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One-parametric selfinjective algebras

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Abstract. In continuation of our papers [**5**], [**6**] we complete the classification of all one-parametric selfinjective algebras over algebraically closed fields which admit simply connected Galois coverings.

0. Introduction.

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules and by D the standard duality $\operatorname{Hom}_K(-,K)$ on $\operatorname{mod} A$. An algebra A is called *selfinjective* if $A \cong \mathrm{D}(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective.

From Drozd's remarkable Tame and Wild theorem [**10**] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parametric families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional K-algebras. Accordingly we may realistically hope to classify the indecomposable finite dimensional modules only for the tame algebras. A special class of tame algebras is formed by the algebras of finite representation type having only finitely many isomorphism classes of indecomposable finite dimensional modules. The representation theory of algebras of finite representation type is presently well understood, and in particular all selfinjective algebras of finite representation type are classified [**8**], [**18**], [**19**]. The representation theory of arbitrary tame algebras is still only emerging.

We are concerned with the problem of classification of all one-parametric selfinjective algebras. Recall that an algebra A of infinite representation type is called *one-parametric* if there exists a $K[x]$-A-bimodule M which is finitely generated and free as left $K[x]$-module and, for any dimension d, all but a finite number of isomorphism classes of indecomposable (right) A-modules of dimensional d are of the form $K[x]/(x-\lambda)^m \otimes M$ for some $\lambda \in K$ and some $m \geqslant 1$. We also mention that the class of one-parametric algebras coincides with the class of algebras having exactly one generic module [**9**]. By general theory, the class of one-parametric selfinjective algebras splits into two classes: the *standard algebras*, having simply connected Galois coverings, and the remaining *nonstandard algebras*. It is expected that the nonstandard one-parametric (even the representation-infinite domestic) selfinjective algebras occur only in characteristic 2 and are geometric deformations of standard one-parametric selfinjective algebras. In [**5**], [**6**] we classified all weakly symmetric standard (selfinjective) algebras, by algebras arising from Brauer graphs. In particular, we proved that the class of all weakly symmetric standard one-parametric

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algebras coincides with the class of all weakly symmetric algebras of Euclidean type with nonsingular Cartan matrix. Recall that a selfinjective algebra A is called a *selfinjective algebra of Euclidean type* if A is isomorphic to an orbit algebra $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra of a tilted algebra B of Euclidean type $\Delta \in \{\widetilde{\boldsymbol{A}}_m, \widetilde{\boldsymbol{D}}_n, \widetilde{\boldsymbol{E}}_6, \widetilde{\boldsymbol{E}}_7, \widetilde{\boldsymbol{E}}_8\}$ and G is an admissible infinite cyclic group of K-automorphisms of $\widehat{B}$. Moreover, a selfinjective algebra A is called *weakly symmetric* if the socle $\operatorname{soc} P$ of any indecomposable projective A-module P is isomorphic to its top $P/\operatorname{rad} P$. Here, we associate to a Brauer graph T with exactly one cycle, a nontrivial rotation σ_s and $\lambda \in K \cdot \{o\}$ (respectively, a Brauer tree T with two distinguished vertices v_1, v_2) a one-parametric selfinjective algebra $\Omega^{(1)}(T,\sigma_s,\lambda)$ of Euclidean type $\widetilde{\boldsymbol{A}}_m$ (respectively, a one-parametric selfinjective algebra $\Omega^{(2)}(T,v_1,v_2)$ of Euclidean type $\widetilde{\boldsymbol{D}}_n$).

The aim of this paper is to prove the following theorem.

THEOREM 1. *Let A be a basic connected selfinjective algebra having a simply connected Galois covering. Then A is one-parametric but not weakly symmetric if and only if A is isomorphic to an algebra of one of the forms $\Omega^{(1)}(T,\sigma_s,\lambda)$ or $\Omega^{(2)}(T,v_1,v_2)$.*

For basic background on the representation theory of algebras we refer to [**4**], [**20**], and on selfinjective algebras to [**11**], [**25**].

1. One-parametric selfinjective algebras of Euclidean type $\widetilde{\boldsymbol{A}}_m$.

It is known (see [**12**], [**22**]) that the class of one-parametric selfinjective algebras of Euclidean type $\widetilde{\boldsymbol{A}}_m$ coincides with the class of one-parametric special biserial selfinjective algebras. Recall that following [**23**] an algebra A is called *special biserial* if it is isomorphic to a bound quiver algebra KQ/I, where the bound quiver (Q,I) satisfies the following conditions:

(SP1) The number of arrows in Q with a prescribed source or sink is at most two,

(SP2) For any arrow α of Q, there is at most one arrow β and at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

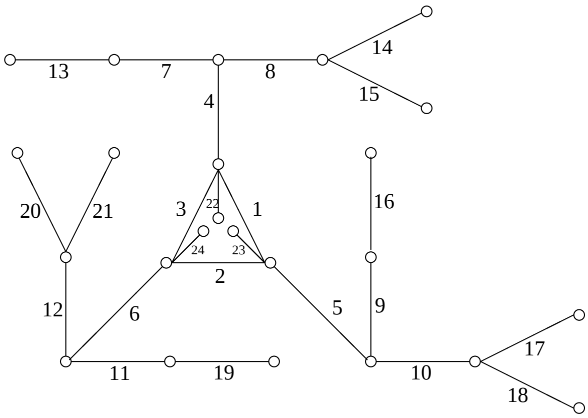
A Brauer graph T is a finite connected undirected graph, where for each vertex there is a fixed circular order on the edges adjacent to it (see [**1**], [**15**], [**17**], [**21**]). In our context we assume that T has at most one cycle (which may be or may not be a loop). We draw T in a plane and agree that the edges adjacent to a given vertex are clockwise ordered. Given a Brauer graph T, this defines a *Brauer quiver* Q_T as follows. The vertices of Q_T are the edges of T and there is an arrow $i \longrightarrow j$ in Q_T if and only if in T j is the direct successor of i in the order around some vertex (to which i and j are both adjacent). We require that every vertex of Q_T belongs to exactly two cycles. Note that this implicitly means that, for every end vertex of T, there is a loop in Q_T.

Let T be a Brauer graph with exactly one cycle $\mathscr{R}_k$, having $k \geqslant 2$ of edges. We draw T in the plane and agree that the vertices and edges of the cycle $\mathscr{R}_k$ are clockwise ordered. Let $v_1, v_2, \ldots, v_k$ be the vertices of $\mathscr{R}_k$ and $e_i = \{v_i, v_{i+1}\}, i = 1,2,\ldots,k$, where $v_{k+1} = v_1$, the edges of $\mathscr{R}_k$. If v is a vertex of the Brauer graph T which is not a vertex of the cycle $\mathscr{R}_k$ then by $n(v)$ we denote the edge incidence to v on the unique walk in T from v to the cycle $\mathscr{R}_k$. Moreover, for $i = 1,2,\ldots,k$, we denote by $n(v_i)$ the edge e_i. For a vertex v of the graph T, we denote by $l(\mathscr{R}_k, v)$ the distance of v to the cycle $\mathscr{R}_k$. Hence $l(\mathscr{R}_k, v) = 0$ if and only if v belongs to $\mathscr{R}_k$. By *an automorphism of the Brauer graph* T we mean an automorphism of the graph T which preserves the fixed circular order on the edges adjacent to any vertex. *A rotation* of the Brauer graph T is an automorphism σ of the Brauer graph T such that, for some integer s with $1 \leqslant s \leqslant k-1$, we have $\sigma(v_i) = v_{i+s}$ for all $i = 1, 2, \ldots k$ (where $k + r = r$ for $r \geqslant 1$), and then we set $\sigma = \sigma_s$. For $k = 2$, we set $\sigma_1(e_1) = e_2$ and $\sigma_1(e_2) = e_1$.

Assume that s is a positive integer such that $1 \leqslant s \leqslant k-1$ and $\gcd(s+2, k) = 1$. We shall define a generalized Brauer quiver Q_{T,σ_s}, obtained from the usual Brauer quiver Q_T of the Brauer graph T by shifting some arrows of Q_T using the rotation σ_s of T. By a σ_s-orbit of a vertex v of T we mean the orbit of v with respect to the action of the cyclic group (σ_s) generated by σ_s on the vertices of T. We note that if two vertices v and w of T belong to the same σ_s-orbit then $l(\mathscr{R}_k, v) = l(\mathscr{R}_k, w)$. Moreover, all σ_s-orbits of vertices of T have the same number of elements, namely k/d, where $d = \gcd(s, k)$. For $m \geqslant 0$, denote by V_m the set of all vertices of T with $l(\mathscr{R}_k, v) = m$. Observe that V_m is a disjoint union of $d|V_m|/k$ σ_s-orbits.

In order to define the generalized Brauer quiver Q_{T,σ_s}, we introduce an order $p(T, \sigma_s)$ of the edges of the Brauer graph T, as the union of $\sum_{m=0}^{\infty}(d|V_m|/k)$ cyclic orders $p(T, \sigma_s, v)$ defined for the representatives v of all pairwise different σ_s-orbits of vertices of T. Let v be a vertex of T. We define the cyclic order $p(T, \sigma_s, v)$ invoking the cyclic orders of edges around the vertices $v, \sigma_s(v), \ldots, \sigma_s^{k/d-1}(v)$ in the Brauer graph T. Let $r \in \{0, 1, \ldots, k/d-1\}$ and i be an edge of T adjacent to the vertex $\sigma_s^r(v)$, and j be the direct successor of i in the cyclic order in T around $\sigma_s^r(v)$. If $j \neq n(\sigma_s^r(v))$, then j is defined to be the direct successor of i in the cyclic order $p(T, \sigma_s, v)$. For $j = n(\sigma_s^r(v))$, $n(\sigma_s^{r+1}(v)) = \sigma_s(n(\sigma_s^r(v)))$ is said to be the direct successor of i in the cyclic order $p(T, \sigma_s, v)$. Therefore, we replaced the cyclic orders around the vertices $\sigma_s^r(v)$, $0 \leqslant r \leqslant k/d-1$, by one (bigger) cyclic order $p(T, \sigma_s, v)$. Observe also that if $e = \{v, w\}$ is an edge of T which is not on the cycle $\mathscr{R}_k$, or e is on the cycle $\mathscr{R}_k$ and $d > 1$, then e belongs to exactly two cyclic orders, namely $p(T, \sigma_s, v)$ and $p(T, \sigma_s, w)$. On the other hand, if $e = \{v, w\}$ is an edge of the cycle $\mathscr{R}_k$ and $d = 1$, then e occurs twice in the cyclic order $p(T, \sigma_s, v) = p(T, \sigma_s, w)$.

EXAMPLE 1.1. Let T be the following Brauer graph with rotation σ_2 defined on the edges as follows: $\sigma_2(1) = 3$, $\sigma_2(2) = 1$, $\sigma_2(3) = 2$, $\sigma_2(4) = 6$, $\sigma_2(5) = 4$, $\sigma_2(6) = 5$, $\sigma_2(7) = 11$, $\sigma_2(8) = 12$, $\sigma_2(9) = 7$, $\sigma_2(10) = 8$, $\sigma_2(11) = 9$, $\sigma_2(12) = 10$, $\sigma_2(13) = 19$, $\sigma_2(14) = 20$, $\sigma_2(15) = 21$, $\sigma_2(16) = 13$, $\sigma_2(17) = 14$, $\sigma_2(18) = 15$, $\sigma_2(19) = 16$, $\sigma_2(20) = 17$, $\sigma_2(21) = 18$, $\sigma_2(22) = 24$, $\sigma_2(23) = 22$, $\sigma_2(24) = 23$,



Then the order $p(T, \sigma_2)$ is the union of the following eight cycles:

(1) $4, 3, 24, 2, 6, 2, 23, 1, 5, 1, 22, 3,$ (2) $4, 7, 8, 6, 11, 12, 5, 9, 10,$
(3) $7, 13, 11, 19, 9, 16,$ (4) $8, 14, 15, 12, 20, 21, 10, 17, 18,$

(5) $13, 19, 16$, (6) $14, 20, 17$,
(7) $15, 21, 18$, (8) $22, 24, 23$.

