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On quasiconformal deformations of transversely holomorphic foliations

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Abstract. Existence of complex codimension-one transverse structure is studied using the complex dilatation. As an application, a version of quasiconformal surgeries of foliations is considered.

1. Introduction.

In the study of foliations of codimension greater than one, it is natural to restrict oneself to foliations which admit transverse geometric structures. In the present paper, we consider transversely holomorphic foliations of complex codimension one, namely, foliations whose holonomy pseudogroups are generated by biholomorphic local diffeomorphisms of $\boldsymbol{C}$. Although the situation seems restrictive, there are many interesting examples. For example, if a holomorphic vector field on $\boldsymbol{C}^2$ with Poincaré type singularities is given, then a transversely holomorphic foliation of S^3 is naturally induced. Although transversely holomorphic foliations of closed 3-manifolds are classified by Brunella [**4**], Ghys [**6**] and Carrière [**5**], it seems difficult to tell if a given flow admits transverse holomorphic structures. In addition, if the ambient manifold is of dimension greater than three, there are very complicated examples [**7**] so that it seems very hard to classify such foliations. Thus it is important to find good criteria for foliations admitting transverse holomorphic structures as well as to find methods to construct such foliations. On open manifolds, a homotopy theoretic approach can be found for example in a work of Haefliger [**8**]. In this paper, we introduce the notion of quasiconformal foliations, and show such foliations are in fact transversely holomorphic. Quasiconformal foliations are foliation version of quasiconformal groups. Indeed, the construction of holomorphic structure is an application of Tukia's method found in [**13**] and the proof of the main theorem presented in the third section is almost identical to Tukia's one for group actions. However, some additional considerations are needed for applying it to foliations and formulating the boundary relative version which is needed for formulating a version of quasiconformal surgeries of foliations. This is almost equivalent to the extension problem of given transverse holomorphic structures on the boundary. Under an additional but natural condition, such an extension is possible if the foliation is quasiconformal. The surgery considered here is closely related to the Julia sets for complex codimension one foliations given by Ghys, Gomez-Mont and Saludes [**7**], and also to characteristic classes of foliations.

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This paper is organized as follows. In Section 2, relevant definitions are given. In Section 3, the main theorem is proved. As a corollary, it is shown that under a natural condition, two transversely holomorphic foliations can be glued possibly after changing the transverse structure on one piece. The relevant tools are complex dilatations and the measurable Riemann mapping theorem (see [**1**] for details). Finally, relation with characteristic classes is discussed in Section 4.

2. Definitions.

Let $\mathscr{F}$ be a transversely oriented, real codimension two foliation of a manifold M. If the boundary of M is nonempty, then assume that $\mathscr{F}$ is transversal to the boundary. Although we are interested in smooth foliations, we only assume that $\mathscr{F}$ is transversely quasiconformal. Roughly speaking, $\mathscr{F}$ is said to be transversely quasiconformal if the holonomy pseudogroup consists of quasiconformal local homeomorphisms of $\boldsymbol{C}$ whose dilatations are uniformly bounded. A more precise definition will be given soon later.

Let $\{U_i\}$ be a locally finite foliation chart so that each U_i is homeomorphic to $V_i \times D_i$, where V_i is an open set of $\boldsymbol{R}^{\dim M-2}$ and D_i is an open disc in $\boldsymbol{R}^2$ (if $\mathscr{F}$ is not smooth, we assume that such a chart exists). Let φ_{ji} be the transition function from U_i to U_j, then every φ_{ji} is of the form (ψ_{ji}, γ_{ji}), where γ_{ji} is a function defined on an open subset of D_i. Fix now for each i an identification of D_i with an open disc in $\boldsymbol{C}$, then each γ_{ji} is a local homeomorphism of $\boldsymbol{C}$. Let T be the disjoint union of D_i's, then T can be considered as an open subset of $\boldsymbol{C}$ and also as a subset of M. We call this T a complete transversal. When we need a measure class of T, we consider the restriction of the Lebesgue measure of $\boldsymbol{C}$ to T. We may assume that this measure class coincides with the natural one induced from M if every γ_{ji} preserves the Lebesgue measure class. This is indeed the case if $\mathscr{F}$ is smooth or every γ_{ji} is a quasiconformal homeomorphism.

