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Essential norms of differences of composition operators on H^∞

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Abstract. We study essential norms of differences of composition operators on the Banach algebra H^∞ of bounded analytic functions on the unit disk.

1. Introduction.

The algebra $H^\infty(= H^\infty(D))$ of the bounded analytic functions on the open unit disk D forms a Banach algebra under the supremum norm $\|f\|_\infty = \sup_{z\in D}|f(z)|$. An analytic self map φ on D induces through composition the bounded linear operator C_φ on H^∞ defined by

$$C_\varphi(f) = f\circ\varphi \quad (f \in H^\infty),$$

and the set of the analytic self maps on D will be denoted by $\mathscr{S}(D)$. Each φ in the closed unit ball $B(H^\infty)$ of H^∞ (except constant functions of modulus one) can be viewed as an analytic self map on D and hence determines C_φ. Let $\mathscr{C}(H^\infty)$ denote the set of all such composition operators equipped with the relative topology as a subset in the algebra of the bounded linear operators on H^∞ with the operator norm.

An important problem in the subject is to determine the topological structure (such as connected components and so on) of the set of all composition operators on the Hardy space H^2 [**1**], [**9**], [**12**]. An H^∞-version equally deserves investigation, and in [**10**] MacCluer, Ohno, and Zhao showed that C_φ and C_ψ sit in the same connected component of $\mathscr{C}(H^\infty)$ if and only if $\|C_\varphi - C_\psi\| < 2$. Indeed, they proved

$$\|C_\varphi - C_\psi\| = \lambda(\sigma(\varphi,\psi)),$$

where $\lambda(t) = \frac{2(1-\sqrt{1-t^2})}{t}, 0 < t \le 1$, and $\sigma(\varphi,\psi)$ is given by

$$\sigma(\varphi,\psi) = \sup_{z\in D}\rho(\varphi(z),\psi(z))$$

with the pseudo-hyperbolic distance

$$\rho(z,w) = |z-w|/|1-\overline{z}w| \quad (z,w\in D).$$

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One of the questions in [**10**] was if isolated points in $\mathscr{C}(H^\infty)$ are essentially isolated in the sence that they are so under the topology induced by the essential (semi-)norm

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\|; K \text{ is compact on } H^\infty\}.$$

Zheng and the authors gave an affirmative answer in [**8**], by showing that $\|C_\varphi - C_\psi\|_e \geq 1$ unless C_φ, C_ψ sit in the same connected component in $\mathscr{C}(H^\infty)$.

A next natural step is to find a handy expression for $\|C_\varphi - C_\psi\|_e$. It is known that if $\|\varphi\|_\infty < 1$ then C_φ is compact, and by [**13**], if $\|\varphi\|_\infty = 1$ then $\|C_\varphi\|_e = 1$. So we are interesting in the case $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. In this paper, as partial results towards this goal, we will characterize a pair (φ, ψ) satisfying $\|C_\varphi - C_\psi\|_e = 2$, and for C_φ, C_ψ in the same connected component we will show

$$\|C_\varphi - C_\psi\|_e = \lambda(\sigma_\infty(\varphi, \psi))$$

with

$$\sigma_\infty(\varphi, \psi) = \limsup_{|\varphi(z)\psi(z)| \to 1} \rho(\varphi(z), \psi(z))$$

under the additional assumption that

$$E_{L^\infty}(|\varphi|) = \{m \in M(L^\infty); |\varphi(m)| = 1\}$$

is a peak set for H^∞. The present work was motivated by the MacCluer-Ohno-Zhao theorem starting that $C_\varphi - C_\psi$ is a compact operator on H^∞ if and only if

$$\limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0,$$

and the following estimate due to Gorkin, Mortini, and Suárez ([**6**]);

$$\max\left\{\limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)), \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z))\right\}$$
$$\leq \|C_\varphi - C_\psi\|_e \leq 4 \max\left\{\limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)), \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z))\right\},$$

provided $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} = 1$.

In Section 2, we will introduce the notion of *σ-asymptotic interpolating sequences* ($\sigma \in (0, 1]$), which is motivated by that of asymptotic interpolating sequences in [**6**], [**8**]. Let $\{(w_n, w_n')\}_n$ be a sequence in $D \times D$ with $|w_n| \to 1$ and $|w_n'| \to 1$. If $\rho(w_n, w_n') \to \sigma$, then the sequence is shown to admit a subsequence $\{(z_k, z_k')\}_k$ such that $\{z_k\}_k \cup \{z_k'\}_k$ is $\lambda(\sigma)/2$-asymptotic interpolating. This technique will enable us to obtain lower bounds for $\|C_\varphi - C_\psi\|_e$. Namely, in Section 3 we show

$$\lambda(\sigma_\infty(\varphi,\psi)) \le \|C_\varphi - C_\psi\|_e$$

under the assumption $\|\varphi\psi\|_\infty = 1$. Hence, we conclude $\|C_\varphi - C_\psi\|_e = 2$ as long as $\|\varphi\psi\|_\infty = 1$ and $\sigma_\infty(\varphi,\psi) = 1$, and we have to deal with the two remaining cases: (i) $\|\varphi\psi\|_\infty < 1$, (ii) $\|\varphi\psi\|_\infty = 1$ and $\sigma_\infty(\varphi,\psi) < 1$. In Section 4, we study upper bounds for $\|C_\varphi - C_\psi\|_e$. In case (i) we show $\|C_\varphi - C_\psi\|_e < 2$. Moreover, if $E_{L^\infty}(|\varphi|)$ and $E_{L^\infty}(|\psi|)$ are peak sets for H^∞ and not so close, then we have $\|C_\varphi - C_\psi\|_e = 1$. In case (ii) we prove $\|C_\varphi - C_\psi\|_e < 2$. Moreover, if $E_{L^\infty}(|\varphi|) = E_{L^\infty}(|\psi|)$ and it is a peak set for H^∞, then we have $\|C_\varphi - C_\psi\|_e \le \lambda(\sigma_\infty(\varphi,\psi))$. It is known that the condition $\|C_\varphi - C_\psi\| < 2$ gives rise to an equivalence relation in $\mathscr{S}(D)$ ([**10**]). We show that it is no longer true for the essential norm $\|\cdot\|_e$. Our analysis indicates that the essential norm $\|C_\varphi - C_\psi\|_e$ is closely related to the quantity

$$\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\|,$$

and these values are calculated in Section 5.

