx)=\delta_{mn}$, $m,n\in\boldsymbol{N}$. Then $A_V(\overline{G}e_m d\nu_0,R)=a_m\int_{\boldsymbol{R}^d} e_m(x)R(dx)$ for any $R\in\mathscr{M}_0(\boldsymbol{R}^d)$. So for any $m\in\boldsymbol{N}$ with $a_m\neq 0$, we may and do assume that $e_m\in\widetilde{C_b}(\boldsymbol{R}^d)$.

Choose and fix a $p>1$ such that $\overline{a}p<1$. Let q be the Hölder conjugate of $p>1$. Then there exists an $N\in\boldsymbol{N}$ such that $\sum_{i=N+1}^{\infty}q^2a_i^2<c_2^2/256c_1^2$. By Hölder's inequality and applying Lemma 7.11 to $V_1(x,y):=q\big(V(x,y)-\sum_{i=1}^{N}a_ie_i(x)e_i(y)\big),x,y\in\boldsymbol{R}^d$, it only remains to prove that there exists an $\varepsilon>0$ such that

$$\sup_{T>0}E^{Q_{\nu_0}}\left[e^{\sum_{i=1}^{N}(p/(2T))\int_0^T\int_0^T a_ie_i(X_t)e_i(X_s)dsdt},A_{T,\varepsilon}\right]<\infty. \tag{7.13}$$

Without loss of generality, we may and do assume that $a_1,\cdots,a_N\geq 0$. In general, we have that for any $\eta>0$, there exist a $m\in\boldsymbol{N}$ and $\xi_i=(\xi_i^1,\cdots,\xi_i^N)\in\boldsymbol{R}^N$, $i=1,\cdots,m$, such that $\|\xi_i\|_{\boldsymbol{R}^d}=1$, $i=1,\cdots,m$, and

$$\bigcap_{i=1}^{m}\left\{x\in\boldsymbol{R}^N:(x,\xi_i)\leq(1+\eta)^{-1/2}\right\}\subset\left\{x\in\boldsymbol{R}^N:\|x\|<1\right\},$$

so

$$\|x\|^2\leq(1+\eta)\max_{i=1,\cdots,m}(x,\xi_i)^2,\qquad x\in\boldsymbol{R}^N.$$

Apply this fact to $\eta=1-p\overline{a}$. Let $\widetilde{e}_i=\sum_{j=1}^{N}\xi_i^je_j$, $i=1,\cdots,m$. Then $(\overline{G}\widetilde{e}_i,\widetilde{e}_i)_{L^2(d\nu_0)}=1$, $\int_{\boldsymbol{R}^d}\widetilde{e}_i(x)\nu_0(dx)=0$, $i=1,\cdots,m$, and

$$\sum_{j=1}^{N}\left(\int_0^T e_j(X_t)dt\right)^2\leq(1+\eta)\max_{i=1,\cdots,m}\left(\int_0^T\widetilde{e}_i(X_t)dt\right)^2.$$

Therefore,

$$E^{Q_{\nu_0}}\left[e^{\sum_{i=1}^{N}(p/(2T))\int_0^T\int_0^T a_ie_i(X_t)e_i(X_s)dsdt},A_{T,\varepsilon}\right]\leq\sum_{i=1}^{m}E^{Q_{\nu_0}}\left[e^{((1-\eta^2)/2)\cdot(1/T)(\int_0^T\widetilde{e}_i(X_t)dt)^2},A_{T,\varepsilon}\right].$$

This combined with Lemma 7.12 yields (7.13), which completes the proof. □

8. Proof of Theorem 2.1.

The proof is similar to that of [**10**], so we only give a sketch.
Let $\widetilde{\Phi}(\nu) = \Phi(\nu) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu - \nu_0)$, $\nu \in \mathscr{M}(\boldsymbol{R}^d)$. Then

$$e^{-\lambda T} E^{P_x}\left[e^{T\Phi((1/T)\int_0^T \delta_{X_t}dt)}, A \,\Big|\, X_T = y\right]$$
$$= \frac{h(x)}{h(y)} E^{Q_x}\left[e^{T\widetilde{\Phi}((1/T)\int_0^T \delta_{X_t}dt)}, A \,\Big|\, X_T = y\right], \qquad \text{for any } A \in \mathscr{F}_T. \tag{8.1}$$

Let $A_{T,\varepsilon} = \{\mathrm{dist}((1/T)\int_0^T \delta_{X_t}dt, \nu_0) < \varepsilon\}$, $T > 0$, $\varepsilon > 0$, as before. So the theorem will be shown if we can show the following two lemmas.