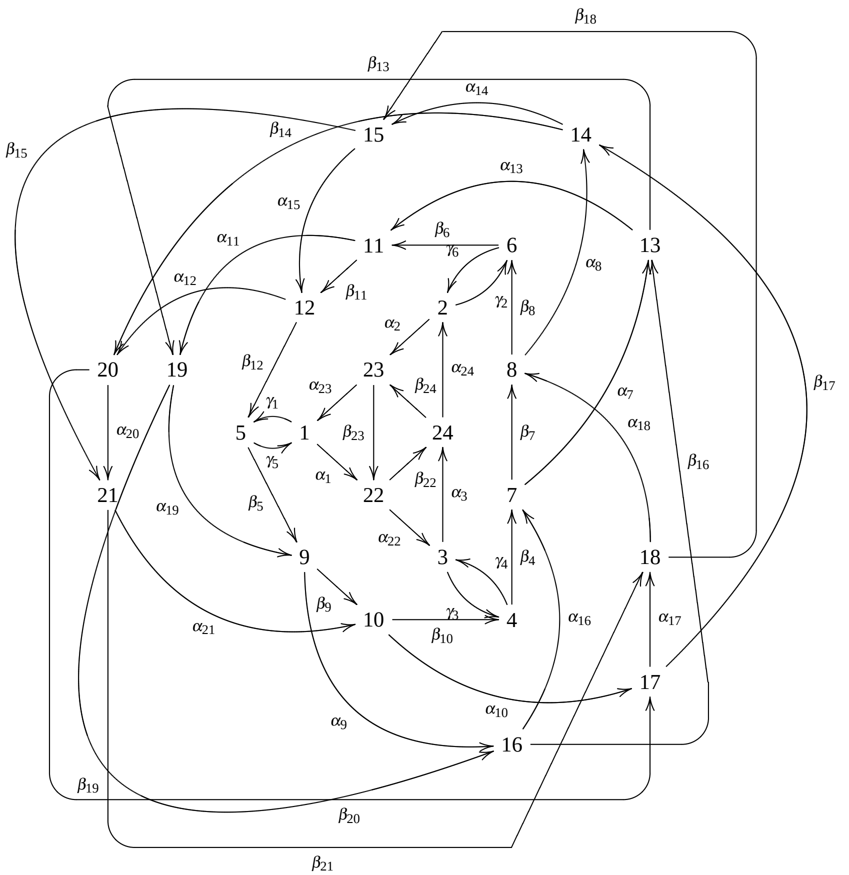
We define *the generalized Brauer quiver* Q_{T,σ_s} as follows. The vertices of Q_{T,σ_s} are the edges of T and there is an arrow $i \longrightarrow j$ in Q_{T,σ_s} if and only if j is the direct successor of i in the order $p(T, \sigma_s)$.

For $\lambda \in K \cdot \{o\}$, we define the algebra $\Omega^{(1)}(T, \sigma_s, \lambda)$ as the bound quiver algebra $KQ_{T,\sigma_s}/\overline{I}^{(1)}(T, \sigma_s, \lambda)$, where KQ_{T,σ_s} is the path algebra of the quiver Q_{T,σ_s} and $\overline{I}^{(1)}(T, \sigma_s, \lambda)$ is the ideal in KQ_{T,σ_s} generated by the elements:

(1) $\alpha\beta$ where $\alpha = i_1 \longrightarrow i_2$, $\beta = i_2 \longrightarrow i_3$ and i_1, i_2, i_3 are not consecutive elements in the cyclic order $p(T, \sigma_s)$.

(2) $C(i, p(T, \sigma_s, v)) - C(i, p(T, \sigma_s, w))$, for $i \neq e_1$ and $C(e_1, p(T, \sigma_s, v)) - \lambda C(e_1, p(T, \sigma_s, w))$, for $i = e_1$, where $i = \{v, w\}$ is an edge of T, $C(i, p(T, \sigma_s, v))$ and $C(i, p(T, \sigma_s, w))$ are the paths from i to $\sigma_s(i)$ in the quiver Q_{T,σ_s}, corresponding to the consecutive elements $i, \dots, \sigma_s(i)$ of the cyclic orders $p(T, \sigma_s, v)$ and $p(T, \sigma_s, w)$, respectively.

EXAMPLE 1.2. For the Brauer graph from Example 1.1, the generalized Brauer quiver Q_{T,σ_2} is of the form



and $\Omega^{(1)}(T,\sigma_2,\lambda)$, for $\lambda \in K\cdot\{o\}$, is given by the above quiver and the ideal $\bar{I}^{(1)}(T,\sigma_2,\lambda)$ in KQ_{T,σ_2} generated by the elements: $\alpha_1\beta_{22}$, $\alpha_2\beta_{23}$, $\alpha_3\beta_{24}$, $\beta_{22}\alpha_{24}$, $\beta_{23}\alpha_{22}$, $\beta_{24}\alpha_{23}$, $\gamma_1\beta_5$, $\beta_{12}\gamma_5$, $\gamma_3\beta_4$, $\beta_{10}\gamma_4$, $\gamma_2\beta_6$, $\beta_8\gamma_6$, $\beta_5\alpha_9$, $\alpha_{19}\beta_9$, $\beta_9\alpha_{10}$, $\alpha_{21}\beta_{10}$, $\beta_4\alpha_7$, $\alpha_{16}\beta_7$, $\beta_7\alpha_8$, $\alpha_{18}\beta_8$, $\beta_6\alpha_{11}$, $\alpha_{13}\beta_{11}$, $\beta_{11}\alpha_{12}$, $\alpha_9\beta_{16}$, $\alpha_7\beta_{13}$, $\beta_{16}\alpha_{13}$, $\alpha_{11}\beta_{19}$, $\beta_{13}\alpha_{19}$, $\alpha_{10}\beta_{17}$, $\beta_{20}\alpha_{17}$, $\alpha_{17}\beta_{18}$, $\beta_{21}\alpha_{18}$, $\alpha_8\beta_{14}$, $\beta_{17}\alpha_{14}$, $\alpha_{14}\beta_{15}$, $\beta_{18}\alpha_{15}$, $\alpha_{12}\beta_{20}$, $\beta_{14}\alpha_{20}$, $\alpha_{20}\beta_{21}$, $\beta_{15}\alpha_{21}$, $\alpha_{15}\beta_{12}$, $\beta_{19}\alpha_{16}$, $\alpha_{23}\alpha_1$, $\alpha_{22}\alpha_3$, $\alpha_{24}\alpha_2$, $\gamma_6\gamma_2$, $\gamma_4\gamma_3$, $\gamma_5\gamma_1$, $\alpha_1\alpha_{22}\gamma_3\gamma_4-\lambda\gamma_1\gamma_5\alpha_1\alpha_{22}$, $\alpha_2\alpha_{23}\gamma_1\gamma_5-\gamma_2\gamma_6\alpha_2\alpha_{23}$, $\alpha_3\alpha_{24}\gamma_2\gamma_6-\gamma_3\gamma_4\alpha_3\alpha_{24}$, $\gamma_4\alpha_3\alpha_{24}\gamma_2-\beta_4\beta_7\beta_8$, $\gamma_5\alpha_1\alpha_{22}\gamma_3-\beta_5\beta_9\beta_{10}$, $\gamma_6\alpha_2\alpha_{23}\gamma_1-\beta_6\beta_{11}\beta_{12}$, $\beta_7\beta_8\beta_6-\alpha_7\alpha_{13}$, $\beta_8\beta_6\beta_{11}-\alpha_8\alpha_{14}\alpha_{15}$, $\beta_9\beta_{10}\beta_4-\alpha_9\alpha_{16}$, $\beta_{10}\beta_4\beta_7-\alpha_{10}\alpha_{17}\alpha_{18}$, $\beta_{11}\beta_{12}\beta_5-\alpha_{11}\alpha_{19}$, $\beta_{12}\beta_5\beta_9-\alpha_{12}\alpha_{20}\alpha_{21}$, $\alpha_{13}\alpha_{11}-\beta_{13}$, $\alpha_{14}\alpha_{15}\alpha_{12}-\beta_{14}$, $\alpha_{15}\alpha_{12}\alpha_{20}-\beta_{15}$, $\alpha_{16}\alpha_7-\beta_{16}$, $\alpha_{17}\alpha_{18}\alpha_8-\beta_{17}$, $\alpha_{18}\alpha_8\alpha_{14}-\beta_{18}$, $\alpha_{19}\alpha_9-\beta_{19}$, $\alpha_{20}\alpha_{21}\alpha_{10}-\beta_{20}$, $\alpha_{21}\alpha_{10}\alpha_{17}-\beta_{21}$, $\alpha_{22}\gamma_3\gamma_4\alpha_3-\beta_{22}$, $\alpha_{23}\gamma_1\gamma_5\alpha_1-\beta_{23}$, $\alpha_{24}\gamma_2\gamma_6\alpha_2-\beta_{24}$.

PROPOSITION 1.3. *Let T be a Brauer graph, σ_s be a rotation of T and $\lambda \in K\cdot\{o\}$ such that the algebra $\Omega^{(1)}(T,\sigma_s,\lambda)$ is defined. Then $\Omega^{(1)}(T,\sigma_s,\lambda)$ is a special biserial one-parametric selfinjective algebra of Euclidean type $\widetilde{\boldsymbol{A}}_m$, and is not weakly symmetric.*

PROOF. It follows from definition that the algebra $\Omega^{(1)}(T,\sigma_s,\lambda)$ is special biserial. Further, the bound quiver $(Q_{T,\sigma_s},\bar{I}^{(1)}(T,\sigma_s,\lambda))$ of $\Omega^{(1)}(T,\sigma_s,\lambda)$ contains a primitive walk (in the sense of [**24**]) of the form

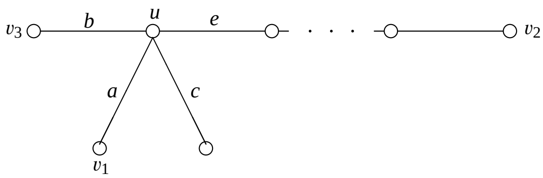
$$\begin{tikzcd}
1 \arrow[r, "+"] \arrow[dd, "+"] & s+2 & s+3 \arrow[l, "+"'] \arrow[r, "+"] & 2(s+2) & 2(s+2)+1 \arrow[l, "+"'] \arrow[d, "+"] \\
 & & & & \vdots \arrow[d, "+"] \\
k(s+2) & \cdots \arrow[l, "+"'] \arrow[r, "+"] & (r+1)(s+2) & r(s+2)+1 \arrow[l, "+"'] \arrow[r, "+"] & r(s+2),
\end{tikzcd}$$

where $\xrightarrow{+}$ is a path of length at least one and the vertex i of the generalized Brauer quiver Q_{T,σ_s} corresponds to the edge e_i of the cycle $\mathscr{R}_k$. In fact, this primitive walk is the unique primitive walk of the bound quiver $\left(Q_{T,\sigma_s},\bar{I}^{(1)}(T,\sigma_s,\lambda)\right)$, because $\gcd(s+2,k)=1$. Consequently $\Omega^{(1)}(T,\sigma_s,\lambda)$ is a one-parametric selfinjective algebra of Euclidean type $\widetilde{\boldsymbol{A}}_m$ (see [**12**], [**22**]). Moreover, for each vertex i of the quiver Q_{T,σ_s}, we have $\operatorname{top}(P(i)) \cong \operatorname{soc}(P(\sigma_s(i))) \ncong \operatorname{soc}(P(i))$, and hence the algebra $\Omega^{(1)}(T,\sigma_s,\lambda)$ is not weakly symmetric. $\square$

2. One-parametric selfinjective algebras of Euclidean type $\widetilde{\boldsymbol{D}}_n$.

Let T be a Brauer tree. Then the simple cycles of the Brauer quiver Q_T may be divided into two camps, α-camps and β-camps, in such a way that any two cycles which intersect nontrivially belong to different camps. We denote by α_i (respectively, β_i) the arrow of the α-camp (respectively, β-camp) of Q_T starting at a vertex i, and by $\alpha(i)$ (respectively, $\beta(i)$) the end vertex of α_i (respectively, β_i). We also denote by A_i (respectively, B_i) the cycle from i to i going once around the α-cycle (respectively, β-cycle) through i.

