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Homotopy classification of twisted complex projective spaces of dimension 4

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Abstract. We study the existence problem of a $2n$ dimensional Poincaré complex whose homology is isomorphic to that of n dimensional complex projective space when $n = 4$.

1. Introduction.

Let M be a simply-connected $2n$ dimensional finite Poincaré complex. Then it is called a *twisted* $\boldsymbol{C}\mathrm{P}^n$ if there is an isomorphism $H_*(M, \boldsymbol{Z}) \cong H_*(\boldsymbol{C}\mathrm{P}^n, \boldsymbol{Z})$. Any twisted $\boldsymbol{C}\mathrm{P}^n$ is a CW complex of the form $M \simeq S^2 \cup e^4 \cup e^6 \cup \cdots \cup e^{2n}$ (up to homotopy equivalence), and it has the homotopy type of $2n$ dimensional closed topological manifolds (see section 8). We note that any twisted $\boldsymbol{C}\mathrm{P}^n$ is homotopy equivalent to the usual $\boldsymbol{C}\mathrm{P}^n$ if $n = 1, 2$. However, if $n = 3$, there are infinitely many twisted $\boldsymbol{C}\mathrm{P}^3$'s of different homotopy types. For example, let us consider CW complexes M_1, M_2, M_3 defined by $M_1 = S^2 \times S^4$, $M_2 = S^2 \vee S^4 \cup_{[i_2, i_4] + i_2 \circ \eta_2^3} e^6$, $M_2 = S^2 \vee S^4 \cup_{[i_2, i_4] + i_4 \circ \eta_4} e^6$. Since $H^*(M_k, \boldsymbol{Z}) \cong E[x_2, x_4]$ (the exterior algebra over $\boldsymbol{Z}$ generated by x_2, x_4 with $\deg x_k = k$), they are twisted $\boldsymbol{C}\mathrm{P}^3$'s, and an easy computation shows that they have different homotopy types. Because the case $n = 3$ was now studied well (e.g. [**13**], [**24**]), in this paper we shall consider the case $n = 4$. More precisely, we shall investigate a simply connected 8 dimensional Poincaré complex M such that

$$(0.1) \quad H_k(M, \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & \text{if } k = 0, 2, 4, 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, if we choose the suitable generators $x_{2k} \in H^{2k}(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$ $(k = 1, 2, 3, 4)$, there is a unique integer $m \geq 0$ such that

$$(0.2) \quad x_2 \cdot x_2 = m x_4,\ x_4 \cdot x_4 = x_8,\ x_2 \cdot x_4 = m x_6,\ x_2 \cdot x_6 = x_8.$$

We note that the conditions $x_2 \cdot x_6 = x_8$ and $x_4 \cdot x_4 = x_8$ hold if and only if the Poincaré duality holds (see [**3**, Proposition 1.2.1]).

DEFINITION 1. Let $m \geq 0$ be an integer. A simply-connected 8 dimensional finite Poincaré complex M is called an *m-twisted* $\boldsymbol{C}\mathrm{P}^4$ if the conditions (0.1) and (0.2) are satisfied.

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For example, $\boldsymbol{C}\mathrm{P}^4$ is a 1-twisted $\boldsymbol{C}\mathrm{P}^4$, and $\boldsymbol{H}\mathrm{P}^2\#(S^2 \times S^6)$ is a 0-twisted $\boldsymbol{C}\mathrm{P}^4$ (# denotes the connected sum). Now we note the following fact.

(0.3) *Let $m \geq 1$ be an integer. If $x_2 \cdot x_2 = mx_4$, $x_2 \cdot x_4 = mx_6$ and $x_2 \cdot x_6 = x_8$ hold, then $x_4 \cdot x_4 = x_8$ also holds.*
(This can be easily obtained by computing $(x_2)^4$.)

Let $\mathscr{M}_m$ denote the set consisting of all homotopy equivalence classes of m-twisted $\boldsymbol{C}\mathrm{P}^4$'s. We would like to investigate how many non-homotopy equivalent m-twisted $\boldsymbol{C}\mathrm{P}^4$'s exist for a fixed integer $m \geq 0$.

In general, if $[M], [N] \in \mathscr{M}_m$, $H^*(M, \boldsymbol{Z}) \cong H^*(N, \boldsymbol{Z})$ as graded rings, but $H^*(M, \boldsymbol{Z}/2)$ and $H^*(N, \boldsymbol{Z}/2)$ are not necessarily isomorphic as $\mathscr{A}_2$-modules, where $\mathscr{A}_p$ denotes the mod p Steenrod algebra. Then for each equivalence class $[M] \in \mathscr{M}_m$, we define its *type* using the concept of $\mathscr{A}_2$-module structure on it and its homotopy type of the 6-skeleton (see Definition 8 in detail). Then our main results are stated as follows.

THEOREM 1.1. *Let $m \geq 0$ be an integer.*

(i) *Let m be an odd integer. Then there exists an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X, 1)$. Conversely, if M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$, it has the type $(X, 1)$. (Hence, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X, 0)$.)*

(ii) *Let $m \equiv 0 \pmod 8$. Then there exists a family of m-twisted $\boldsymbol{C}\mathrm{P}^4$'s, $\mathscr{F}_m = \{M^X_{m,0}, M^X_{m,1}, M^Y_{m,0}\}$, such that each $M^T_{m,\varepsilon} \in \mathscr{F}_m$ has the type (T, ε) and any two of them are not homotopy equivalent each other, where $(T, \varepsilon) \in \{X, Y\} \times \boldsymbol{Z}/2$. However, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Y, 1)$.*

THEOREM 1.2. *Let $m \geq 0$ be an even integer. Then there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Z, ε) for any $\varepsilon \in \boldsymbol{Z}/2$. Moreover, if m is not divisible by 8, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$.*

Let $\mathrm{card}(V)$ denote the cardinality of a set V and let (a, b) be the greatest common divisor of integers a and b.

COROLLARY 1.3. *Let $m \geq 0$ be an integer.*

(i) *If $m \equiv 1 \pmod 2$, then $1 \leq \mathrm{card}(\mathscr{M}_m) \leq m(m, 3)$.*
(ii) *If $m \equiv 0 \pmod 8$ and $m \neq 0$, then $3 \leq \mathrm{card}(\mathscr{M}_m) \leq 2^5 \cdot 3 \cdot m(m, 3)$.*
(iii) *If $m = 0$, then $3 \leq \mathrm{card}(\mathscr{M}_m) \leq 2^7 \cdot 3^2$.*
(iv) *If m is an even integer and not divisible by 8, $\mathscr{M}_m = \varnothing$.*

REMARK. We remark that $\mathscr{M}_m$ is a finite set. However, because the estimate of $\mathrm{card}(\mathscr{M}_m)$ is very rough, we would like to investigate this number very carefully in the subsequent paper [**27**].

The principal motivation of this paper is as follows. Originally we would like to classify the homotopy types of highly connected Poincaré complexes of even dimension. Because the homotopy type of $(n-k)$-connected $2n$ dimensional Poincaré complexes was already classified well (e.g. [**7**], [**19**], [**20**]) if $k \leq 2$, we would like to study the homotopy types of $(n-3)$-connected $2n$ dimensional Poincaré complexes. Because a twisted $\boldsymbol{C}\mathrm{P}^4$ is one of typical examples of such ones for $n = 4$, it may be worth-while to study the homotopy type classification problem of twisted $\boldsymbol{C}\mathrm{P}^4$'s.

Next, it is very interesting to study a twisted $\boldsymbol{C}\mathrm{P}^4$ from the point of view of surgery theory. In fact, since $\mathscr{M}_m = \varnothing$ if $2 \leq (m, 8) \leq 4$ (see Corollary 1), this strongly suggests that the existence problem of twisted $\boldsymbol{C}\mathrm{P}^4$'s would be related to the problem of surgery obstructions, although we cannot solve this problem at the moment.

Finally, we can show that there exists a finitely many non-trivial twisted $\boldsymbol{C}\mathrm{P}^n$'s for each odd integer n by using the technique of the transformation group theory (cf. [**2**]). However, it is not known whether a non-trivial twisted $\boldsymbol{C}\mathrm{P}^n$ exists or not if $n \geq 4$ is an even integer. So it seems valuable to investigate the existence problem of twisted $\boldsymbol{C}\mathrm{P}^4$'s as its first step.

This paper is organized as follows. In section 2, we compute homotopy groups of 2-cell complexes L_m and $\mathrm{P}^4(m)$. In section 3, we study a 2-connective covering of L_m and in section 4, we determine the homotopy types of the 6-skeletons of m-twisted $\boldsymbol{C}\mathrm{P}^4$'s. In sections 5 and 6, we compute the homotopy groups of the 6-skeleton of an m-twisted $\boldsymbol{C}\mathrm{P}^4$ and study an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (X, ε) or of type (Y, ε). In section 7, we study an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Z, ε) and give the proofs of our main results. Finally in section 8, we show that any twisted $\boldsymbol{C}\mathrm{P}^n$ has the homotopy type of closed topological manifold using a standard surgery theory.

2. Homotopy groups $\pi_*(L_m)$ and $\pi_*(\mathrm{P}^4(m))$.

Let $\iota_n \in \pi_n(S^n)$, and $\eta_2 \in \pi_3(S^2)$ or $\nu_4 \in \pi_7(S^4)$ be the oriented generator and the Hopf maps, respectively. We take $\eta_n = E^{n-2}\eta_2 \in \pi_{n+1}(S^n)$, $\eta_n^2 = \eta_n \circ \eta_{n+1} \in \pi_{n+2}(S^n)$, $\eta_n^3 = \eta_n \circ \eta_{n+1} \circ \eta_{n+2} \in \pi_{n+3}(S^n)$ for $n \geq 2$ and $\nu_n = E^{n-4}\nu_4 \in \pi_{n+3}(S^n)$ for $n \geq 5$, where E^k denotes the k-fold iterated suspension. Similarly, let $\omega \in \pi_6(S^3)$ be Blackers-Massey element.

LEMMA 2.1 ([**18**]).

(i) $\pi_n(S^n) = \boldsymbol{Z} \cdot \iota_n$ for $n \geq 1$, $\pi_3(S^2) = \boldsymbol{Z} \cdot \eta_2$ and $\pi_{n+1}(S^n) = \boldsymbol{Z}/2 \cdot \eta_n$ for $n \geq 3$.

(ii) $\pi_{n+2}(S^n) = \boldsymbol{Z}/2 \cdot \eta_n^2$ for $n \geq 2$, $\pi_5(S^2) = \boldsymbol{Z}/2 \cdot \eta_2^3$, $\pi_6(S^3) = \boldsymbol{Z}/12 \cdot \omega$, $\pi_7(S^4) = \boldsymbol{Z} \cdot \nu_4 \oplus \boldsymbol{Z}/12 \cdot E\omega$ and $\pi_{n+3}(S^n) = \boldsymbol{Z}/24 \cdot \nu_n$ for $n \geq 5$.

(iii) $\pi_7(S^2) = \boldsymbol{Z}/2 \cdot \eta_2 \circ \omega \circ \eta_6$, $\pi_7(S^3) = \boldsymbol{Z}/2 \cdot \omega \circ \eta_6$, $\pi_8(S^3) = \boldsymbol{Z}/2 \cdot \omega \circ \eta_6^2$ and $\pi_8(S^4) = \boldsymbol{Z}/2 \cdot \nu_4 \circ \eta_7 \oplus \boldsymbol{Z}/2 \cdot E\omega \circ \eta_7$.

DEFINITION 2. For an integer $m \geq 0$, let L_m denote the mapping cone defined by $L_m = S^2 \cup_{m\eta_2} e^4$. Let $a_m \in \pi_4(L_m, S^2)$ be the characteristic map of the top cell e^4 in L_m, and

$$S^3 \xrightarrow{m\eta_2} S^2 \xrightarrow{i} L_m \xrightarrow{q_m} S^4 \tag{1}$$

be the induced cofiber sequence. We denote by W_m the total space of S^2-bundle over S^4 with its characteristic element $c(W_m) = m\rho \in \pi_3(SO_3) = \boldsymbol{Z} \cdot \rho$.

LEMMA 2.2 ([**8**], [**9**], [**11**]). *There is some element $b_m \in \pi_5(L_m)$ such that $W_m \simeq L_m \cup_{b_m} e^6$ (up to homotopy) and that $i_{L*}(b_m) = [a_m, \iota_2]_r$, where $[\ ,\ ]_r$ denotes the relative Whitehead product, $i_L : (L_m, *) \to (L_m, S^2)$ is an inclusion and $i_{L*} : \pi_5(L_m) \to \pi_5(L_m, S^2) = \boldsymbol{Z} \cdot [a_m, \iota_2]_r \oplus a_{m*}\pi_5(D^4, S^3) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$ denote the induced homomorphism.*

LEMMA 2.3 ([**24**]).

(i) $\pi_4(L_m, S^2) = \boldsymbol{Z} \cdot a_m$, $\pi_2(L_m) = \boldsymbol{Z} \cdot i$, $\pi_3(L_m) = \boldsymbol{Z}/m \cdot i_*(\eta_2)$.

(ii) $\pi_4(L_m) = \begin{cases} 0 & \textit{if } m \equiv 1 \pmod 2, \\ \mathbf{Z}/2 \cdot i_*(\eta_2^2) & \textit{if } m \equiv 0 \pmod 2, m \neq 0, \\ \mathbf{Z} \cdot i'_4 \oplus \mathbf{Z}/2 \cdot i_*(\eta_2^2) & \textit{if } m = 0. \end{cases}$

Here $i_4 : S^4 \to S^2 \vee S^4 = L_0$ *denotes the natural inclusion.*

(iii) $\pi_5(L_m) = \begin{cases} \mathbf{Z} \cdot b_m & \textit{if } m \equiv 1 \pmod 2, \\ \mathbf{Z} \cdot b_m \oplus \mathbf{Z}/4 \cdot \gamma_m & \textit{if } m \equiv 2 \pmod 4, \\ \mathbf{Z} \cdot b_m \oplus \mathbf{Z}/2 \cdot \gamma_m \oplus \mathbf{Z}/2 \cdot i_*(\eta_2^3) & \textit{if } m \equiv 0 \pmod 4, \end{cases}$

where we take $b_m = [i, i_4]$ *and* $\gamma_m = i_4 \circ \eta_4$ *if* $m = 0$, *and* $2\gamma_m = i_*(\eta_2^3)$ *if* $m \equiv 2 \pmod 4$.

DEFINITION 3. Let $\mathrm{P}^{k+1}(m)$ be the Moore space $\mathrm{P}^{k+1}(m) = S^k \cup_{m\iota_k} e^{k+1}$, and let $\alpha_m \in \pi_4(\mathrm{P}^4(m), S^3)$ be the characteristic map of the top cell e^4.

Let us consider the cofiber sequence

$$S^3 \xrightarrow{m\iota_3} S^3 \xrightarrow{i'} \mathrm{P}^4(m) = S^3 \cup_{m\iota_3} e^4 \xrightarrow{q'_m} S^4. \tag{2}$$

Because $(2\iota_3) \circ \eta_3 = 0$, there is a coextension $\tilde{\eta}_3 \in \pi_5(\mathrm{P}^4(2))$ of η_3 such that

$$q'_2 \circ \tilde{\eta}_3 = \eta_4. \tag{3}$$

Moreover, when $m \equiv 0 \pmod 2$, there is a map $f'_m : \mathrm{P}^4(2) \to \mathrm{P}^4(m)$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} S^3 & \xrightarrow{2\iota_3} & S^3 & \xrightarrow{i'} & \mathrm{P}^4(2) & \xrightarrow{q'_2} & S^4 \\ \| & & \downarrow{\scriptstyle (m/2)\iota_3} & & \downarrow{\scriptstyle f'_m} & & \| \\ S^3 & \xrightarrow{m\iota_3} & S^3 & \xrightarrow{i'} & \mathrm{P}^4(m) & \xrightarrow{q'_m} & S^4. \end{array}$$

LEMMA 2.4 ([**5**], [**6**]).

