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Generic smooth maps with sphere fibers

Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

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Abstract. In this paper, we study various topological properties of generic smooth maps between manifolds whose regular fibers are disjoint unions of homotopy spheres. In particular, we show that if a closed 4-manifold admits such a generic map into a surface, then it bounds a 5-manifold with nice properties. As a corollary, we show that each regular fiber of such a generic map of the 4-sphere into the plane is a homotopy ribbon 2-link and that any spun 2-knot of a classical knot can be realized as a component of a regular fiber of such a map.

1. Introduction.

Let $f : M \to N$ be a smooth map between manifolds of dimensions n and p with $n \geq p$. Suppose that M is compact and has no boundary. If f is a submersion, then it is a fiber bundle and the topology of the source manifold M can be studied by using the classical theory of fiber bundles. However, if f is not a submersion, then the topology of such a map is not easy to understand because of the presence of singularities.

The simplest one among the generic singularities is the definite fold point, and a smooth map with only this kind of singularities is called a *special generic map*. This type of maps was first studied by Burlet and de Rham [**1**] and then a systematic study was made by the first named author in [**24**]. One of the most important features of special generic maps is that a regular fiber of such a map is always diffeomorphic to a disjoint union of standard spheres.

In this paper, we carry out a systematic study of the topology of smooth maps whose regular fibers are unions of homotopy spheres. Such maps are said to be *spherical* or have *sphere fibers*. This is a generalization of the class of special generic maps. The smooth maps which we consider will have (not necessarily definite) fold points and cusp points as their singularities. Since these singularities are more complicated than the definite fold singularity, the same argument cannot be directly applied. In this paper, we mainly concentrate on such maps of 4-dimensional manifolds into surfaces. We also assume that they are generic, or more precisely C^∞ stable.

The paper is organized as follows. In §2 we recall some basic concepts necessary for the global study of generic maps. One of the most important tools is the *Stein factorization* or the *quotient space*. For a special generic map this quotient space is a

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smooth manifold with boundary, and this fact enabled us to obtain various interesting properties of such maps in [**24**]. In our case of generic maps, the quotient space is a 2-dimensional polyhedron, and its local forms were classified by Porto and Furuya [**21**]. As an easy application, we give an Euler characteristic formula for generic maps of sphere fibers.

In §3, before going to the 4-dimensional case, we study spherical Morse functions on 3-manifolds. We give a complete list of those closed 3-manifolds which admit such functions. We will see that this list completely coincides with that of those closed 3-manifolds which admit special generic maps into the plane. This is not a coincidence, and we will explain the reason using orthogonal projections of the plane onto the line. As a byproduct, we give a new proof of a result of Burlet and de Rham [**1**].

In §4 we first state our main theorem (Theorem 4.1): for a generic map $f : M \to N$ of a 4-manifold into a surface with sphere fibers, the quotient map $q_f : M \to W_f$ in its Stein factorization can be extended to a map $r_f : V_f \to W_f$ from a 5-manifold V_f with $\partial V_f = M$ such that the fibers of r_f are all contractible and that r_f is a homotopy equivalence. This result has various interesting corollaries. For example, this shows that those 4-manifolds which admit such spherical maps should be null-cobordant. Furthermore, the quotient map $q_f : M \to W_f$ induces an isomorphism between the fundamental groups (this result itself has been obtained by the second named author in [**27**]). We also show that if the source manifold M is the 4-sphere, then every regular fiber should be a homotopy ribbon 2-link in the sense of [**2**]. Moreover, by combining a result obtained in [**12**], we give a characterization of the standard 4-sphere in terms of spherical maps.

In §5, we prove the main theorem. We first decompose the quotient space W_f into some pieces by using its simplicial structure and then construct 5-dimensional manifolds piece by piece. Finally we glue them together to obtain the desired 5-manifold.

In §6, we give a systematic method to construct spherical smooth maps. As an example, we show that every spun 2-knot in S^4 of a classical knot in S^3 can be realized as a component of a regular fiber of a generic spherical map of S^4 into the plane.

Throughout the paper, we mainly work in the smooth category. The homology and cohomology groups are with integer coefficients unless otherwise indicated. For a topological space X, id_X denotes the identity map of X. For an n-dimensional closed connected manifold X and a positive integer k, $X_{(k)}$ denotes the manifold X with k open n-disks removed, where we assume that the closures of the n-disks do not intersect with each other. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

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2. Preliminaries.

In this section, we recall some fundamental definitions and properties of generic

[1]Some of the results in this paper were obtained in the second author's PhD thesis [**27**] under the supervision of Yukio Matsumoto.

smooth maps between manifolds. We also specify the class of maps which we study in this paper.

Let M and N be smooth manifolds of dimensions n and p respectively, and $f : M \to N$ a smooth map. We set

$$S(f) = \{x \in M \mid \operatorname{rank} df_x < \min\{n, p\}\},$$

which is called the *singular set* of f. A point of $S(f)$ is called a *singular point* of f.

DEFINITION 2.1. Suppose that $n \geq p$. A singular point $x \in S(f)$ of f is called a *fold singular point* (or a *fold point*) if there exist local coordinates $(x_1, x_2, \dots, x_n)$ around x and $(y_1, y_2, \dots, y_p)$ around $f(x)$ such that f has the form

$$y_i \circ f = \begin{cases} x_i, & 1 \leq i \leq p-1, \\ -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2, & i = p, \end{cases}$$

for some λ with $0 \leq \lambda \leq [(n-p+1)/2]$, where for $\alpha \in \boldsymbol{R}$, $[\alpha]$ denotes the greatest integer not exceeding α (see, for example, [**3**], [**4**], [**5**]). We call λ the (reduced) *index* of x. We say that x is a *definite fold singular point* (or a *definite fold point*) if $\lambda = 0$; otherwise, it is an *indefinite fold singular point* (or an *indefinite fold point*).

A smooth map $f : M \to N$ is called a *fold map* if $S(f)$ consists only of fold singular points. A fold map is called a *special generic map* if $S(f)$ consists only of definite fold singular points (see [**1**], [**24**]).

For example, a nondegenerate critical point of a smooth function on a manifold is a fold singular point and its reduced index is equal to $\min\{\lambda, n-\lambda\}$, where λ is the index as a nondegenerate critical point of a function and n is the dimension of the manifold. In particular, a Morse function $f : M \to \boldsymbol{R}$ on a manifold M is always a fold map.

Note that a fold singular point can be characterized in terms of its jets just as a nondegenerate critical point of a smooth function is characterized in terms of the associated Hessian matrix. Accordingly, a fold map can be characterized in terms of its jet extension as follows.

Let Σ_r denote the submanifold of the 1-jet bundle $J^1(M, N)$ consisting of the jets of corank r, where the *corank* of a smooth map f between manifolds of dimensions n and p means $\min\{n, p\} - \operatorname{rank} df$. Then a smooth map $f : M \to N$ with $n \geq p$ is a fold map if and only if the following conditions are satisfied.

(1) The 1-jet extension $j^1 f\colon M \to J^1(M, N)$ does not hit Σ_r with $r \geq 2$.
(2) The smooth map $j^1 f\colon M \to J^1(M, N)$ is transverse to Σ_1 so that $S(f) = (j^1 f)^{-1}(\Sigma_1)$ is a smooth submanifold of M (of dimension $p-1$).
(3) The restriction $f|_{S(f)} : S(f) \to N$ is an immersion.

(For details, see [**7**, Chapter III, §4]. Note that a fold map is called a *submersion with folds* in [**7**]).

Fold singularities are the simplest, i.e. have the smallest codimension, among the generic singularities of corank one, i.e. among the Morin singularities [**20**].

Let us now consider smooth maps into surfaces.

Definition 2.2. Let M be a manifold of dimension $n \geq 2$ and $f : M \to N$ a smooth map into a surface N. A singular point $x \in S(f)$ of f is called a *cusp singular point* (or a *cusp point*) if there exist local coordinates $(x_1, x_2, \ldots, x_n)$ around x and (y_1, y_2) around $f(x)$ such that f has the form

$$y_i \circ f = \begin{cases} x_1, & i = 1, \\ x_1 x_2 + x_2^3 \pm x_3^2 \pm \cdots \pm x_n^2, & i = 2. \end{cases}$$

A smooth map $f : M \to N$ is called a *generic map* if $S(f)$ consists only of fold and cusp singular points.