Let T be a Brauer tree with two (different) distinguished vertices v_1 and v_2 such that v_1 is the end of exactly one edge a. Let the edge b be the direct successor of the edge a and c be the direct predecessor of the edge a in the cyclic order of edges at the end vertex u of a different from v_1. The vertices v_1, v_2 and edges b, c determine a subtree



of the Brauer tree T, where possibly $u = v_2$, $v_2 = v_3$, $b = e$, $c = e$, $b = c = e$, but every time $a \neq b$ and $a \neq c$. We assume that the Brauer quiver Q_T has exactly one exceptional cycle (with multiplicity two) given by the edges of T converging at the exceptional vertex v_2. Moreover, we assume that the cycle in Q_T corresponding to the vertex u is an α-cycle.

We define the algebra $\Omega^{(2)}(T,v_1,v_2)$ as the bound quiver algebra $K\overline{Q}_T^{(2)}/\overline{I}^{(2)}(T,v_1,v_2)$, where $K\overline{Q}_T^{(2)}$ is the path algebra of the quiver

$$\overline{Q}_T^{(2)} = ((Q_T)_0 \cup \{w\}, (Q_T)_1 \cup \{\gamma_1 : c \longrightarrow w, \gamma_2 : w \longrightarrow b\} \setminus \{\beta_a : a \longrightarrow a\})$$

and $\overline{I}^{(2)}(T,v_1,v_2)$ is the ideal in $K\overline{Q}_T^{(2)}$ generated by the elements:

(1) $\alpha_i\beta_{\alpha(i)}$, for all vertices i of Q_T different from c,

(2) $\beta_i\alpha_{\beta(i)}$, for all vertices i of Q_T different from a,

(3) $A_j - B_j$, if the both α-cycle and β-cycle through the vertex j are not exceptional,

(4) $A_j^2 - B_j$, if the α-cycle through the vertex j is exceptional but the β-cycle through j is not exceptional,

(5) $A_j - B_j^2$, if the α-cycle through the vertex j is not exceptional but the β-cycle through the vertex j is exceptional,

(6) $\gamma_2\beta_b$, $\beta_{\beta^{-1}(c)}\gamma_1$,

(7) $\gamma_2\alpha_b\dots\alpha_{\alpha^{-1}(c)}\gamma_1$, A_a ($\gamma_2\gamma_1$, A_a, if $b = c = e$), if the α-cycle through the vertex a is not exceptional,

(8) $\gamma_2A_b\alpha_b\dots\alpha_{\alpha^{-1}(c)}\gamma_1$, A_a^2 ($\gamma_2A_b\gamma_1$, A_a^2, if $b = c = e$), if the α-cycle through the vertex a is exceptional,

(9) $\alpha_c\alpha_a - \gamma_1\gamma_2$.

In order to prove the main proposition of this section we recall the description of exceptional tilted algebras of Euclidean type $\widetilde{\boldsymbol{D}}_n$ presented in [**6**]. Let B be a representation-infinite tilted algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$ and $\{e_1, e_2, \dots, e_m\}$ $(m = n+1)$ a complete set of primitive orthogonal idempotents of B such that $1_B = e_1 + e_2 + \dots + e_m$. Recall that the *repetitive algebra* $\widehat{B}$ of B is the locally finite dimensional algebra without identity [**14**]

$$\widehat{B} = \bigoplus_{k \in \boldsymbol{Z}} (B_k \oplus Q_k),$$

where $B_k = B$ and $Q_k = D(B)$ for all $k \in \boldsymbol{Z}$, and the multiplication in $\widehat{B}$ is defined by

$$(a_k, f_k)_k \cdot (b_k, g_k)_k = (a_k b_k, a_k g_k + f_k b_{k-1})_k$$

for $a_k, b_k \in B_k$, $f_k, g_k \in Q_k$. Then we have the canonical set $\mathscr{E} = \{e_{k,i} | 1 \leqslant i \leqslant m, k \in \boldsymbol{Z}\}$ of primitive orthogonal idempotents of the repetitive algebra $\widehat{B}$ such that $e_{k,1} + e_{k,2} + \ldots + e_{k,m}$ is the identity of the diagonal algebra $B_k = B$ of $\widehat{B}$. By an *automorphism* of $\widehat{B}$ we mean a K-algebra automorphism of $\widehat{B}$ which fixes the set $\mathscr{E}$. A group G of automorphisms of $\widehat{B}$ is called *admissible* if G acts freely on the set $\mathscr{E}$ and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ is defined (see [**13**] for details) and is a (finite dimensional) selfinjective algebra. The action of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ on the set $\mathscr{E}$ is given by $\nu_{\widehat{B}}(e_{k,i}) = e_{k+1,i}$ for $(k,i) \in \boldsymbol{Z} \times \{1, 2, \ldots, m\}$, the infinite cyclic group $(\nu_{\widehat{B}})$ is admissible, and $\widehat{B}/(\nu_{\widehat{B}})$ is isomorphic to the trivial extension $T(B) = B \ltimes D(B)$. An automorphism η of $\widehat{B}$ is said to be *rigid* (see [**22**]) if for any $(k,i) \in \boldsymbol{Z} \times \{1, 2, \ldots, m\}$ there exists $j \in \{1, 2, \ldots, m\}$ such that $\eta(e_{k,i}) = e_{k,j}$. Moreover, an automorphism ρ of $\widehat{B}$ is said to be *nontrivial* if $\rho(e_{k,i}) \neq e_{k,i}$ for some $(k,i) \in \boldsymbol{Z} \times \{1, 2, \ldots, m\}$.

Denote by Q_B the (Gabriel) quiver of B with the set of vertices $\{1, 2, \ldots, m\}$. For each vertex i of Q_B, denote by $P_B(i)$ the indecomposable projective B-module $e_i B$ and by $I_B(i)$ the indecomposable injective B-module $D(Be_i)$. Then, for a sink i of Q_B, the *reflection $S_i^+ B$ of B at i* is the quotient of the one-point extension $B[I_B(i)]$ by the two-sided ideal generated by e_i. The quiver $\sigma_i^+ Q_B$ of $S_i^+ B$ is called the *reflection of Q_B at i*. Observe that the sink i of Q_B is replaced in $\sigma_i^+ Q_B$ by a source i'. Moreover, we have $\widehat{B} \cong \widehat{S_i^+ B}$. A *reflection sequence of sinks* is a sequence $i_1, i_2, \ldots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \ldots \sigma_{i_1}^+ Q_B$ for $1 \leqslant s \leqslant t$ (see [**14**, (2.8)]). Following [**22**] the tilted algebra B is said to be *exceptional* if there exists a reflection sequence $i_1, i_2, \ldots, i_t$ of sinks such that $t < m$ and $B \cong S_{i_t}^+ \ldots S_{i_1}^+ B$. Recall from [**22**, Proposition 2.13] that the tilted algebra B is exceptional if and only if there exists an automorphism φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho \nu_{\widehat{B}}$, for some rigid automorphism ρ of $\widehat{B}$.

The following known fact (see [**4**, Section4] and [**22**, Section 2]) explains our interest in the exceptional tilted algebras of type $\widetilde{\boldsymbol{D}}_n$.

PROPOSITION 2.1. *Let A be a selfinjective algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$. Then A is one-parametric if and only if $A \cong \widehat{B}/(\varphi)$ for an exceptional tilted algebra B of type $\widetilde{\boldsymbol{D}}_n$ and an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \rho \nu_{\widehat{B}}$, for a rigid automorphism ρ of $\widehat{B}$. Moreover, $A \cong \widehat{B}/(\varphi)$ is weakly symmetric if and only if $\varphi^2 = \nu_{\widehat{B}}$ (ρ is trivial).*

Recall from [**20**, (4.9)] that an algebra B is a representation-infinite tilted algebra of an Euclidean type $\widetilde{\boldsymbol{D}}_n$ if and only if B is a tubular extension or a tubular coextension of tubular type $(2, 2, n-2)$ of a tame concealed algebra of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$, for some $1 \leqslant p < n$ and $4 \leqslant q \leqslant n$. Moreover, we know from [**2**, Propositions 2.6 and 3.5] that the class of repetitive algebras $\widehat{B}$ of tilted algebras B of Euclidean types $\widetilde{\boldsymbol{D}}_n$, $n \geqslant 4$, coincides with the class of repetitive algebras $\widehat{B}$ of tubular extensions (equivalently, tubular coextensions) B of tubular types $(2, 2, n-2)$ of tame concealed algebras of types $\widetilde{\boldsymbol{A}}_p$ and $\widetilde{\boldsymbol{D}}_q$, $p \geqslant 1$, $q \geqslant 4$. A tubular extension B of a tame concealed algebra C of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$, $p \geqslant 1$, $q \geqslant 4$, by a finite sequence of pairwise nonisomorphic simple regular, but not simple, C-modules and a finite family of branches is called a *special tubular extension of C*. We describe first all exceptional special tubular extensions of tubular type $(2, 2, n-2)$, $n \geqslant 4$, of tame concealed algebras of types $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$. We abbreviate by $s \xleftarrow{(m,\alpha_i)} t$

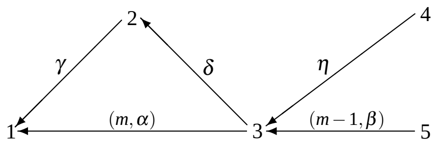
the quiver of the form

$$s \xleftarrow{\alpha_{i,s,1}} (s,1) \xleftarrow{\alpha_{i,s,2}} (s,2) \longleftarrow \cdots \longrightarrow (s,m-1) \xleftarrow{\alpha_{i,s,m}} t\,,$$

where for $m=1$ $s \xleftarrow{(m,\alpha_i)} t$ is the arrow $s \xleftarrow{\alpha_{i,s,1}} t$, and for $m=0$ $s \xleftarrow{(m,\alpha_i)} t$ is the point $s=t$.

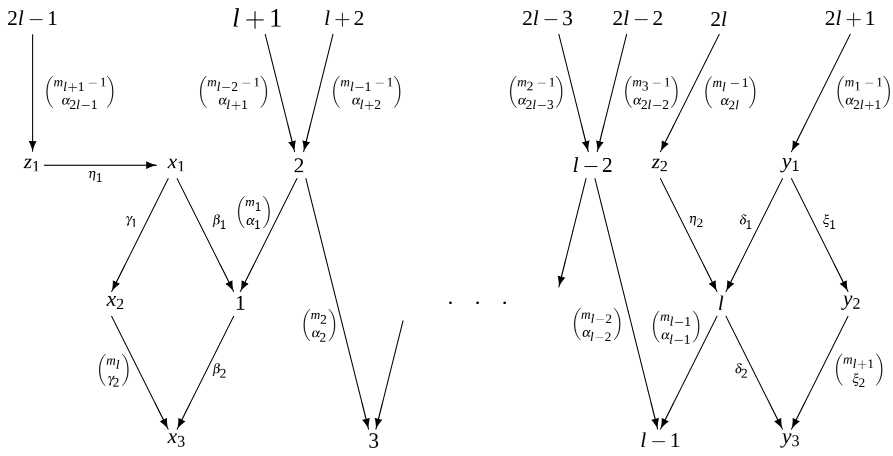
Consider the following families of algebras $\Theta^{(i)}(l,\underline{m})$, $0 \leqslant i \leqslant 8$:

(0) $\Theta^{(0)}(l,\underline{m}) = KQ^{(0)}(l,\underline{m})/I^{(0)}(l,\underline{m})$, where $l=1$, $m \geqslant 2$ and l-tuple $\underline{m}=(m)$, $Q^{(0)}(l,\underline{m})$ is of the form



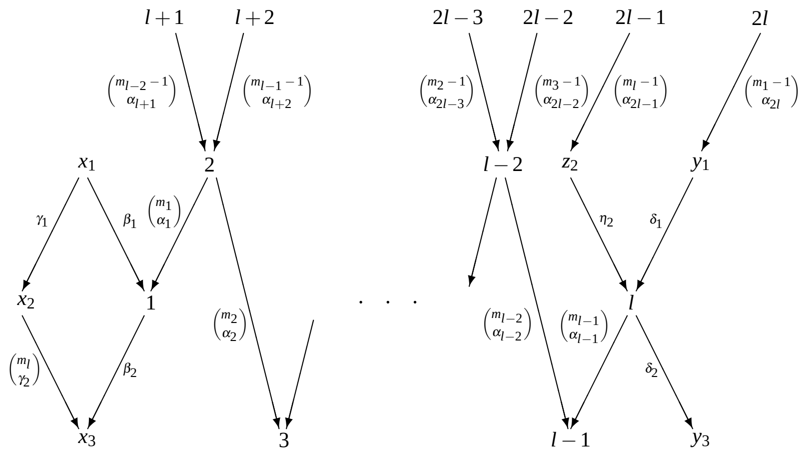
and the ideal $I^{(0)}(l,\underline{m})$ is generated by $\beta_{3,1}\alpha_{1,m}$, $\eta\delta\gamma - \eta\alpha_{1,m}\alpha_{1,m-1}\ldots\alpha_{1,2}\alpha_{1,1}$.