(i) $\pi_3(\mathrm{P}^4(m)) = \mathbf{Z}/m \cdot i'$, $\pi_4(\mathrm{P}^4(m), S^3) = \mathbf{Z} \cdot \alpha_m$.

(ii)

$$\pi_4(\mathrm{P}^4(m)) = \begin{cases} 0 & \textit{if } m \equiv 1 \pmod 2, \\ \mathbf{Z}/2 \cdot i' \circ \eta_3 & \textit{if } m \equiv 0 \pmod 2. \end{cases}$$

(iii)

$$\pi_5(\mathrm{P}^4(m)) = \begin{cases} 0 & \textit{if } m \equiv 1 \pmod 2, \\ \mathbf{Z}/4 \cdot f'_m \circ \tilde{\eta}_3 & \textit{if } m \equiv 2 \pmod 4, \\ \mathbf{Z}/2 \cdot f'_m \circ \tilde{\eta}_3 \oplus \mathbf{Z}/2 \cdot i' \circ \eta_3^2 & \textit{if } m \equiv 0 \pmod 4, m \neq 0, \\ \mathbf{Z}/2 \cdot i'_4 \circ \eta_4 \oplus \mathbf{Z}/2 \cdot i' \circ \eta_3^2 & \textit{if } m = 0, \end{cases}$$

where $2 f'_m \circ \tilde{\eta}_3 = i' \circ \eta_3^2$ *if* $m \equiv 2 \pmod 4$, *and* $i'_4 : S^4 \to S^3 \vee S^4 = \mathrm{P}^4(0)$ *denotes the natural inclusion if* $m = 0$.

LEMMA 2.5 ([**24**]). *Let $j' : L_m \to W_m = L_m \cup_{b_m} e^6$ denote the inclusion.*
(i) *If $m \equiv 1 \pmod 2$, then $\pi_6(W_m) = \mathbf{Z}/(m,3) \cdot (j' \circ i)_*(\eta_2 \circ \omega)$.*
(ii) *If $0 \neq m = 2m \equiv 0 \pmod 2$, then*

$$\pi_6(W_m) = \mathbf{Z}/(m',12) \cdot (j' \circ i)_*(\eta_2 \circ \omega) \oplus \mathbf{Z}/2 \cdot j'_*(\tilde{\eta}_3 \circ \eta_5).$$

DEFINITION 4. If $k, l \geq 1$ and $s \geq 0$ are integers and $k = 2^s \cdot l$ with $l \equiv 1 \pmod 2$, we write $s = \nu_2(k)$, and let (a,b) denote the greatest common divisor of integers a and b. Let V_m denote the S^3-bundle over S^4 with characteristic element $c(V_m) = m\iota_3 \in \mathbf{Z} \cdot \iota_3 \oplus \mathbf{Z} \cdot \rho = \pi_3(S^3) \oplus \pi_3(SO_3) \cong \pi_3(SO_4)$.

PROPOSITION 2.6 ([**17**]).

(i) *There exists some element $\sigma \in \pi_6(\mathrm{P}^4(m))$ such that $i'_*(\sigma) = [\alpha_m, \iota_3]_r$ and that there is a homotopy equivalence $V_m \simeq \mathrm{P}^4(m) \cup_\sigma e^7$, where $i'_* : \pi_6(\mathrm{P}^4(m)) \to \pi_6(\mathrm{P}^4(m), S^3) = \mathbf{Z}/m \cdot [\alpha_m, \iota_3]_r \oplus \alpha_{m*}\pi_6(D^4, S^3) \cong \mathbf{Z}/m \oplus \mathbf{Z}/2$ denotes the induced homomorphism.*
(ii) *If $m \equiv 1 \pmod 2$, $\pi_6(\mathrm{P}^4(m)) = \mathbf{Z}/(m,3) \cdot i' \circ \omega \oplus \mathbf{Z}/m \cdot \sigma$.*
(iii) *If $\nu_2(m) \geq 3$,*

$$\pi_6(\mathrm{P}^4(m)) = \mathbf{Z}/(m,12) \cdot i' \circ \omega \oplus \mathbf{Z}/m \cdot \sigma \oplus \mathbf{Z}/2 \cdot f'_m \circ \tilde{\eta}_3 \circ \eta_5.$$

(iv) *If $1 \leq \nu_2(m) \leq 2$,*

$$\pi_6(\mathrm{P}^4(m)) = \mathbf{Z}/m' \cdot \lambda_m \oplus \mathbf{Z}/2m \cdot \sigma \oplus \mathbf{Z}/2 \cdot f'_m \circ \tilde{\eta}_3 \circ \eta_5,$$

where we take $m' = (m,12)/2$ and $\lambda_m = 2m/(12,m)\sigma + i' \circ \omega$. In particular, the order of $i' \circ \omega \in \pi_6(\mathrm{P}^4(m))$ is $(12,m)$.

LEMMA 2.7.

(i) *If $m \equiv 1 \pmod 2$, $\pi_7(\mathrm{P}^4(m), S^3) = \alpha_{m*}\pi_7(D^4, S^3) \cong \mathbf{Z}/12$.*
(ii) *If $m \equiv 0 \pmod 2$,*

$$\pi_7(\mathrm{P}^4(m), S^3) = \mathbf{Z}/2 \cdot [\alpha_m, \eta_3]_r \oplus \alpha_{m*}\pi_7(D^4, S^3) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/12.$$

PROOF. It follows from [**8**] that there is an isomorphism $\pi_7(\mathrm{P}^4(m), S^3) = \langle [\alpha_m, \eta_3]_r \rangle \oplus \alpha_{m*}\pi_7(D^4, S^3)$. Hence, it suffices to show that $[\alpha_m, \eta_3]_r = 0$ if and only if $m \equiv 1 \pmod 2$. Then by [**8**], there is a commutative diagram

$$\begin{tikzcd}
\pi_8(S^4) \arrow[r, "H\alpha"] \arrow[d, "H"'] & \pi_4(S^3) = \mathbf{Z}/2 \cdot \eta_3 \arrow[r, "Q"] & \pi_7(\mathrm{P}^4(m), S^3) \\
\mathbf{Z}/2 \cdot \eta_7 = \pi_8(S^7) & \pi_4(S^3) = \mathbf{Z}/2 \cdot \eta_3 \arrow[l, "E^4"', "\cong"] \arrow[u, "(m\iota_3)_*"'] &
\end{tikzcd}$$

where the horizontal sequence is exact, $H : \pi_8(S^4) \to \pi_8(S^7)$ denotes the Hopf homomorphism and we take $Q(\eta_3) = [\alpha_m, \eta_3]_r$.

Hence, $[\alpha_m, \eta_3]_r = 0$ if and only if H_α is surjective. However, since (for $\nu_4 \circ \eta_7 \in \pi_8(S^4)$), $H(\nu_4 \circ \eta_4) = H(\nu_4) \circ \eta_7 = \iota_4 \circ \eta_7 = \eta_7$, H is surjective. So $[\alpha_m, \eta_3]_r = 0$ if and only if $(m\iota_3)_*$ is an isomorphism. But, because S^3 is an H-space, it is equivalent to the condition $m \equiv 1 \pmod 2$. □

DEFINITION 5. For a connected CW complex X, let $X^{(k)}$ be its k-skeleton. For a CW-pair (X,A), let $(X,A)_\infty$ denote the relative James reduced product defined by B. Gray [**4**]. It's known [**4**] that there is a homotopy equivalence $(X,A)_\infty \simeq F$, where F denotes the homotopy fiber of the pinch map $q : X \cup CA \to \Sigma X$. We denote by F_m the homotopy fiber of the pinch map $q'_m : \mathrm{P}^4(m) \to S^4$. Then there is a homotopy equivalence $F_m \simeq (\mathrm{P}^4(m), S^3)_\infty$ and there is a fibration sequence

$$F_m \longrightarrow \mathrm{P}^4(m) \xrightarrow{q'_m} S^4. \tag{4}$$

LEMMA 2.8. *There is a homotopy equivalence* $F_m^{(9)} \simeq (S^3 \vee S^6) \cup_{m[i_3,i_6]} e^9$, *where* $i_k : S^k \to S^3 \vee S^9$ *are the natural inclusions* $(k = 3,6)$, *and* $[\ ,\ ]$ *denotes the Whitehead product.*

PROOF. From now on, we identify $(\mathrm{P}^4(m), S^3)_\infty = F_m$. It follows from [**4**] that $F_m^{(9)} \simeq S^3 \cup_g e^6 \cup_f e^9$ (This can be also obtained using Serre spectral sequence induced from (4)). Since S^3 is an H-space, by [**4**, Corollary 5.8], $g = [\iota_3, \iota_3] = 0 \in \pi_5(S^3)$. Hence, $F_m^{(9)} \simeq (S^3 \vee S^6) \cup_f e^9$ for some $f \in \pi_8(S^3 \vee S^6) = \boldsymbol{Z}/2 \cdot i_3 \circ \omega \circ \eta_6^2 \oplus \boldsymbol{Z}/2 \cdot i_6 \circ \eta_6^2 \oplus \boldsymbol{Z} \cdot [i_3, i_6]$. Then it follows from the definition of the relative James reduced product that the top cell e^9 is attached by the map $f = [i_3, mi_6] = m[i_3, i_6]$. Hence, we have $F_m^{(9)} \simeq (S^3 \vee S^6) \cup_{m[i_3,i_6]} e^9$. □

Consider the exact sequence induced from (4)

$$\cdots \longrightarrow \pi_k(F_m) \xrightarrow{\tilde{i}'_*} \pi_k(\mathrm{P}^4(m)) \xrightarrow{q'_{m*}} \pi_k(S^4) \xrightarrow{\Delta'_k} \pi_{k-1}(F_m) \longrightarrow \cdots . \tag{5}$$

If $1 \le k \le 8$, we can identify $\Delta'_k : \pi_k(S^4) \to \pi_{k-1}(S^3 \vee S^6)$.

PROPOSITION 2.9. (i) *If* $m \equiv 1 \pmod 2$, $\pi_7(\mathrm{P}^4(m)) = \boldsymbol{Z}/(3,m) \cdot \omega_m$.
(ii) *If* $m \equiv 0 \pmod 4$ *and* $m \neq 0$,

$$\pi_7(\mathrm{P}^4(m)) = \boldsymbol{Z}/4 \cdot \widetilde{\nu'} \oplus \boldsymbol{Z}/2 \cdot \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot i'_*(\omega \circ \eta_6) \oplus \boldsymbol{Z}/(m,3) \cdot \omega_m.$$

(iii) *If* $m \equiv 2 \pmod 4$,

$$\pi_7(\mathrm{P}^4(m)) = \boldsymbol{Z}/2 \cdot \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot \tilde{\eta}_3 \circ \eta_5^2 \oplus \boldsymbol{Z}/(m,3) \cdot \omega_m.$$

PROOF. (i) We suppose $m \equiv 1 \pmod 2$ and consider the exact sequence

$$\pi_8(\mathrm{P}^4(m), S^3) \xrightarrow{\partial'_8} \pi_7(S^3) \to \pi_7(\mathrm{P}^4(m)) \to \pi_7(\mathrm{P}^4(m), S^3) \xrightarrow{\partial'_7} \pi_6(S^3).$$

Because there is a commutative diagram

$$\begin{array}{ccc}
\pi_8(\mathrm{P}^4(m), S^3) & \xrightarrow{\partial'_8} & \pi_7(S^3) = \mathbf{Z}/2 \cdot \omega \circ \eta_7 \\
{\scriptstyle \alpha_{m*}} \big\uparrow & & {\scriptstyle (m\iota_3)_*} \big\uparrow \cong \\
\pi_8(D^4, S^3) & \underset{\cong}{\xrightarrow{\partial'}} & \pi_7(S^3) = \mathbf{Z}/2 \cdot \omega \circ \eta_7,
\end{array}$$

∂'_8 is surjective, and we have the exact sequence

$$0 \longrightarrow \pi_7(\mathrm{P}^4(m)) \longrightarrow \pi_7(\mathrm{P}^4(m), S^3) \xrightarrow{\partial'_7} \pi_6(S^3).$$

Similarly, if we consider the boundary homomorphism

$$\partial'_7 : \alpha_{m*}\pi_7(D^4, S^3) = \pi_7(\mathrm{P}^4(m), S^3) \to \pi_6(S^3) = \mathbf{Z}/12 \cdot \omega,$$

we have $\mathrm{Im}\,\partial'_7 = m \cdot \pi_6(S^3)$ and $\pi_7(\mathrm{P}^4(m)) = \ker \partial'_7 = \mathbf{Z}/(3,m) \cdot \omega_m$ for some $\omega_m \in \pi_7(\mathrm{P}^4(m))$.