Note that the set of generic maps is open and dense in the mapping space $C^\infty(M, N)$ equipped with the Whitney C^∞ topology (for example, see [**7**]). It is known that for a generic map $f : M \to N$, the singular set $S(f)$ is a 1-dimensional closed submanifold of M, that the cusp points are discrete, and that $f|_{S(f)}$ is an immersion except exactly at cusp singular points.

Recall that a smooth map $f : M \to N$ between smooth manifolds is a (C^∞) *stable map* if there exists an open neighborhood U of f in the mapping space $C^\infty(M, N)$ such that every $g \in U$ is C^∞ right-left equivalent to f; i.e. there exist diffeomorphisms $\Phi : M \to M$ and $\varphi : N \to N$ such that $g \circ \Phi = \varphi \circ f$.

The following characterization of stable maps into surfaces is well-known (for example, see [**7**]).

Proposition 2.3. *Let $f : M \to N$ be a proper smooth map of an n-dimensional manifold M with $n \geq 2$ into a surface N. Then f is stable if and only if f is generic and the following two conditions are satisfied.*

(1) *For a cusp point $x \in S(f)$, we have $f^{-1}(f(x)) \cap S(f) = \{x\}$.*
(2) *The map $f|_{(S(f) \setminus \{\text{cusp points}\})}$ is an immersion with normal crossings.*

Let us recall the following notion of a Stein factorization, which will play an essential role in this paper.

Definition 2.4. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')$ and x and x' belong to the same connected component of a fiber of f. We define $W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation and $q_f : M \to W_f$ the quotient map. Then it is easy to see that there exists a unique continuous map $\bar{f} : W_f \to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {}_{q_f}\searrow \quad \nearrow_{\bar{f}} & \\ & W_f & \end{array}$$

commutes. The space W_f or the above commutative diagram is called the *Stein factorization* of f (see [**15**]).

It is known that if f is a proper stable map, then W_f is a polyhedron and all the maps appearing in the above diagram are triangulable (for details, see [**9**]).

The Stein factorization is a very useful tool for studying topological properties of fold maps or generic maps. For example, for special generic maps, the following is known (see, for example, [**1**], [**24**]).

LEMMA 2.5. *Let $f : M \to N$ be a proper special generic map of an n-dimensional manifold into a p-dimensional manifold with $n > p$. Then we have the following.*

(1) *The quotient space W_f admits a structure of a smooth p-dimensional manifold with boundary such that $\bar{f} : W_f \to N$ is an immersion.*
(2) *The quotient map $q_f : M \to W_f$ is a smooth map such that $q_f|_{S(f)}$ is a diffeomorphism onto ∂W_f.*
(3) *For each connected component W of W_f with $\partial W \neq \varnothing$, $q_f|_{q_f^{-1}(\mathrm{Int}\, W)} : q_f^{-1}(\mathrm{Int}\, W) \to \mathrm{Int}\, W$ is a smooth fiber bundle with fiber the standard $(n-p)$-dimensional sphere.*

Furthermore, for stable maps of closed 3-manifolds into the plane, the local structures of their Stein factorizations have been completely determined (see [**14**], [**15**]). See also [**12**] for stable maps of higher dimensional manifolds into the plane.

Now let us specify the class of maps which we consider in this paper.

DEFINITION 2.6. Let $f : M \to N$ be a proper smooth map between smooth manifolds of dimensions n and p with $n \geq p$. For $y \in f(M) \setminus f(S(f))$, $f^{-1}(y)$ is a smooth submanifold of M of dimension $n-p$. If every component of $f^{-1}(y)$ is homotopy equivalent to the $(n-p)$-dimensional sphere S^{n-p} for all y, then we say that f is *spherical* (or f has *sphere fibers*). Note that a proper special generic map which has no submersion component is always spherical (for example, see [**1**], [**21**], [**24**]). Furthermore, if $\dim M = \dim N + 1$, then a proper smooth map $f : M \to N$ is always spherical.

In this paper, we mainly study spherical maps of 4-dimensional manifolds into surfaces. For the Stein factorizations of such maps we have the following lemma, which is a direct consequence of a result of Porto and Furuya [**21**], and which plays an essential role in this paper. For a detailed proof, see [**6**]. (One can also find a detailed argument for general dimensions in [**12**].) Note also that this can be considered as a generalization of the description due to Kushner, Levine and Porto [**14**], [**15**] for the Stein factorizations of stable maps of 3-manifolds into the plane.

LEMMA 2.7. *Let $f : M \to N$ be a proper stable map with sphere fibers of a 4-manifold into a surface. Then W_f has the structure of a 2-dimensional polyhedron, and for every point $x \in M$, $q_f(x) \in W_f$ has one of the regular neighborhoods as depicted in Figure 1.*

We call a point $q_f(x) \in W_f$ whose neighborhood is as in the last figure of Figure 1 a *trident*.

To end this section, let us give an immediate application of the above lemma. In the following, χ will denote the Euler characteristic.

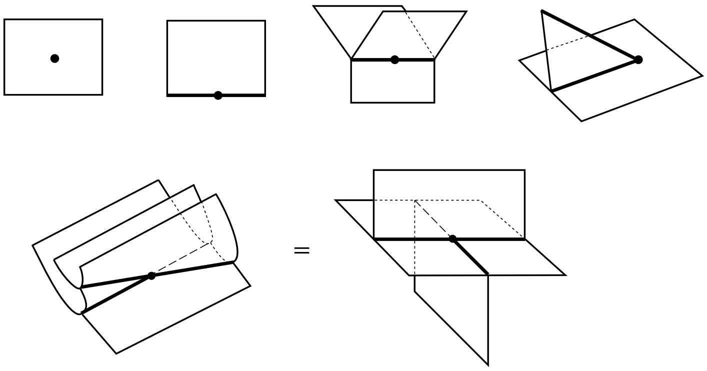

Figure 1. Neighborhoods of $q_f(x) \in W_f$.

Proposition 2.8. *Let $f : M \to N$ be a spherical stable map of a closed 4-manifold into a surface. Then we have $\chi(M) = 2\chi(W_f)$.*

Proof. Let us consider the decomposition of $\Sigma = q_f(S(f)) \subset W_f$, $\Sigma = (C \cup T) \cup (F_d \cup F_i)$, where C is the q_f-image of cusp points, T is the set of tridents, F_d is the q_f-image of definite fold points, and F_i is the q_f-image of indefinite fold points minus T. Note that C and T are finite set of points and that F_d and F_i are finite unions of open arcs and circles, where the end points of the open arcs are in $C \cup T$.

Let us introduce the following notation:

v_c^0 : number of cusp points,
v_t^0 : number of tridents,
v_d^1 : number of open arcs in F_d,
v_{cc}^1 : number of open arcs γ in F_i such that $\overline{\gamma} \subset F_i \cup C$,
v_{ct}^1 : number of open arcs γ in F_i such that $\overline{\gamma} \cap C \neq \varnothing \neq \overline{\gamma} \cap T$,
v_{tt}^1 : number of open arcs γ in F_i such that $\overline{\gamma} \subset F_i \cup T$.

It is easy to see that

$$v_c^0 = 2v_d^1 = 2v_{cc}^1 + v_{ct}^1, \tag{2.1}$$

$$4v_t^0 = v_{ct}^1 + 2v_{tt}^1. \tag{2.2}$$

Let $N(\Sigma)$ be a regular neighborhood of $\Sigma = (C \cup T) \cup (F_d \cup F_i)$ in W_f (for a detailed description of $N(\Sigma)$, see the proof of Theorem 4.1 in §5). Then we see easily that

$$\chi(N(\Sigma)) = \chi(\Sigma) = v_c^0 + v_t^0 - v_d^1 - v_{cc}^1 - v_{ct}^1 - v_{tt}^1. \tag{2.3}$$

On the other hand, by carefully examining the fibers of q_f over the points of Σ, we see that

$$\begin{aligned}\chi\big(q_f^{-1}(N(\Sigma))\big) &= 2v_c^0 + 4v_t^0 - v_d^1 - 3v_{cc}^1 - 3v_{ct}^1 - 3v_{tt}^1 \\ &= 2\big(v_c^0 + v_t^0 - v_d^1 - v_{cc}^1 - v_{ct}^1 - v_{tt}^1\big) \\ &\quad + \frac{1}{2}v_{ct}^1 + v_{tt}^1 + v_{cc}^1 + \frac{1}{2}v_{ct}^1 - v_{cc}^1 - v_{ct}^1 - v_{tt}^1 \\ &= 2\chi(N(\Sigma))\end{aligned}$$

holds by (2.1), (2.2) and (2.3). Since f has sphere fibers, we also have

$$\chi(q_f^{-1}(\overline{W_f \setminus N(\Sigma)})) = 2\chi(\overline{W_f \setminus N(\Sigma)}).$$

Hence, we have the conclusion, since $\chi(M) = \chi(q_f^{-1}(N(\Sigma))) + \chi(q_f^{-1}(\overline{W_f \setminus N(\Sigma)}))$ and $\chi(W_f) = \chi(N(\Sigma)) + \chi(\overline{W_f \setminus N(\Sigma)})$. □

Later we will give another proof of the above proposition (see Remark 4.3).