(1) $\Theta^{(1)}(l,\underline{m}) = KQ^{(1)}(l,\underline{m})/I^{(1)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m}=(m_1,m_2,\ldots,m_{l+1})$ is an $(l+1)$-tuple of positive integers, $Q^{(1)}(l,\underline{m})$ is of the form



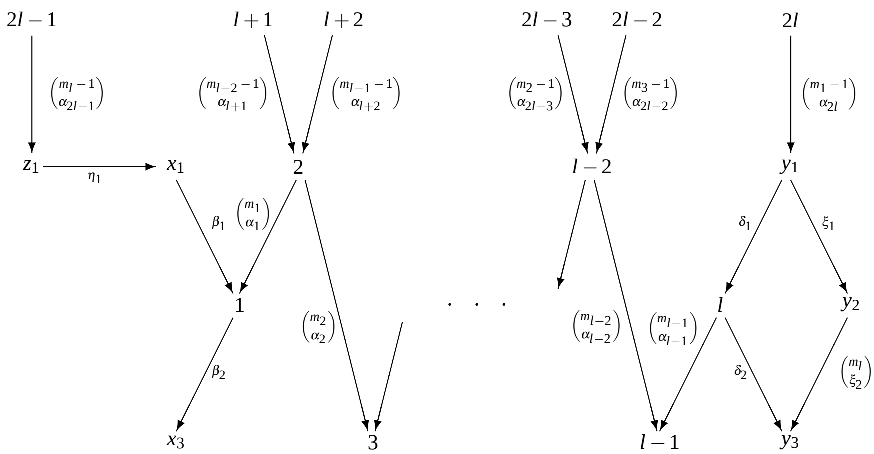
and the ideal $I^{(1)}(l,\underline{m})$ is generated by $\gamma_1\gamma_{2,x_3,m_l}\gamma_{2,x_3,m_l-1}\ldots\gamma_{2,x_3,1} - \beta_1\beta_2$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\ldots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$, $\eta_1\gamma_1$, $\eta_2\alpha_{l-1,l-1,m_{l-1}}$, $\alpha_{2l+1,y_1,1}\xi_1$, $\delta_1\delta_2 - \xi_1\xi_{2,y_3,m_{l+1}}\xi_{2,y_3,m_{l+1}-1}\ldots\xi_{2,y_3,1}$;

(2) $\Theta^{(2)}(l,\underline{m}) = KQ^{(2)}(l,\underline{m})/I^{(2)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m} = (m_1, m_2, \ldots, m_l)$ is an l-tuple of positive integers, $Q^{(2)}(l,\underline{m})$ is of the form



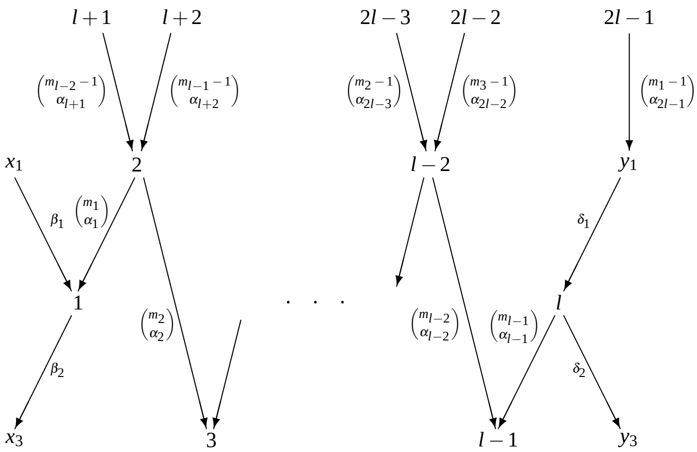
and the ideal $I^{(2)}(l,\underline{m})$ is generated by $\eta_2\alpha_{l-1,l-1,m_{l-1}}$, $\gamma_1\gamma_{2,x_3,m_l}\gamma_{2,x_3,m_l-1}\cdots\gamma_{2,x_3,1} - \beta_1\beta_2$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\cdots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$, $\alpha_{2l,y_1,1}\delta_1\delta_2$;

(3) $\Theta^{(3)}(l,\underline{m}) = KQ^{(3)}(l,\underline{m})/I^{(3)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m} = (m_1, m_2, \ldots, m_l)$ is an l-tuple of positive integers, $Q^{(3)}(l,\underline{m})$ is of the form



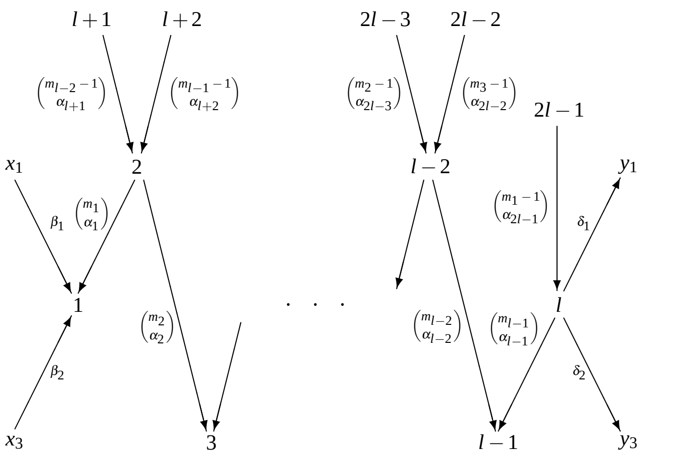
and the ideal $I^{(3)}(l,\underline{m})$ is generated by $\eta_1\beta_1\beta_2$, $\delta_1\delta_2 - \xi_1\xi_{2,y_3,m_l}\xi_{2,y_3,m_l-1}\ldots\xi_{2,y_3,1}$, $\alpha_{2l,y_1,1}\xi_1$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\cdots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$;

(4) $\Theta^{(4)}(l,\underline{m}) = KQ^{(4)}(l,\underline{m})/I^{(4)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m} = (m_1, m_2, \dots, m_{l-1})$ is an $(l-1)$-tuple of positive integers, $Q^{(4)}(l,\underline{m})$ is of the form



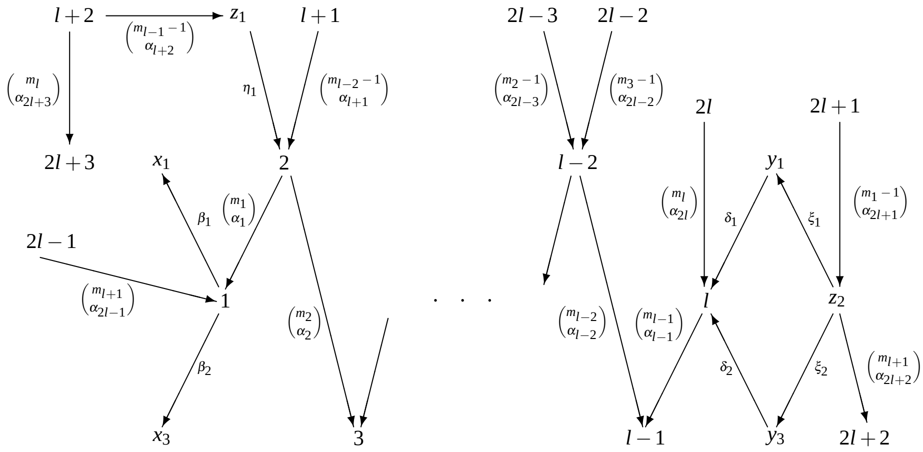
and the ideal $I^{(4)}(l,\underline{m})$ is generated by $\alpha_{2l-1,y_1,1}\delta_1\delta_2$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\cdots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$;

(5) $\Theta^{(5)}(l,\underline{m}) = KQ^{(5)}(l,\underline{m})/I^{(5)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m} = (m_1, m_2, \dots, m_{l-1})$ is an $(l-1)$-tuple of positive integers, $Q^{(5)}(l,\underline{m})$ is of the form



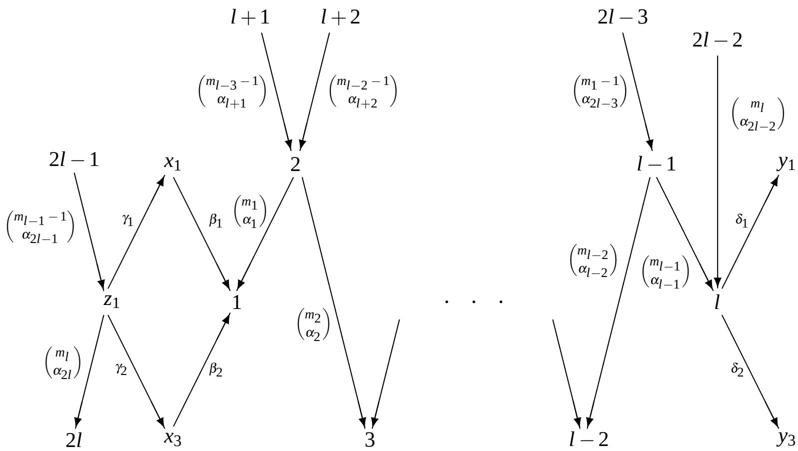
and the ideal $I^{(5)}(l,\underline{m})$ is generated by $\alpha_{2l-1,l,1}\delta_1$, $\alpha_{2l-1,l,1}\delta_2$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\cdots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$;

(6) $\Theta^{(6)}(l,\underline{m}) = KQ^{(6)}(l,\underline{m})/I^{(6)}(l,\underline{m})$, where $l \geqslant 2$ is even, $\underline{m} = (m_1, m_2, \ldots, m_{l+1})$ is an $(l+1)$-tuple of integers with $m_1 \geqslant 1$, $m_2 \geqslant 1$, …, $m_{l-1} \geqslant 1$, $m_l \geqslant 0$, $m_{l+1} \geqslant 0$, $Q^{(6)}(l,\underline{m})$ is of the form