(ii), (iii): Assume $m \equiv 0 \pmod 2$. If we consider the exact sequence

$$\mathbf{Z} \cdot \iota_4 = \pi_4(S^4) \to \pi_3(S^3 \vee S^6) = \mathbf{Z} \cdot i_3 \to \pi_3(\mathrm{P}^4(m)) = \mathbf{Z}/m \to 0,$$

we have $\Delta'_4(\iota_4) = mi_3$. Next, consider the exact sequence

$$\begin{array}{ccccccc}
\pi_7(S^4) & \xrightarrow{\Delta'_7} & \pi_6(S^3 \vee S^6) & \xrightarrow{\tilde{i}'_*} & \pi_6(\mathrm{P}^4(m)) & \xrightarrow{q'_{m*}} & \pi_6(S^4). \\
& & \| & & & & \| \\
& & \mathbf{Z} \cdot i_6 \oplus \mathbf{Z}/12 \cdot i_3 \circ \omega & & & & \mathbf{Z}/2 \cdot \eta_4^2
\end{array}$$

Because the order of $\eta'' = f'_m \circ \tilde{\eta}_3 \circ \eta_5 \in \pi_6(\mathrm{P}^4(m))$ is 2 and $q'_{m*}(\eta'') = \eta_4^2$, this induces the exact sequence $\pi_7(S^4) \xrightarrow{\Delta'_7} \pi_6(S^3 \vee S^6) \longrightarrow H_m \longrightarrow 0$, where $\pi_7(S^4) = \mathbf{Z} \cdot \nu_4 \oplus \mathbf{Z}/12 \cdot E\omega$ and

$$H_m = \begin{cases} \mathbf{Z}/(m,12) \cdot i' \circ \omega \oplus \mathbf{Z}/m \cdot \sigma & \text{if } \nu_2(m) \geq 3, \\ \mathbf{Z}/m' \cdot \lambda_m \oplus \mathbf{Z}/2m \cdot \sigma & \text{if } 1 \leq \nu_2(m) \leq 2, m' = (m,12)/2. \end{cases}$$

Since $\Delta'_7(E\omega) = \Delta'_4(\iota_4) \circ \omega = m(i_3 \circ \omega)$, for some $n \in \mathbf{Z}/12$, we have

$$(\dagger_1) \quad \Delta'_7(\nu_4) = \begin{cases} \pm mi_6 + n \cdot i_3 \circ \omega & \text{if } \nu_2(m) \geq 3, \\ \pm 2mi_6 + (m,12)/2 \cdot i_3 \circ \omega & \text{if } 1 \leq \nu_2(m) \leq 2. \end{cases}$$

$$(\dagger_2) \quad \mathrm{Ker}\,\Delta'_7 = \langle E\omega \rangle \cong \mathbf{Z}/(m,12),\ \tilde{i}'_*(i_6) = \sigma,\ \tilde{i}'_*(i_3 \circ \omega) = i' \circ \omega.$$

If we consider the boundary homomorphism

$$\Delta'_8 : \pi_8(S^4) \to \pi_7(S^3 \vee S^6) = \boldsymbol{Z}/2 \cdot i_3 \circ \omega \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot i_6 \circ \eta_6,$$

we have $\Delta'_8(E\omega \circ \eta_7) = \Delta'_7(E\omega) \circ \eta_6 = m(i_3 \circ \omega) \circ \eta_6 = 0$. Next we compute $\Delta'_8(\nu_4 \circ \eta_7)$. If $\nu_2(m) \geq 3$,

$$\Delta'_8(\nu_4 \circ \eta_7) = \Delta'_7(\nu_4) \circ \eta_6 = (\pm m i_6 + n(m,12) i_3 \circ \omega) \circ \eta_6 = 0.$$

If $1 \leq \nu_2(m) \leq 2$,

$$\begin{aligned}\Delta'_8(\nu_4 \circ \eta_7) &= \Delta'_7(\nu_4) \circ \eta_6 = (\pm 2m i_6 + (m,12)/2 \cdot i_3 \circ \omega) \circ \eta_6 \\ &= \begin{cases} 0 & \text{if } \nu_2(m) = 2, \\ i_3 \circ \omega \circ \eta_6 & \text{if } \nu_2(m) = 1. \end{cases}\end{aligned}$$

Now we remark that $\mathrm{P}^4(m) \simeq \mathrm{P}^4(n) \vee \mathrm{P}^4(k)$ if $m = nk$ with $(n,k) = 1$, and that we already showed the case $m \equiv 1 \pmod 2$. Because the extension problem is trivial for the odd torsion, the assertions (ii) and (iii) easily follows from the assertions for the case $m = 2^k$ $(k \geq 1)$. So from now on, we assume $m = 2^k$ $(k \geq 1)$. Let $\nu' \in \pi_6(S^3)_{(2)} \cong \boldsymbol{Z}/4$ denote the 2-primary component of ω. Then it follows from the above computations that we have:

(a) If $k \geq 2$, there is an exact sequence

$$0 \longrightarrow \pi_7(S^3 \vee S^6) \xrightarrow{\tilde{i}'_*} \pi_7(\mathrm{P}^4(2^k)) \xrightarrow{q'_{m*}} \boldsymbol{Z}/4 \cdot E\nu' \longrightarrow 0.$$

(b) If $k = 1$, $\langle 2E\nu' \rangle \cong \boldsymbol{Z}/2$ and there is an exact sequence

$$0 \longrightarrow \boldsymbol{Z}/2 \cdot i_6 \circ \eta_6 \xrightarrow{\tilde{i}'_*} \pi_7(\mathrm{P}^4(2)) \xrightarrow{q'_{m*}} \langle 2E\nu' \rangle \longrightarrow 0.$$

First, consider the case $m = 2^k$ with $k \geq 2$. It is known that there is an element $\widetilde{\nu'} \in \{i', 2^k \iota_3, \nu'\}$ of order 4 such that $q'_{m*}(\widetilde{\nu'}) = E\nu'$. Hence, if $k \geq 2$, using $(\dagger_3)$ we have $\pi_7(\mathrm{P}^4(2^k)) = \boldsymbol{Z}/4 \cdot \widetilde{\nu'} \oplus \boldsymbol{Z}/2 \cdot \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot i'_*(\omega \circ \eta_6)$.

A similar method also obtains $\pi_7(\mathrm{P}^4(2)) = \boldsymbol{Z}/2 \cdot \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot \tilde{\eta}_3 \circ \eta_5^2$, and this completes the proof. □

3. 2-connective covering of L_m.

We denote by $q' : L_m \to K(\boldsymbol{Z},2)$ the map which represents the generator of $\boldsymbol{Z} \cong [L_m, K(\boldsymbol{Z},2)] \cong H^2(L_m, \boldsymbol{Z})$. If A_m denotes the homotopy fiber of q', there is a fibration sequence $A_m \xrightarrow{\tilde{j}} L_m \xrightarrow{q'} K(\boldsymbol{Z},2)$. Remark that $\tilde{j} : A_m \to L_m$ is a 2-connective covering of L_m with fiber S^1 (up to homotopy). The following result was taught by Jie Wu [**23**].

PROPOSITION 3.1 (J. Wu). *There is a homotopy equivalence* $\mathrm{P}^4(m) \vee S^5 \xrightarrow{\simeq} A_m$.

PROOF. This easily follows from the main result given in [**26**]. □

DEFINITION 6. Define the maps $f_m : \mathrm{P}^4(m) \to L_m$ and $g_m : S^5 \to L_m$ by

$$\begin{cases} f_m : \mathrm{P}^4(m) \xrightarrow{\tilde{i_1}} \mathrm{P}^4(m) \vee S^5 \simeq A_m \xrightarrow{\tilde{j}} L_m, \\ g_m : S^5 \xrightarrow{\tilde{i_2}} \mathrm{P}^4(m) \vee S^5 \simeq A_m \xrightarrow{\tilde{j}} L_m, \end{cases}$$

where $\mathrm{P}^4(m) \xrightarrow{\tilde{i_1}} \mathrm{P}^4(m) \vee S^5 \xleftarrow{\tilde{i_2}} S^5$ denote the inclusion to the first or second factor, respectively. It is easy to see that $g_m \in \pi_5(L_m)$ represents the generator of torsion free part $\boldsymbol{Z}$ of it. Hence, without loss of generalities, we may assume that $g_m = b_m$ and we have the 2-connective covering with fiber S^1,

$$(f_m, b_m) : \mathrm{P}^4(m) \vee S^5 \to L_m. \tag{6}$$

COROLLARY 3.2. *If $k \geq 3$, $(f_m, b_m)_* : \pi_k(\mathrm{P}^4(m) \vee S^5) \xrightarrow{\cong} \pi_k(L_m)$ is an isomorphism.*

LEMMA 3.3.

(i) *There is a commutative diagram (up to homotopy equivalences)*

$$\begin{array}{ccccccc} S^3 & \xrightarrow{m\iota_3} & S^3 & \xrightarrow{i'} & \mathrm{P}^4(m) & \xrightarrow{q'_m} & S^4 \\ \| & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle f_m} & & \| \\ S^3 & \xrightarrow{m\eta_2} & S^2 & \xrightarrow{i} & L_m & \xrightarrow{q_m} & S^4 \end{array}$$

(ii) *If $m \equiv 0 \pmod 2$, we can take*

$$\gamma_m = f_m \circ f'_m \circ \tilde{\eta}_3 \in \pi_5(L_m). \tag{7}$$

PROOF. If (i) is true, (ii) easily follows from the definition of γ_m (see page 313 in [**24**]). So it remains to show (i). Since $f_{m*} : \boldsymbol{Z}/m \cdot i' = \pi_3(\mathrm{P}^4(m)) \xrightarrow{\cong} \pi_3(L_m) = \boldsymbol{Z}/m \cdot i \circ \eta_2$ is bijective, $f_m \circ i' = \varepsilon \cdot i_*(\eta_2)$ for some unit $\varepsilon \in (\boldsymbol{Z}/m)^\times$. So we may assume that $f_m \circ i' = i_*(\eta_2)$ (up to homotopy equivalence). Moreover, it follows from the Puppe exact sequence (induced from (2))

$$0 \longrightarrow \pi_4(S^4) \xrightarrow{(m\iota_4)^*} \pi_4(S^4) \xrightarrow{q_m'^*} [\mathrm{P}^4(m), S^4] \longrightarrow 0$$

that $[\mathrm{P}^4(m), S^4] = \boldsymbol{Z}/m \cdot q'_m$. Hence, there is some $\varepsilon' \in \boldsymbol{Z}/m$ such that $q_m \circ f_m = \varepsilon' \cdot q'_m$. It is sufficient to show that $\varepsilon' \in (\boldsymbol{Z}/m)^\times$. It follows from the computation of the spectral sequence given in the proof of Proposition 3.1 that $f_m^* : \boldsymbol{Z} \cong H^4(L_m, \boldsymbol{Z}) \to H^4(\mathrm{P}^4(m), \boldsymbol{Z}) \cong \boldsymbol{Z}/m$ can be identified with the natural projection $pr : \boldsymbol{Z} \to \boldsymbol{Z}/m$ (up to unit in $(\boldsymbol{Z}/m)^\times$). Similarly $q_m^* : \boldsymbol{Z} = H^4(S^4, \boldsymbol{Z}) \to H^4(\mathrm{P}^4(m), \boldsymbol{Z}) = \boldsymbol{Z}/m$ can be also identified with the natural projection pr. So if we consider the commutative diagram

$$\begin{array}{ccc} \boldsymbol{Z} = H^4(S^4, \boldsymbol{Z}) & \underset{\cong}{\xrightarrow{q_m^*}} & H^4(L_m, \boldsymbol{Z}) = \boldsymbol{Z} \\ \downarrow{\scriptstyle q_m^*} & & \downarrow{\scriptstyle f_m^*} \\ \boldsymbol{Z}/m = H^4(\mathrm{P}^4(m), \boldsymbol{Z}) & \xrightarrow{\times \varepsilon'} & H^4(\mathrm{P}^4(m), \boldsymbol{Z}) = \boldsymbol{Z}/m \end{array}$$

we have $\varepsilon' \in (\boldsymbol{Z}/m)^\times$. □

It follows from Proposition 2.6 that we have

COROLLARY 3.4. (i) *If* $m \equiv 1 \pmod 2$,

$$\pi_6(L_m) = \boldsymbol{Z}/(m,3) \cdot i_*(\eta_2 \circ \omega) \oplus \boldsymbol{Z}/m \cdot f_m \circ \sigma \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5.$$

(ii) *If* $\nu_2(m) \geq 3$,

$$\pi_6(L_m) = \boldsymbol{Z}/(m,12) \cdot i_*(\eta_2 \circ \omega) \oplus \boldsymbol{Z}/m \cdot f_m \circ \sigma \oplus \boldsymbol{Z}/2 \cdot \gamma_m \circ \eta_5 \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5.$$

(iii) *If* $1 \leq \nu_2(m) \leq 2$,

$$\pi_6(L_m) = \boldsymbol{Z}/m' \cdot f_m \circ \lambda_m \oplus \boldsymbol{Z}/2m \cdot f_m \circ \sigma \oplus \boldsymbol{Z}/2 \cdot \gamma_m \circ \eta_5 \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5,$$

where $m' = (m,12)/2 = 2^{\nu_2(m)-1}(m,3)$ *and* $\lambda_m = (2m/(12,m))\sigma + i' \circ \omega$.
(iv) *If* $m = 0$, $L_0 = S^2 \vee S^4$ *and*

$$\pi_6(L_0) = \boldsymbol{Z}/12 \cdot i_*(\eta_2 \circ \omega) \oplus \boldsymbol{Z}/2 \cdot i_4 \circ \eta_4^2 \oplus \boldsymbol{Z}/2 \cdot [i, i_4 \circ \eta_4] \oplus \boldsymbol{Z}/2 \cdot [i_*(\eta_2), i_4].$$

REMARK. It is known that the order of $i_*(\eta_2 \circ \omega) \in \pi_6(L_m)$ is $(12,m)$ ([**15**]). This can be also obtained using Proposition 1.

Similarly, using Propositions 2.9 and 3.1 we have

COROLLARY 3.5. (i) *If* $m \equiv 1 \pmod 2$,

$$\pi_7(L_m) = \boldsymbol{Z}/(m,3) \cdot f_m \circ \omega_m \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5^2 \oplus \boldsymbol{Z}/m \cdot [b_m, i_*(\eta_2)].$$

(ii) *If* $m \equiv 0 \pmod 4$ *and* $m \neq 0$,

$$\begin{aligned}\pi_7(L_m) = \boldsymbol{Z}/4 \cdot f_m \circ \widetilde{\nu'} \oplus \boldsymbol{Z}/2 \cdot f_m \circ \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot i_*(\eta_2 \circ \omega \circ \eta_6) \\ \oplus \boldsymbol{Z}/(m,3) \cdot f_m \circ \omega_m \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5^2 \oplus \boldsymbol{Z}/m \cdot [b_m, i_*(\eta_2)].\end{aligned}$$

(iii) *If* $m \equiv 2 \pmod 4$,

$$\begin{aligned}\pi_7(L_m) = \boldsymbol{Z}/2 \cdot f_m \circ \sigma \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot f_m \circ \tilde{\eta}_3 \circ \eta_5^2 \oplus \boldsymbol{Z}/(m,3) \cdot f_m \circ \omega_m \\ \oplus \boldsymbol{Z}/2 \cdot b_m \circ \eta_5^2 \oplus \boldsymbol{Z}/m \cdot [b_m, i_*(\eta_2)].\end{aligned}$$

(iv) *If* $m = 0$,

$$\begin{aligned}\pi_7(L_0) = \boldsymbol{Z}/2 \cdot i_*(\eta_2 \circ \omega \circ \eta_6) \oplus \boldsymbol{Z} \cdot i_4 \circ \nu_4 \oplus \boldsymbol{Z}/12 \cdot i_4 \circ E\omega \\ \oplus \boldsymbol{Z}/2 \cdot [i, i_4 \circ \eta_4^2] \oplus \boldsymbol{Z}/2 \cdot [i_*(\eta_2), i_4 \circ \eta_4] \oplus \boldsymbol{Z}/2 \cdot [i_*(\eta_2^2), i_4].\end{aligned}$$

4. The 6-skeleton of an m-twisted $\boldsymbol{C}\mathrm{P}^4$.

We would like to study the 6-skeleton of an m-twisted $\boldsymbol{C}\mathrm{P}^4$. For this purpose, define the spaces X_m, Y_m and Z_m as follows.

DEFINITION 7. Define the space X_m by

$$X_m = L_m \cup_{m \cdot b_m} e^6. \tag{8}$$

When $m \equiv 0$ (mod 2), define the spaces Y_m and Z_m by

$$Y_m = L_m \cup_{m \cdot b_m + i_*(\eta_2^3)} e^6, \text{ and } Z_m = L_m \cup_{m \cdot b_m + \gamma_m} e^6. \tag{9}$$

REMARK. If $m \equiv 0$ (mod 2), it is known [**24**] that there are isomorphisms

$$\begin{cases} H^*(X_m, \boldsymbol{Z}) \cong H^*(Y_m, \boldsymbol{Z}) \cong H^*(Z_m, \boldsymbol{Z}) & \text{(as graded rings)} \\ H^*(X_m, \boldsymbol{Z}/p) \cong H^*(Y_m, \boldsymbol{Z}/p) & \text{(as } \mathscr{A}_p\text{-modules)} \end{cases}$$

for any prime p. However, since $\pi_6(\Sigma X_m) \neq \pi_6(\Sigma Y_m)$, X_m and Y_m are not homotopy equivalent. It is also known that $H^*(X_m, \boldsymbol{Z}/2)$ and $H^*(Z_m, \boldsymbol{Z}/2)$ are not isomorphic as $\mathscr{A}_2$-modules. Hence, any two of $\{X_m, Y_m, Z_m\}$ are not homotopy equivalent.