3. Spherical Morse functions on 3-manifolds.

In this section, we give a characterization of those closed 3-manifolds which admit spherical fold maps into $\boldsymbol{R}$, or equivalently spherical Morse functions. As a corollary, we will give a new proof of a result of Burlet and de Rham [**1**] about special generic maps of closed 3-manifolds into $\boldsymbol{R}^2$.

In the following, $S^1 \tilde{\times} S^2$ denotes the total space of the unique nontrivial (and hence nonorientable) S^2-bundle over S^1.

Recall that a Morse function on a closed manifold is stable if and only if the values at its critical points are all distinct (for example, see [**7**]). Such a function is called a *stable Morse function*.

Proposition 3.1. *A connected closed 3-manifold M admits a spherical stable Morse function $f : M \to \boldsymbol{R}$ if and only if it is diffeomorphic to*

$$(\sharp^k(S^1 \times S^2))\sharp(\sharp^\ell(S^1 \tilde{\times} S^2)) \tag{3.1}$$

for some $k, \ell \geq 0$, where the connected sum over the empty set is understood to be the 3-sphere S^3.

Proof. It is easy to see that the 3-manifolds (3.1) admit spherical stable Morse functions. For example, see §6. See also the proof of Theorem 3.2 below.

Let $f : M \to \boldsymbol{R}$ be a spherical stable Morse function on a connected closed 3-manifold M, i.e. f is a stable Morse function whose regular fiber always consists of a finite disjoint union of 2-spheres. Let $c_1 < c_2 < \cdots < c_\ell$ be its critical values. Take real numbers t_i, $i = 0, 1, \ldots, \ell+1$, such that $t_0 < c_0 < t_1 < c_1 < t_2 < \cdots < t_\ell < c_\ell < t_{\ell+1}$ and set $M_i = f^{-1}[t_0, t_i]$, $i = 1, 2, \ldots, \ell+1$. Since f is spherical, M_i is a compact 3-manifold such that each connected component of ∂M_i is diffeomorphic to the 2-sphere. Let us denote by $\widehat{M_i}$ the closed 3-manifold obtained by attaching 3-disks to M_i along all the boundary components.

Let us show, by induction on i, that each connected component of $\widehat{M}_i$ is diffeomorphic to a 3-manifold as in (3.1).

For $i = 1$, this is clear, since M_1 is diffeomorphic to the 3-disk by the standard Morse theory.

Suppose that each connected component of $\widehat{M}_i$ is diffeomorphic to a 3-manifold as in (3.1). First note that M_{i+1} is obtained by attaching $f^{-1}[t_i, t_{i+1}]$ to M_i along the union of 2-spheres $f^{-1}(t_i)$. Furthermore, since f has exactly one critical point on $f^{-1}[t_i, t_{i+1}]$, it is diffeomorphic to the disjoint union of $S^3_{(1)} \cong D^3$ (or $S^3_{(3)}$) and some copies of $S^3_{(2)} \cong S^2 \times [0,1]$ (for the notation, see the end of §1).

Attaching a copy of $S^3_{(2)}$ along a 2-sphere does not affect the diffeomorphism type of M_i. As to $S^3_{(1)}$, either it does not change the diffeomorphism type of $\widehat{M}_i$ or it adds S^3 to $\widehat{M}_i$.

As to $S^3_{(3)}$, it is attached to M_i either along a 2-sphere or along a union of two 2-spheres. In the former case, the number of boundary components of M_{i+1} is larger than that of M_i by one, but $\widehat{M}_{i+1}$ is diffeomorphic to $\widehat{M}_i$. In the latter case, we have two subcases. If the two attaching 2-spheres belong to the same connected component of M_i, then $\widehat{M}_{i+1}$ is diffeomorphic to $\widehat{M}_i \sharp (S^1 \times S^2)$ or $\widehat{M}_i \sharp (S^1 \widetilde{\times} S^2)$, which can be proved by an elementary 3-manifold theory technique (for example, see [**8**]). If the two attaching 2-spheres belong to different connected components of M_i, then $\widehat{M}_{i+1}$ is diffeomorphic to the closed 3-manifold obtained from $\widehat{M}_i$ by taking connected sum of two of the components.

In any case, each component of $\widehat{M}_{i+1}$ is diffeomorphic to a 3-manifold as in (3.1). Since $\widehat{M}_{\ell+1} = M$, we have the desired conclusion. □

As an interesting corollary, we obtain a new proof of the following characterization of those closed 3-manifolds which admit a special generic map into the plane, originally due to Burlet and de Rham [**1**] (see also [**24**]).

Theorem 3.2 ([**1**]). *A connected closed 3-manifold admits a special generic map into $\boldsymbol{R}^2$ if and only if it is diffeomorphic to*

$$(\sharp^k (S^1 \times S^2)) \sharp (\sharp^\ell (S^1 \widetilde{\times} S^2)) \tag{3.2}$$

for some $k, \ell \geq 0$, where the connected sum over the empty set is understood to be the 3-sphere S^3.

Proof. It is easy to see that the 3-manifolds (3.2) admit special generic maps into $\boldsymbol{R}^2$ (for example, see [**1**] or [**24**]).

Conversely, let $f : M \to \boldsymbol{R}^2$ be a special generic map of a connected closed 3-manifold M into the plane. By choosing an orthogonal projection $\pi : \boldsymbol{R}^2 \to \boldsymbol{R}$ generically, we may assume that the composition $g = \pi \circ f : M \to \boldsymbol{R}$ is a stable Morse function (see [**16**] and [**5**]). Take a regular value $y \in g(M) \smallsetminus g(S(g))$. Since $g = \pi \circ f = \pi \circ \bar{f} \circ q_f$, we have $g^{-1}(y) = q_f^{-1}(\bar{f}^{-1}(\pi^{-1}(y)))$. Note that $\pi^{-1}(y)$ is a line in $\boldsymbol{R}^2$ and that $\bar{f} : W_f \to \boldsymbol{R}^2$ is an immersion of a compact surface with boundary such that $\bar{f}|_{\partial W_f}$ is transverse to $\pi^{-1}(y)$. Therefore, each connected component γ of $\bar{f}^{-1}(\pi^{-1}(y))$ is a properly embedded arc in W_f. Furthermore, the map $q_f|_{q_f^{-1}(\gamma)} : q_f^{-1}(\gamma) \to \gamma \cong [0,1]$ is a Morse function with exactly one maximum and one minimum. Hence $q_f^{-1}(\gamma)$ is diffeomorphic to S^2. Therefore, $g : M \to \boldsymbol{R}$ is a spherical stable Morse function. Hence by Proposition 3.1, we see that M is diffeomorphic to a closed 3-manifold as in (3.2). This completes the proof. $\square$

By an argument similar to that in the above proof, we can also show the following.

PROPOSITION 3.3. *If a closed n-dimensional manifold with $n \geq 3$ admits a special generic map into $\boldsymbol{R}^2$, then it admits a spherical stable Morse function.*

Note that those closed manifolds which admit special generic maps into $\boldsymbol{R}^2$ have been completely determined in [**24**] (see also [**21**]). Probably, we can give a new proof of this result by using Proposition 3.3.

The above construction of a spherical fold map into $\boldsymbol{R}$ will be generalized in a more general setting in §6 (see Proposition 6.1 and Corollary 6.2).

We end this section by posing a problem.

PROBLEM 3.4. Let $f : M \to \boldsymbol{R}$ be an arbitrary spherical stable Morse function on a closed 3-manifold and $\pi : \boldsymbol{R}^2 \to \boldsymbol{R}$ an orthogonal projection. Then, does there exist a special generic map $\widetilde{f} : M \to \boldsymbol{R}^2$ such that $f = \pi \circ \widetilde{f}$?