LEMMA 8.1.

$$\limsup_{T\to\infty} \frac{1}{T} \log E^{Q_x}\left[\exp\left(T\widetilde{\Phi}\left(\frac{1}{T}\int_0^T \delta_{X_t}dt\right)\right), A^c_{T,\varepsilon} \,\Big|\, X_T = y\right] < 0$$

for any $\varepsilon > 0$ and $x, y \in \boldsymbol{R}^d$.

LEMMA 8.2. *For any $x, y \in \boldsymbol{R}^d$, there exists an $\varepsilon > 0$ such that*

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} E^{Q_x}\left[\exp\left(T\widetilde{\Phi}\left(\frac{1}{T}\int_0^T \delta_{X_t}dt\right)\right), A_{T,\varepsilon} \,\Big|\, X_T = y\right]$$
$$= \exp\left\{\frac{1}{2}\int_{\boldsymbol{R}^d} \overline{G}_x \Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(u,u)} \nu_0(du)\right\} \times det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.$$

Lemma 8.1 is easy from large deviation principle. We prove Lemma 8.2. First, we have the following by Proposition 7.1:

LEMMA 8.3. *There exist constants $p > 1$ and $\varepsilon > 0$ such that*

$$\sup_{T>0} E^{Q_x}\left[e^{pT\widetilde{\Phi}((1/T)\int_0^T \delta_{X_t}dt)}, A_{T,\varepsilon} | X_T = y\right] < \infty.$$

PROOF. Let $R(\nu_0, \cdot)$ be the third remainder of the Taylor expansion of Φ around ν_0, *i.e.*, $R(\nu_0, \nu) = \widetilde{\Phi}(\nu) - (1/2)D^2\Phi(\nu_0)(\nu - \nu_0, \nu - \nu_0)$. Also, define $\widetilde{\Phi^{(2)}}$ on $\boldsymbol{R}^d \times \boldsymbol{R}^d$ by

$$\widetilde{\Phi^{(2)}}(x,y) = \Phi^{(2)}(\nu_0; x, y) - \int_{\boldsymbol{R}^d} \Phi^{(2)}(\nu_0; x, z)\nu_0(dz) - \int_{\boldsymbol{R}^d} \Phi^{(2)}(\nu_0; z, y)\nu_0(dz)$$
$$+ \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} \Phi^{(2)}(\nu_0; z_1, z_2)\nu_0(dz_1)\nu_0(dz_2), \qquad x, y \in \boldsymbol{R}^d.$$

Then

$$D^2\Phi(\nu_0)(L_T - \nu_0, L_T - \nu_0) = \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} \widetilde{\Phi^{(2)}}(\cdot,\cdot)dL_T dL_T.$$

By A5, there exist $p, r > 1$ such that all of the eigenvalues of $pr\widetilde{\Phi^{(2)}}\big|_{H\times H}$ are smaller than 1. Therefore, by Proposition 7.1, there exists an $\varepsilon_0 > 0$ such that

$$\sup_{T>1} E^{Q_x}\left[\exp\left(prD^2\Phi(\nu_0)(L_T-\nu_0, L_T-\nu_0)\right), A_{T,\varepsilon}\Big| X_T = y\right] < \infty, \qquad \text{for any } \varepsilon \le \varepsilon_0.$$

Let s be the Hölder conjugate of r. Then by A6, we have by re-choosing $\varepsilon_0 > 0$ if necessary

$$\sup_{T>1} E^{Q_x}\left[\exp\left(psR(\nu_0, L_T)\right), A_{T,\varepsilon}\Big| X_T = y\right] < \infty, \qquad \text{for any } \varepsilon \le \varepsilon_0.$$