Definition 2.1. Let $T = \amalg D_i$ be a complete transversal and consider it as an open subset of $\boldsymbol{C}$. Set $\Gamma_1 = \{\gamma_{ji}\}$, where γ_{ji} is as above. We denote by Γ the holonomy pseudogroup associated with T, namely, let Γ be the pseudogroup generated by Γ_1. If we denote by Γ_n the set of local homeomorphisms of $\boldsymbol{C}$ obtained as the composition of at most n elements of Γ_1, then $\Gamma = \bigcup \Gamma_n$. For an element γ of Γ, the domain of γ is denoted by $\operatorname{dom}\gamma$ and the range of γ is denoted by $\operatorname{range}\gamma$.

In order to introduce transversely quasiconformal foliations, recall the notion of complex dilatation.

Definition 2.2. For an orientation preserving quasiconformal local homeomorphism f of $\boldsymbol{C}$, we denote by $\mu_f(z)$ the complex dilatation (Beltrami coefficient) of f, namely, we set

$$\mu_f(z) = \frac{f_{\bar{z}}(z)}{f_z(z)},$$

where $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}$ and $f_z = \frac{\partial f}{\partial z}$. Such an f is said to be K-quasiconformal for $K \geq 1$ if $\|\mu_f\| \leq \frac{1-K}{1+K}$, where $\|\mu_f\|$ denotes the essential supremum of $|\mu_f|$ on the domain of f.

It is known that the partial derivatives are well-defined almost everywhere for quasiconformal local homeomorphisms. A quasiconformal local homeomorphism f is biholomorphic if and only if it is 1-quasiconformal, or equivalently, $|\mu_f(z)| = 0$ a.e. z.

DEFINITION 2.3. Let $\mathscr{F}$ be a real codimension two foliation of a manifold M. If $\partial M \neq \varnothing$, assume that $\mathscr{F}$ is transversal to ∂M. Let T be a complete transversal for $\mathscr{F}$ and let Γ be the holonomy pseudogroup associated with T. Then, $\mathscr{F}$ is said to be K-quasiconformal with respect to T if every element of Γ is K-quasiconformal.

Since the foliation is assumed to be transversely oriented, $1 > |\mu_\gamma(z)| \geq 0$ for $\gamma \in \Gamma$.

REMARK 2.4. The notion of K-quasiconformality depends on the choice of complete transversals. However, provided that M is compact and $\mathscr{F}$ is smooth, if $\mathscr{F}$ is K-quasiconformal for some choice, then $\mathscr{F}$ is K'-quasiconformal for other choices with some $K' \geq 1$.

The notion of transversely quasiconformal foliations is a foliation version of quasiconformal groups. Let G be a group of quasiconformal self-homeomorphisms of an open subset U of $\boldsymbol{C}P^1$ and assume that the action is orientation preserving. The group G is said to be a quasiconformal group if there is a constant $k < 1$ such that $|\mu_g(z)| \leq k$ for any $g \in G$ and a.e. z. A theorem of Tukia [**13**] shows then that there is a K-quasiconformal homeomorphism $f : U \to \boldsymbol{C}P^1$ such that the action of $f \circ G \circ f^{-1}$ on $f(U)$ is holomorphic. The main theorem in this paper is a foliation version of his theorem.

DEFINITION 2.5. Let $\mathscr{F}$ be a transversely quasiconformal foliation of a manifold M which is transversely orientable and of real codimension two. Let $\{V_i \times D_i, (\psi_{ji}, \gamma_{ji})\}$ be a foliation chart as above. Set $T = \amalg D_i$ and consider T as an open subset of $\boldsymbol{C}$, and let Γ be the holonomy pseudogroup associated with T. Let f be a K-quasiconformal homeomorphism from T to its image, then one can form a new foliation $\mathscr{F}'$ whose foliation chart is given by $\{V_i \times f(D_i), (\psi_{ji}, f \circ \gamma_{ji} \circ f^{-1})\}$. The foliation $\mathscr{F}'$ is called a K-quasiconformal conjugate of $\mathscr{F}$. The complex dilatation μ_f of f is called the transverse complex dilatation of the conjugacy.