PROPOSITION 4.1 ([**24**]). *Let M be an m-twisted $\boldsymbol{C}\mathrm{P}^4$.*

(i) *If $m \equiv 1$ (mod 2), then $Sq^2 : H^4(M, \boldsymbol{Z}/2) \to H^6(M, \boldsymbol{Z}/2)$ is trivial and there is a homotopy equivalence $M^{(6)} \simeq X_m$.*
(ii) *If $m \equiv 0$ (mod 2) and $Sq^2 : H^4(M, \boldsymbol{Z}/2) \to H^6(M, \boldsymbol{Z}/2)$ is trivial, then $M^{(6)}$ is homotopy equivalent to X_m or Y_m.*
(iii) *If $m \equiv 0$ (mod 2) and $Sq^2 : H^4(M, \boldsymbol{Z}/2) \to H^6(M, \boldsymbol{Z}/2)$ is non-trivial, then there is a homotopy equivalence $M^{(6)} \simeq Z_m$.*

PROOF. This easily follows from Theorem 6.9 of [**24**]. □

DEFINITION 8. Let M be an m-twisted $\boldsymbol{C}\mathrm{P}^4$.
In this case, we note that $H^6(M, \boldsymbol{Z}/2) \cong H^8(M, \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$. Then if $y_{2k} \in H^{2k}(M, \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$ denotes the generator ($k = 3, 4$), there exists a unique number $\varepsilon \in \{0, 1\} = \boldsymbol{Z}/2$ such that

$$Sq^2(y_6) = \varepsilon \cdot y_8. \tag{10}$$

Then M is called an m-twisted $\boldsymbol{C}\mathrm{P}^4$ *of type* (X, ε) if there is a homotopy equivalence $M^{(6)} \simeq X_m$ and it satisfies the condition (10).

When $m \equiv 0$ (mod 2), M is called an m-twisted $\boldsymbol{C}\mathrm{P}^4$ *of type* (Y, ε) if there is a homotopy equivalence $M^{(6)} \simeq Y_m$ and satisfies the condition (10). Similarly, M is called an m-twisted $\boldsymbol{C}\mathrm{P}^4$ *of type* (Z, ε) if there is a homotopy equivalence $M^{(6)} \simeq Z_m$ which satisfies the condition (10). If M and N are m-twisted $\boldsymbol{C}\mathrm{P}^4$'s of different types, clearly M and N have the different homotopy types.

Here we remark the following two general facts.

LEMMA 4.2. *If $m \equiv 1 \pmod 2$ and M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$, it is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X,1)$. Hence, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X,0)$ in this case.*

PROOF. It follows from Proposition 4.1 that it suffices to show that $Sq^2 : H^6(M,\boldsymbol{Z}/2) \xrightarrow{\cong} H^8(M,\boldsymbol{Z}/2)$ is an isomorphism. First, it follows from (0.2) and Proposition 4.1 that we have $y_2 \cdot y_4 = y_6$, $(y_2)^2 = y_4$, $(y_4)^2 = y_8$ and $Sq^2(y_4) = 0$, where $y_{2k} \in H^{2k}(M,\boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$ ($k = 1,2,3,4$) denotes the generator. Then if we write $Sq^2(y_6) = \varepsilon \cdot y_8$ ($\varepsilon \in \boldsymbol{Z}/2$), we have $\varepsilon = 1$ by using

$$\varepsilon \cdot y_8 = Sq^2(y_6) = Sq^2(y_2 \cdot y_4) = Sq^2(y_2) \cdot y_4 = (y_2)^2 \cdot y_4 = (y_4)^2 = y_8. \qquad \square$$

LEMMA 4.3. *If $m \equiv 0 \pmod 2$ and M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Z,ε), then $\varepsilon = 0$. Hence there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z,1)$ in this case.*

PROOF. We suppose that M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z,1)$. Then because $Sq^2 : H^{2k}(M,\boldsymbol{Z}/2) \to H^{2k+2}(M,\boldsymbol{Z}/2)$ is an isomorphism for $k = 2$ or $k = 3$ (by Proposition 4.1), $Sq^2Sq^2 : H^4(M,\boldsymbol{Z}/2) \to H^8(M,\boldsymbol{Z}/2)$ is an isomorphism. However, if we use the Adem relation $Sq^2Sq^2 = Sq^1Sq^2Sq^2$, Sq^2Sq^2 is trivial and this is a contradiction. $\square$

REMARK. In section 7, we shall prove that there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z,0)$ (see Theorem 7.4).

DEFINITION 9. Let $\beta_m \in \pi_6(X_m,L_m)$, $\beta'_m \in \pi_6(Y_m,L_m)$ and $\beta''_m \in \pi_6(Z_m,L_m)$ be the corresponding characteristic maps of the top cell e^6's in X_m, Y_m or Z_m, respectively. If we denote by $\eta'_k \in \pi_{k+2}(D^{k+1},S^k) \cong \boldsymbol{Z}/2$ the generator for $k \geq 3$, it is easy to see:

LEMMA 4.4 ([**8**]). *There are isomorphisms*

$$\begin{cases} \pi_6(X_m,L_m) = \boldsymbol{Z}\cdot\beta_m,\ \pi_6(Y_m,L_m) = \boldsymbol{Z}\cdot\beta'_m,\ \pi_6(Z_m,L_m) = \boldsymbol{Z}\cdot\beta''_m, \\ \pi_7(X_m,L_m) = \boldsymbol{Z}\cdot[\beta_m,i]_r \oplus \boldsymbol{Z}/2\cdot\beta_m\circ\eta'_5, \\ \pi_7(Y_m,L_m) = \boldsymbol{Z}\cdot[\beta'_m,i]_r \oplus \boldsymbol{Z}/2\cdot\beta'_m\circ\eta'_5, \\ \pi_7(Z_m,L_m) = \boldsymbol{Z}\cdot[\beta''_m,i]_r \oplus \boldsymbol{Z}/2\cdot\beta''_m\circ\eta'_5. \end{cases}$$

Finally in this section we recall the following useful result for checking whether M is a twisted $\boldsymbol{C}\mathrm{P}^4$ or not.

THEOREM 4.5. *Let $m \geq 1$ be an integer and $\varepsilon \in \boldsymbol{Z}/2$ be the number mod 2.*

(i) *Let $\varphi \in \pi_7(X_m)$ be an element and $j_{1*} : \pi_7(X_m) \to \pi_7(X_m,L_m)$ be the induced homomorphism. Then the mapping cone $M = X_m \cup_\varphi e^8$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (X,ε) if and only if $j_{1*}(\varphi) = \pm[\beta_m,i]_r + \varepsilon\cdot\beta_m\circ\eta'_5$.*

(ii) *Let $\varphi \in \pi_7(Y_m)$ be an element and $j_{2*} : \pi_7(Y_m) \to \pi_7(Y_m,L_m)$ be the induced homomorphism. Then the mapping cone $M = Y_m \cup_\varphi e^8$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Y,ε) if and only if $j_{2*}(\varphi) = \pm[\beta'_m,i]_r + \varepsilon\cdot\beta'_m\circ\eta'_5$.*

(iii) *Let $\varphi \in \pi_7(Y_m)$ be an element and $j_{3*} : \pi_7(Z_m) \to \pi_7(Z_m,L_m)$ be the induced homomorphism. Then the mapping cone $M = Z_m \cup_\varphi e^8$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Z,ε) if and only if $j_{3*}(\varphi) = \pm[\beta''_m,i]_r + \varepsilon\cdot\beta''_m\circ\eta'_5$.*

PROOF. Since the proof is analogous, we only give the proof of the case (i). Let $\varphi \in \pi_7(X_m)$ be an element such that $j_{1*}(\varphi) = n[\beta_m, i]_r + \varepsilon' \cdot \beta_m \circ \eta'_5$ $(n \in \boldsymbol{Z}, \varepsilon' \in \boldsymbol{Z}/2)$ and we take $M = X_m \cup_\varphi e^8$. Let $x_{2k} \in H^{2k}(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$ $(k = 1,2,3,4)$ be the corresponding generator. Then it follows from [**25**] that $x_2 \cdot x_6 = \pm n x_8$. Moreover, because the 6-skeleton of M is X_m, without loss of generalities we may assume that the equations $x_2 \cdot x_2 = m x_4$ and $x_2 \cdot x_4 = m x_6$ hold. Hence, by using (0.3), M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ if and only if $n = \pm 1$. So from now on we assume $n = \pm 1$ and it suffices to show that the equality $Sq^2(y_6) = \varepsilon \cdot y_8$ holds if and only if the equality

$$j_{1*}(\varphi) = \pm[\beta_m, i]_r + \varepsilon \cdot \beta_m \circ \eta'_5 \tag{$*$}$$

holds, where $y_{2k} \in H^{2k}(M, \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$ $(k = 1,2,3,4)$ denotes the corresponding generator.

First, we assume that $(*)$ holds. Let $q : M \to M/S^2 = N$ denote the pinch map, and consider the commutative diagram (for $k = 1$ or 2)

$$\begin{array}{ccc} H^4(M, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q^*} & H^4(M/S^2, \boldsymbol{Z}/2) \\ {\scriptstyle Sq^{2k}}\big\downarrow & & {\scriptstyle Sq^{2k}}\big\downarrow \\ H^{2k+4}(M, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q^*} & H^{2k+4}(M/S^2, \boldsymbol{Z}/2). \end{array} \tag{$\dagger$}_k$$

Since $Sq^2 : H^4(M, \boldsymbol{Z}/2) \to H^6(M, \boldsymbol{Z}/2)$ is trivial (by Proposition 4.1), it follows from the diagram $(\dagger)_1$ that Sq^2 is trivial on $H^4(M/S^2, \boldsymbol{Z}/2)$. Hence, the 4-skeleton of $N = M/S^2$ is homotopy equivalent to $S^4 \vee S^6$, and it has the CW-structure (up to homotopy)

$$N = M/S^2 \simeq S^4 \vee S^6 \cup_f e^8 \quad (f \in \pi_7(S^4 \vee S^6)).$$

Since M is an m-twisted $\boldsymbol{C}\mathrm{P}^4$, by using (0.2), $Sq^4(y_4) = (y_4)^2 = y_8$. Hence, by using the diagram $(\dagger)_2$, $Sq^4 : H^4(M/S^2, \boldsymbol{Z}/2) \xrightarrow{\cong} H^8(M/S^2, \boldsymbol{Z}/2)$ is an isomorphism. Thus we can write $f = \pm i_4 \circ \nu_4 + k \cdot i_4 \circ E\omega + \varepsilon'' \cdot i_6 \circ \eta_7$ for some $(k, \varepsilon'') \in \boldsymbol{Z}/12 \times \boldsymbol{Z}/2$. However, if we use $(*)$, we have $\varepsilon = \varepsilon''$ and

$$N = M/S^2 \simeq S^4 \vee S^6 \cup_{\pm i_4 \circ \nu_4 + k \cdot i_4 \circ E\omega + \varepsilon \cdot i_6 \circ \eta_7} e^8. \tag{11}$$

Now let $q' : N \to N/S^4$ be the pinch map, and consider the 2-cell complex N/S^4. By using (11), there is a homotopy equivalence $N/S^4 \simeq S^6 \cup_{\varepsilon \cdot \eta_7} e^8$. Hence, if we consider the commutative diagram

$$\begin{array}{ccccc} H^6(M, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q^*} & H^6(M/S^2, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q'^*} & H^6(S^6 \cup_{\varepsilon \cdot \eta_7} e^8, \boldsymbol{Z}/2) \\ {\scriptstyle Sq^2}\big\downarrow & & {\scriptstyle Sq^2}\big\downarrow & & {\scriptstyle Sq^2}\big\downarrow \\ H^8(M, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q^*} & H^8(M/S^2, \boldsymbol{Z}/2) & \xleftarrow[\cong]{q'^*} & H^8(S^6 \cup_{\varepsilon \cdot \eta_7} e^8, \boldsymbol{Z}/2) \end{array}$$

we have $Sq^2(y_6) = \varepsilon \cdot y_8$. Hence, if $(*)$ is satisfied, $Sq^2(y_6) = \varepsilon \cdot y_8$ holds. A similar method shows the opposite direction and this completes the proof. $\square$

5. An m-twisted CP^4 of type (X,ε).

Consider the cofiber sequence

$$S^5 \xrightarrow{mb_m} L_m \xrightarrow{j} X_m \xrightarrow{p_m} S^6. \tag{12}$$

Now we try to compute the Whitehead product $[b_m, i] \in \pi_6(L_m)$.

PROPOSITION 5.1. *Let $m \geq 1$ be an integer.*

(i) *If $m \equiv 1 \pmod 2$ or $m \equiv 0 \pmod 8$, there is a unit $x_m \in (\mathbf{Z}/m)^\times$ and $\varepsilon_m \in \mathbf{Z}/2$ such that $[b_m, i] = x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5$.*

(ii) *If $1 \leq \nu_2(m) \leq 2$, there is a unit $x_m \in (\mathbf{Z}/2m)^\times$ and $\varepsilon_m \in \mathbf{Z}/2$ such that $[b_m, i] = x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5$.*

REMARK. If $m \equiv 1 \pmod 2$, we can show $\varepsilon_m = 1$. This will be proved in Corollary 5.11.