These and Hölder's inequality complete the proof. □

PROOF OF LEMMA 8.2. As in Kusuoka-Tamura [**12**], Q_x has the strong mixing property, so X_T and $\sqrt{T}(L_T - \nu_0)$ are asymptotically independent under Q_x as $T \to \infty$ for any $x \in \boldsymbol{R}^d$, also,

$$E^{Q_x}\left[e^{\sqrt{-1}\sqrt{T}\int_{\boldsymbol{R}^d} u(x)((1/T)\int_0^T \delta_{X_t}dt-\nu_0)(dx)}\right] \to e^{-(1/2)\int_{\boldsymbol{R}^d} u(y)\overline{G}u(y)\nu_0(dy)}, \qquad \text{as } T\to\infty$$

for any $u \in L^2(\boldsymbol{R}^d, d\nu_0)$.

Take a separable Hilbert space H_1 such that the set $\{\overline{G}u d\nu_0 \,|\, \int_{\boldsymbol{R}^d} u\overline{G}u d\nu_0 < \infty\}$ is a dense linear subspace of H_1, and the inclusion map is a Hilbert-Schmidt operator. Let W be an H_1-valued random variable such that

$$E\left[\exp(\sqrt{-1}(u,W))\right] = \exp\left(-\frac{1}{2}\int_{\boldsymbol{R}^d} u(y)\overline{G}u(y)\nu_0(dy)\right)$$

for any $u \in H_1^*$. (Write the distribution of W as ζ).

Then by the central limit theorem for Hilbert space valued random variables, the distribution of $(X_T, \sqrt{T}(L_T-\nu_0))$ under Q_x converges weakly to $\nu_0 \otimes \zeta$ as $T \to \infty$.

As claimed before, $D^2\Phi(\nu_0)(\cdot,\cdot)\big|_{H\times H}$ is a Hilbert-Schmidt function. Write the eigenvalues and the corresponding eigenvectors as a_m and $\overline{G}e_m d\nu_0$, $m \in \boldsymbol{N}$. Then $\sum_{m=1}^N a_m\left((e_m,W)^2-1\right)$ converges in $L^2(d\zeta)$ as $N\to\infty$. Use $: D^2\Phi(\nu_0)(W,W) :$ to denote the $L^2(d\zeta)$-limit of $\sum_{m=1}^N a_m\left((e_m,W)^2-1\right)$.

It is easy to see that

$$\frac{1}{T}\int_0^T\int_0^T \sum_{m=1}^N a_m e_m(X_s)e_m(X_t)dsdt - \frac{1}{T}\int_0^T \sum_{m=1}^N a_m e_m(X_s)\overline{G}e_m(X_s)ds \to \sum_{m=1}^N a_m\left((e_m,W)^2-1\right)$$

under Q_x in distribution for any $N \in \boldsymbol{N}$, and

$$\begin{aligned}&\sup_{T>0} E^{Q_x}\Bigg[\Bigg\{\left(\frac{1}{T}\int_0^T\int_0^T \Phi^{(2)}(\nu_0;X_t,X_s)dsdt - \frac{1}{T}\int_0^T \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)}ds\right)\\ &\quad -\left(\frac{1}{T}\int_0^T\int_0^T\sum_{m=1}^N a_m e_m(X_s)e_m(X_t)dsdt - \frac{1}{T}\int_0^T\sum_{m=1}^N a_m e_m(X_s)\overline{G}e_m(X_s)ds\right)\Bigg\}^2\Bigg]\\ &\quad \to 0\end{aligned}$$

as $N \to \infty$. Therefore,

$$\frac{1}{T}\int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s)\,ds\,dt - \frac{1}{T}\int_0^T \overline{G}_x \Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)} ds \to: D^2\Phi(\nu_0)(W,W):$$

in distribution as $T \to \infty$. Also,

$$\frac{1}{T}\int_0^T \overline{G}_x \Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)} ds \to \int_{\boldsymbol{R}^d} \overline{G}_x \Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(u,u)} \nu_0(du)$$

Q_x-almost surely as $T \to \infty$, and

$$TR\left(\nu_0, \frac{1}{T}\int_0^T \delta_{X_t}\,dt\right) \to 0$$

under Q_x in distribution as $T \to \infty$. Therefore,

$$T\widetilde{\Phi}\left(\frac{1}{T}\int_0^T \delta_{X_t}\,dt\right) \to: D^2\Phi(\nu_0)(W,W): + \int_{\boldsymbol{R}^d} \overline{G}_x \Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(u,u)} \nu_0(du)$$

in distribution as $T \to \infty$. This together with Lemma 8.3 give us Lemma 8.2. □

9. Examples.

In this section, we will give some examples of $\{P_t\}_{t\geq 0}$ that satisfy our assumptions A1 and A2 in Section 2.