Definition 2.5 is reduced to a more natural form for transversely holomorphic foliations.

DEFINITION 2.6. Let $\mathscr{F}$ be a transversely holomorphic foliation of M, of complex codimension one. The transverse complex dilatation $\mu_f^\pitchfork$ of a foliation preserving diffeomorphism f of M into itself is defined to be the complex dilatation of f in the transverse direction with respect to the transverse holomorphic structure of $\mathscr{F}$. If $\left\|\mu_f^\pitchfork\right\|_\infty \leq \frac{K-1}{K+1}$, f is said to be transversely K-quasiconformal.

Finally, we introduce the complex dilatation for germs of elements of Γ.

DEFINITION 2.7. Let Γ be a topological groupoid acting on an open subset T of $\boldsymbol{C}$. Suppose that Γ is generated by orientation preserving quasiconformal local homeomorphisms of T. We denote by $[\gamma]_x$ the germ of element γ of Γ at x if $x \in \operatorname{dom}\gamma$, and set $\Gamma_x = \{[\gamma]_x \mid \gamma \in \Gamma, x \in \operatorname{dom}\gamma\}$. For an element $[\gamma]_x$ in Γ_x, we choose its representative γ' and set $\mu_{[\gamma]_x}(x) = \mu_{\gamma'}(x)$. By abuse of notation, $\mu_{[\gamma]_x}(x)$ is denoted again by $\mu_\gamma(x)$.

Note that $\mu_\gamma(x)$ is well-defined for a.e. x and any γ. We have $\Gamma_{\gamma x}[\gamma]_x = \{[\gamma'\gamma]_x \,|\, \gamma' \in \Gamma_{\gamma x}\} = \Gamma_x$ if $\gamma \in \Gamma_x$. This property plays an important role in proving the main theorem. In what follows, we denote $[\gamma]_x$ simply by γ, because only the germ of elements of Γ is relevant.

3. Main Theorem.

Definition 3.1. Let $\boldsymbol{D}$ be the Poincaré disc and let X be a subset of $\boldsymbol{D}$ bounded in the Poincaré metric. Let $P(X)$ be the center of the unique hyperbolic ball $D(x,r)$ with the properties that 1) $D(x,r) \supset X$ and 2) if $D(y,r') \supset X$ and $y \neq x$, then $r' > r$, where $D(x,r)$ denotes the hyperbolic ball centered at x and of radius r. We call $P(X)$ the hyperbolic mean of X.

The existence of such a $D(x,r)$ is shown in [**13**].
We adopt the following notations.

Notation 3.2. Let $\mathscr{F}$ be a transversely orientable, real codimension two foliation of a manifold M. Let W be a codimension zero submanifold of M and suppose that ∂W is transversal to $\mathscr{F}$. Denote by T_W a complete transversal for $\mathscr{F}|_W$, and denote by Γ_W the holonomy pseudogroup of $\mathscr{F}|_W$ associated with T_W. Choose a complete transversal T of $\mathscr{F}$ such that T contains T_W and that $T \setminus T_W$ is a complete transversal for $\mathscr{F}|_{M\setminus W}$. Let Γ be the holonomy pseudogroup associated with T. Set then $\widetilde{\Gamma}_W = \{\gamma \in \Gamma \,|\, \operatorname{dom}\gamma \subset T_W, \operatorname{range}\gamma \subset T_W\}$.

Elements of $\widetilde{\Gamma}_W$ represent holonomies along leaf paths connecting points of W but not necessarily contained in W.
The main theorem is as follows.

Theorem 3.3.