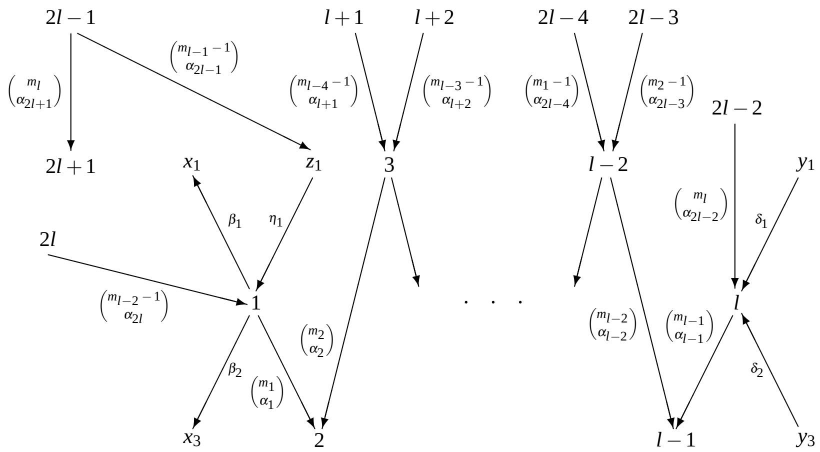
and the ideal $I^{(6)}(l,\underline{m})$ is generated by $\eta_1\alpha_{2,3,m_2}$ (if $l \geqslant 4$), $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+3,4,1}\alpha_{3,3,m_3}$, $\alpha_{l+4,4,1}\alpha_{4,5,m_4}$, $\cdots$, $\alpha_{2l-3,l-2,1}\alpha_{l-3,l-3,m_{l-3}}$, $\alpha_{2l-2,l-2,1}\alpha_{l-2,l-1,m_{l-2}}$, $\alpha_{2l+1,z_2,1}\alpha_{2l+2,2l+2,m_{l+1}}$, $\alpha_{2l-1,1,1}\beta_1$, $\alpha_{2l-1,1,1}\beta_2$, $\xi_1\delta_1 - \xi_2\delta_2$;

(7) $\Theta^{(7)}(l,\underline{m}) = KQ^{(7)}(l,\underline{m})/I^{(7)}(l,\underline{m})$, where $l \geqslant 1$ is odd, $\underline{m} = (m_1, m_2, \ldots, m_l)$ is an l-tuple of integers with $m_1 \geqslant 1$, $m_2 \geqslant 1$, …, $m_{l-1} \geqslant 1$, $m_l \geqslant 0$, $Q^{(7)}(l,\underline{m})$ is of the form



and the ideal $I^{(7)}(l,\underline{m})$ is generated by $\alpha_{2l-1,z_1,1}\alpha_{2l,2l,m_{2l}}$, $\gamma_1\beta_1 - \gamma_2\beta_2$, $\alpha_{l+1,2,1}\alpha_{1,1,m_1}$, $\alpha_{l+2,2,1}\alpha_{2,3,m_2}$, $\cdots$, $\alpha_{2l-5,l-3,1}\alpha_{l-4,l-4,m_{l-4}}$, $\alpha_{2l-4,l-3,1}\alpha_{l-3,l-2,m_{l-3}}$, $\alpha_{2l-3,l-1,1}\alpha_{l-1,l,m_{l-1}}$, $\alpha_{2l-2,l,1}\delta_1$, $\alpha_{2l-2,l,1}\delta_2$;

(8) $\Theta^{(8)}(l,\underline{m}) = KQ^{(8)}(l,\underline{m})/I^{(8)}(l,\underline{m})$, where $l \geqslant 3$ is odd, $\underline{m} = (m_1, m_2, \ldots, m_l)$ is an l-tuple of integers with $m_1 \geqslant 1$, $m_2 \geqslant 1$, …, $m_{l-1} \geqslant 1$, $m_l \geqslant 0$, $Q^{(8)}(l,\underline{m})$ is of the form



and the ideal $I^{(8)}(l,\underline{m})$ is generated by $\alpha_{2l,1,1}\beta_1$, $\alpha_{2l,1,1}\beta_2$, $\eta_1\alpha_{1,2,m_1}$, $\alpha_{l+1,3,1}\alpha_{2,2,m_2}$, $\alpha_{l+2,3,1}\alpha_{3,4,m_3}, \cdots, \alpha_{2l-4,l-2,1}\alpha_{l-3,l-3,m_{l-3}}, \alpha_{2l-3,l-2,1}\alpha_{l-2,l-1,m_{l-2}}, \alpha_{2l-2,l,1}\alpha_{l-1,l-1,m_{l-1}}$.

The algebras $\Theta^{(0)}(l,\underline{m})$ are tubular extensions of tubular type $(2,2,n-2)$ of hereditary algebras of types $\widetilde{\boldsymbol{A}}_p$, while the algebras $\Theta^{(i)}(l,\underline{m})$, $1 \leqslant i \leqslant 8$, are tubular extensions of tubular type $(2,2,n-2)$ of tame concealed algebras of types $\widetilde{\boldsymbol{D}}_q$. Then we have the following consequences of [**6**, Proposition 2.3, 2.7 and Corollary 2.9].

PROPOSITION 2.2. *An algebra B is an exceptional special tubular extension of tubular type $(2,2,n-2)$, $n \geqslant 4$, of a tame concealed algebra C of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$ if and only if $B \cong \Theta^{(i)}(l,\underline{m})$ for some i with $0 \leqslant i \leqslant 8$, $l \geqslant 1$, and a tuple $\underline{m}$ such that the algebra $\Theta^{(i)}(l,\underline{m})$ is defined.*

PROPOSITION 2.3. *Let B be an exceptional algebra which is a special tubular extension of tubular type $(2,2,n-2)$, $n \geqslant 4$, of a tame concealed algebra C of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$. Then there exists an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \nu_{\widehat{B}}$.*

Let $\Theta^{(i)}(l,\underline{m})$, with $0 \leqslant i \leqslant 8$, be an exceptional special tubular extension of tubular type $(2,2,n-2)$ of a tame concealed algebra C of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$. Take a family $\mathscr{S}$ of one-dimensional simple regular C-modules lying in the stable tube of Γ_C, used in the special tubular extension $\Theta^{(i)}(l,\underline{m})$ of C, and a family $\mathscr{B}$ of branches (in the sense of [**20**, (4.4)]) indexed by $\mathscr{S}$. We denote by $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ the tubular extension of $\Theta^{(i)}(l,\underline{m})$ using the modules from $\mathscr{S}$ and the associated branches from $\mathscr{B}$. Observe that $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ is a tubular extension of C of tubular type $(2,2,r-2)$, for some $r \geqslant n$. Clearly, $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}) = \Theta^{(i)}(l,\underline{m})$ if $\mathscr{S}$ and $\mathscr{B}$ are empty. We also note that $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ is not exceptional if $\mathscr{S}$ and $\mathscr{B}$ are nonempty. But we have the following fact proved in [**6**, Proposition 2.11].

PROPOSITION 2.4. *There is a unique exceptional tubular extension $\Theta^{(i)}(l,\underline{m},\mathscr{B})$, $0 \leqslant$*

$i \leqslant 8$, of C of tubular type $(2,2,s-2)$, $s \geqslant n$, containing $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ as a convex subalgebra. Moreover, there exists an automorphism φ of $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $\varphi^2 = \nu_{\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}}$.

We recall the construction of the algebras $\Theta^{(i)}(l,\underline{m},\mathscr{B})$ presented in the proof of [**6**, Proposition 2.11]. Let $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}) \cong K\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})/\overline{I}^{(i)}(l,\underline{m},\mathscr{B})$, $C \cong KQ_C/I_C$, where Q_C is a convex subquiver of $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})$ and $I_C = \overline{I}^{(i)}(l,\underline{m},\mathscr{B}) \cap KQ_C$. We set

$$k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big) = \big|\big(\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})\big)_0\big| - |(Q_C)_0|.$$

We will construct the exceptional algebras $\Theta^{(i)}(l,\underline{m},\mathscr{B})$ by induction on the number $k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big)$. If $k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big) = 0$, then there exists an exceptional algebra $\Theta^{(i)}(l,\underline{m},\mathscr{B})$, containing $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ as a convex subalgebra. Moreover, it follows from Proposition 2.3 that there exists an automorphism φ of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $\varphi^2 = \nu_{\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}}$.

Let $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B})$ be an algebra with $k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big) \geqslant 1$. Assume that, for all algebras $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}^*)$ such that $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B}^*)$ is a subquiver of $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})$ and $0 \leqslant k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}^*),C\big) < k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big)$, there exist an exceptional algebra $\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)$, containing $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}^*)$ as a convex subalgebra, and an automorphism φ_1 of the algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}$ such that $\varphi_1^2 = \nu_{\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}}$. Since $k\big(\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}),C\big) \geqslant 1$ there exists a vertex w_1 of the quiver $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})$, but not of Q_C, such that either w_1 is the source of exactly one arrow ξ or w_1 is the target of exactly one arrow ξ. Observe that the vertex w_1 (respectively, arrow ξ) is the vertex (respectively, arrow) of some branch from the family $\mathscr{B}$. Let $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B}^*)$ be the quiver obtained from $\overline{Q}^{(i)}(l,\underline{m},\mathscr{B})$ by deleting w_1 and ξ. Then, by our inductive assumption, there exist an exceptional algebra $\Theta^{(i)}(l,\underline{m},\mathscr{B}^*) = KQ^{(i)}(l,\underline{m},\mathscr{B}^*)/I^{(i)}(l,\underline{m},\mathscr{B}^*)$, containing $\overline{\Theta}^{(i)}(l,\underline{m},\mathscr{B}^*)$ as a convex subalgebra, and an automorphism φ_1 of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}$ such that $\varphi_1^2 = \nu_{\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}}$. For an arrow α of the bound quiver algebra $\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)$ and $k \in \boldsymbol{Z}$, we denote by (k,α) the arrow of the k-part $\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)_k$ of the repetitive bound quiver algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}$ corresponding to the arrow α. We have two cases to consider.

(1) Assume that w_1 is the source of the arrow ξ. Let w_2 be the target of the arrow ξ, and $\eta_1 \dots \eta_t$ be the unique nonzero path of maximal length with the first arrow $\eta_1 = \xi$ in the bound quiver $\big(\overline{Q}^{(i)}(l,\underline{m},\mathscr{B}),\overline{I}^{(i)}(l,\underline{m},\mathscr{B})\big)$. Denote by w_3 the end vertex of the arrow η_t.

(a) Assume that $w_2 = w_3$. Let w_4 be such that $e_{0,w_4} = \varphi_1(e_{0,w_2})$. We add the new vertices w_1 and w_6, and new arrows $\xi : w_1 \longrightarrow w_2$ and $\delta : w_4 \longrightarrow w_6$, to the quiver $Q^{(i)}(l,\underline{m},\mathscr{B}^*)$. Then we define an algebra

$$\Theta^{(i)}(l,\underline{m},\mathscr{B}) = KQ^{(i)}(l,\underline{m},\mathscr{B})/I^{(i)}(l,\underline{m},\mathscr{B}),$$

where

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_0 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_0 \cup \{w_1,w_6\}$$

and

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_1 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_1 \cup \{\xi,\delta\}.$$

The ideal $I^{(i)}(l,\underline{m},\mathscr{B})$ in $KQ^{(i)}(l,\underline{m},\mathscr{B})$ is generated by the following elements:

- all generators of the ideal $I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\gamma\delta$, if there exists the arrow γ such that $t(\gamma)=w_4$,
- $\xi\beta$, if there exists the arrow β such that $s(\beta)=w_2$.

The required automorphism φ of the algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ is determined by the following equalities: $\varphi(e_{k,w_1}) = e_{k+1,w_6}$, $\varphi(e_{k,w_6}) = e_{k,w_1}$ and $\varphi(e_{k,r}) = \varphi_1(e_{k,r})$ for all remaining indices r and k. Let (k,ξ') and (k,δ') be the arrows of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $s((k,\xi')) = (k+1,w_2)$, $t((k,\xi')) = (k,w_1)$, $s((k,\delta')) = (k+1,w_6)$ and $t((k,\delta')) = (k,w_4)$. Then we have $\varphi((k,\xi)) = (k,\delta')$, $\varphi((k,\delta)) = (k,\xi')$, $\varphi((k,\xi')) = (k+1,\delta)$, $\varphi((k,\delta')) = (k+1,\xi)$.