Let U and b be any pair of functions satisfying the following:

- E0 $U \in C^\infty(\boldsymbol{R}^d;\boldsymbol{R})$ with $\int_{\boldsymbol{R}^d} e^{-U(x)}dx < \infty$, and $b \in C^\infty(\boldsymbol{R}^d;\boldsymbol{R}^d)$,
- E1 There exist constants $\gamma_1 > 1$ and $\gamma_2 \in [\gamma_1, \gamma_1 + (1/2)(\gamma_1 - 1))$ such that $(1/2)\Delta + (b - \nabla U)\cdot\nabla \in H_2(\gamma_1,\gamma_2)$,
- E2 There exist constants $\gamma_1' > 1$ and $\gamma_2' \in [\gamma_1', \gamma_1' + (1/2)(\gamma_1' - 1))$ such that $(1/2)\Delta - b\cdot\nabla \in H_2(\gamma_1',\gamma_2')$,
- E3 There exists a $\theta_0 \in [0, (\gamma_1'-1)/2 - (\gamma_2' - \gamma_1'))$ such that $\varphi_0 := \nabla U\cdot(b - (1/2)\nabla U) - \mathrm{div}(b - (1/2)\nabla U) \in B^0_{\theta_0}$.

Let $\{P_t\}_{t>0}$ be the semi-group of continuous linear operators on $C_b(\boldsymbol{R}^d)$ corresponding to

$$L_0 = \frac{1}{2}\Delta - \nabla U\cdot\nabla + b\cdot\nabla.$$

Let μ be the invariant measure of $\{P_t\}_{t>0}$, which exists uniquely with all moments finite. Let $P_t^{*\mu}$ denote the dual operator of P_t in $L^2(d\mu)$.

Let $\{S_t\}_{t>0}$ be the semi-group of continuous linear operators on $C_b(\boldsymbol{R}^d)$ corresponding to generator $(1/2)\Delta - b\cdot\nabla$. We show the following:

LEMMA 9.1. *$P_t^{*\mu} = Q(R)_t^{\varphi_0}$ for any $t > 0$.*

Lemma 9.1 means that $\{P_t\}_{t>0}$ satisfies A1 and A2 of Section 2. Let us prove it from now on.

First, let μ_0 be the finite measure on $\boldsymbol{R}^d$ given by

$$\mu_0(dx) = e^{-U(x)}dx.$$

Without loss of generality, we may and do assume that μ_0 is a probability measure. Before giving the proof of Lemma 9.1, we first show the following:

LEMMA 9.2. *For any $t > 0$, the dual operator $P_t^{*\mu_0}$ of P_t in $L^2(d\mu_0)$ is given by $P_t^{*\mu_0} = S_t^{\varphi_0}$.*

PROOF. The generator of $\{P_t^{*\mu_0}\}_{t>0}$ is the dual operator $L_0^{*\mu_0}$ of L_0 in $L^2(d\mu_0)$. Note that

$$\begin{aligned}\int_{\boldsymbol{R}^d} L_0^{*\mu_0} g(x) f(x)\mu_0(dx) &= \int_{\boldsymbol{R}^d} g(x) L_0 f(x) e^{-U(x)}dx \\ &= \int_{\boldsymbol{R}^d} f(x)\left(\frac{1}{2}\Delta - b\cdot\nabla + \left(\frac{1}{2}\Delta U - \frac{1}{2}|\nabla U|^2 + b\cdot\nabla U - \operatorname{div} b\right)\right) g(x) e^{-U(x)}dx\end{aligned}$$

for any $f, g \in C_0^\infty(\boldsymbol{R}^d)$. So

$$L_0^{*\mu_0} = \frac{1}{2}\Delta - b\cdot\nabla + \left(\frac{1}{2}\Delta U - \frac{1}{2}|\nabla U|^2 + b\cdot\nabla U - \operatorname{div} b\right).$$