4. Main results.

In this section, we will state our main result and its consequences.

THEOREM 4.1. *Let $f : M \to N$ be a spherical stable map of a closed 4-manifold into a surface. Then there exist a smooth compact 5-manifold V_f with $\partial V_f = M$ and a continuous map $r_f : V_f \to W_f$ with $r_f|_{\partial V_f} = q_f : M \to W_f$ satisfying the following properties.*

1. *For every point $z \in W_f \setminus q_f(S(f))$, $r_f^{-1}(z)$ is diffeomorphic to D^3.*
2. *The composition $\bar{f} \circ r_f : V_f \to N$ is a smooth submersion.*
3. *There exist a smooth triangulation of V_f and a triangulation of W_f such that r_f is a simplicial map.*
4. *Each fiber of r_f collapses to a point and r_f is a homotopy equivalence.*
5. *The PL 5-manifold V_f collapses to a 2-dimensional polyhedron $\widetilde{W}_f$ such that $r_f|_{\widetilde{W}_f} : \widetilde{W}_f \to W_f$ is a PL homeomorphism outside of a neighborhood of the tridents.*

We will prove Theorem 4.1 in §5.

REMARK 4.2. We can prove a similar result for “simple” fold maps of closed orientable n-dimensional manifolds into $(n-1)$-dimensional manifolds. For details see [**23**, Proposition 3.12].

REMARK 4.3. By using Theorem 4.1, we can easily prove Proposition 2.8, since $\chi(V_f) = \chi(W_f)$ and $2\chi(V_f) - \chi(M) = 0$.

Remark 4.4. For spherical stable Morse functions on closed 3-manifolds, we also have a result similar to Theorem 4.1. Using this, we can prove Proposition 3.1 more easily, since for a spherical stable Morse function $f : M \to \boldsymbol{R}$ on a closed 3-manifold M, V_f is a compact 4-manifold consisting only of 0- and 1-handles, and hence its boundary M is easily seen to be diffeomorphic to a manifold as in (3.1).

Remark 4.5. In Theorem 4.1, the inclusion map $i_M : M \to V_f$ induces a monomorphism $i_M^* : H^i(V_f; \boldsymbol{Z}_2) \to H^i(M; \boldsymbol{Z}_2)$ for $i \leq 2$, since $H^i(V_f, M; \boldsymbol{Z}_2) \cong H_{5-i}(V_f; \boldsymbol{Z}_2)$ vanishes. Therefore, if M is orientable, then so is V_f, since $i_M^* w_1(V_f) = w_1(M)$, where w_i denotes the i-th Stiefel-Whitney class. Furthermore, if M is spin (i.e. if $w_2(M) = 0$), then so is V_f, since $i_M^* w_2(V_f) = w_2(M)$.

Remark 4.6. The smooth map $\bar{f} \circ r_f : V_f \to N$ is a stable map, since $f : M \to N$ is a stable map (for example, see [**25**]). Thus the map $\bar{f} \circ r_f$ is a *nonsingular stable map* in the terminology of [**25**].

As an immediate corollary to Theorem 4.1 and Remark 4.5, we have the following.

Corollary 4.7. *Let M be a closed 4-manifold. If it admits a spherical stable map into a surface, then it is null-cobordant. If, in addition, M is oriented, then it is oriented null-cobordant and its signature vanishes.*

Corollary 4.8. *Let $f : M \to N$ be a spherical stable map of a closed 4-manifold into a surface. Then $q_{f*} : \pi_1(M) \to \pi_1(W_f)$ is an isomorphism.*

Proof. Let $i_M : M \to V_f$ be the inclusion map. Since we have $q_f = r_f \circ i_M$ and r_f is a homotopy equivalence, we have only to show that $i_{M*} : \pi_1(M) \to \pi_1(V_f)$ is an isomorphism. Since V_f collapses to a 2-dimensional polyhedron, it admits a (PL) handlebody decomposition consisting of 0-, 1- and 2-handles. Dualizing the handles, we see that V_f can be obtained from $M \times [0, 1]$ by attaching 3-, 4- and 5-handles along $M \times \{1\}$. Hence, $i_{M*} : \pi_1(M) \to \pi_1(V_f)$ is an isomorphism. This completes the proof. □

Remark 4.9. When f is a spherical fold map, Corollary 4.8 has been obtained by the second named author [**27**] by using a different method.

Corollary 4.10. *Let $f : M \to N$ be a spherical stable map of a closed 4-manifold into a surface. Then $H_2(W_f)$ is free of rank* $\operatorname{rank} H_2(M)/2$.

Proof. We have

$$\chi(W_f) = b_0(W_f) - b_1(W_f) + b_2(W_f),$$
$$\chi(M) = 2b_0(M) - 2b_1(M) + b_2(M),$$

where b_i denotes the rank of the i-th homology group with integer coefficients. By Proposition 2.8, we have $2\chi(W_f) = \chi(M)$, and by Corollary 4.8, we have $b_1(M) = b_1(W_f)$. Furthermore, we have $b_0(M) = b_0(W_f)$. Combining these equalities, we easily obtain $b_2(W_f) = b_2(M)/2$. Furthermore, since W_f is a 2-dimensional polyhedron, its 2nd homology group is torsion free. Hence the required result follows. □

Lemma 4.11. *Let $f : M \to N$ be a spherical stable map of a closed 4-manifold into a surface. Then M is a homotopy 4-sphere if and only if W_f is contractible.*

Proof. Suppose that M is a homotopy 4-sphere. By Corollary 4.8, W_f is a simply connected 2-dimensional polyhedron. Furthermore, by Corollary 4.10, $H_2(W_f) = 0$. Hence, W_f is contractible.

Conversely, if W_f is contractible, then M is simply connected with $b_2(M) = 0$. Hence, M is a homotopy 4-sphere. □

Compare the above result with [**24**, Proposition 4.1].

Let us introduce the following notion, which is originally due to Cochran [**2**].

Definition 4.12. A finite disjoint union of smoothly embedded 2-spheres in S^4 is called a 2-*link*. When it is connected, we also call it a 2-*knot*. A 2-link or a 2-knot is said to be *homotopy ribbon* if it bounds disjoint smoothly embedded 3-disks in D^5 whose exterior has a handlebody decomposition with only 0-, 1- and 2-handles.

Note that any sublink of a homotopy ribbon 2-link is again homotopy ribbon, which is a direct consequence of the above definition.

Corollary 4.13. *Let $f : S^4 \to N$ be a spherical stable map into a surface. Then for every $y \in f(S^4) \setminus f(S(f))$, $f^{-1}(y)$ is a homotopy ribbon 2-link in S^4.*

Proof. Let V_f be the 5-manifold as in Theorem 4.1 associated with the spherical stable map f. By Lemma 4.11, W_f is contractible, and hence so is V_f. Therefore, V_f is diffeomorphic to the 5-dimensional disk D^5 (for example, see [**17**]).

By Theorem 4.1, $r_f^{-1}(z)$ is a 3-disk for every $z \in W_f \setminus q_f(S(f))$. Therefore, $f^{-1}(y) \subset S^4$ bounds the disjoint union of 3-disks $r_f^{-1}(\bar{f}^{-1}(y))$ in $V_f \cong D^5$. Furthermore, the exterior of the 3-disks collapses to a 2-dimensional polyhedron of the form $\widetilde{W_f} \setminus \operatorname{Int} U$, where U is a small regular neighborhood of the finite set of points in $\widetilde{W_f}$ corresponding to $\bar{f}^{-1}(y)$ in W_f. Therefore, the exterior of the 3-disks admits a handlebody decomposition consisting only of 0-, 1- and 2-handles. Hence $f^{-1}(y) \subset S^4$ is a homotopy ribbon 2-link. □

For explicit examples, see §6.

Remark 4.14. For an arbitrary 2-link L in S^4, there exists a stable map $f : S^4 \to \boldsymbol{R}^2$ (not necessarily with sphere fibers) and a regular value y such that $f^{-1}(y)$ contains L. This can be seen by first constructing a map on a tubular neighborhood of L, extending it arbitrarily to the whole S^4, and then approximating it by a stable map.