PROOF. Since the proof is completely analogous, we shall prove only the case $m \equiv 0 \pmod 8$. In this case, it follows from Corollary 3.4 that we can take

$$[b_m, i] = y \cdot i_*(\eta_2 \circ \omega) + x_m \cdot f_m \circ \sigma + \varepsilon' \cdot \gamma_m \circ \eta_5 + \varepsilon_m \cdot b_m \circ \eta_5 \tag{†}$$

for some $(y, x_m) \in \mathbf{Z}/(12, m) \times \mathbf{Z}/m$ and $\varepsilon', \varepsilon_m \in \mathbf{Z}/2$. It suffices to show that $x_m \in (\mathbf{Z}/m)^\times$ and $y = \varepsilon' = 0$. Consider the cofiber sequence

$$S^6 \xrightarrow{\sigma} \mathrm{P}^4(m) \xrightarrow{j''} V_m = \mathrm{P}^4(m) \cup_\sigma e^7. \tag{††}$$

If we denote by $\overline{\sigma} \in \pi_7(V_m, \mathrm{P}^4(m))$ the characteristic map of the top cell e^7 in V_m, then $\pi_7(V_m, \mathrm{P}^4(m)) = \mathbf{Z} \cdot \overline{\sigma}$. Moreover, it follows from Lemma 2.5 that $j' \circ f_m \circ \sigma = 0$. Hence, it follows from the cofiber sequence (††) that there is a map $\overline{f}_m : V_m \to W_m = L_m \cup_{b_m} e^6$ such that $\overline{f}_m \circ j'' = j' \circ f_m$. Now consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{Z} \cdot \overline{\sigma} = \pi_7(V_m, \mathrm{P}^4(m)) & \xrightarrow{\partial^*} & \pi_6(\mathrm{P}^4(m)) \\
{\scriptstyle \overline{f}_{m*}} \big\downarrow & & \big\downarrow {\scriptstyle f_{m*}} \\
\mathbf{Z} \cdot [\beta, i]_r \oplus \mathbf{Z}/2 \cdot \beta \circ \eta'_5 = \pi_7(W_m, L_m) & \xrightarrow{\partial^*_7} & \pi_6(L_m),
\end{array}$$

where $\beta \in \pi_6(W_m, L_m) \cong \mathbf{Z}$ denotes the characteristic map of the top cell e^6 in W_m. We note that $\pi_7(W_m, L_m) = \mathbf{Z} \cdot [\beta, i]_r \oplus \mathbf{Z}/2 \cdot \beta \circ \eta'_5$ (by [**1**], [**8**]). So there exists an integer $x \in \mathbf{Z}$ and $\varepsilon \in \mathbf{Z}/2$ such that

$$\overline{f}_{m*}(\overline{\sigma}) = x[\beta, i]_r + \varepsilon \cdot \beta \circ \eta'_5. \tag{13}$$

Since $\partial^*(\overline{\sigma}) = \sigma$ and $\partial^*_6(\beta) = b_m$,

$$\begin{aligned} f_m\circ\sigma =& f_{m*}\circ\partial^*(\overline{\sigma})=\partial_7^*\circ\overline{f}_{m*}(\overline{\sigma})=\partial_7^*(x[\beta,i]_r+\varepsilon\cdot\beta\circ\eta_5')\quad\text{(by (13))}\\ =& x\cdot\partial_7^*([\beta,i]_r)+\varepsilon\cdot b_m\circ\eta_5=-x[\partial_6^*(\beta),i]+\varepsilon\cdot b_m\circ\eta_5\\ =& -x[b_m,i]+\varepsilon\cdot b_m\circ\eta_5\\ =& -x(y\cdot i_*(\eta_2\circ\omega)+x_m\cdot f_m\circ\sigma+\varepsilon'\cdot\gamma_m\circ\eta_5+\varepsilon_m\cdot b_m\circ\eta_5)+\varepsilon\cdot b_m\circ\eta_5\quad\text{(by (†))}\\ =& -xy\cdot i_*(\eta_2\circ\omega)-xx_m\cdot f_m\circ\sigma+x\varepsilon'\cdot\gamma_m\circ\eta_5+(\varepsilon+x\varepsilon_m)\cdot b_m\circ\eta_5\in\pi_6(L_m). \end{aligned}$$

Hence, $xy=0\in\boldsymbol{Z}/(m,12)$, $1=-xx_m\in\boldsymbol{Z}/m$ and $x\varepsilon'=0\in\boldsymbol{Z}/2$. An easy computation shows that $x,x_m\in(\boldsymbol{Z}/m)^\times$ and $y=\varepsilon'=0$. □

THEOREM 5.2. (i) *If* $m\equiv 1 \pmod 2$,

$$\pi_6(X_m)=\boldsymbol{Z}/(m,3)\cdot j_*(i_*(\eta_2\circ\omega))\oplus\boldsymbol{Z}/m\cdot j_*(f_m\circ\sigma).$$

(ii) *If* $\nu_2(m)\geq 3$,

$$\begin{aligned}\pi_6(X_m)=&\boldsymbol{Z}/(m,12)\cdot j_*(i_*(\eta_2\circ\omega))\oplus\boldsymbol{Z}/m\cdot j_*(f_m\circ\sigma)\\ &\oplus\boldsymbol{Z}/2\cdot j_*(\gamma_m\circ\eta_5)\oplus\boldsymbol{Z}/2\cdot j_*(b_m\circ\eta_5).\end{aligned}$$

(iii) *If* $1\leq\nu_2(m)\leq 2$ *and* $m'=2m/(m,12)=2^{\nu_2(m)-1}(m,3)$,

$$\begin{aligned}\pi_6(X_m)=&\boldsymbol{Z}/m'\cdot j_*(f_m\circ\lambda_m)\oplus\boldsymbol{Z}/m\cdot j_*(f_m\circ\sigma)\\ &\oplus\boldsymbol{Z}/2\cdot j_*(\gamma_m\circ\eta_5)\oplus\boldsymbol{Z}/2\cdot j_*(b_m\circ\eta_5).\end{aligned}$$

PROOF. Consider the exact sequence

$$\boldsymbol{Z}\cdot[\beta_m,i]_r\oplus\boldsymbol{Z}/2\cdot\beta_m\circ\eta_5'=\pi_7(X_m,L_m)\xrightarrow{\partial_7}\pi_6(L_m)\xrightarrow{j_*}\pi_6(X_m)\longrightarrow 0.$$

Then we remark that $\begin{cases}\partial_7(\beta_m\circ\eta_5')=(mb_m)\circ\eta_5=m(b_m\circ\eta_5),\\ \partial_7([\beta_m,i]_r)=-[mb_m,i]=-m[b_m,i].\end{cases}$

(i) Assume $m\equiv 1 \pmod 2$. Then because

$$\begin{cases}\partial_7(\beta_m\circ\eta_5')=m(b_m\circ\eta_5)=b_m\circ\eta_5,\\ \partial_7([\beta_m,i]_r)=-m[b_m,i]=-m(x_m\cdot f_m\circ\sigma+\varepsilon_m\cdot b_m\circ\eta_5)=\varepsilon_m\cdot b_m\circ\eta_5,\end{cases}$$

we have Im $\partial_7=\boldsymbol{Z}/2\cdot b_m\circ\eta_5$ and the assertion (i) follows.

(ii) We suppose $\nu_2(m)\geq 3$. The same method as above shows

$$\begin{cases}\partial_7(\beta_m\circ\eta_5')=m(b_m\circ\eta_5)=0,\\ \partial_7([\beta_m,i]_r)=-m[b_m,i]=-m(x_m\cdot f_m\circ\sigma+\varepsilon_m\cdot b_m\circ\eta_5)=0,\end{cases}$$

and Im $\partial_7=0$. So the assertion easily follows.

(iii) Finally consider the case $1 \leq \nu_2(m) \leq 2$. In this case, because the order of $f_m \circ \sigma$ is $2m$ and $x_m \in (\boldsymbol{Z}/2m)^\times$, $mx_m \cdot f_m \circ \sigma \neq 0$ and its order is 2. Then if we remark that

$$\begin{cases} \partial_7(\beta_m \circ \eta_5') = m(b_m \circ \eta_5) = 0, \\ \partial_7([\beta_m, i]_r) = -m[b_m, i] = -m(x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5) = mx_m \cdot f_m \circ \sigma, \end{cases}$$

we have $\operatorname{Im} \partial_7 = \langle m \cdot f_m \circ \sigma \rangle \cong \boldsymbol{Z}/2$. Then the assertion easily follows from the above exact sequence. □

COROLLARY 5.3. *Let $\partial_7 : \pi_7(X_m, L_m) \to \pi_6(L_m)$ be the boundary operator.*

(i) *If* $m \equiv 1 \pmod 2$, $\operatorname{Ker} \partial_7 = \langle [\beta_m, i]_r + \varepsilon_m \cdot \beta_m \circ \eta_5' \rangle \cong \boldsymbol{Z}$.
(ii) *If* $\nu_2(m) \geq 3$, $\operatorname{Ker} \partial_7 = \pi_7(X_m, L_m) = \boldsymbol{Z} \cdot [\beta_m, i]_r \oplus \boldsymbol{Z}/2 \cdot \beta_m \circ \eta_5'$.
(iii) *If* $1 \leq \nu_2(m) \leq 2$, $\operatorname{Ker} \partial_7 = \langle 2[\beta_m, i]_r \rangle \oplus \boldsymbol{Z}/2 \cdot \beta_m \circ \eta_5' \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$.

PROOF. The assertion easily follows from the proof of Theorem 5.2. □

COROLLARY 5.4. *If $1 \leq \nu_2(m) \leq 2$, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (X, ε) for any $\varepsilon \in \boldsymbol{Z}/2$.*

PROOF. The assertion follows from Theorem 4.5, Corollary 5.3 and the equality $\operatorname{Im} [j_{1*} : \pi_7(X_m) \to \pi_7(X_m, L_m)] = \langle 2[\beta_m, i]_r \rangle \oplus \boldsymbol{Z}/2 \cdot \beta_m \circ \eta_5' \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$. □

DEFINITION 10. Let $\tilde{q}_m \in [X_m, K(\boldsymbol{Z}, 2)] \cong H^2(X_m, \boldsymbol{Z}) \cong \boldsymbol{Z}$ denote the map which represents a generator, and let $\tilde{X}_m$ be the homotopy fiber of the map $\tilde{q}_m$. Then $\tilde{X}_m$ is a 2-connective covering of X_m and there is a fibration sequence

$$S^1 \to \tilde{X}_m \to X_m. \tag{14}$$

PROPOSITION 5.5. *Let $m \geq 1$ be an integer. Then if* $m \equiv 1 \pmod 2$ *or* $m \equiv 0 \pmod 8$, *there is a homotopy equivalence $\tilde{X}_m \simeq \mathrm{P}^4(m) \vee \mathrm{P}^6(m) \vee S^7$.*

PROOF. If we consider the Serre spectral sequence

$$E_2^{s,t} = H^s(X_m, \boldsymbol{Z}) \otimes H^t(S^1, \boldsymbol{Z}) \Rightarrow H^{s+t}(\tilde{X}_m, \boldsymbol{Z})$$

associated to (14), we have $H^k(\tilde{X}_m, \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z}/m & \text{if } k = 4, 6, \\ \boldsymbol{Z} & \text{if } k = 0, 7, \\ 0 & \text{otherwise.} \end{cases}$

Hence, there is a homotopy equivalence $\tilde{X}_m \simeq \mathrm{P}^4(m) \vee \mathrm{P}^6(m) \cup_\theta e^7$ for some $\theta \in \pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m))$. It suffices to show that $\theta = 0$. First, we remark that

$$\pi_6(\mathrm{P}^6(m)) = \begin{cases} 0 & \text{if } m \equiv 1 \pmod 2, \\ \boldsymbol{Z}/2 \cdot i'' \circ \eta_5 & \text{if } m \equiv 0 \pmod 2, \end{cases} \tag{15}$$

where $i'' : S^5 \to \mathrm{P}^6(m)$ denotes the inclusion of the bottom cell.

Hence, by using Proposition 2.6, we have

$$\begin{aligned}\pi_6(\mathrm{P}^4(m)\vee \mathrm{P}^6(m)) &\cong \pi_6(\mathrm{P}^4(m))\oplus \pi_6(\mathrm{P}^6(m))\\ &\cong \begin{cases}\boldsymbol{Z}/(m,3)\oplus \boldsymbol{Z}/m & \text{if } m\equiv 1 \text{ (mod 2)},\\ \boldsymbol{Z}/(m,12)\oplus \boldsymbol{Z}/m\oplus (\boldsymbol{Z}/2)^2 & \text{if } \nu_2(m)\geq 3.\end{cases}\end{aligned}$$

So it follows from Theorem 5.2 that there is an isomorphism $\pi_6(X_m)\cong \pi_6(\mathrm{P}^4(m)\vee \mathrm{P}^6(m))$. However, because $\pi_6(X_m)\cong \pi_6(\tilde{X}_m)\cong \pi_6(\mathrm{P}^4(m)\vee \mathrm{P}^6(m))/(\theta)$, we have $\theta=0$. □

We recall $\pi_7(\mathrm{P}^6(m))\cong \begin{cases}0 & \text{if } m\equiv 1 \text{ (mod 2)},\\ \boldsymbol{Z}/2\oplus \boldsymbol{Z}/2 & \text{if } m\equiv 0 \text{ (mod 8)}.\end{cases}$

COROLLARY 5.6.

$$\pi_7(X_m)\cong \begin{cases}\boldsymbol{Z}\oplus \boldsymbol{Z}/(m,3)\oplus \boldsymbol{Z}/m & \text{if } m\equiv 1 \text{ (mod 2)},\\ \boldsymbol{Z}\oplus \boldsymbol{Z}/4\oplus (\boldsymbol{Z}/2)^4\oplus \boldsymbol{Z}/m\oplus \boldsymbol{Z}/(m,3) & \text{if } m\equiv 0 \text{ (mod 8)}.\end{cases}$$

PROOF. The assertions easily follow from [**8**] and Proposition 5.5. □

LEMMA 5.7. *If* $m\equiv 0$ (mod 2), *there exists a coextension* $\tilde{\eta}_5\in \pi_7(X_m)$ *of* η_5 *such that*

$$p_m\circ\tilde{\eta}_5=\eta_6\in\pi_7(S^6) \textit{ with } 2\tilde{\eta}_5=\begin{cases}0 & \textit{if } m\equiv 0 \text{ (mod 4)},\\ j_*(b_m\circ\eta_5^2) & \textit{if } m\equiv 2 \text{ (mod 4)}.\end{cases}$$

PROOF. Consider the cofiber sequence (12). Since $(mb_m)\circ\eta_5=0$, there exists a coextension $\tilde{\eta}_5\in\pi_7(X_m)$ such that $p_m\circ\tilde{\eta}_5=\eta_6$. Moreover, it follows from [**18**, Corollary 3.7] that we have

$$\{mb_m,\eta_5,2\iota_6\}\supset b_m\circ\{m\iota_5,\eta_5,2\iota_6\}=\begin{cases}\equiv 0 & \text{if } m\equiv 0 \text{ (mod 4)},\\ \ni b_m\circ\eta_5^2 & \text{if } m\equiv 2 \text{ (mod 4)}.\end{cases}$$

On the other hand, using [**18**, Proposition 1.8], we have

$$2\tilde{\eta}_5=\tilde{\eta}_5\circ(2\iota_7)\in -j\circ\{mb_m,\eta_5,2\iota_6\},$$

where the indeterminacy of $\{mb_m,\eta_5,2\iota_6\}$ is $mb_m\circ\pi_7(S^5)+2\pi_7(L_m)=2\pi_7(L_m)$. Hence,

$$2\tilde{\eta}_5\equiv\begin{cases}0 & \text{mod } 2j_*(\pi_7(L_m)) & \text{if } m\equiv 0 \text{ (mod 4)},\\ j_*(b_m\circ\eta_5^2) & \text{mod } 2j_*(\pi_7(L_m)) & \text{if } m\equiv 2 \text{ (mod 4)}.\end{cases}$$

So if we choose $\tilde{\eta}_5$ properly, the assertions are satisfied. □

THEOREM 5.8. *Let* $j_{1*}:\pi_7(X_m)\to\pi_7(X_m,L_m)$ *denote the induced homomorphism.*

(i) *If* $m \equiv 1 \pmod 2$, *there exists some element* $\varphi_m \in \pi_7(X_m)$ *such that*

$$j_{1*}(\varphi_m) = [\beta_m, i]_r + \varepsilon_m \cdot \beta_m \circ \eta_5', \tag{16}$$

and there is an isomorphism

$$\pi_7(X_m) = \boldsymbol{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \oplus \boldsymbol{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \boldsymbol{Z} \cdot \varphi_m.$$

(ii) *If* $m \equiv 0 \pmod 8$ *and* $m \neq 0$, *there exists some element* $\varphi_m \in \pi_7(X_m)$ *such that*

$$j_{1*}(\varphi_m) = [\beta_m, i]_r \tag{17}$$

and there is an isomorphism

$$\begin{aligned}\pi_7(X_m) = {} & \boldsymbol{Z} \cdot \varphi_m \oplus \boldsymbol{Z}/4 \cdot j_*(f_m \circ \widetilde{\nu'}) \oplus \boldsymbol{Z}/2 \cdot j_*(f_m \circ \sigma \circ \eta_6) \\ & \oplus \boldsymbol{Z}/2 \cdot (j \circ i)_*(\eta_2 \circ \omega \circ \eta_6) \oplus \boldsymbol{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \\ & \oplus \boldsymbol{Z}/2 \cdot j_*(b_m \circ \eta_5^2) \oplus \boldsymbol{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \boldsymbol{Z}/2 \cdot \tilde{\eta}_5.\end{aligned}$$

(iii) *If* $m = 0$, *then* $X_0 = S^2 \vee S^4 \vee S^6$ *and there is an isomorphism*

$$\begin{aligned}\pi_7(X_0) = {} & \boldsymbol{Z} \cdot j_4 \circ \nu_4 \oplus \boldsymbol{Z} \cdot [j_2, j_6] \oplus \boldsymbol{Z}/2 \cdot j_2 \circ \eta_2 \circ \omega \circ \eta_6 \oplus \boldsymbol{Z}/2 \cdot j_6 \circ \eta_6 \\ & \oplus \boldsymbol{Z}/12 \cdot j_4 \circ E\omega \oplus \boldsymbol{Z}/2 \cdot [j_2, j_4 \circ \eta_4^2] \oplus \boldsymbol{Z}/2 \cdot [j_2 \circ \eta_2, j_4 \circ \eta_4] \\ & \oplus \boldsymbol{Z}/2 \cdot [j_2 \circ \eta_2^2, j_4],\end{aligned}$$

where $j_k : S^k \to S^2 \vee S^4 \vee S^6$ $(k = 2, 4, 6)$ *denote the corresponding inclusions.*

PROOF. We take $\mathrm{Im}\ j_* = \mathrm{Im}\ [j_* : \pi_7(X_m) \to \pi_7(X_m, L_m)]$.