This gives us our assertion. □

LEMMA 9.3. $\Lambda^{S,\varphi_0} = 0$. *Therefore, for any $\alpha > 0$, there exists a unique* (*up to constant multiplication*) *positive $\widetilde{h} \in B_\alpha^0$ such that $\widetilde{h} = P_t^{*\mu_0}\widetilde{h}$ and $Q(R)_t^{\varphi_0} = \widetilde{h} P_t^{*\mu_0}\widetilde{h}^{-1}$ for any $t > 0$.*

PROOF. We have by Lemma 9.2, E2, E3 and Lemma 4.3 that for any $\alpha > 0$, there exists a unique (up to constant multiplication) positive $\widetilde{h} \in B_\alpha^0$ such that $\widetilde{h} = e^{-\Lambda^{S,\varphi_0}t} P_t^{*\mu_0}\widetilde{h}$ for any $t > 0$. We show that $\Lambda^{S,\varphi_0} = 0$.

First, we show that $\widetilde{h}$ is μ_0-integrable. By E1 and Lemma 3.2, there exists an $r > 0$ such that $\inf_{x\in\boldsymbol{R}^d} P_x(|X_t| \le r) > 1/2$. Let f be a positive continuous function with compact support satisfying $\inf_{B_r} f > 0$, where B_r means the ball with center 0 and radius r as before. Then $\inf_{x\in\boldsymbol{R}^d} P_t f(x) \ge \inf_{B_r} f \times \inf_{x\in\boldsymbol{R}^d} P_x(|X_t| \le r) > 0$ for any $t > 0$. On the other hand,

$$\int_{\boldsymbol{R}^d} \widetilde{h} P_t f\, d\mu_0 = \int_{\boldsymbol{R}^d} P_t^{*\mu_0}\widetilde{h} f\, d\mu_0 = e^{\Lambda^{S,\varphi_0}t}\int_{\boldsymbol{R}^d}\widetilde{h} f\, d\mu_0, \tag{9.1}$$

which is finite, since $f \in C_0(\boldsymbol{R}^d)$ and $\widetilde{h}$ is continuous. Therefore, $\widetilde{h}$ is μ_0-integrable.

So the left hand side of (9.1) converges to $\int_{\boldsymbol{R}^d} f\,d\mu \times \int_{\boldsymbol{R}^d}\widetilde{h}\,d\mu_0$ as $t \to \infty$. This gives us that $\Lambda^{S,\varphi_0} = 0$. □

PROOF OF LEMMA 9.1. Let $h = d\mu/d\mu_0$, which is well-defined and positive since $\operatorname{supp}\mu = \operatorname{supp}\mu_0 = \boldsymbol{R}^d$, and both of them are absolutely continuous with respect to Lebesgue measure with positive density. Then $\int P_t^{*\mu} fg\,d\mu = \int P_t^{*\mu_0}(fh)gh^{-1}d\mu$ for any $f, g \in B_\alpha^0$. So

$$P_t^{*\mu} f = h^{-1} P_t^{*\mu_0}(hf), \qquad \text{for any } t > 0, f \in B_\alpha^0. \tag{9.2}$$

Therefore, by Lemma 9.3, it is sufficient to show $h = const \times \widetilde{h}$. We do this from now on. Since $P_t^{*\mu} 1 = 1$, (9.2) and Lemma 9.1 give us that

$$h = S_t^{\varphi_0} h. \tag{9.3}$$

For any $A > 0$, consider $Ah \wedge \widetilde{h}$. It is trivial that $Ah \wedge \widetilde{h} \in B_\alpha^0$. Also, since $S_t^{\varphi_0}$ is a monotone non-decreasing operator, by (9.3) and Lemma 9.3, $Ah \wedge \widetilde{h} = S_t^{\varphi_0} ah \wedge S_t^{\varphi_0} \widetilde{h} \geq S_t^{\varphi_0}(ah \wedge \widetilde{h})$. Therefore, there exists a b_A such that

$$Ah \wedge \widetilde{h} = b_A \widetilde{h}, \qquad \text{for any } A > 0. \tag{9.4}$$