Remark 4.15. In the situation of Corollary 4.13, let $\alpha : [0,1] \to \boldsymbol{R}^2$ be a smooth embedding transverse to f such that $\alpha(0) = y$, $\alpha(1) \notin f(S^4)$ and α does not pass through the image of the cusp points or the double points of $f|_{S(f)}$. Then $F = f^{-1}(\alpha([0,1]))$ is a compact orientable 3-manifold embedded in S^4 such that $\partial F = f^{-1}(y)$. Since the map $g = \alpha^{-1} \circ f|_F : F \to [0,1]$ is a stable Morse function whose regular fibers are all disjoint unions of 2-spheres, we can show, by an argument similar to that in the proof of Proposition 3.1, that each connected component of $\widehat{F}$ is diffeomorphic to S^3 or to the connected sum of some copies of $S^1 \times S^2$, where $\widehat{F}$ is the closed 3-manifold obtained by attaching 3-disks to F along all the boundary components. In particular, if $f^{-1}(y)$ is connected, then it bounds $(\sharp^k(S^1 \times S^2))_{(1)}$ in S^4 as a Seifert hypersurface for some $k \geq 0$. However, we do not know if $f^{-1}(y)$ is a ribbon 2-knot in general. See [**31**], [**10**].

In [**12**, §7], it has been shown the following. Let $f : M \to \boldsymbol{R}^2$ be a spherical stable fold map of a closed simply connected 4-manifold. If $H_2(W_f) = 0$ and W_f contains no trident, then M is diffeomorphic to the standard 4-sphere. Hence, by Corollary 4.10, we have the following.

Corollary 4.16. *Let $f : \Sigma^4 \to \boldsymbol{R}^2$ be a spherical stable fold map of a homotopy 4-sphere Σ^4 into the plane. If W_f contains no trident, then Σ^4 is diffeomorphic to the standard 4-sphere S^4.*

Remark 4.17. Let $S^2 \tilde{\times} S^2$ be the total space of the unique nontrivial S^2-bundle over S^2 (for example, see [**11**]), and set $M = (\sharp^k(S^2 \times S^2))\sharp(\sharp^\ell(S^2 \tilde{\times} S^2))$ for some $k, \ell \geq 0$. (We will see in Example 6.6 that M admits a spherical stable fold map into the plane.) Suppose that we are given a spherical stable map $f : M \to N$ into a surface. Then we can show that the 5-manifold V_f is diffeomorphic to $(\natural^k(S^2 \times D^3))\natural(\natural^\ell(S^2 \tilde{\times} D^3))$, where $S^2 \tilde{\times} D^3$ is the total space of the unique nontrivial D^3-bundle over S^2. This follows from a result of Kreck [**13**] on 5-dimensional h-cobordisms together a result of Wall [**28**] on self-diffeomorphisms of M.

We end this section by posing a problem.

Problem 4.18. Let M be the Moishezon-Teicher surface with zero signature [**18**], [**19**]. It is known that it is homeomorphic to the connected sum of (a big number of) copies of $S^2 \times S^2$, but not diffeomorphic to it. Does M admit a spherical stable map into a surface? Or more generally, if a simply connected closed 4-manifold admits a spherical stable map into a surface, then is it diffeomorphic to the connected sum of some copies of $S^2 \times S^2$ and some copies of $S^2 \tilde{\times} S^2$?

5. Proof of Theorem 4.1.

In this section, we will prove Theorem 4.1. For this, we will first consider a decomposition of W_f into certain nice pieces and then construct a 5-manifold together with a map corresponding to r_f for each piece. Finally, we will piece them together to get the required 5-manifold V_f and the map $r_f : V_f \to W_f$.

Proof of Theorem 4.1. We divide the proof into five steps as follows.

Step 1. Trident.

Let $t \in W_f$ be a trident. We take its regular neighborhood $N(t)$ in W_f as in Figure 2. More precisely, $N(t)$ is a component of $\bar{f}^{-1}(N(\bar{f}(t)))$ containing t, where $N(\bar{f}(t)) \cong I \times J$ ($I = J = [0, 1]$) is a neighborhood of $\bar{f}(t)$ in N as depicted in Figure 3 (for a more detailed description, see [**15**, §1.2, Proposition 1]). Let $p : N(t) \to I$ (resp. $h : N(t) \to J$) be the composition of $\bar{f}|_{N(t)} : N(t) \to N(\bar{f}(t))$ and the projection $N(\bar{f}(t)) \to [0, 1]$ to the first (resp. second) factor as in Figure 2.

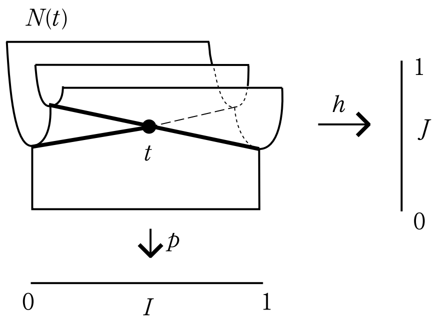

Figure 2. Neighborhood of a trident in W_f.

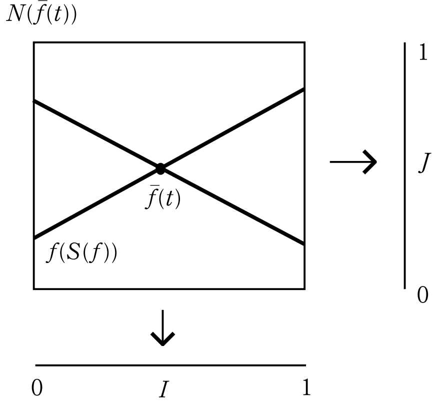

Figure 3. Neighborhood $N(\bar{f}(t))$ of $\bar{f}(t)$ in N.

Then we see that $H = p \circ q_f|_{q_f^{-1}(N(t))} : q_f^{-1}(N(t)) \to I$ is a submersion (or more precisely, a trivial bundle), and that for each $s \in I$, $L_s = H^{-1}(s)$ is diffeomorphic to $S^3_{(4)}$. Furthermore,

$$f_s = h \circ q_f|_{L_s} : L_s \to J, \quad s \in I, \tag{5.1}$$

defines a one parameter family of Morse functions such that each f_s has exactly two critical points of index 2. This family of Morse functions corresponds to exchanging the two critical values.

We see easily that the family of Morse functions (5.1) can be extended to a family of submersions $\widetilde{f}_s : \widetilde{L}_s \to J$, $s \in I$, such that

(1) for each regular value $u \in J$ of f_s, $\widetilde{f}_s^{-1}(u)$ is a disjoint union of one, two or three 3-disks,
(2) $\widetilde{f}_s^{-1}(0)$ is a 3-disk and $\widetilde{f}_s^{-1}(1)$ is a disjoint union of three 3-disks,
(3) $\widetilde{L}_s$ is a smooth 4-manifold with corners along $\partial(\widetilde{f}_s^{-1}(0))$ and $\partial(\widetilde{f}_s^{-1}(1))$,
(4) $\widetilde{L}_s$ is diffeomorphic to D^4 after the corners being smoothed.

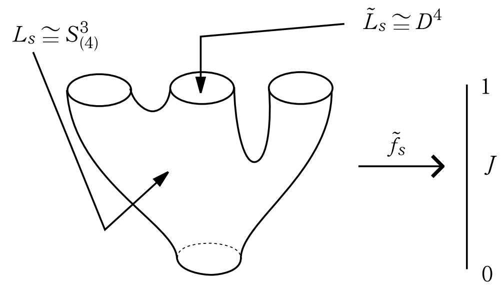

Figure 4. Submersion $\widetilde{f}_s$ for a trident.

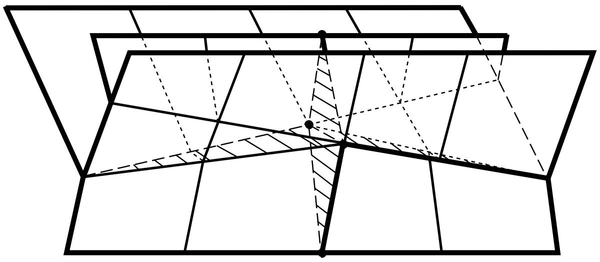

Figure 5. The 2-dimensional polyhedron $\widetilde{N}(t)$.

(See Figure 4.)