(i) We assume $m \equiv 1 \pmod 2$, and we take $\delta_m = [\beta_m, i]_r + \varepsilon_m \cdot \beta_m \circ \eta_5'$. Then it follows from Corollary 5.3 that there is an exact sequence

$$0 \to \mathrm{Im}\ j_* \longrightarrow \pi_7(X_m) \xrightarrow{j_{1*}} \boldsymbol{Z} \cdot \delta_m \longrightarrow 0.$$

If we choose an element $\varphi_m \in \pi_7(X_m)$ such that $j_{1*}(\varphi_m) = \delta_m$, then $\pi_7(X_m) = \boldsymbol{Z} \cdot \varphi_m \oplus \mathrm{Im}\ j_*$, and it remains to show that

$$\mathrm{Im}\ j_* = \boldsymbol{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \oplus \boldsymbol{Z}/m \cdot j_*([b_m, i_*(\eta_2)]). \tag{$\dagger_1$}$$

First, by using Corollary 5.6, we have

$$\mathrm{Im}\ j_* \cong \boldsymbol{Z}/m \oplus \boldsymbol{Z}/(m,3). \tag{$\dagger_2$}$$

Then consider the exact sequence, $\pi_8(X_m, L_m) \xrightarrow{\partial_8} \pi_7(L_m) \xrightarrow{j_*} \mathrm{Im}\ j_* \to 0$.

Let $\widehat{\eta_5}^2 \in \pi_8(D^6, S^5) \cong \boldsymbol{Z}/2$ be the generator and consider the element $\beta_m \circ \widehat{\eta_5}^2 \in$

$\pi_8(X_m, L_m)$. Then because

$$\partial_8(\beta_m \circ \widehat{\eta_5}^2) = (mb_m) \circ \eta_5^2 = m(b_m \circ \eta_5^2) = b_m \circ \eta_5^2 \neq 0 \in \pi_7(L_m),$$

$j_*(b_m \circ \eta_5^2) = 0$. Then if we consider the group structure of $\pi_7(L_m)$ (see Corollary 3.5), there is a surjective homomorphism,

$$\boldsymbol{Z}/(m,3) \cdot f_m \circ \omega_m \oplus \boldsymbol{Z}/m \cdot [b_m, i_*(\eta_2)] \xrightarrow{j_*} \mathrm{Im}\ j_*.$$

However, if we recall $(\dagger_2)$, j_* is an isomorphism. Hence, $(\dagger_1)$ is proved.

(ii) We suppose $m \equiv 0 \pmod 8$ and $m \neq 0$. Then using Corollary 5.3, we have the exact sequence

$$0 \longrightarrow \mathrm{Im}\ j_* \longrightarrow \pi_7(X_m) \xrightarrow{j_{1*}} \boldsymbol{Z} \cdot [\beta_m, i]_r \oplus \boldsymbol{Z}/2 \cdot \beta_m \circ \eta_5' \longrightarrow 0.$$

If we choose $\varphi_m \in \pi_7(X_m)$ such that $j_{1*}(\varphi_m) = [\beta_m, i]_r$, we have the isomorphism $\pi_7(X_m) = \boldsymbol{Z} \cdot \varphi_m \oplus \mathrm{Tor}(\pi_7(X_m))$ and the exact sequence

$$0 \to \mathrm{Im}\ j_* \to \mathrm{Tor}(\pi_7(X_m)) \xrightarrow{p_{m*}} \pi_7(S^6) = \boldsymbol{Z}/2 \cdot \eta_6 \to 0.$$

Now consider the element $\tilde{\eta}_5 \in \pi_7(X_m)$. Then because the order of $\tilde{\eta}_5$ is 2 and $p_{m*}(\tilde{\eta}_5) = \eta_6$ (by Lemma 5.7), we have

$$\pi_7(X_m) = \boldsymbol{Z} \cdot \varphi_m \oplus \boldsymbol{Z}/2 \cdot \tilde{\eta}_5 \oplus \mathrm{Im}\ j_*. \tag{18}$$

If we recall $\pi_7(X_m) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/4 \oplus (\boldsymbol{Z}/2)^4 \oplus \boldsymbol{Z}/m \oplus \boldsymbol{Z}/(m,3)$ (by Corollary 5.6) and (18), there is an isomorphism $\mathrm{Im}\ j_* \cong \boldsymbol{Z}/4 \oplus (\boldsymbol{Z}/2)^3 \oplus \boldsymbol{Z}/m \oplus \boldsymbol{Z}/(m,3)$. Now consider the surjective homomorphism $j_* : \pi_7(L_m) \to \mathrm{Im}\ j_*$.

Because $\pi_7(L_m) \cong \boldsymbol{Z}/4 \oplus (\boldsymbol{Z}/2)^3 \oplus \boldsymbol{Z}/m \oplus \boldsymbol{Z}/(m,3)$ (by Corollary 3.5), $j_* : \pi_7(L_m) \xrightarrow{\cong} \mathrm{Im}\ j_*$ is an isomorphism. Hence, we have

$$\begin{aligned}\mathrm{Im}\ j_* = {} & \boldsymbol{Z}/4 \cdot j_*(f_m \circ \widetilde{\nu'}) \oplus \boldsymbol{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \oplus \boldsymbol{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \\ & \oplus \boldsymbol{Z}/2 \cdot j_*(i_*(\eta_2 \circ \omega \circ \eta_6)) \oplus \boldsymbol{Z}/2 \cdot j_*(b_m \circ \eta_5^2) \oplus \boldsymbol{Z}/2 \cdot j_*(f_m \circ \sigma \circ \eta_6),\end{aligned}$$

and the assertion (ii) follows from (18).

(iii) Finally, since $X_0 = S^2 \vee S^4 \vee S^6$, the assertion (iv) also follows from the Hilton-Milnor Theorem [**22**]. □

Now we can prove the following key result.

THEOREM 5.9. *Let $m \geq 0$ be an integer.*

(i) *If $m \equiv 1 \pmod 2$, there exists $M_{m,1}^X$ which is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X,1)$. Conversely, if there is an m-twisted $\boldsymbol{C}\mathrm{P}^4$, it has the type $(X,1)$.*

(ii) *If* $m \equiv 0 \pmod 8$, *for each* $\varepsilon \in \{0,1\} = \boldsymbol{Z}/2$ *there exists* $M^X_{m,\varepsilon}$ *which is an* m*-twisted* $\boldsymbol{C}\mathrm{P}^4$ *of type* (X,ε).

PROOF. Let $j_{1*} : \pi_7(X_m) \to \pi_7(X_m, L_m)$ be the induced homomorphism.

(i) Assume $m \equiv 1 \pmod 2$. Then it follows from Lemma 4.2 that it suffices to show that there exists an m-twisted $\boldsymbol{C}\mathrm{P}^4$. For this purpose, consider the mapping cone

$$M^X_{m,1} = X_m \cup_{\varphi_m} e^8. \tag{19}$$

Then by Theorem 4.5 and Theorem 5.8, $M^X_{m,1}$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$.

(ii) Next we assume that $m \equiv 0 \pmod 8$. First, consider the case $m \neq 0$. Then it follows from Theorem 5.8 that $j_{1*}(\varphi_m + \varepsilon \cdot \tilde{\eta}_5) = [\beta_m, i]_r + \varepsilon \cdot \beta_m \circ \eta'_5$ for $\varepsilon \in \boldsymbol{Z}/2$. We define the mapping cone $M^X_{m,\varepsilon}$ by

$$M^X_{m,\varepsilon} = X_m \cup_{\varphi_m + \varepsilon \cdot \tilde{\eta}_5} e^8. \tag{20}$$

Then it follows from Theorem 4.5 that $M^X_{m,\varepsilon}$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (X,ε). Next consider the case $m = 0$. In this case, for $\varepsilon \in \boldsymbol{Z}/2$, define the space $M^X_{0,\varepsilon}$ by

$$M^X_{0,\varepsilon} = X_0 \cup_{j_4 \circ \nu_4 + [j_2, j_6] + \varepsilon \cdot j_6 \circ \eta_6} e^8 = S^2 \vee S^4 \vee S^6 \cup_{j_4 \circ \nu_4 + [j_2, j_6] + \varepsilon \cdot j_6 \circ \eta_6} e^8. \tag{21}$$

Then an easy diagram chasing shows that $M^X_{0,\varepsilon}$ is a 0-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (X,ε) and this completes the proof. □

EXAMPLE. Since $\boldsymbol{H}\mathrm{P}^2 \# (S^2 \times S^6) \simeq (S^2 \vee S^4 \vee S^6) \cup_{[j_2, j_6] + j_4 \circ \nu_4} e^8$ (**[21]**), it is a 0-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X,0)$.

COROLLARY 5.10. *Let* $\varphi \in \pi_7(X_0)$ *be an element such that*

$$\begin{aligned} \varphi = {} & n_1 j_1 \circ \nu_4 + n_2 [j_2, j_6] + \varepsilon \cdot j_6 \circ \eta_6 + \varepsilon_1 \cdot j_2 \circ \eta_2 \circ \omega \circ \eta_6 + \varepsilon_2 \cdot [j_2, j_4 \circ \eta_4^2] \\ & + \varepsilon_3 \cdot [j_2 \circ \eta_2, j_4 \circ \eta_4] + \varepsilon_4 \cdot [j_2, \eta_2^2, j_4] + a \cdot j_4 \circ E\omega, \end{aligned}$$

where $n_1, n_2 \in \boldsymbol{Z}$; $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \boldsymbol{Z}/2$; $a \in \boldsymbol{Z}/12$. *Then* $M = X_0 \cup_\varphi e^8$ *is a* 0*-twisted* $\boldsymbol{C}\mathrm{P}^4$ *of type* (X,ε) *if and only if* $n_1, n_2 \in \{\pm 1\}$ *and* $\varepsilon = \varepsilon_0$.

PROOF. This can be proved using a tedious diagram chasing. □

COROLLARY 5.11. *If* $m \equiv 1 \pmod 2$, *then there exists a unit* $x_m \in (\boldsymbol{Z}/m)^\times$ *such that,* $[b_m, i] = x_m \cdot f_m \circ \sigma + b_m \circ \eta_5 \in \pi_6(L_m)$.

PROOF. It follows from Proposition 5.1 that $[b_m, i] = x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5$ for $x_m \in (\boldsymbol{Z}/m)^\times$, $\varepsilon_m \in \boldsymbol{Z}/2 = \{0,1\}$. So it suffices to show that $\varepsilon_m = 1$.

Let $M = M^X_{m,1} = X_m \cup_{\varphi_m} e^8$ be the mapping cone. Then it follows from the proof of Theorem 5.8 that it is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(X,1)$. Hence, $Sq^2 : H^6(M, \boldsymbol{Z}/2) \xrightarrow{\cong} H^8(M, \boldsymbol{Z}/2)$ is an isomorphism. Moreover, it follows from Proposition 4.1 that $Sq^2 : H^4(M, \boldsymbol{Z}/2) \to H^6(M, \boldsymbol{Z}/2)$ is trivial.

Hence, since $j_{1*}(\varphi_m) = [\beta_m, i]_r + \varepsilon_m \cdot \beta_m \circ \eta_7'$, using the analogous proof of Theorem 4.5 (c.f. (11)) there is a homotopy equivalence

$$M/S^2 \simeq S^4 \vee S^6 \cup_{\pm i_4 \circ \eta_4 + k \cdot i_4 \circ E\omega + \varepsilon_m \cdot i_6 \circ \eta_7} e^8.$$

Thus, if $z_{2k} \in H^{2k}(M/S^2, \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$ ($k = 3, 4$) denotes the generator, an easy diagram chasing shows that $Sq^2(z_6) = \varepsilon_m \cdot z_8$. So it remains to show that $Sq^2 : H^6(M/S^2, \boldsymbol{Z}/2) \to H^8(M/S^2, \boldsymbol{Z}/2)$ is an isomorphism. Here we remark that $Sq^2 : H^6(M, \boldsymbol{Z}/2) \xrightarrow{\cong} H^8(M, \boldsymbol{Z}/2)$ is an isomorphism (by Lemma 4.2). Hence, this can be easily obtained by the following commutative diagram

$$\begin{array}{ccc} H^6(M/S^2; \boldsymbol{Z}/2) & \underset{\cong}{\xrightarrow{q^*}} & H^6(M, \boldsymbol{Z}/2) \\ {\scriptstyle Sq^2}\downarrow & & \downarrow{\scriptstyle Sq^2}\,\cong \\ H^8(M/S^2; \boldsymbol{Z}/2) & \underset{\cong}{\xrightarrow{q^*}} & H^8(M, \boldsymbol{Z}/2). \end{array}$$

□

6. An m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Y, ε).

Throughout this section, we assume that $m \geq 0$ is an even integer. Let us consider the cofiber sequence

$$S^5 \xrightarrow{mb_m + i_*(\eta_2^3)} L_m \xrightarrow{j'} Y_m \xrightarrow{\tilde{p}_m} S^6. \tag{22}$$

THEOREM 6.1. (i) *If* $\nu_2(m) \geq 3$,

$$\begin{aligned} \pi_6(Y_m) = \boldsymbol{Z}/(m,6) \cdot j'_*(i_*(\eta_2 \circ \omega)) \oplus \boldsymbol{Z}/m \cdot j'_*(f_m \circ \sigma) \\ \oplus \boldsymbol{Z}/2 \cdot j'_*(\gamma_m \circ \eta_5) \oplus \boldsymbol{Z}/2 \cdot j'_*(b_m \circ \eta_5). \end{aligned}$$

(ii) *If* $1 \leq \nu_2(m) \leq 2$ *and* $m' = (m, 12)/2 = 2^{\nu_2(m)-1}(m, 3)$,

$$\begin{aligned} \pi_6(Y_m) = \boldsymbol{Z}/m' \cdot j'_*(f_m \circ \lambda_m) \oplus \boldsymbol{Z}/m \cdot j'_*(f_m \circ \sigma) \\ \oplus \boldsymbol{Z}/2 \cdot j'_*(\gamma_m \circ \eta_5) \oplus \boldsymbol{Z}/2 \cdot j'_*(b_m \circ \eta_5). \end{aligned}$$