For any $x_0 \in \boldsymbol{R}^d$, there exists a $A > 0$ such that $Ah(x_0) < \widetilde{h}(x_0)$. So (9.4) gives us $Ah(x_0) = b_A \widetilde{h}(x_0)$, hence $b = (ah(x_0))/\widetilde{h}(x_0) < 1$. Since $\widetilde{h} \neq 0$, this combined with (9.4) give us $Ah = b_A \widetilde{h}$. This completes the proof. □

Finally, let us give some concrete examples that satisfy E0, E1, E2 and E3. For example, let $d = 1$, let $A > 0$, $\delta > 0$, $\eta > 0$, $\xi > \delta/2 \vee (\eta - \delta - 2)$ be any constants, let $y(x) \in C_b^\infty(\boldsymbol{R})$ such that $y(x) = |x|^{-\xi-2} x$ for any $|x| \geq 2$ and $y(x) = x$ for any $|x| \leq 1$, and let

$$\begin{aligned} \nabla U(x) &= |x|^\delta x - Ay(x)|x|^\eta, \\ b(x) &= \frac{1}{2}\left(|x|^\delta x - Ay(x)|x|^\eta\right) + y(x). \end{aligned}$$

Then b and U satisfy the conditions of this section with $\gamma_1 = \gamma_2 = \gamma_1' = \gamma_2' = 1 + \delta$ and $\theta_0 = \delta - \xi < \delta/2 = (\gamma_1' - 1)/2 - (\gamma_2' - \gamma_1')$.

References

[1] E. Bolthausen, Laplace approximations for sums of independent random vectors, Probab. Theory Related Fields, **72** (1986), no. 2, 305–318.

[2] E. Bolthausen, J.-D. Deuschel and U. Schmock, Convergence of path measures arising from a mean field and polaron type interaction., Probab. Theory Related Fields, No. 95 (1993), pp. 283–310.

[3] E. Bolthausen, J.-D. Deuschel and Y. Tamura, Laplace approximations for large deviations of nonreversible Markov processes. The nondegenerate case., Ann. Probab., Vol. 23 (1995), No. 1, pp. 236–267.

[4] M. D. Donsker and S. R. S. Varadhan, Asymptotic Evaluation of Certain Markov Process Expectations for Large Time–III, Commun. Pure Appl. Math., Vol. XXIX (1976), 389–461.

[5] N. Dunford and J. T. Schwartz, Linear Operators Part I: General Theory, Interscience Publishers, Inc., New York, 1964.

[6] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin-Heidelberg-New York, 1977.

[7] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodansha, 1981.

[8] N. V. Krylov and M. V. Safonov, On a Problem Suggested by A. D. Wenzell, The Dynkin Festchrift, Progr. Probab., **34**, Birkhuser Boston, Boston, 1994, 209–219.

[9] S. Kusuoka and S. Liang, Laplace approximations for sums of independent random vectors, Probab. Theory Related Fields, **116** (2000), no. 2, 221–238.

[10] S. Kusuoka and S. Liang, Laplace Approximations for Diffusion Processes on Torus, nondegenerate case, J. Math. Sci. Univ. Tokyo, **8** (2001), 43–70.
[11] S. Kusuoka and S. Liang, On an Ergodic Property of Diffusion Semigroups on Euclidean Space, J. Math. Sci. Univ. Tokyo, **10** (2003), 538–553.
[12] S. Kusuoka and Y. Tamura, Symmetric Markov processes with mean field potentials., J. Fac. Sci. Univ. Tokyo Sect IA, Vol. 34 (1987), No. 2, pp. 371–389.
[13] S. Liang, Laplace approximations for sums of independent random vectors—the degenerate case, J. Math. Sci. Univ. Tokyo, **7** (2000), no. 2, 195–220.
[14] S. Liang, Large Deviation Principles for a type of Diffusion Processes on Euclidean Space, J. Math. Sci. Univ. Tokyo, **10** (2003), 555–578.
[15] S. Liang, A Bounded Property for Gradients of Diffusion Semigroups on Euclidean Spaces, J. Funct. Anal., **216** (2004), 71–85.
[16] S. Liang and J. J. Liu, On the Precise Laplace Approximation for Large Deviations of Markov Chain—the Non-degenerate Case, J. Math. Sci. Univ. Tokyo, **10** (2003), 421–454.

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