1) *Let $\mathscr{F}$ be a real codimension two foliation of a manifold M. If $\partial M \neq \varnothing$, then assume that $\mathscr{F}$ is transversal to ∂M. If $\mathscr{F}$ is K-quasiconformal with respect to a complete transversal T, then $\mathscr{F}$ admits a transverse holomorphic structure after taking a transverse K-quasiconformal conjugate of $\mathscr{F}$.*
2) *Let W be a codimension zero submanifold of M and assume that ∂W is transversal to $\mathscr{F}$. Assume that $\mathscr{F}$ is K-quasiconformal and that a transverse holomorphic structure is given to $\mathscr{F}|_W$. Define T_W, T, Γ_W and $\widetilde{\Gamma}_W$ as in Definition* 3.2. *Suppose now that $\widetilde{\Gamma}_W$ is an extension of Γ_W by biholomorphic local diffeomorphisms of T_W, then the transverse holomorphic structure of $\mathscr{F}|_W$ extends to a transverse holomorphic structure of $\mathscr{F}$ on M after taking a transverse K-quasiconformal conjugate of $\mathscr{F}$ which is transversely holomorphic on W.*

Before giving a proof, we explain the condition assumed in 2). Suppose that the given transverse holomorphic structure of $\mathscr{F}|_W$ extends to the whole manifold by modifying T by a quasiconformal homeomorphism f which is biholomorphic on T_W. Then, $f \circ \Gamma \circ f^{-1}$ is generated by biholomorphic local diffeomorphisms. Let γ be an element of Γ which corresponds to a leaf path connecting points of W but not necessarily contained in W. Then $f \circ \gamma \circ f^{-1}$ is biholomorphic. Since f is biholomorphic when restricted to W, the mapping γ itself should be biholomorphic. Hence the assumption in 2) is indispensable. The easiest case where the compatibility condition is satisfied is that $\mathscr{F}$ is in fact a flow and that each orbit meets the boundary at most once.

If the compatibility condition is dropped in Theorem 3.3, there is a counterexample as follows.

EXAMPLE 3.4. Let $\mathscr{F}$ be the foliation of $T^2 \times [0,1]$ by the intervals $\{p\} \times [0,1]$, $p \in T^2$. Give $T^2 \times \{0\}$ and $T^2 \times \{1\}$ two distinct complex structures, and extend them trivially to $W_0 = T^2 \times [0,\epsilon)$ and $W_1 = T^2 \times (1-\epsilon, 1]$, respectively, where ϵ is a small positive real number. It is then obvious that the transverse holomorphic structure on $W_0 \cup W_1$ cannot be extended to any transverse holomorphic structure of $\mathscr{F}$ on the whole $T^2 \times [0,1]$.

PROOF OF THEOREM 3.3. First we show 1). This part is essentially due to Tukia. We repeat his proof with necessary adaptations, basically following the notations in [**13**].

Denote by $\boldsymbol{D} \subset \boldsymbol{C}$ the Poincaré disc. For $x \in T$ and $\gamma \in \Gamma_x$, define a Möbius transformation of $\boldsymbol{D}$ by the formula

$$T_\gamma(x)(z) = \frac{\gamma_{\bar z}(x) + \overline{\gamma_z(x)}z}{\gamma_z(x) + \overline{\gamma_{\bar z}(x)}z},$$

where $z \in \boldsymbol{D}$. If $\gamma' \in \Gamma_{\gamma x}$ is a quasiconformal homeomorphism with the complex dilatation $\mu_{\gamma'}$, then $T_\gamma(x)(\mu_{\gamma'}(\gamma x)) = \mu_{\gamma'\gamma}(x)$, where $\gamma \in \Gamma_x$. For $x \in T$, set $M_x = \{\mu_\gamma(x) \mid \gamma \in \Gamma_x\}$, then we have

$$\begin{aligned} T_\gamma(x)(M_{\gamma x}) &= \{T_\gamma(x)(\mu_{\gamma'}(\gamma x)) \mid \gamma' \in \Gamma_{\gamma x}\} \\ &= \{\mu_{\gamma'\gamma}(x) \mid \gamma' \in \Gamma_{\gamma x}\} \\ &= \{\mu_{\gamma'\gamma}(x) \mid (\gamma'\gamma) \in \Gamma_x\} \\ &= M_x. \end{aligned}$$