In fact, we can construct a smooth compact 5-manifold $V_t \cong D^4 \times I$ with corners and a map $r_t : V_t \to N(t)$ such that $(p \circ r_t)^{-1}(s) = \widetilde{L}_s$ and $h \circ r_t|_{\widetilde{L}_s} = \widetilde{f}_s$ for all $s \in I$, and that $r_t|_{q_f^{-1}(N(t))} = q_f|_{q_f^{-1}(N(t))}$. Note that there is also a 2-dimensional polyhedron $\widetilde{N}(t) \subset V_t$ such that V_t collapses to $\widetilde{N}(t)$ and that $r_t|_{\widetilde{N}(t)} : \widetilde{N}(t) \to N(t)$ is a PL homeomorphism outside of a 1-dimensional polyhedron containing t. (See Figure 5. Note that the shadowed 2-simplices are mapped to 1-dimensional simplices of $N(t)$ and that outside of them, the map $r_t|_{\widetilde{N}(t)}$ is a homeomorphism.)

STEP 2. Cusp.

Let $c \in W_f$ be the q_f-image of a cusp point of f. We take $N(c)$, $N(\bar{f}(c))$, p, h, H, L_s, etc. as in Step 1 (see Figure 6). Then L_s is diffeomorphic to $S^3_{(2)} \cong S^2 \times [0,1]$ for each $s \in I$. Furthermore, $f_s : L_s \to J$, $s \in I$, defines a one parameter family of functions which are Morse functions except for a unique parameter $s_0 \in I$. This family corresponds to a birth or a death of critical points of Morse functions.

By an argument similar to that in Step 1, we can extend the family of functions f_s, $s \in I$, to a family of submersions $\widetilde{f}_s : \widetilde{L}_s \to J$, $s \in I$, such that

(1) for each regular value $u \in J$ of f_s, $\widetilde{f}_s^{-1}(u)$ is a 3-disk or a disjoint union of two 3-disks,
(2) $\widetilde{f}_s^{-1}(i)$ is a 3-disk for $i = 0, 1$,
(3) $\widetilde{L}_s$ is a smooth 4-manifold with corners along $\partial(\widetilde{f}_s^{-1}(0))$ and $\partial(\widetilde{f}_s^{-1}(1))$,
(4) $\widetilde{L}_s$ is diffeomorphic to D^4 after the corners being smoothed.

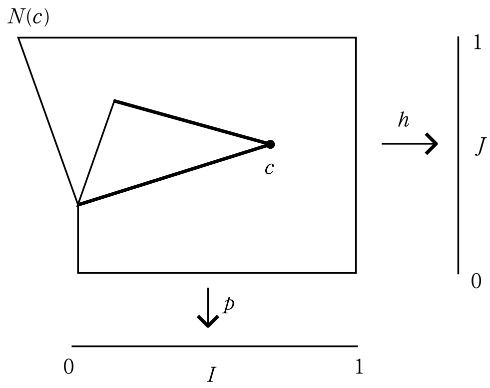

Figure 6. Neighborhood of the q_f-image of a cusp point in W_f.

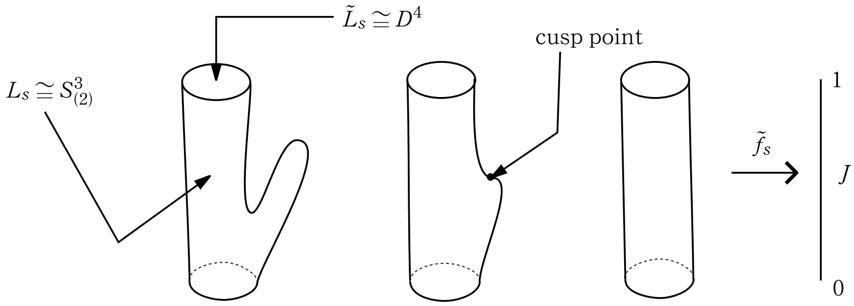

Figure 7. Submersion $\widetilde{f}_s$ for a cusp.

(See Figure 7.)

In fact, we can construct a smooth compact 5-manifold $V_c \cong D^4 \times I$ and a map $r_c : V_c \to N(c)$ such that $(p \circ r_c)^{-1}(s) = \widetilde{L}_s$ and $h \circ r_c|_{\widetilde{L}_s} = \widetilde{f}_s$ for all $s \in I$, and that $r_c|_{q_f^{-1}(N(c))} = q_f|_{q_f^{-1}(N(c))}$. Note that there is also a 2-dimensional polyhedron $\widetilde{N}(c) \subset V_c$ such that V_c collapses to $\widetilde{N}(c)$ and that $r_c|_{\widetilde{N}(c)} : \widetilde{N}(c) \to N(c)$ is a PL homeomorphism.

Step 3. Definite fold.

In Steps 1 and 2, we can choose $N(t)$ and $N(c)$ for each trident t and for each q_f-image c of a cusp point so small that they are mutually disjoint. Let N_{tc} denote the union of all $N(t)$ and $N(c)$.

Let α be a component of $\overline{q_f(S(f)) \smallsetminus N_{tc}}$ which consists of q_f-images of definite fold points. Note that α is either an arc or a circle. Let $N(\alpha) \cong \alpha \times [0,1]$ be a small regular neighborhood of α in $\overline{W_f \smallsetminus N_{tc}}$. Note that α corresponds to $\alpha \times \{0\}$.

For each $s \in \alpha$, $K_s = q_f^{-1}(\{s\} \times [0,1])$ is diffeomorphic to D^3, and $g_s = q_f|_{K_s} : K_s \to \{s\} \times [0,1]$ is a Morse function with exactly one minimum. Then g_s, $s \in \alpha$, can be extended to a family of submersions $\widetilde{g}_s : \widetilde{K}_s \to \{s\} \times [0,1]$, $s \in \alpha$, such that

(1) for each regular value $u \in \{s\} \times [0,1]$ of g_s, $\widetilde{g}_s^{-1}(u)$ is a 3-disk,

(2) $\widetilde{g}_s^{-1}(0)$ is a point and $\widetilde{g}_s^{-1}(1)$ is a 3-disk,
(3) $\widetilde{K}_s$ is a 4-manifold with corners along $\partial(\widetilde{g}_s^{-1}(1))$,
(4) $\widetilde{K}_s$ is diffeomorphic to D^4 after the corners being smoothed.

In fact, we can construct a smooth compact 5-manifold V_α and a map $r_\alpha : V_\alpha \to N(\alpha)$ such that $r_\alpha^{-1}(\{s\} \times [0,1]) = \widetilde{K}_s$ and $r_\alpha|_{\widetilde{K}_s} = \widetilde{g}_s$ for all $s \in \alpha$, and that $r_\alpha|_{q_f^{-1}(N(\alpha))} = q_f|_{q_f^{-1}(N(\alpha))}$. Note that there is also a 2-dimensional polyhedron $\widetilde{N}(\alpha) \subset V_\alpha$ such that V_α collapses to $\widetilde{N}(\alpha)$ and that $r_\alpha|_{\widetilde{N}(\alpha)} : \widetilde{N}(\alpha) \to N(\alpha)$ is a PL homeomorphism.

Note that V_α is diffeomorphic to $D^4 \times [0,1]$ if α is an arc, and is diffeomorphic to the total space of a D^4-bundle over S^1 if α is a circle.

Suppose α is an arc and let us denote by a_0 and a_1 its end points. Furthermore, let us denote by c_0 and c_1 the q_f-images of cusp points corresponding to the end points a_0 and a_1 of α respectively. Then, by the construction of V_c and r_c in Step 2, we see that there exist embeddings $\varphi_i : \widetilde{K}_{a_i} \to \partial V_{c_i}$ such that $r_{c_i} \circ \varphi_i = r_\alpha|_{\widetilde{K}_{a_i}}$ and $\varphi_i(\widetilde{K}_{a_i} \cap \widetilde{N}(\alpha)) = \widetilde{N}(c_i)$, $i = 0, 1$. Therefore, we can attach V_α to V_{c_i} by φ_i to obtain the 5-manifold

$$V_\alpha \cup_{\varphi_0} V_{c_0} \cup_{\varphi_1} V_{c_1} \tag{5.2}$$

which admits the map

$$r_\alpha \cup r_{c_0} \cup r_{c_1} \tag{5.3}$$

onto $N(\alpha) \cup N(c_0) \cup N(c_1)$. Furthermore, $\widetilde{N}(\alpha) \cup \widetilde{N}(c_0) \cup \widetilde{N}(c_1)$ is a 2-dimensional polyhedron such that the 5-manifold (5.2) collapses to it and that the map (5.3) restricted to it is a PL homeomorphism outside of a neighborhood of the tridents.