(iii) *In particular, if* $m = 0$,

$$\begin{aligned} \pi_6(Y_0) = \boldsymbol{Z}/6 \cdot j'_*(i \circ \eta_2 \circ \omega) \oplus \boldsymbol{Z}/2 \cdot j'_*(i_4 \circ \eta_4^2) \\ \oplus \boldsymbol{Z}/2 \cdot j'_*([i_2, i_4 \circ \eta_4]) \oplus \boldsymbol{Z} \cdot j'_*([i_2 \circ \eta_2, i_4]). \end{aligned}$$

PROOF. Consider the exact sequence

$$\pi_7(Y_m, L_m) \xrightarrow{\partial_7} \pi_6(L_m) \xrightarrow{j'_*} \pi_6(Y_m) \longrightarrow 0. \tag*{$(*)_1$}$$

Since $[\eta_2^3, \iota_2] = 0$, using Proposition 5.1 we have

$$\begin{aligned}
\partial_7([\beta'_m, i]_r) &= -[mb_m + i_*(\eta_2^3), i] = -m[b_m, i] - i \circ [\eta_2^3, \iota_2] \\
&= -m(x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5) \\
&= \begin{cases} 0 & \text{if } \nu_2(m) \geq 3 \text{ or } m = 0, \\ m(f_m \circ \sigma) \neq 0 & \text{if } 1 \leq \nu_2(m) \leq 2. \end{cases}
\end{aligned}$$

Moreover, since the order of $i_*(\eta_2 \circ \omega)$ is $(12, m)$,

$$\begin{aligned}
\partial_7(\beta'_m \circ \eta'_5) =& (mb_m + i_*(\eta_2^3)) \circ \eta_5 = m(b_m \circ \eta_5) + i_*(\eta_2^4) = i_*(\eta_2^4) \\
=& 2i_*(\eta_2 \circ \nu') = \begin{cases} i_*(\eta_2^4) \neq 0 & \text{if } \nu_2(m) \geq 2 \text{ or } m = 0, \\ 0 & \text{if } \nu_2(m) = 1. \end{cases}
\end{aligned}$$

Hence,

$$\operatorname{Im} \partial_7 = \begin{cases} \boldsymbol{Z}/2 \cdot i_*(\eta_2^4) & \text{if } \nu_2(m) \geq 3 \text{ or } m = 0, \\ \langle m(f_m \circ \sigma) \rangle \oplus \boldsymbol{Z}/2 \cdot i_*(\eta_2^4) \cong (\boldsymbol{Z}/2)^2 & \text{if } \nu_2(m) = 2, \\ \langle m(f_m \circ \sigma) \rangle \cong \boldsymbol{Z}/2 & \text{if } \nu_2(m) = 1, \end{cases} \tag{23}$$

and the assertions easily follow from the exact sequence $(*)_1$. □

COROLLARY 6.2. *If $\partial_7 : \pi_7(Y_m, L_m) \to \pi_6(L_m)$ denotes the boundary operator,*

$$\operatorname{Ker} \partial_7 = \begin{cases} \boldsymbol{Z} \cdot [\beta'_m, i]_r & \text{if } \nu_2(m) \geq 3 \text{ or } m = 0, \\ \langle 2[\beta'_m, i]_r \rangle \cong \boldsymbol{Z} & \text{if } \nu_2(m) = 2, \\ \langle 2[\beta'_m, i]_r \rangle \oplus \boldsymbol{Z}/2 \cdot \beta'_m \circ \eta'_5 \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2 & \text{if } \nu_2(m) = 1. \end{cases}$$

PROOF. This also follows from the proof of Theorem 6.1. □

COROLLARY 6.3.

(i) *If $1 \leq \nu_2(m) \leq 2$, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Y, ε) for any $\varepsilon \in \boldsymbol{Z}/2$.*
(ii) *If $m \equiv 0 \pmod 8$ and $m \geq 0$, there exists some element $\varphi'_m \in \pi_7(Y_m)$ satisfying the condition $j_{2*}(\varphi'_m) = [\beta'_m, i]_r$ and there is the isomorphism $\pi_7(Y_m) = \boldsymbol{Z} \cdot \varphi'_m \oplus \operatorname{Im} j'_*$, where we take $\operatorname{Im} j'_* = \operatorname{Im} [j'_* : \pi_7(L_m) \to \pi_7(X_m)]$ and $j_{2*} : \pi_7(Y_m) \to \pi_7(Y_m, L_m)$ denotes the induced homomorphism.*

PROOF. The assertion (i) follows from Theorem 4.5 and the analogous proof of Corollary 5.4, and (ii) also easily follows from the homotopy exact sequence of the pair (Y_m, L_m) and Corollary 6.2. □

DEFINITION 11. We note that $\pi_6(\mathrm{P}^6(m)) = \boldsymbol{Z}/2 \cdot i''_*(\eta_5)$ and that $\pi_7(\mathrm{P}^6(m)) = \boldsymbol{Z}/2 \cdot \tilde{\eta}_6 \oplus \boldsymbol{Z}/2 \cdot i''_*(\eta_5^2)$ if $m \equiv 0 \pmod 4$, where $i'' : S^5 \to \mathrm{P}^6(m)$ denotes the inclusion. Let $\tilde{q}' \in [Y_m, K(\boldsymbol{Z}, 2)] \cong H^2(Y_m, \boldsymbol{Z}) \cong \boldsymbol{Z}$ denote the map which represents the generator and let $\tilde{Y}_m$ be the homotopy fiber of the map $\tilde{q}'$. Then $\tilde{Y}_m$ is a 2-connective covering of Y_m and there is a fibration sequence

$$S^1 \to \tilde{Y}_m \to Y_m. \tag{24}$$

When $m \equiv 0 \pmod 8$ and $m \geq 8$, we define the spaces T_m, T'_m, Q_m, Q'_m by

$$\begin{cases} T_m = \mathrm{P}^4(m) \cup_{i'_*(\eta_3^3)} e^7, & T'_m = \mathrm{P}^4(m) \cup_{i'_*(\eta_3^3)+f'_m \circ \tilde{\eta}_3 \circ \eta_5} e^7, \\ Q_m = \mathrm{P}^4(m) \vee \mathrm{P}^6(m) \cup_{\theta_1} e^7, & Q'_m = \mathrm{P}^4(m) \vee \mathrm{P}^6(m) \cup_{\theta_2} e^7, \end{cases} \tag{25}$$

where we identify $\pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m)) = \pi_6(\mathrm{P}^4(m)) \oplus \pi_6(\mathrm{P}^6(m))$ and we take

$$\theta_1 = i'_*(\eta_3^3) + i''_*(\eta_5),\ \theta_2 = i'_*(\eta_3^3) + f'_m \circ \tilde{\eta}_3 \circ \eta_5 + i''_*(\eta_5) \in \pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m)).$$

LEMMA 6.4. *If $m \equiv 0 \pmod 8$ and $m \geq 8$, the following isomorphisms hold:*

(i) $\pi_7(T_m) \cong \pi_7(T'_m) \cong \mathbf{Z} \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus (\mathbf{Z}/2)^2$.
(ii) $\pi_7(Q_m) \cong \pi_7(Q'_m) \cong \mathbf{Z} \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^3$.
(iii) $\pi_7(T_m \vee \mathrm{P}^6(m)) \cong \pi_7(T'_m \vee \mathrm{P}^6(m)) \cong \mathbf{Z} \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^4$.

PROOF. (i), (ii): We only show that $\pi_7(Q_m) \cong \mathbf{Z} \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^3$, because a similar method shows the another cases of (i) and (ii). Let $\mathrm{P}^{4,6}(m) = \mathrm{P}^4(m) \vee \mathrm{P}^6(m)$ and $\alpha \in \pi_7(Q_m, \mathrm{P}^{4,6}(m)) \cong \mathbf{Z}$ be the characteristic map of the top cell e^7 in Q_m. Then $\pi_7(Q_m, \mathrm{P}^4(m) \vee \mathrm{P}^6(m)) = \mathbf{Z} \cdot \alpha$ and consider the boundary operator $\partial_k : \pi_k(Q_m, \mathrm{P}^{4,6}(m)) \to \pi_{k-1}(\mathrm{P}^{4,6}(m))$ associated to the exact sequence of the pair $(Q_m, \mathrm{P}^{4,6}(m))$ for $k = 7, 8$.

Since $\partial_7(\alpha) = \theta_1$ is the element of order 2, $\mathrm{Ker}\, \partial_7 = \langle 2\alpha \rangle \cong \mathbf{Z}$. Now we recall the commutative diagram

$$\begin{tikzcd} \pi_8(Q_m, \mathrm{P}^{4,6}(m)) \arrow[r, "\partial_8"] & \pi_7(\mathrm{P}^{4,6}(m)) \\ \pi_8(D^7, S^6) \arrow[u, "\alpha_*", "\cong"'] \arrow[r, "\partial'", "\cong"'] & \pi_7(S^6) = \mathbf{Z}/2 \cdot \eta_6, \arrow[u, "\theta_{1*}"'] \end{tikzcd}$$

the equality $\theta_{1*}(\eta_6) = (i'_*(\eta_3^3) + i''_*(\eta_5)) \circ \eta_6 = i''_*(\eta_5^2) \neq 0 \in \pi_7(\mathrm{P}^6(m))$, and the isomorphism

$$\begin{aligned} \pi_7(\mathrm{P}^{4,6}(m)) \cong& \pi_7(\mathrm{P}^4(m)) \oplus \pi_7(\mathrm{P}^6(m) \oplus [\pi_3(\mathrm{P}^4(m), \pi_5(\mathrm{P}^6(m))] \\ \cong& \pi_7(\mathrm{P}^4(m)) \oplus \pi_7(\mathrm{P}^6(m) \oplus \mathbf{Z}/m. \end{aligned}$$

Then by using Proposition 2.9, there is an isomorphism $\pi_7(\mathrm{P}^{4,6})/\mathrm{Im}\, \partial_8 \cong \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^3$. Hence there is an exact sequence $0 \to \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^3 \to \pi_7(Q_m) \to \mathbf{Z} \to 0$, and we have $\pi_7(Q_m) \cong \mathbf{Z} \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/(m,3) \oplus \mathbf{Z}/m \oplus (\mathbf{Z}/2)^3$.

(iii) Since $\pi_3(T_m) \cong \pi_5(\mathrm{P}^6(m)) \cong \mathbf{Z}/m$, the assertion (iii) easily follows from (i) and the Hilton-Milnor Theorem. $\square$

LEMMA 6.5. *Let Q be the 2-cell complex defined by $Q = S^2 \cup_{\eta_2^3} e^6$.*

(i) $\pi_2(Q) \cong \pi_3(Q) \cong \boldsymbol{Z}, \pi_4(Q) \cong \boldsymbol{Z}/2, \pi_5(Q) = 0$ *and* $\pi_6(Q) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/6$.
(ii) $\pi_7(Q) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$.

PROOF. (i) By using the homotopy exact sequence of the pair (Q, S^2) and the James's isomorphism [**8**, (2.7)], we can show (i) easily. So it remains to show (ii). We remark that we cannot use the James's isomorphism for $\pi_7(Q)$ because the dimension exceeds the range that James's isomorphism holds. So let us consider the 2-connective covering $\tilde{Q}$ of Q. Then it follows from the main result given in [**26**] that there is a homotopy equivalence

$$\tilde{Q} \simeq S^3 \cup_{\eta_3^2} e^6 \vee S^7. \tag{26}$$

Hence, $\pi_7(Q) \cong \pi_7(\tilde{Q}) \cong \pi_7(S^3 \cup_{\eta_3^2} e^6 \vee S^7) \cong \boldsymbol{Z} \oplus \pi_7(S^3 \cup_{\eta_3^2} e^6)$. So it remains to show that $\pi_7(S^3 \cup_{\eta_3^2} e^6) \cong \boldsymbol{Z}/2$. However, if we consider the homotopy exact sequence of the pair $(S^3 \cup_{\eta_3^2} e^6, S^3)$, we can show this easily. $\square$

REMARK. If we use the result of Gray [**4**], we can show that the 8-skeleton of the homotopy fiber of the pinch map $S^2 \cup_{\eta_2^3} e^6 \to S^6$ is $S^2 \vee S^7$ (up to homotopy). By using this fact, we can also show $\pi_7(S^2 \cup_{\eta_2^3} e^6) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$.

THEOREM 6.6. *Let $m \geq 0$ be an integer with $m \equiv 0 \pmod 8$, and let $j_2 : \pi_7(Y_m) \to \pi_7(Y_m, L_m)$ be the induced homomorphism. Then there exists some element $\varphi'_m \in \pi_7(Y_m)$ such that*

$$j_{2*}(\varphi'_m) = [\beta'_m, i]_r \tag{27}$$

and there is an isomorphism

$$\begin{aligned}
\pi_7(Y_m) = \boldsymbol{Z} \cdot \varphi'_m &\oplus \boldsymbol{Z}/4 \cdot j'_*(f_m \circ \widetilde{\nu'}) \oplus \boldsymbol{Z}/2 \cdot j'_*(f_m \circ \sigma \circ \eta_6) \\
&\oplus \boldsymbol{Z}/2 \cdot j'_*(i_*(\eta_2 \circ \omega \circ \eta_6)) \oplus \boldsymbol{Z}/(m,3) \cdot j'_*(f_m \circ \omega_m) \\
&\oplus \boldsymbol{Z}/2 \cdot j'_*(b_m \circ \eta_5^2) \oplus \boldsymbol{Z}/m \cdot j'_*([b_m, i_*(\eta_2)]) \qquad \text{if } m \neq 0,
\end{aligned}$$

$$\begin{aligned}
\pi_7(Y_0) = \boldsymbol{Z} \cdot j'_*(i_4 \circ \nu_4) &\oplus \boldsymbol{Z} \cdot \varphi'_0 \oplus \boldsymbol{Z}/2 \cdot j'_*([i, i_4 \circ \eta_4^2]) \oplus \boldsymbol{Z}/12 \cdot j'_*(i_4 \circ E\omega) \\
&\oplus \boldsymbol{Z}/2 \cdot j'_*([i_*(\eta_2), i_4 \circ \eta_4]) \oplus \boldsymbol{Z}/2 \cdot j'_*([i_*(\eta_2^2), i_4]) \\
&\oplus \boldsymbol{Z}/2 \cdot j'_*(\eta_2 \circ \omega \circ \eta_6) \qquad \text{if } m = 0.
\end{aligned}$$

PROOF. If we remark Corollary 3.5 and Corollary 6.3, it suffices to show that $j'_* : \pi_7(L_m) \to \pi_7(Y_m)$ is injective. First, we assume $m \neq 0$, and consider the 2-connective covering of Y_m. By using the computation of the Serre spectral sequence associated to the fibration (24), we can show that there is a homotopy equivalence

$$\tilde{Y}_m \simeq \mathrm{P}^4(m) \vee \mathrm{P}^6(m) \cup_\theta e^7 \tag{28}$$

for some $\theta \in \pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m)) \cong \pi_6(\mathrm{P}^4(m)) \oplus \pi_6(\mathrm{P}^6(m))$.