For $x \in T$, we set $\mu(x) = P(M_x)$ if M_x is bounded, where $P(M_x)$ is the hyperbolic mean of M_x, and $\mu(x) = 0$ if either M_x is unbounded or $x \notin T$. Although foliations are considered, μ is still measurable. To see this, recall that Γ is generated by Γ_1, which is countable. We give an order to elements of Γ_1 and denote by γ_i the i-th element. Set $G_i = \{\gamma_1, \gamma_2, \ldots, \gamma_i\}$ and let Γ'_n be the subset of Γ_n which consists of the composition of elements of G_n. Then clearly Γ'_n is finite and $\bigcup \Gamma'_n = \Gamma$. Let M'_x be the subset of $\boldsymbol{D}$ obtained by collecting $\mu_{\gamma'}(x)$, where $\gamma' \in \Gamma'_n \cap \Gamma_x$. We set $\mu_n(x) = P(M'_x)$, where $P(\varnothing)$ is set to be 0. An elementary argument shows that the sequence $\{\mu_n(x)\}$ converges to $\mu(x)$ if M_x is bounded. As $\mu_n(x)$ is the unique point determined in a measurable way as in Definition 3.1, μ_n is a measurable function. Notice that the boundary of the domain of each element is of Lebesgue measure zero, since quasiconformal maps preserve the Lebesgue measure class. Hence μ is also a measurable function.

Finally let f be the quasiconformal mapping with $\mu_f(x) = \mu(x)$ a.e. x given by the measurable Riemann mapping theorem, then

$$\begin{aligned}\mu_f(x) &= P(M_x)\\ &= P(T_\gamma(x)(M_{\gamma x}))\\ &= T_\gamma(x)(P(M_{\gamma x}))\\ &= T_\gamma(x)(\mu_f(\gamma x))\\ &= \mu_{f\gamma}(x)\end{aligned}$$

for a.e. $x \in T$ and every $\gamma \in \Gamma_x$. This implies that $f \circ \Gamma \circ f^{-1}$ acts as holomorphic transformations on $f(T)$. The dilatation of f can be estimated exactly as in [**13**].

The second part is shown as follows. Let $\widetilde{W}$ be the saturation of W in M and let $T_{\widetilde{W}}$ be the corresponding subset of T, then $T_{\widetilde{W}}$ is open subset of T invariant under the action of Γ. We define a measurable function μ' on T instead of μ as follows. For $x \in T_{\widetilde{W}}$, set $M'_x = \{\mu_\gamma(x) \,|\, \gamma \in \Gamma_x, \gamma x \in T_W\}$. If $\gamma \in \Gamma_x$, then we have

$$\begin{aligned}T_\gamma(x)(M'_{\gamma x}) &= \{T_\gamma(x)(\mu_{\gamma'}(\gamma x)) \,|\, \gamma' \in \Gamma_{\gamma x}, \gamma'\gamma x \in T_W\}\\ &= \{\mu_{\gamma'\gamma}(x) \,|\, \gamma' \in \Gamma_{\gamma x}, \gamma'\gamma x \in T_W\}\\ &= \{\mu_{\gamma'\gamma}(x) \,|\, (\gamma'\gamma) \in \Gamma_x, \gamma'\gamma x \in T_W\}\\ &= M'_x.\end{aligned}$$

The compatibility condition on Γ_W and $\widetilde{\Gamma}_W$ implies that $M'_x = \{0\}$ if $x \in T_W$. Set now

$$\mu'(x) = \begin{cases} P(M'_x) & \text{if } x \in T_{\widetilde{W}} \text{ and } M'_x \text{ is bounded},\\ P(M_x) & \text{if } x \notin T_{\widetilde{W}} \text{ and } M_x \text{ is bounded}.\end{cases}$$

As in the case 1), $\mu'(x)$ is essentially bounded, measurable and invariant under the action of Γ. Since $\mu'(x) = 0$ for $x \in T_W$, the conjugacy is transversely holomorphic on W. This completes the proof. □

Remark 3.5. Sullivan also made a similar construction in [**11**] involving the barycenter of M_x instead of the hyperbolic mean.