Step 4. Indefinite fold.

Let α be a component of $\overline{q_f(S(f)) \setminus N_{tc}}$ which consists of q_f-images of indefinite fold points. Note that α is either an arc or a circle. Let $N(\alpha)$ be a small regular neighborhood of α in $\overline{W_f \setminus N_{tc}}$. Note that $N(\alpha)$ is homeomorphic to the total space of a Y-bundle over α, where

$$Y = \{r \exp(2\pi\sqrt{-1}\theta) \in \boldsymbol{C} \,|\, 0 \le r \le 1,\ \theta = 0, 1/3, 2/3\}.$$

Let $\tau : Y \to Y$ be the involution defined by the complex conjugation. When α is a circle, $N(\alpha)$ is homeomorphic either to $Y \times S^1$ or to the total space E of the Y-bundle over S^1 whose monodromy is given by τ, i.e.

$$E = Y \times [0,1]/(z,1) \sim (\tau(z), 0). \tag{5.4}$$

Then, by an argument similar to those in Steps 1, 2 and 3, we can construct a smooth compact 5-manifold V_α, a map $r_\alpha : V_\alpha \to N(\alpha)$ and a 2-dimensional polyhedron $\widetilde{N}(\alpha) \subset V_\alpha$ which satisfy the similar properties. Furthermore, these are compatible with those constructed in Steps 1 and 2 so that we can attach them consistently.

STEP 5. Regular part.

Now we carry out the constructions of Steps 3 and 4 for each connected component of $\overline{q_f(S(f)) \smallsetminus N_{tc}}$ and then glue all of them to those constructed in Steps 1 and 2. In this way, we obtain a smooth compact 5-manifold V_S, a map $r_S : V_S \to N_S$ and a 2-dimensional polyhedron $\widetilde{N}_S \subset V_S$ with some nice properties, where N_S is the union of N_{tc} and the regular neighborhood of $\overline{q_f(S(f)) \smallsetminus N_{tc}}$ in $\overline{W_f \smallsetminus N_{tc}}$.

Set $R = \overline{W_f \smallsetminus N_S}$, which is a compact surface with boundary. Since f has sphere fibers, $q_f|_{q_f^{-1}(R)} : q_f^{-1}(R) \to R$ is a smooth S^2-bundle over R. Since the diffeomorphism group of S^2 is homotopy equivalent to the orthogonal group $O(3)$ (see [**26**]), the structure group of this S^2-bundle can be reduced to $O(3)$. Let $r_R : V_R \to R$ be the D^3-bundle associated with the S^2-bundle, and $\widetilde{R} \subset V_R$ the zero section.

Since $r_S|_{r_S^{-1}(\partial R)}$ is a smooth D^3-bundle over the 1-dimensional manifold ∂R, and it admits a zero section $\widetilde{N}_S \cap r_S^{-1}(\partial R)$, its structure group can also be reduced to $O(3)$. Then it is easy to check that we can glue V_S and V_R together to give a smooth compact 5-manifold V_f, and the maps $r_S : V_S \to N_S$ and $r_R : V_R \to R$ (resp. the 2-dimensional polyhedrons $\widetilde{N}_S$ and $\widetilde{R}$) glue together to give the map $r_f : V_f \to W_f$ (resp. the 2-dimensional polyhedron $\widetilde{W}_f$). From the construction, it is now easy to verify all the required conditions stated in Theorem 4.1. This completes the proof. □

REMARK 5.1. With a little bit more effort, we can take $\widetilde{W}_f$ inside $M = \partial V_f$. This can be considered to be a kind of a section of $q_f : M \to W_f$.

REMARK 5.2. We can prove a result similar to Theorem 4.1 for spherical stable maps $f : M \to N$ of a closed manifold of an even dimension $n \geq 6$ into a surface. However, the manifold V_f is only a topological manifold and is not a smooth manifold in general. Compare this with a result in [**24**] for special generic maps.

REMARK 5.3. We do not know if we can generalize Theorem 4.1 to generic maps with sphere fibers into p-dimensional manifolds with $p \geq 3$.

6. Examples.

In this section, we give a systematic method to construct spherical smooth maps.

PROPOSITION 6.1. *Let $f : M \to N$ be a special generic map and $\pi : N \to P$ a submersion, where M, N and P are smooth manifolds with* $\dim M > \dim N > \dim P$. *We assume that*

(1) *M is a closed manifold and that*
(2) *for every $y \in \pi \circ f(M) \smallsetminus \pi \circ f(S(\pi \circ f|_{S(f)}))$, each component of $(\pi \circ \bar{f})^{-1}(y) \subset W_f$ is contractible.*

Then $\pi \circ f : M \to P$ has sphere fibers.

PROOF. Set $g = \pi \circ f$. Since π is a submersion, we have $S(g) = S(g|_{S(f)})$. Take a point $y \in g(M) \smallsetminus g(S(g))$ as in (2) above. Since $g = \pi \circ f = \pi \circ \bar{f} \circ q_f$, we have $g^{-1}(y) = q_f^{-1}((\pi \circ \bar{f})^{-1}(y))$. By our assumption, every component of $W_y = (\pi \circ \bar{f})^{-1}(y)$

is contractible. Note that $\pi \circ \bar{f} : W_f \to P$ is a submersion and y is a regular value of $\pi \circ \bar{f}|_{\partial W_f}$. Therefore, W_y is a proper smooth submanifold of W_f which is transverse to ∂W_f. Then it is easy to see that $q_f|_{q_f^{-1}(W_y)} : q_f^{-1}(W_y) \to W_y$ is a special generic map. Note that its Stein factorization can be identified with W_y. Then by [**24**], each connected component of $g^{-1}(y) = q_f^{-1}(W_y)$ is a homotopy sphere, since each connected component of the Stein factorization W_y is contractible. Hence g has sphere fibers. □

We have the following direct consequence of the above proposition.

Corollary 6.2. *Every smooth closed n-dimensional manifold that admits a special generic map into $\boldsymbol{R}^p$ with $2 \le p < n$ admits a smooth spherical map into $\boldsymbol{R}^{p-1}$.*

Remark 6.3. The converse of the above corollary does not hold in general. For example, there is a stable Morse function $f : \boldsymbol{C}P^2 \to \boldsymbol{R}$ with exactly three critical points, whose indices are equal to 0, 2 and 4. It is easy to see that f has sphere fibers. However, $\boldsymbol{C}P^2$ does not admit any special generic map into $\boldsymbol{R}^2$ according to [**24**].

The following lemma is implicitly proved in [**5**].

Lemma 6.4. *Let $f : M \to N$ be a fold map and $\pi : N \to P$ a submersion, where M, N and P are smooth manifolds with $\dim M \ge \dim N > \dim P$. If $\pi \circ f|_{S(f)} : S(f) \to P$ is a fold map, then $g = \pi \circ f : M \to P$ is also a fold map and $S(g) = S(\pi \circ f|_{S(f)})$.*

As a corollary, we have the following.

Corollary 6.5. *Let $f : M \to N$ be a special generic map and $\pi : N \to P$ a submersion, where M, N and P are smooth manifolds with $\dim M > \dim N > \dim P$. We assume that*

(1) *M is a closed manifold, that*
(2) *the map $\pi \circ f|_{S(f)} : S(f) \to P$ is a fold map, and that*
(3) *for every $y \in \pi \circ f(M) \smallsetminus \pi \circ f(S(\pi \circ f|_{S(f)}))$, each component of $(\pi \circ \bar{f})^{-1}(y) \subset W_f$ is contractible.*

Then $\pi \circ f : M \to P$ is a spherical fold map.

Example 6.6. Let us give a typical application of Proposition 6.1.

Let W be a compact connected orientable 3-manifold with boundary. Then it can be immersed into $\boldsymbol{R}^3$ (see [**29**]). Let $\pi : W \to \boldsymbol{R}^2$ be the composition of an immersion $\eta : W \looparrowright \boldsymbol{R}^3$ and an orthogonal projection $pr : \boldsymbol{R}^3 \to \boldsymbol{R}^2$. By choosing the projection pr generically, we may assume that $\pi|_{\partial W} : \partial W \to \boldsymbol{R}^2$ is a stable map (for this, refer to [**16**], [**30**]). In the terminology of [**25**], the resulting map π is a *nonsingular stable map.* Note that for every point $y \in \pi(W) \smallsetminus \pi(S(\pi|_{\partial W}))$, $\pi^{-1}(y)$ is a finite disjoint union of closed arcs. In particular, each of its connected components is contractible.