(b) Assume that $w_2 \neq w_3$. Let w_4 and w_5 be such that $e_{0,w_2} = \varphi_1(e_{0,w_4})$ and $e_{0,w_5} = \varphi_1(e_{0,w_3})$, and $\delta \in (Q^{(i)}(l,\underline{m},\mathscr{B}^*))_1$ such that $\varphi_1((0,\delta)) = (0,\eta_t')$ for the arrow $(0,\eta_t')$ of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}$ such that $s((0,\eta_t')) = (1,w_3)$ and $t((0,\eta_t')) = (0,w_2)$. We add the new vertices w_1 and w_6, new arrow $\xi : w_1 \longrightarrow w_2$, and replace the arrow $\delta : w_5 \longrightarrow w_4$ by new arrows $\delta_1 : w_6 \longrightarrow w_4$ and $\delta_2 : w_5 \longrightarrow w_6$, in the quiver $Q^{(i)}(l,\underline{m},\mathscr{B}^*)$. Then we define an algebra

$$\Theta^{(i)}(l,\underline{m},\mathscr{B}) = KQ^{(i)}(l,\underline{m},\mathscr{B})/I^{(i)}(l,\underline{m},\mathscr{B}),$$

where

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_0 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_0 \cup \{w_1,w_6\}$$

and

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_1 = \big(\big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_1 \backslash \{\delta\}\big) \cup \{\xi,\delta_1,\delta_2\}.$$

The ideal $I^{(i)}(l,\underline{m},\mathscr{B})$ in $KQ^{(i)}(l,\underline{m},\mathscr{B})$ is generated by the following elements:

- all generators of the ideal $I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\gamma\delta_1$, if there exists the arrow γ such that $\gamma\delta \in I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\xi\beta$, if there exists the arrow β such that $\xi\beta \in \bar{I}^{(i)}(l,\underline{m},\mathscr{B})$.

The required automorphism φ of the algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ is determined by the following equalities: $\varphi(e_{k,w_1}) = e_{k+1,w_6}$, $\varphi(e_{k,w_6}) = e_{k,w_1}$ and $\varphi(e_{k,r}) = \varphi_1(e_{k,r})$ for all remaining indices r and k. Let (k,ξ') be the arrow of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $s((k,\xi')) = (k+1,w_3)$ and $t((k,\xi')) = (k,w_1)$. Then we have $\varphi((k,\xi)) = (k+1,\delta_1)$, $\varphi((k,\delta_1)) = (k,\xi)$, $\varphi((k,\delta_2)) = (k,\xi')$, $\varphi((k,\xi')) = (k+1,\delta_2)$.

(2) Assume that w_1 is the target of the arrow ξ. Let w_2 be the source of ξ and $\eta_1\ldots\eta_t$ be the unique nonzero path of maximal length with the last arrow $\eta_t = \xi$ in the bound quiver

$(\overline{Q}^{(i)}(l,\underline{m},\mathscr{B}),\bar{I}^{(i)}(l,\underline{m},\mathscr{B}))$. Denote by w_3 the source of the arrow η_1.

(a) Assume that $w_2 = w_3$. Let w_4 be such that $e_{0,w_2} = \varphi_1(e_{0,w_4})$. We add the new vertices w_1 and w_6, and new arrows $\xi : w_2 \longrightarrow w_1$ and $\delta : w_6 \longrightarrow w_4$, to the quiver $Q^{(i)}(l,\underline{m},\mathscr{B}^*)$. Then we define an algebra

$$\Theta^{(i)}(l,\underline{m},\mathscr{B}) = KQ^{(i)}(l,\underline{m},\mathscr{B})/I^{(i)}(l,\underline{m},\mathscr{B}),$$

where

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_0 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_0 \cup \{w_1,w_6\}$$

and

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_1 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_1 \cup \{\xi,\delta\}.$$

The ideal $I^{(i)}(l,\underline{m},\mathscr{B})$ in $KQ^{(i)}(l,\underline{m},\mathscr{B})$ is generated by the following elements:

- all generators of the ideal $I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\delta\gamma$, if there exists the arrow γ such that $s(\gamma) = w_4$,
- $\beta\xi$, if there exists the arrow β such that $t(\beta) = w_2$.

The required automorphism φ of the algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ is determined by the following equalities: $\varphi(e_{k,w_1}) = e_{k,w_6}$, $\varphi(e_{k,w_6}) = e_{k+1,w_1}$ and $\varphi(e_{k,r}) = \varphi_1(e_{k,r})$ for all remaining indices r and k. Let (k,ξ') and (k,δ') be the arrows of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $s((k,\xi')) = (k+1,w_1)$, $t((k,\xi')) = (k,w_2)$, $s((k,\delta')) = (k+1,w_4)$ and $t((k,\delta')) = (k,w_6)$. Then we have $\varphi((k,\xi)) = (k,\delta')$, $\varphi((k,\delta)) = (k,\xi')$, $\varphi((k,\xi')) = (k+1,\delta)$, $\varphi((k,\delta')) = (k+1,\xi)$.

(b) Assume that $w_2 \neq w_3$. Let w_4 and w_5 be such that $e_{0,w_4} = \varphi_1(e_{0,w_2})$ and $e_{0,w_3} = \varphi_1(e_{0,w_5})$, and $\delta \in \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_1$ such that $\varphi_1((0,\delta)) = (0,\eta'_{t-1})$ for the arrow $(0,\eta'_{t-1})$ of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B}^*)}$ such that $s((0,\eta'_{t-1})) = (1,w_2)$ and $t((0,\eta'_{t-1})) = (0,w_3)$. We add the new vertices w_1 and w_6, new arrow $\xi : w_2 \longrightarrow w_1$, and replace the arrow $\delta : w_4 \longrightarrow w_5$ by new arrows $\delta_1 : w_6 \longrightarrow w_5$ and $\delta_2 : w_4 \longrightarrow w_6$, in the quiver $Q^{(i)}(l,\underline{m},\mathscr{B}^*)$. Then we define an algebra

$$\Theta^{(i)}(l,\underline{m},\mathscr{B}) = KQ^{(i)}(l,\underline{m},\mathscr{B})/I^{(i)}(l,\underline{m},\mathscr{B}),$$

where

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_0 = \big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_0 \cup \{w_1,w_6\}$$

and

$$\big(Q^{(i)}(l,\underline{m},\mathscr{B})\big)_1 = \big(\big(Q^{(i)}(l,\underline{m},\mathscr{B}^*)\big)_1 \setminus \{\delta\}\big) \cup \{\xi,\delta_1,\delta_2\}.$$

The ideal $I^{(i)}(l,\underline{m},\mathscr{B})$ in $KQ^{(i)}(l,\underline{m},\mathscr{B})$ is generated by the following elements:

- all generators of the ideal $I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\delta_1\gamma$, if there exists the arrow γ such that $\delta\gamma \in I^{(i)}(l,\underline{m},\mathscr{B}^*)$,
- $\beta\xi$, if there exists the arrow β such that $\beta\xi \in \bar{I}^{(i)}(l,\underline{m},\mathscr{B})$.

The required automorphism φ of the algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ is determined by the following equalities: $\varphi(e_{k,w_1}) = e_{k,w_6}$, $\varphi(e_{k,w_6}) = e_{k+1,w_1}$ and $\varphi(e_{k,r}) = \varphi_1(e_{k,r})$ for all remaining indices r and k. Let (k,ξ') be the arrow of the quiver of the repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ such that $s((k,\xi')) = (k+1,w_1)$ and $t((k,\xi')) = (k,w_3)$. Then we have $\varphi((k,\xi)) = (k,\delta_2)$, $\varphi((k,\delta_1)) = (k,\xi')$, $\varphi((k,\delta_2)) = (k+1,\xi)$, $\varphi((k,\xi')) = (k+1,\delta_1)$.

Further, the following fact proved in [**6**, Theorem 3] gives a complete description of the repetitive algebras of exceptional tilted algebras of types $\widetilde{\boldsymbol{D}}_n$.

PROPOSITION 2.5. *Let B be a tilted algebra of Euclidean type. Then B is exceptional if and only if $\widehat{B}$ is isomorphic to a repetitive algebra $\widehat{\Theta^{(i)}(l,\underline{m},\mathscr{B})}$ for some i with $0 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches.*

PROPOSITION 2.6. *Let B be an exceptional special tubular extension of tubular type $(2,2,n-2)$, $n \geqslant 4$, of a tame concealed algebra C of type $\widetilde{\boldsymbol{A}}_p$ or $\widetilde{\boldsymbol{D}}_q$. Then there exists an automorphism φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$, if and only if $B \cong \Theta^{(i)}(l,\underline{m})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, and a tuple $\underline{m}$.*

PROOF. Assume that $B \cong \Theta^{(i)}(l,\underline{m})$ for some i with $5 \leqslant i \leqslant 8$. Then there exist exactly two automorphisms φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$, and are determined by the following equalities (see [**6**, Lemma 2.8 and Corollary 2.9]):

(1) If $B \cong \Theta^{(5)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_3}) = e_{k+1,y_1}$, $\varphi(e_{k,y_1}) = e_{k,x_1}$, $\varphi(e_{k,y_3}) = e_{k,x_3}$, (respectively, $\varphi(e_{k,x_1}) = e_{k+1,y_1}$, $\varphi(e_{k,x_3}) = e_{k+1,y_3}$, $\varphi(e_{k,y_1}) = e_{k,x_3}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$), $\varphi(e_{k,2i-1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,l/2$, $k \in \boldsymbol{Z}$.

(2) If $B \cong \Theta^{(6)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k,y_1}$, $\varphi(e_{k,x_3}) = e_{k,y_3}$, $\varphi(e_{k,y_1}) = e_{k+1,x_3}$, $\varphi(e_{k,y_3}) = e_{k+1,x_1}$, (respectively, $\varphi(e_{k,x_1}) = e_{k,y_3}$, $\varphi(e_{k,x_3}) = e_{k,y_1}$, $\varphi(e_{k,y_1}) = e_{k+1,x_1}$, $\varphi(e_{k,y_3}) = e_{k+1,x_3}$), $\varphi(e_{k,z_1}) = e_{k+1,l}$, $\varphi(e_{k,z_2}) = e_{k+1,1}$, $\varphi(e_{k,1}) = e_{k,z_2}$, $\varphi(e_{k,l}) = e_{k,z_1}$, $\varphi(e_{k,2i+1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,(l/2)-1$, $k \in \boldsymbol{Z}$.

(3) If $B \cong \Theta^{(7)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_3}) = e_{k+1,y_1}$, $\varphi(e_{k,y_1}) = e_{k,x_1}$, $\varphi(e_{k,y_3}) = e_{k,x_3}$, (respectively, $\varphi(e_{k,x_1}) = e_{k+1,y_1}$, $\varphi(e_{k,x_3}) = e_{k+1,y_3}$, $\varphi(e_{k,y_1}) = e_{k,x_3}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$), $\varphi(e_{k,z_1}) = e_{k+1,l}$, $\varphi(e_{k,l}) = e_{k,z_1}$, $\varphi(e_{k,2i-1}) = e_{k,l+1-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l-2i}$ for $i = 1,2,\dots,(l-1)/2$, $k \in \boldsymbol{Z}$.

(4) If $B \cong \Theta^{(8)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k,y_1}$, $\varphi(e_{k,x_3}) = e_{k,y_3}$, $\varphi(e_{k,y_1}) = e_{k+1,x_3}$, $\varphi(e_{k,y_3}) = e_{k+1,x_1}$, (respectively, $\varphi(e_{k,x_1}) = e_{k,y_3}$, $\varphi(e_{k,x_3}) = e_{k,y_1}$, $\varphi(e_{k,y_1}) = e_{k+1,x_1}$, $\varphi(e_{k,y_3}) = e_{k+1,x_3}$), $\varphi(e_{k,z_1}) = e_{k+1,l}$, $\varphi(e_{k,l}) = e_{k,z_1}$, $\varphi(e_{k,2i-1}) = e_{k+1,l+1-2i}$, $\varphi(e_{k,2i}) = e_{k,l-2i}$ for $i = 1,2,\dots,(l-1)/2$, $k \in \boldsymbol{Z}$.