Remark 3.6. Even if the foliation is not transversely orientable, one can find an invariant transverse conformal structure under the same condition. After conjugation, the holonomy pseudogroup is generated by biholomorphic and bi-antiholomorphic local diffeomorphisms of $\boldsymbol{C}$.

A version of quasiconformal surgery is formulated as follows. Consider the following situation: let M_1 and M_2 be manifolds with boundaries ∂M_1 and ∂M_2. Let $\mathscr{F}_i$ be a transversely holomorphic foliation of M_i transversal to the boundary ($i = 1, 2$). Let N_1 and N_2 be the union of several components of ∂M_1 and ∂M_2, respectively. Assume that there is a foliation preserving, transversely quasiconformal homeomorphism φ from $(N_1, \mathscr{F}_1|_{N_1})$ to $(N_2, \mathscr{F}_2|_{N_2})$. If one tries to glue M_1 and M_2 by φ, a situation as in the part 2) of Theorem 3.3 occurs. Pulling back the structure by φ, $\mathscr{F}_1$ is given a transverse holomorphic structure on a collar neighborhood W of N_1, because $\mathscr{F}_1$ is transversal to the boundary. The problem is if this structure can be extended to the whole M_1. Let T_W, Γ_W and $\widetilde{\Gamma}_W$ as in Definition 3.2. The latter should be an extension of Γ_W by biholomorphic local diffeomorphisms of T_W. This is sufficient if one more condition is fulfilled.

COROLLARY 3.7. *Suppose that $\widetilde{\Gamma}_W$ is an extension of Γ_W by biholomorphic local diffeomorphisms of T_W and that φ is transversely K-quasiconformal. Then $M_1 \cup_\varphi M_2$ admits a transversely holomorphic foliation which is the same as $\mathscr{F}_2$ on M_2 and which is transversely K-quasiconformal conjugate to $\mathscr{F}_1$ on M_1.*

PROOF. Let ℓ be a leaf path, then we may assume that ℓ is transversal to ∂W. If ℓ comes into a component of W and goes out of W, then by pushing ℓ slightly into the interior $\mathrm{int}(M \setminus W)$ of $M \setminus W$, ℓ can be modified so that ℓ stays in $\mathrm{int}(M \setminus W)$ because W is a collar. Hence we may assume that ℓ satisfies one of the following conditions; 1) ℓ stays in $\mathrm{int}(M \setminus W)$, 2) ℓ connects a point in W and a point in $\mathrm{int}(M \setminus W)$ meeting ∂W once, or 3) ℓ connects two points in W. Now by assumption, $\mathscr{F}_1$ is transversely holomorphic when restricted respectively to W and to $\mathrm{int}(M \setminus W)$, and $\widetilde{\Gamma}_W$ is an extension of Γ_W by biholomorphic local diffeomorphisms of T_W. It follows that the holonomy along ℓ is holomorphic in the cases 1) and 3), and that the distortion of the holonomy along ℓ is bounded by the distortion of φ, which is bounded by assumption in the case 2). Hence 2) of Theorem 3.3 can be applied so that one can find a transverse complex structure on M_1. □

REMARK 3.8.

1) The gluing map φ is a priori transversely K-quasiconformal for some K if N_1 is compact and φ is smooth.
2) If $\mathscr{F}$ is a flow, then N_1 and N_2 are complex lines and $\varphi|_{N_1}$ is seen to be a mapping between these lines. The transverse complex dilatation of φ is then equal to the complex dilatation of the mapping $\varphi|_{N_1}$.
3) As easily seen, the surgery given by Corollary 3.7 need not produce a new foliation.

REMARK 3.9. This kind of surgeries of transversely holomorphic foliations are considered in [**8**] when the gluing mappings are transversely holomorphic. Corollary 3.7 shows that these mappings need not be transversely holomorphic if one is allowed to modify the transverse holomorphic structure on one piece.