In this case, we note that $\pi_6(Y_m) \cong \pi_6(\tilde{Y}_m) \cong \pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m))/\langle\theta\rangle$.

On the other hand, it follows from Proposition 2.6, (15) and Theorem 6.1 that there are isomorphisms

$$\begin{cases} \pi_6(\mathrm{P}^4(m) \vee \mathrm{P}^6(m)) \cong \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^2 \oplus \boldsymbol{Z}/4, \\ \pi_6(Y_m) \cong \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^3. \end{cases}$$

Then if we compare above isomorphisms and consider the 2-components $\pi_6(\mathrm{P}^4(m))_{(2)}$ and $\pi_6(\mathrm{P}^6(m))_{(2)}$, we can write

$$\theta = i_*(\eta_3^3) + \varepsilon_1 \cdot f'_m \circ \tilde{\eta}_3 \circ \eta_5 + \varepsilon_2 \cdot i''_*(\eta_5) \quad \text{for some } \varepsilon_1, \varepsilon_2 \in \boldsymbol{Z}/2.$$

Now we prove that $\varepsilon_2 = 1$. For this purpose, we assume that $\varepsilon_2 = 0$. Then $\tilde{Y}_m$ is homotopy equivalent to $T_m \vee \mathrm{P}^6(m)$ (if $\varepsilon_1 = 0$) or to $T'_m \vee \mathrm{P}^6(m)$ (if $\varepsilon_1 = 1$). Then using Lemma 6.4, we have $\pi_7(Y_m) \cong \pi_7(\tilde{Y}_m) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^4$. So it follows from Corollary 6.3 that

$$\mathrm{Im}\ j'_* = \mathrm{Im}\,[j'_* : \pi_7(L_m) \to \pi_7(Y_m)] \cong \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^4.$$

However, since $\pi_7(L_m) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^3$, the order of Im j'_* is bigger than that of $\pi_7(L_m)$, which is a contradiction. Hence, we have $\varepsilon_2 = 1$. Therefore, $\tilde{Y}_m$ is homotopy equivalent to Q_m (for $\varepsilon_1 = 0$) or to Q'_m (for $\varepsilon_2 = 1$). By using Lemma 6.4, $\pi_7(Y_m) \cong \pi_7(\tilde{Y}_m) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^3$, and Im $j'_* \cong \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/(m,3) \oplus \boldsymbol{Z}/m \oplus (\boldsymbol{Z}/2)^3$. Hence, $\pi_7(L_m) \cong$ Im j'_*. Thus, $j'_* : \pi_7(L_m) \to \pi_7(Y_m)$ is injective, and the case $m \neq 0$ is proved.

Next, consider the case $m = 0$. In this case, we note that $Y_0 = S^4 \vee Q$, where $Q = S^2 \cup_{\eta_2^3} e^6$. Then by using Hilton-Milnor Theorem,

$$\begin{aligned} \pi_7(Y_0) = \pi_7(Q \vee S^4) &\cong \pi_7(Q) \oplus \pi_7(S^4) \oplus [\pi_4(Q), \pi_4(S^4)] \\ &\oplus [\pi_3(Q), \pi_5(S^4)] \oplus [\pi_2(Q), \pi_6(S^4)] \cong \boldsymbol{Z}^2 \oplus \boldsymbol{Z}/12 \oplus (\boldsymbol{Z}/2)^4. \end{aligned}$$

Because $\pi_7(Y_0) = \boldsymbol{Z} \cdot \varphi'_0 \oplus \mathrm{Im}\ j'_*$ (by Corollary 6.3), Im $j'_* \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/12 \oplus (\boldsymbol{Z}/2)^4$, which is also isomorphic to $\pi_7(L_0)$ (by Corollary 3.5). Hence $j'_* : \pi_7(L_m) \to \pi_7(Y_0)$ is injective and the case $m = 0$ is also proved. □

THEOREM 6.7. *If $m \equiv 0$ (mod 8), there exists a space $M^Y_{m,0}$ which is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Y,0)$. However, there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Y,1)$.*

PROOF. Let $M^Y_{m,0}$ denote the mapping cone defined by

$$M^Y_{m,0} = \begin{cases} Y_m \cup_{\varphi'_m} e^8 & \text{if } m \neq 0, \\ Y_0 \cup_{j'_*(i_4 \circ \nu_4) + \varphi'_0} e^8 & \text{if } m = 0. \end{cases} \tag{29}$$

Then it follows from Theorem 4.5 that $M^Y_{m,0}$ is an m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Y,0)$. However, since

$j_{2*}(\varphi) \neq \pm[\beta'_m, i]_r + \beta'_m \circ \eta'_5$ for any $\varphi \in \pi_7(Y_m)$ (by Theorem 6.6), by Theorem 4.5 there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Y, 1)$. □

7. An m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z, 0)$.

Throughout this section, we assume that $m \geq 0$ is an integer such that $m \equiv 0 \pmod 2$. We note that there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z, 1)$ (by Lemma 4.3), and in this section we shall prove that there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type $(Z, 0)$, too.

LEMMA 7.1. *If $m \geq 0$ be an integer such that $m \equiv 0 \pmod 2$, then*

$$[\gamma_m, i] \equiv b_m \circ \eta_5 \neq 0 \mod (2, m/2) i_*(\eta_2 \circ \nu').$$

PROOF. First, we remark that $[\gamma_m, i]$ is at most the element of order 2. (In fact, if $m \equiv 0 \pmod 4$, the order of γ_m is 2 and the this is clear. If $m \equiv 2 \pmod 4$, since $[\eta_2^3, \iota_2] = 0$ and $2\gamma = i_*(\eta_2^3)$, $2[\gamma_m, i] = [i_*(\eta_2^3), i] = i \circ [\eta_2^3, \iota_2] = 0$.)

Now consider the induced homomorphism $i_{L*} : \pi_k(L_m) \to \pi_k(L_m, S^2)$ for $k = 5, 6$. Then because $i_{L*}(\gamma_m) = a_m \circ \eta'_3$ by [**24**, (2.14)],

$$\begin{aligned} i_{L*}([\gamma_m, i]) &= [i_{L*}(\gamma_m), \iota_2]_r \quad \text{(by [\textbf{9}, (2.2)])} \\ &= [a_m \circ \eta'_3, \iota_2]_r = [a_m, \iota_2]_r \circ \eta'_4 \quad \text{(by [\textbf{14}, Lemma 1.6])} \\ &= i_{L*}(b_m) \circ \eta'_4 \quad \text{(by Lemma 2.2)} \\ &= i_{L*}(b_m \circ \eta_5) \quad \text{(by [\textbf{14}, Lemma 1.2])}. \end{aligned}$$

Hence, $[\gamma_m, i] \equiv b_m \circ \eta_5 \mod i_*(\pi_6(S^2))$. Since $[\gamma_m, i]$ and $b_m \circ \eta_5$ are elements of at most order 2, the assertion follows from Corollary 3.4. □

Consider the cofiber sequence

$$S^5 \xrightarrow{mb_m + \gamma_m} L_m \xrightarrow{j''} Z_m \xrightarrow{p'_m} S^6. \tag{30}$$

Consider the boundary operators $\partial_7 : \pi_7(Z_m, L_m) \to \pi_6(L_m)$.

LEMMA 7.2. *If $m \equiv 0 \pmod 2$, $\partial_7(\beta'_m \circ \eta'_5) = \gamma_m \circ \eta_5 \neq 0 \in \pi_6(L_m)$, and*

$$\partial_7([\beta''_m, i]_r) = \begin{cases} [\gamma_m, i] & \textit{if } \nu_2(m) \geq 3, \\ m \cdot (f_m \circ \sigma) + [\gamma_m, i] & \textit{if } 1 \leq \nu_2(m) \leq 2. \end{cases}$$

PROOF. First, we have $\partial_7(\beta''_m \circ \eta'_5) = (mb_m + \gamma_m) \circ \eta_5 = \gamma_m \circ \eta_5$. Similarly, we also obtain:

$$\begin{aligned} \partial_7([\beta''_m, i]_r) &= -[mb_m + \gamma_m, i] = -m[b_m, i] - [\gamma_m, i] \\ &= -m(x_m \cdot f_m \circ \sigma + \varepsilon_m \cdot b_m \circ \eta_5) - [\gamma_m, i] \quad \text{(by Proposition 5.1)} \\ &= \begin{cases} -mx_m \cdot (f_m \circ \sigma) - [\gamma_m, i] & \text{if } 1 \leq \nu_2(m) \leq 2, \\ -[\gamma_m, i] = [\gamma_m, i] & \text{if } \nu_2(m) \geq 3. \end{cases} \end{aligned}$$

Hence, if $\nu_2(m) \geq 3$ the assertion is satisfied. So we assume $1 \leq \nu_2(m) \leq 2$. In this case, since $x_m \in (\boldsymbol{Z}/2m)^{\times}$, $-mx_m = m \in \boldsymbol{Z}/2m$ and we have

$$\partial_7([\beta''_m, i]_r) = -mx_m \cdot (f_m \circ \sigma) - [\gamma_m, i] = m \cdot (f_m \circ \sigma) + [\gamma_m, i]. \qquad \square$$

COROLLARY 7.3. *If $\partial_7 : \pi_7(Z_m, L_m) \to \pi_6(L_m)$ denotes the boundary operator,*

$$\mathrm{Ker}\ \partial_7 = \langle 2[\beta''_m, i]_r \rangle \cong \boldsymbol{Z}.$$

THEOREM 7.4. *Let $m \geq 0$ be an even integer. Then there is no m-twisted $\boldsymbol{C}\mathrm{P}^4$ of type (Z, ε) for any $\varepsilon \in \boldsymbol{Z}/2$.*

PROOF. This easily follows from Theorem 4.5 and Corollary 7.3. $\square$

Now we complete the proof of the main results.

PROOFS OF THEOREMS 1.1 AND 1.2 Theorem 1.1 follows from Theorem 5.8 and Theorem 6.7 and Theorem 1.2 also follows from Theorem 5.8, Theorem 6.7 and Theorem 7.4.

PROOF OF COROLLARY 1.3. Since (iv) is trivial, it suffices to show (i), (ii), (iii).

(i) Assume that $m \equiv 1 \pmod 2$. We note that there is a homotopy equivalence $X_m \cup_{\varphi} e^8 \simeq X_m \cup_{-\varphi} e^8$ for any $\varphi \in \pi_7(X_m)$. Then it follows from Theorem 4.5 and Corollary 5.11 that we obtain

$$\begin{aligned} 1 \leq \mathrm{card}(\mathscr{M}_m) &\leq \mathrm{card}(\{\varphi \in \pi_7(X_m) : j_{1*}(\varphi) = [\beta_m, i]_r + \beta_m \circ \eta'_5\}) \\ &= \mathrm{card}(\mathrm{Im}\ [j_* : \pi_7(L_m) \to \pi_7(X_m)]) = m(m,3). \end{aligned}$$

(ii), (iii): We suppose $m \equiv 0 \pmod 8$. Since $\mathrm{card}(\mathscr{M}_m) \geq 3$ is clear, it remains to consider the upper bound of $\mathrm{card}(\mathscr{M}_m)$. First, we consider the case $m \neq 0$. Define the set $\mathscr{F}^X_m$ and $\mathscr{F}^Y_m$ by

$$\begin{cases} \mathscr{F}^X_m = \{\varphi \in \pi_7(X_m) : j_{1*}(\varphi) = [\beta_m, i]_r + \varepsilon \cdot \beta_m \circ \eta'_5, \varepsilon \in \boldsymbol{Z}/2\}, \\ \mathscr{F}^Y_m = \{\varphi \in \pi_7(Y_m) : j_{2*}(\varphi) = [\beta'_m, i]_r + \varepsilon \cdot \beta'_m \circ \eta'_5, \varepsilon \in \boldsymbol{Z}/2\}. \end{cases}$$

In this case, a similar method (as in (i)) also shows

$$\begin{aligned} \mathrm{card}(\mathscr{M}_m) \leq \mathrm{card}(\mathscr{F}^X_m) + \mathrm{card}(\mathscr{F}^Y_m) &= \mathrm{card}(\mathrm{Im}\ j_*) + \mathrm{card}(\mathrm{Im}\ j'_*) \\ &= 2^5 \cdot 3m(m,3). \end{aligned}$$

Next, consider the case $m = 0$. Then, for $\varphi \in \pi_7(X_0)$, $M = X_0 \cup_{\varphi} e^8$ is a 0-twisted $\boldsymbol{C}\mathrm{P}^4$ if and only if $\varphi \equiv \pm j_4 \circ \nu_4 \pm [j_2, j_6] \bmod \mathrm{Tor}(\pi_7(X_0))$. Similarly, for $\psi \in \pi_7(Y_0)$, $M = Y_0 \cup_{\psi} e^8$ is a 0-twisted $\boldsymbol{C}\mathrm{P}^4$ if and only if $\psi \equiv \pm j'_*(i_4 \circ \nu_4) \pm \varphi'_0 \bmod \mathrm{Tor}(\pi_7(Y_0))$. Hence,

$$3 \leq \mathrm{card}(\mathscr{M}_m) \leq 2(\mathrm{card}(\mathrm{Tor}(\pi_7(X_0))) + \mathrm{card}(\mathrm{Tor}(\pi_7(Y_0)))) = 2^7 \cdot 3^2. \qquad \square$$

8. Spivak normal fibrations.

We recall the standard surgery theory ([**3**], [**16**]). Let SG_n be the topological monoid consisting of all self-homotopy equivalence of S^n with degree one. Similarly, let STop_n denote the topological monoid of orientation preserving homeomorphisms $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $f(0) = 0$. A suspension induces natural inclusions of monoids, $SG_n \to SG_{n+1}$ and $\mathrm{STop}_n \to \mathrm{STop}_{n+1}$. We denote by SG and STop the topological monoids defined by $SG = \lim_n SG_n$ and $\mathrm{STop} = \lim_n \mathrm{STop}_n$. A natural inclusion $\mathrm{STop} \to SG$ induces a map of classifying spaces, $Bi : B\mathrm{STop} \to BSG$.

PROPOSITION 8.1. *If M is a twisted $\boldsymbol{C}\mathrm{P}^n$, it has the homotopy type of $2n$ dimensional closed topological manifolds.*

PROOF. If we choose a sufficiently larger number N, there is a unique (up to homotopy equivalence) Spivak normal fibration over the base space M with fiber S^{N-1} ([**3**]). By a result of Stasheff, this is classified by a map $f_M : M \to BSG$. Let us consider whether f_M lifts to BSTop or not. In this case, the obstructions lie in the groups $H^k(M, \pi_{k-1}(SG/\mathrm{STop}))$. Since $\pi_k(SG/\mathrm{STop}) = 0$ if $k \equiv 1 \pmod 2$ ([**16**]), $H^k(M, \pi_{k-1}(SG/\mathrm{STop})) = 0$ for any $k \geq 1$. So f_M can be factorized through BSTop. It follows from a theorem of Browder ([**16**]) that there is a closed topological $2n$ dimensional manifold L such that $M \simeq L$ (homotopy equivalence). □

REMARK. Let M be an m-twisted $\boldsymbol{C}\mathrm{P}^4$ and $f : M \to BSG$ the classifying map of its Spivak spherical fibration. Because $\pi_k(SG/SO) = 0$ for any $k \in \{1, 3, 5, 7\}$ ([**16**]), f can be factorized through BSO. So if we can compute the surgery obstruction, we can determine whether M has the homotopy type of 8 dimensional closed smooth manifolds or not. It seems very interesting to study this problem.

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