Then $\varphi^2(e_{k,x_1}) = e_{k+1,x_3}$, $\varphi^2(e_{k,x_3}) = e_{k+1,x_1}$, $\varphi^2(e_{k,y_1}) = e_{k+1,y_3}$, $\varphi^2(e_{k,x_3}) = e_{k+1,y_1}$ for $k \in \boldsymbol{Z}$, and $\varphi^2(e_{k,r}) = e_{k+1,r}$ for all remaining indices r and k. The rigid automorphism ρ is determined by the following equalities: $\rho(e_{k,x_1}) = e_{k,x_3}$, $\rho(e_{k,x_3}) = e_{k,x_1}$, $\rho(e_{k,y_1}) = e_{k,y_3}$, $\rho(e_{k,y_3}) = e_{k,y_1}$ for $k \in \boldsymbol{Z}$, and $\rho(e_{k,r}) = e_{k,r}$ for all remaining indices r and k.

Now, assume that $B \cong \Theta^{(i)}(l,\underline{m})$ for some i with $0 \leqslant i \leqslant 4$. Then there exists exactly one automorphism φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some rigid automorphism ρ of $\widehat{B}$, and it is determined by the following equalities (see [**6**, Lemma 2.8 and Corollary 2.9]):

(1) If $B \cong \Theta^{(0)}(l,\underline{m})$, then $\varphi(e_{k,1}) = e_{k,3}$, $\varphi(e_{k,2}) = e_{k,4}$, $\varphi(e_{k,3}) = e_{k+1,1}$, $\varphi(e_{k,4}) = e_{k+1,2}$, $\varphi(e_{k,5}) = e_{k+1,(1,m-1)}$, $\varphi(e_{k,(1,j)}) = e_{k,(3,j)}$, $\varphi(e_{k,(3,j)}) = e_{k+1,(1,j)}$, for $j = 1,2,\dots,m-2$.

(2) If $B \cong \Theta^{(1)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_2}) = e_{k,z_2}$, $\varphi(e_{k,x_3}) = e_{k,l}$, $\varphi(e_{k,y_1}) = e_{k+1,1}$, $\varphi(e_{k,y_2}) = e_{k,z_1}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$, $\varphi(e_{k,z_1}) = e_{k+1,y_2}$, $\varphi(e_{k,z_2}) = e_{k+1,x_2}$, $\varphi(e_{k,1}) = e_{k,y_1}$, $\varphi(e_{k,l}) = e_{k+1,x_3}$, $\varphi(e_{k,2i+1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,(l/2)-1$, $k \in \boldsymbol{Z}$.

(3) If $B \cong \Theta^{(2)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_2}) = e_{k,z_2}$, $\varphi(e_{k,x_3}) = e_{k,l}$, $\varphi(e_{k,y_1}) = e_{k+1,1}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$, $\varphi(e_{k,z_2}) = e_{k+1,x_2}$, $\varphi(e_{k,1}) = e_{k,y_1}$, $\varphi(e_{k,l}) = e_{k+1,x_3}$, $\varphi(e_{k,2i+1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,(l/2)-1$, $k \in \boldsymbol{Z}$.

(4) If $B \cong \Theta^{(3)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_3}) = e_{k,l}$, $\varphi(e_{k,y_1}) = e_{k+1,1}$, $\varphi(e_{k,y_2}) = e_{k,z_1}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$, $\varphi(e_{k,z_1}) = e_{k+1,y_2}$, $\varphi(e_{k,1}) = e_{k,y_1}$, $\varphi(e_{k,l}) = e_{k+1,x_3}$, $\varphi(e_{k,2i+1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,(l/2)-1$, $k \in \boldsymbol{Z}$.

(5) If $B \cong \Theta^{(4)}(l,\underline{m})$, then $\varphi(e_{k,x_1}) = e_{k+1,y_3}$, $\varphi(e_{k,x_3}) = e_{k,l}$, $\varphi(e_{k,y_1}) = e_{k+1,1}$, $\varphi(e_{k,y_3}) = e_{k,x_1}$, $\varphi(e_{k,1}) = e_{k,y_1}$, $\varphi(e_{k,l}) = e_{k+1,x_3}$, $\varphi(e_{k,2i+1}) = e_{k,l-2i}$, $\varphi(e_{k,2i}) = e_{k+1,l+1-2i}$ for $i = 1,2,\dots,(l/2)-1$, $k \in \boldsymbol{Z}$.

An easy checking shows that in this case we have $\varphi^2 = \nu_{\widehat{B}}$ (ρ is trivial). $\square$

PROPOSITION 2.7. *Let B be a representation-infinite exceptional tilted algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$ which is a tubular extension of a tame concealed algebra C. Then there exists an automorphism φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$, if and only if $B \cong \Theta^{(i)}(l,\underline{m},\mathscr{B})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches.*

PROOF. Assume that there exists an automorphism φ of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$. It follows from [**6**, Lemma 2.10], that there exists a subset $\mathscr{D}$ of $\{e_1,e_2,\dots,e_m\}$ such that $e_{\mathscr{D}}Be_{\mathscr{D}}$ is an exceptional convex subalgebra of B and a special tubular extension of tubular type $(2,2,n_1-2)$, for some $n_1 \leqslant n$, of the tame concealed algebra C. Denote by φ_1 the restriction of φ to $e_{\mathscr{D}}Be_{\mathscr{D}}$. Since $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$, then $\varphi_1^2 = \rho_1\nu_{\widehat{e_{\mathscr{D}}Be_{\mathscr{D}}}}$, for some nontrivial rigid automorphism ρ_1 of $\widehat{e_{\mathscr{D}}Be_{\mathscr{D}}}$. Thus from Proposition 2.6 follows that $e_{\mathscr{D}}Be_{\mathscr{D}} \cong \Theta^{(i)}(l,\underline{m})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, and a tuple $\underline{m}$. Then Proposition 2.4 implies that $B \cong \Theta^{(i)}(l,\underline{m},\mathscr{B})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches.

Assume that $B \cong \Theta^{(i)}(l,\underline{m},\mathscr{B})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches. It follows from Proposition 2.4 that there exists an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \nu_{\widehat{B}}$. We define an automorphism $\overline{\varphi}$ of the algebra automorphism $\widehat{B}$ as follows:

(1) If $B \cong \Theta^{(5)}(l,\underline{m},\mathscr{B})$, then $\overline{\varphi}(e_{k,x_1}) = e_{k+1,y_3}$, $\overline{\varphi}(e_{k,x_3}) = e_{k+1,y_1}$, $\overline{\varphi}(e_{k,y_1}) = e_{k,x_1}$, $\overline{\varphi}(e_{k,y_3}) = e_{k,x_3}$ for $k \in \boldsymbol{Z}$, and $\overline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(2) If $B \cong \Theta^{(6)}(l,\underline{m},\mathscr{B})$, then $\overline{\varphi}(e_{k,x_1}) = e_{k,y_1}$, $\overline{\varphi}(e_{k,x_3}) = e_{k,y_3}$, $\overline{\varphi}(e_{k,y_1}) = e_{k+1,x_3}$, $\overline{\varphi}(e_{k,y_3}) = e_{k+1,x_1}$ for $k \in \boldsymbol{Z}$, and $\overline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(3) If $B \cong \Theta^{(7)}(l,\underline{m},\mathscr{B})$, then $\overline{\varphi}(e_{k,x_1}) = e_{k+1,y_3}$, $\overline{\varphi}(e_{k,x_3}) = e_{k+1,y_1}$, $\overline{\varphi}(e_{k,y_1}) = e_{k,x_1}$, $\overline{\varphi}(e_{k,y_3}) = e_{k,x_3}$ for $k \in \boldsymbol{Z}$, and $\overline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(4) If $B \cong \Theta^{(8)}(l,\underline{m},\mathscr{B})$, then $\overline{\varphi}(e_{k,x_1}) = e_{k,y_1}$, $\overline{\varphi}(e_{k,x_3}) = e_{k,y_3}$, $\overline{\varphi}(e_{k,y_1}) = e_{k+1,x_3}$, $\overline{\varphi}(e_{k,y_3}) = e_{k+1,x_1}$ for $k \in \boldsymbol{Z}$, and $\overline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

Then $\overline{\varphi}^2(e_{k,x_1}) = e_{k+1,x_3}$, $\overline{\varphi}^2(e_{k,x_3}) = e_{k+1,x_1}$, $\overline{\varphi}^2(e_{k,y_1}) = e_{k+1,y_3}$, $\overline{\varphi}^2(e_{k,x_3}) = e_{k+1,y_1}$ for $k \in \boldsymbol{Z}$, and $\overline{\varphi}^2(e_{k,r}) = e_{k+1,r}$ for all remaining indices r and k. A direct checking shows that $\overline{\varphi}^2 = \rho\nu_{\widehat{B}}$, for the nontrivial rigid automorphism ρ of $\widehat{B}$ determined by the following equalities: $\rho(e_{k,x_1}) = e_{k,x_3}$, $\rho(e_{k,x_3}) = e_{k,x_1}$, $\rho(e_{k,y_1}) = e_{k,y_3}$, $\rho(e_{k,y_3}) = e_{k,y_1}$ for $k \in \boldsymbol{Z}$, and $\rho(e_{k,r}) = e_{k,r}$ for all remaining indices r and k. □

PROPOSITION 2.8. *Let B be a representation-infinite exceptional tilted algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$ which is a tubular extension of a tame concealed algebra C such that there exists an automorphism φ of the repetitive algebra $\widehat{B}$ with $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$. Then $\widehat{B}/(\varphi) \cong \Omega^{(2)}(T,v_1,v_2)$, for some Brauer graph T and its vertices v, v_1, v_2.*

PROOF. Let φ be an automorphism of the repetitive algebra $\widehat{B}$ such that $\varphi^2 = \rho\nu_{\widehat{B}}$, for some nontrivial rigid automorphism ρ of $\widehat{B}$. It follows from Proposition 2.7 that $B \cong \Theta^{(i)}(l,\underline{m},\mathscr{B})$ for some i with $5 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches. We define the automorphism $\underline{\varphi}$ of $\widehat{B}$ as follows:

(1) If $B \cong \Theta^{(5)}(l,\underline{m},\mathscr{B})$, then $\underline{\varphi}(e_{k,x_1}) = e_{k+1,y_1}$, $\underline{\varphi}(e_{k,x_3}) = e_{k+1,y_3}$, $\underline{\varphi}(e_{k,y_1}) = e_{k,x_1}$, $\underline{\varphi}(e_{k,y_3}) = e_{k,x_3}$ for $k \in \boldsymbol{Z}$, and $\underline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(2) If $B \cong \Theta^{(6)}(l,\underline{m},\mathscr{B})$, then $\underline{\varphi}(e_{k,x_1}) = e_{k,y_1}$, $\underline{\varphi}(e_{k,x_3}) = e_{k,y_3}$, $\underline{\varphi}(e_{k,y_1}) = e_{k+1,x_1}$, $\underline{\varphi}(e_{k,y_3}) = e_{k+1,x_3}$ for $k \in \boldsymbol{Z}$, and $\underline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(3) If $B \cong \Theta^{(7)}(l,\underline{m},\mathscr{B})$, then $\underline{\varphi}(e_{k,x_1}) = e_{k+1,y_1}$, $\underline{\varphi}(e_{k,x_3}) = e_{k+1,y_3}$, $\underline{\varphi}(e_{k,y_1}) = e_{k,x_1}$, $\underline{\varphi}(e_{k,y_3}) = e_{k,x_3}$ for $k \in \boldsymbol{Z}$, and $\underline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

(4) If $B \cong \Theta^{(8)}(l,\underline{m},\mathscr{B})$, then $\underline{\varphi}(e_{k,x_1}) = e_{k,y_1}$, $\underline{\varphi}(e_{k,x_3}) = e_{k,y_3}$, $\underline{\varphi}(e_{k,y_1}) = e_{k+1,x_1}$, $\underline{\varphi}(e_{k,y_3}) = e_{k+1,x_3}$ for $k \in \boldsymbol{Z}$, and $\underline{\varphi}(e_{k,r}) = \varphi(e_{k,r})$ for all remaining indices r and k.

A direct checking shows that $\underline{\varphi}^2 = \nu_{\widehat{B}}$.