4. Remarks from the viewpoint of characteristic classes.

The above results can be explained in terms of the classifying spaces as follows. Let $\Gamma_1^{\boldsymbol{C}}$ and Γ_2^{qc} be the pseudogroups generated by local biholomorphic diffeomorphisms of $\boldsymbol{C}$ and by orientation preserving local quasiconformal homeomorphisms of $\boldsymbol{R}^2$, respectively. Let $B\Gamma_1^{\boldsymbol{C}}$ and $B\Gamma_2^{\mathrm{qc}}$ be the classifying spaces for $\Gamma_1^{\boldsymbol{C}}$-structures and Γ_2^{qc}-structures, respectively. Denote by π the natural mapping from $B\Gamma_1^{\boldsymbol{C}}$ to $B\Gamma_2^{\mathrm{qc}}$.

DEFINITION 4.1. A mapping $f\colon M \to B\Gamma_2^{\mathrm{qc}}$ is said to be bounded if there is a real number $K \geq 1$ such that the corresponding pseudogroup consists of K-quasiconformal mappings.

Note that if the mapping f as above admits a lift to $B\Gamma_1^{\boldsymbol{C}}$, it is bounded. Given a transversely (K-)quasiconformal foliation of a manifold M, there is a classifying map from M to $B\Gamma_2^{\mathrm{qc}}$. Such a classifying mapping is also bounded. In considering classifying spaces, the homotopy classes of mappings are relevant. However, we do not know any good criteria for mappings to $B\Gamma_2^{\mathrm{qc}}$ being homotopic to bounded ones.

In this line, we have the following

Theorem 4.2.

1) *A mapping $f : M \to B\Gamma_2^{\mathrm{qc}}$ admits a lift to $B\Gamma_1^{\boldsymbol{C}}$ if the mapping f is bounded. The lift is not unique in general.*
2) *The image of the mapping $\pi^* : H^3(B\Gamma_2^{\mathrm{qc}}; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_1^{\boldsymbol{C}}; \boldsymbol{C}/\boldsymbol{Z})$ is $\{0\}$.*
3) *The imaginary part of the Bott class is neither invariant under transversely quasiconformal homeomorphisms, nor well-defined in the category of transversely quasiconformal foliations.*

Proof. The first part of 1) is a reformulation of Theorem 3.3, 1). In order to show 2), let $B\Gamma_2^1$ and $B\overline{\Gamma}_2^1$ be the classifying spaces for real codimension-two transversely oriented C^1-foliations and for real codimension-two transversely oriented C^1-foliations with trivialized normal bundles, respectively. The mapping $B\Gamma_1^{\boldsymbol{C}} \to B\Gamma_2^{\mathrm{qc}}$ is then decomposed as $B\Gamma_1^{\boldsymbol{C}} \to B\Gamma_2^1 \to B\Gamma_2^{\mathrm{qc}}$. Hence the above mapping π^* is also decomposed as $H^3(B\Gamma_2^{\mathrm{qc}}; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_2^1; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_1^{\boldsymbol{C}}; \boldsymbol{C}/\boldsymbol{Z})$. On the other hand, it is shown in [**12**] that $B\overline{\Gamma}_2^1$ is contractible, which implies that $H^3(B\Gamma_2^1; \boldsymbol{C}/\boldsymbol{Z}) = \{0\}$. The second part of 1) and 3) follow from the following Example 4.4. □

Remark 4.3. The part 2) of Theorem 4.2 implies that the Bott class is not well-defined in the category of foliations whose holonomy pseudogroup is generated by quasiconformal local homeomorphisms. The first part of 3) is stronger than this. The second claim in 3) do not necessarily imply the first part. For example, it is known that the Godbillon-Vey class is invariant under foliation preserving diffeomorphisms of class C^1 even though it is not well-defined in the category of foliations of class C^1 [**10**], [**12**], [**2**].

Example 4.4. Let $\alpha \in \boldsymbol{C}$ and define a mapping $f : \boldsymbol{C} \to \boldsymbol{C}$ by setting $f_\alpha(z) = e^{2\pi\sqrt{-1}\alpha} z$. Note that α and $\alpha+1$ give the same mapping. Assume that $\alpha, \beta \in \boldsymbol{C} \setminus \boldsymbol{R}$, and define a homeomorphism φ of $\boldsymbol{C}$ to itself by setting $\varphi(z) = z\,|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}$. We also assume that $(\operatorname{Im}\beta)/(\operatorname{Im}\alpha) > 0$ so that φ is orientation preserving. The homeomorphism φ is in fact a quasiconformal homeomorphism and $\varphi \circ f_\alpha = f_\beta \circ \varphi$. The complex dilatation of φ is given by $\frac{\varphi_{\bar{z}}}{\varphi_z}(z) = \frac{\alpha-\beta}{\bar{\alpha}-\beta}\frac{z}{\bar{z}}$. Note that $\bar{\alpha} - \beta \neq 0$ because $(\operatorname{Im}\beta)/(\operatorname{Im}\alpha) > 0$.