Let E be the total space of an arbitrary D^2-bundle over W whose structure group is $O(2)$, and set $M = \partial E$, which is a closed 4-manifold. Then by [**24**], there exists a special generic map $f : M \to \boldsymbol{R}^3$ such that W_f is identified with W and that $\bar{f} : W_f \to \boldsymbol{R}^3$ is identified with the immersion $\eta : W \to \boldsymbol{R}^3$.

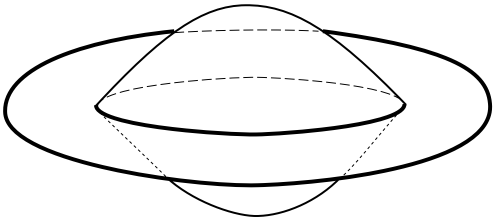

Figure 8. Stein factorization W_g of the spherical fold map g.

Therefore, according to Proposition 6.1, $g = \pi \circ f : M \to \boldsymbol{R}^2$ is a spherical stable map. Furthermore, if the stable map $\pi|_{\partial W} : \partial W \to \boldsymbol{R}^2$ is a fold map, then g is a spherical fold map by Corollary 6.5.

Note that the Stein factorization W_g of the spherical stable map g constructed above is naturally identified with the Stein factorization W_π of the nonsingular stable map $\pi : W \to \boldsymbol{R}^2$. For a local characterization of Stein factorizations of nonsingular stable maps of 3-manifolds into surfaces, refer to [**25**].

As an explicit example, let us consider the standard embedding $\eta : S^2 \times [0,1] \to \boldsymbol{R}^3$. Then $\pi|_{\partial(S^2\times[0,1])} : \partial(S^2 \times [0,1]) \to \boldsymbol{R}^2$ is a fold map. Let V be the total space of the D^2-bundle over $S^2 \times [0,1]$ with Euler number $e \in \boldsymbol{Z}$ and set $M = \partial V$. Note that M is diffeomorphic to $S^2 \times S^2$ if e is even and is diffeomorphic to $S^2 \widetilde{\times} S^2$ if e is odd. Then the map $g : M \to \boldsymbol{R}^2$ constructed above is a spherical fold map. Note that its Stein factorization W_g is a polyhedron as depicted in Figure 8.

By the connected sum construction along the definite folds, we can also show that any connected sum of some copies of $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$ admits a spherical fold map into $\boldsymbol{R}^2$ (see [**24**, Lemma 5.4]).

Remark 6.7. Not all spherical stable map of a closed 4-manifold into $\boldsymbol{R}^2$ can be obtained as in Example 6.6. If a spherical stable map is obtained as in the example, then its Stein factorization is homeomorphic to that of the nonsingular stable map π. As remarked in [**25**], such a Stein factorization cannot contain the total space E of the nontrivial Y-bundle over S^1 defined in (5.4). A spherical fold map whose Stein factorization contains E does exist. An explicit example is implicitly constructed in [**27**, Theorem 4.4].

Recall the following definition of a spun 2-knot.

Definition 6.8. Let $k \subset S^3$ be a classical knot. Take a point $x \in k$ and its small disk neighborhood D_x in S^3 such that $(D_x, D_x \cap k)$ is diffeomorphic to the standard disk pair (D^3, D^1). Set $B = S^3 \setminus \operatorname{Int} D_x$. Note that ∂B is naturally identified with S^2. We denote the two end points of $B \cap k \subset S^2$ by p_0 and p_1. Then $(B \times S^1) \cup (S^2 \times D^2)$ is diffeomorphic to S^4 and

$$\widetilde{k} = ((B \cap k) \times S^1) \cup (\{p_0, p_1\} \times D^2)$$

gives a 2-knot in S^4, which is called the (untwisted) *spun 2-knot* of k (for more details, see [**22**], for example).

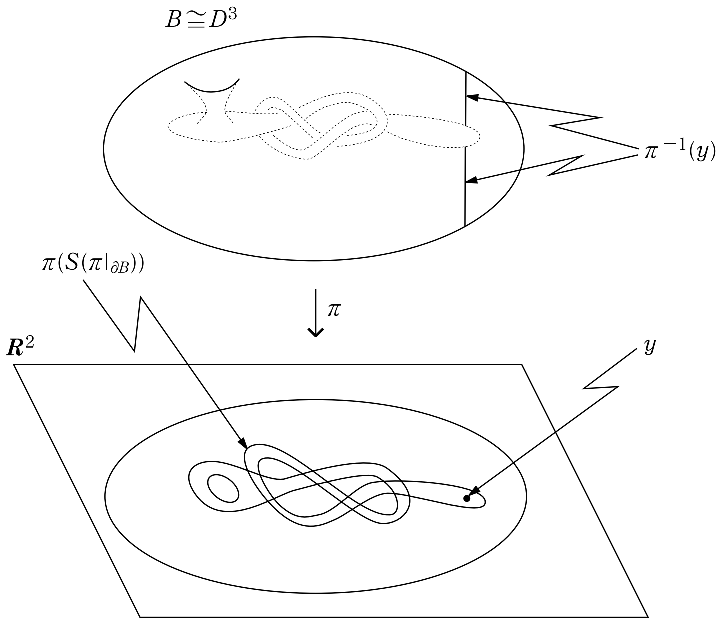

Figure 9. Nonsingular stable map $\pi : B \to \boldsymbol{R}^2$.

Note that $(S^4, \widetilde{k})$ can be identified with $\partial(B \times D^2, (B \cap k) \times D^2)$. The exterior of $B \cap k$ in B is a compact 3-manifold with nonempty boundary and hence admits a handlebody decomposition with 0-, 1- and 2-handles. Hence the exterior of the 3-disk $(B \cap k) \times D^2$ in the 5-disk $B \times D^2$ has a handlebody decomposition with 0-, 1- and 2-handles. Therefore, a spun 2-knot is always homotopy ribbon. (In fact, it is known that a spun 2-knot is always a ribbon 2-knot and hence is homotopy ribbon.)

The following theorem gives a plenty of examples of Corollary 4.13.

Theorem 6.9. *For every knot k in S^3, there exists a spherical stable fold map $g : S^4 \to \boldsymbol{R}^2$ and a point $y \in g(S^4) \setminus g(S(g))$ such that a component of $g^{-1}(y) \subset S^4$ is isotopic to the spun 2-knot $\widetilde{k}$ of k.*

Proof. Let B be as in Definition 6.8. Let us first construct a nonsingular stable map $\pi : B \to \boldsymbol{R}^2$ with $B \cong D^3$ such that a component of $\pi^{-1}(y)$ coincides with $B \cap k$ for a point $y \in \pi(B) \setminus \pi(S(\pi|_{\partial B}))$.

First, embed the 3-disk B standardly in $\boldsymbol{R}^3$. Then we isotope it in $\boldsymbol{R}^3$ so that we make a "knotted hole" along $B \cap k$. Furthermore, we can arrange so that a vertical line passing through a bottom point of the "hole" cuts B in two line segments ℓ_1 and ℓ_2 such that ℓ_1 coincides with $B \cap k$. (See Figure 9 for the case of the figure eight knot k.) We can further arrange so that an appropriate orthogonal projection $\boldsymbol{R}^3 \to \boldsymbol{R}^2$ restricted to ∂B is a stable fold map. Then the projection restricted to B gives a required nonsingular stable map. Note that for a point $y \in \pi(B) \setminus \pi(S(\pi|_{\partial B}))$, we have $\pi^{-1}(y) = \ell_1 \cup \ell_2$.

Then apply the construction for $M = S^4 = \partial(B \times D^2)$ as described in Example 6.6. The resulting spherical stable fold map $g : S^4 \to \boldsymbol{R}^2$ and the point y satisfy the required conditions. This completes the proof. □

REMARK 6.10. We do not know if in Theorem 6.9, there exists a spherical stable (fold) map $h : S^4 \to \boldsymbol{R}^2$ and a point $z \in h(S^4) \smallsetminus h(S(h))$ such that $h^{-1}(z)$ coincides with $\widetilde{k}$. Note that in the above proof, $g^{-1}(y)$ is a two component 2-link each of whose component is isotopic to the spun 2-knot $\widetilde{k}$.

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