Let T be a Brauer tree and v_1, v_2 its vertices such that the algebra $\Omega^{(2)}(T,v_1,v_2)$ is defined. We define the symmetric algebra (see [**6**, Proposition 1.4]) $\Gamma^{(2)}(T,v_1,v_2)$ as the bound quiver algebra $KQ_T^{(2)}/I^{(2)}(T,v_1,v_2)$, where $KQ_T^{(2)}$ is the path algebra of the quiver

$$Q_T^{(2)} = ((Q_T)_0 \cup \{w\}, (Q_T)_1 \cup \{\gamma_1 : c \longrightarrow w, \gamma_2 : w \longrightarrow b, \gamma_3 : w \longrightarrow w\})$$

and $I^{(2)}(T,v_1,v_2)$ is the ideal in $KQ_T^{(2)}$ generated by the elements:

(1) $\alpha_i\beta_{\alpha(i)}$, $\beta_i\alpha_{\beta(i)}$, for all vertices i of Q_T,

(2) $A_j - B_j$, if the both α-cycle and β-cycle through the vertex j are not exceptional,

(3) $A_j^2 - B_j$, if the α-cycle through the vertex j is exceptional but the β-cycle through j is not exceptional,

(4) $A_j - B_j^2$, if the α-cycle through the vertex j is not exceptional but the β-cycle through the vertex j is exceptional,

(5) $\gamma_2\beta_b$, $\beta_{\beta^{-1}(c)}\gamma_1$, $\gamma_1\gamma_3$, $\gamma_3\gamma_2$,

(6) $\gamma_2\alpha_b\ldots\alpha_c$, $\alpha_a\alpha_b\ldots\alpha_{\alpha^{-1}(c)}\gamma_1$ ($\gamma_2\alpha_b$, $\alpha_a\gamma_1$, if $b = c = e$), if the α-cycle through the vertex a is not exceptional,

(7) $\gamma_2\alpha_b\dots\alpha_{\alpha^{-1}(c)}\gamma_1\gamma_2\alpha_b\dots\alpha_c$, $\alpha_a\alpha_b\dots\alpha_{\alpha^{-1}(c)}\alpha_c\alpha_a\alpha_b\dots\alpha_{\alpha^{-1}(c)}\gamma_1$ ($\gamma_2\gamma_1\gamma_2\alpha_b$, $\alpha_a\alpha_b\alpha_a\gamma_1$, if $b=c=e$), if the α-cycle through the vertex a is exceptional,

(8) $\alpha_c\alpha_a-\gamma_1\gamma_2$,

(9) $\gamma_2\alpha_b\alpha_{\alpha(b)}\dots\alpha_{\alpha^{-1}(c)}\gamma_1-\gamma_3$, if the α-cycle through the vertex a is not exceptional,

(10) $(\gamma_2\alpha_b\alpha_{\alpha(b)}\dots\alpha_{\alpha^{-1}(c)}\gamma_1)^2-\gamma_3$, if the α-cycle through the vertex a is exceptional.

It follows from [**6**, Proposition 2.11] that there exists a Brauer tree T and its vertices v_1 and v_2 such that $\widehat{B}/(\underline{\varphi})\cong\Gamma^{(2)}(T,v_1,v_2)$. Then we have $\widehat{B}/(\varphi)\cong\Omega^{(2)}(T,v_1,v_2)$. $\square$

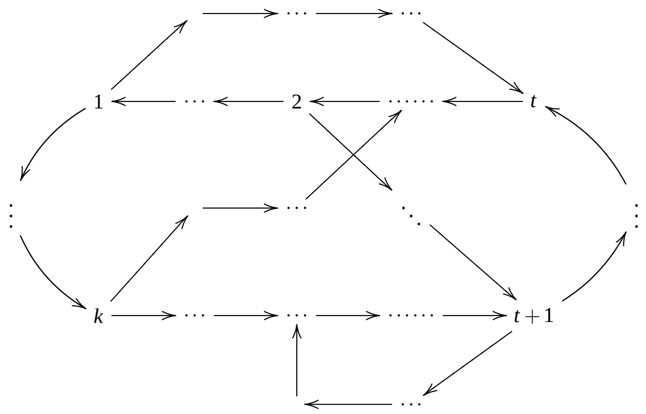
PROPOSITION 2.9. *Let T be a Brauer tree such that the algebra $\Omega^{(2)}(T,v_1,v_2)$ is defined. Then $\Omega^{(2)}(T,v_1,v_2)$ is a one-parametric selfinjective algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$, and is not weakly symmetric.*

PROOF. It follows from Theorem 3.1 and Lemma 2.8 in [**6**] that $\Gamma^{(2)}(T,v_1,v_2)\cong\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}/(\varphi)$, for some i with $5\leqslant i\leqslant 8$, $l\geqslant 1$, a tuple $\underline{m}$, a family $\mathscr{B}$ of branches, and a square root φ of the Nakayama automorphism $\nu_{\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}}$ of $\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}$. We define an automorphism $\overline{\varphi}$ of the algebra $\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}$ in the same way as in proof of Proposition 2.7. A direct checking shows that $\overline{\varphi}^2=\rho\nu_{\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}}$, for some nontrivial rigid automorphism ρ of $\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}$ of order 2, and we have $\Omega^{(2)}(T,v_1,v_2)\cong\Theta^{(i)}\widehat{(l,\underline{m},\mathscr{B})}/(\overline{\varphi})$. Hence $\Omega^{(2)}(T,v_1,v_2)$ is a one-parametric selfinjective algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$.

Since $\operatorname{top}(P(w))\cong\operatorname{soc}(P(a))\ncong\operatorname{soc}(P(w))$ and $\operatorname{top}(P(a))\cong\operatorname{soc}(P(w))\ncong\operatorname{soc}(P(a))$, the algebra $\Omega^{(2)}(T,v_1,v_2)$ is not weakly symmetric. We also note that, for all vertices i of $Q_T^{(2)}$ different from w and a, we have $\operatorname{top}(P(i))\cong\operatorname{soc}(P(i))$. $\square$

3. Proof of Theorem 1.

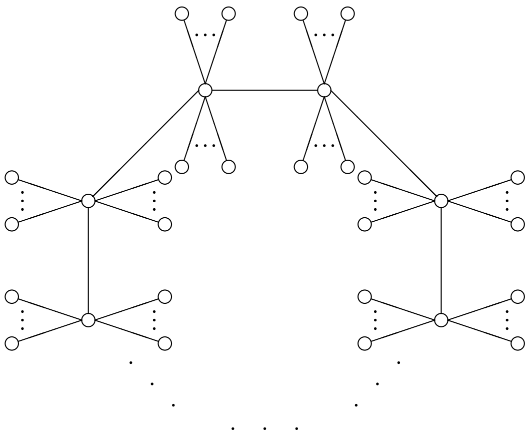
The aim of this section is to complete the proof of Theorem 1. Let A be a basic connected selfinjective algebra having a simply connected Galois covering. Assume that A is a one-parametric but not weakly symmetric algebra. Then invoking [**2**], [**16**] and [**22**], we conclude that $A\cong\widehat{B}/(\varphi)$, where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\boldsymbol{A}}_m$ or $\widetilde{\boldsymbol{D}}_n$ having all indecomposable injective modules located in the unique preinjective component, and φ is an automorphism of $\widehat{B}$ such that $\varphi^2=\rho\nu_{\widehat{B}}$ for a nontrivial rigid automorphism ρ of $\widehat{B}$. Then by [**20**], B is a tubular extension of a tame concealed algebra H. Assume first that B is a tilted algebra of Euclidean type $\widetilde{\boldsymbol{A}}_m$. Then it follows from [**3**] that $\widehat{B}$ is special biserial, and hence A is selfinjective and special biserial. Further, since $\varphi^2=\rho\nu_{\widehat{B}}$, it follows from [**12**] and [**22**] that the stable Auslander-Reiten quiver Γ_A^s of A consists of one component of the form $\boldsymbol{Z}\widetilde{\boldsymbol{A}}_m$ and a $\boldsymbol{P}_1(K)$-family of stable tubes. Moreover, the one-parameter families of indecomposable modules are given by the images of the one-parameter families of indecomposable modules over the hereditary algebra H of type $\widetilde{\boldsymbol{A}}_p$ by the push-down functor $F_\lambda:\operatorname{mod}\widehat{B}\longrightarrow\operatorname{mod}A$ associated to the canonical Galois covering $F:\widehat{B}\longrightarrow\widehat{B}/(\varphi)=A$. In fact, the bound quiver, say (Q,I), of A admits a unique primitive walk (in the sense of [**24**]) being the image of the unique cycle (with underlying graph $\widetilde{\boldsymbol{A}}_p$) of the Gabriel quiver of B. This primitive walk in (Q,I) is formed by the corresponding paths of the bound quiver (Q^*,I^*), for a subquiver Q^* of Q of the form



and the ideal $I^* = KQ^* \cap I$ generated by the elements:

(1) $A_j B_{j-1} - \lambda_j B_j A_{j+t-1}$, for $\lambda_j \in K \cdot \{o\}$, $j = 1,2,\ldots,k$,
(2) $\alpha_{n_j,j}\alpha_{1,j-1}$, $\beta_{m_j,j}\beta_{m_{j+t-1},1}$, for $j = 1,2,\ldots,k$,
(3) $\alpha_{i,j}\alpha_{i+1,j}\ldots\alpha_{n_j,j}B_{j-1}\alpha_{1,j+t-2}\ldots\alpha_{i-1,j+t-2}\alpha_{i,j+t-2}$, for $i = 1,2,\ldots,n_j$, $j = 1,2,\ldots,k$,
(4) $\beta_{i,j}\beta_{i+1,j}\ldots\beta_{m_j,j}A_{j+t-1}\beta_{1,j+t-2}\ldots\beta_{i-1,j+t-2}\beta_{i,j+t-2}$, for $i = 1,2,\ldots,m_j$, $j = 1,2,\ldots,k$,

where A_i is the path from i to $i+1$ and B_i is the path from i to $i+t-1$, n_j is the number of arrows on the path A_j, m_j is the number of arrows on the path B_j, $n_j = n_{j+t-2}$, $m_j = m_{j+t-2}$, $\alpha_{i,j}$ is the arrow on the path A_j starting at the vertex i and $\beta_{i,j}$ is the arrow on the path B_j starting at the vertex i. The above algebra is an algebra of the form $\Omega^{(1)}(T_0,\overline{\sigma}_s,\lambda)$, for some $\lambda \in K\cdot\{o\}$, $s = t-2$, and the Brauer graph T_0 of the form



.

Moreover, if the unique cycle of T_0 has k edges, then $k \geqslant 2$, $1 \leqslant s \leqslant k-1$ and $gcd(s+2,k) = 1$, because (Q,I) admits exactly one primitive walk. Since $A = KQ/I$ is special biserial, and (Q,I) contains exactly one primitive walk (described above), we deduce that $Q = Q_{T,\sigma_s}$ and $I = \overline{I}^{(1)}(T,\sigma_s,\lambda)$ for a Brauer graph T with exactly one cycle, containing the Brauer graph T_0 as a full convex subgraph, and the rotation σ_s is an extension of the automorphism $\overline{\sigma}_s$.

Now assume that B is a tilted algebra of Euclidean type $\widetilde{\boldsymbol{D}}_n$. Applying Propositions 2.5 and 2.7, we conclude that $B \cong \Theta^{(i)}(l, \underline{m}, \mathscr{B})$ for some i, $5 \leqslant i \leqslant 8$, a positive integer l, a tuple $\underline{m}$, and a family $\mathscr{B}$ of branches. Therefore, it follows from Proposition 2.8 that $A \cong \Omega^{(2)}(T, v_1, v_2)$ for some Brauer tree T and its vertices v_1, v_2.

Finally, if A is isomorphic to an algebra of one of the forms $\Omega^{(1)}(T, \sigma_s, \lambda)$ or $\Omega^{(2)}(T, v_1, v_2)$, then if follows from Propositions 1.3 and 2.9 that A is a one-parametric but not weakly symmetric algebra.

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