Let $R = \left\{(t,z) \in \boldsymbol{R} \times \boldsymbol{C} \,\middle|\, |z| \leq e^{2\pi \operatorname{Im}\alpha t}\right\}$ and set $H_\alpha = R/(t+1, z) \sim (t, f_\alpha(z))$. The foliation of $\boldsymbol{R} \times \boldsymbol{C}$ by the lines $\boldsymbol{R} \times \{z\}$ naturally induces a transversely holomorphic foliation of H_α and the orientation of the leaves. Since $\alpha \in \boldsymbol{C} \setminus \boldsymbol{R}$, this foliation is transversal to the boundary. Hence ∂H_α is naturally a complex torus. According to Corollary 3.7, a new transversely holomorphic structure will be defined if we modify the complex structure of ∂H_α. For example, if we construct ∂H_β in a parallel way and replace the complex structure of ∂H_α with that of ∂H_β, the corresponding deformation is given by the mapping $\widetilde{\varphi} : H_\alpha \to H_\beta$ defined by $\widetilde{\varphi}(t, z) = (t, \varphi(z))$.

As firstly remarked, $H_\alpha = H_{\alpha+1}$. The value α modulo $\boldsymbol{Z}$ can be detected by the residue of the Bott class [**3**] (see also [**9**]). In this case it is equal to $\alpha \in H^1(S^1; \boldsymbol{C}/\boldsymbol{Z}) \cong \boldsymbol{C}/\boldsymbol{Z}$, where S^1 is the unique closed orbit in the torus.

One can obtain S^3 by gluing H_α and $H_{1/\alpha}$ in the standard way, and the induced transversely holomorphic flow (or its complex conjugate) is given by the holomorphic vector field $X = z\frac{\partial}{\partial z} + \alpha w \frac{\partial}{\partial w}$ of $\boldsymbol{C}^2$, where S^3 is considered as the unit sphere. By varying α, one obtains a family of transversely holomorphic flows on S^3. The Bott class of this flow is well-known to be $\alpha + (1/\alpha) \in H^3(S^3; \boldsymbol{C}/\boldsymbol{Z}) \cong \boldsymbol{C}/\boldsymbol{Z}$. On the other hand, the quasiconformal deformation on H_α as above naturally extends to a quasiconformal deformation on S^3. Hence desired deformations are obtained. The quasiconformal deformations obtained in this way always fix the two closed orbits, and the complex dilatations of these deformations have singularities there. It is closely related with the fact that the Julia set defined in [**7**] is also the union of two closed orbits.

Remark 4.5. By considering the glueing of H_α as above, one can obtain transversely holomorphic flows on the Lens spaces and calculate their Bott classes.

Remark 4.6. As in Example 4.4, transversely holomorphic flows on S^3 can be obtained from holomorphic vector fields of $\boldsymbol{C}^2$ having a Poincaré type singularity at the origin. Note that the orbits are naturally oriented. Assume that there are two closed orbits, then it is easy to see that these orbits are positively linked. On the other hand, taking the complex conjugate of Example 4.4 in the transverse direction, one can obtain a transversely holomorphic flow on S^3 which has two closed orbits which are negatively linked. This flow can be obtained by using the vector field $z\frac{\partial}{\partial z} + \alpha \bar{w} \frac{\partial}{\partial \bar{w}}$ on $\boldsymbol{C}^2$, or by gluing H_α and $H_{1/\alpha}$ also in a standard way by Corollary 3.7 but after turning $H_{1/\alpha}$ so that the direction of the longitude is reversed.

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