2. Asymptotic interpolating sequences.

First, we give some definitions and notations used in this paper. Let σ be a positive number with $\sigma \le 1$. A sequence $\{z_k\}_k$ in D is called *σ-asymptotic interpolating* if for every sequence of complex numbers $\{a_k\}_k$ with $|a_k| \le \sigma$ for every k, there exists $h \in B(H^\infty)$ satisfying $|h(z_k) - a_k| \to 0$ as $k \to \infty$, see [**6**], [**8**].

We denote by $\mathscr{M}(H^\infty)$ the maximal ideal space of H^∞, which is the set of nonzero multiplicative linear functionals on H^∞. With the weak*-topology, $\mathscr{M}(H^\infty)$ is a compact Hausdorff space. For a subset E in $\mathscr{M}(H^\infty)$, we denote by $\overline{E}$ the closure of E in $\mathscr{M}(H^\infty)$. We identify a function f in H^∞ with its Gelfand transform; $\hat{f}(\zeta) = \zeta(f)$, $\zeta \in \mathscr{M}(H^\infty)$. For each point z in D, the evaluation of functions f in H^∞: $f \to f(z)$ is a nonzero multiplicative linear functional, so that we think of D as an open subset in $\mathscr{M}(H^\infty)$. The well known corona theorem says that D is dense in $\mathscr{M}(H^\infty)$, see [**2**]. We also identify a function in H^∞ and its boundary function, and we think of H^∞ as a supremum norm closed subalgebra of L^∞, where L^∞ is the usual Lebesgue space on the unit circle ∂D. We may think of $\mathscr{M}(L^\infty)$ as a closed subset in $\mathscr{M}(H^\infty)$. It is known that $\mathscr{M}(L^\infty)$ is the Shilov boundary of H^∞, see [**7**].

We denote by $C(\mathscr{M}(H^\infty))$ the algebra of continuous functions on $\mathscr{M}(H^\infty)$. For a function f in $C(\mathscr{M}(H^\infty))$, we define

$$E_{H^\infty}(f) = \{m \in \mathscr{M}(H^\infty); f(m) = 1\}.$$

For a function f in $C(\mathscr{M}(L^\infty))$, we define

$$E_{L^\infty}(f) = \{m \in \mathscr{M}(L^\infty); f(m) = 1\}.$$

A nonempty closed subset E of $\mathscr{M}(L^\infty)$ is called a peak set for H^∞ if there exists $\varphi \in H^\infty$ such that $\varphi = 1$ on E and $|\varphi| < 1$ on $\mathscr{M}(L^\infty) \setminus E$. In this case, φ is called a peaking function for E.

For a point z_0 in D and a positive number r with $r < 1$, we write $\Delta(z_0, r)$ for the pseudo-hyperbolic disk with center z_0 and radius r,

$$\Delta(z_0, r) = \{w \in D; \rho(z_0, w) \leq r\}.$$

The pseudo-hyperbolic disk $\Delta(z_0, r)$ is a Euclidean disk with center c and radius R, where

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2}\, z_0, \quad R = r\, \frac{1 - |z_0|^2}{1 - r^2|z_0|^2}.$$

By Schwarz and Pick's lemma, $f(\Delta(z_0, r)) \subset \Delta(f(z_0), r)$ for every f in $B(H^\infty)$, see [**4**, pp. 2–3].

For distinct two points z, w in D, it is known that

$$\sup\{|f(z) - f(w)|; f \in B(H^\infty)\} = \frac{2\bigl(1 - \sqrt{1 - \rho(z, w)^2}\bigr)}{\rho(z, w)}.$$

In this paper, this equation plays an important role, so we introduce a function $\lambda(t)$ defined by

$$\lambda(t) = \frac{2\bigl(1 - \sqrt{1 - t^2}\bigr)}{t}, \quad 0 < t \leq 1.$$

It is not difficult to see that λ is an increasing function on $(0, 1]$, and $\lim_{t \to 0^+} \lambda(t) = 0$, so we define $\lambda(0) = 0$. Thus we get

$$|f(z) - f(w)| \leq \lambda(\rho(z, w)) \tag{2.1}$$

for every $f \in B(H^\infty)$, and there exists a function $g \in B(H^\infty)$ satisfying $|g(z) - g(w)| = \lambda(\rho(z, w))$, see [**4**, p. 42]. By the definition of the function λ, for every complex numbers a, b with $\max\{|a|, |b|\} \leq \lambda(\rho(z, w))/2$, there is $h \in B(H^\infty)$ with $h(z) = a$ and $h(w) = b$. This fact leads to the following lemma.

LEMMA 2.1. *Let σ be a positive number with $\sigma \leq 1$, and $\{z_n\}_n$, $\{z'_n\}_n$ be sequences in D with $\rho(z_n, z'_n) \to \sigma$. For two sequences of complex numbers $\{a_n\}_n$, $\{a'_n\}_n$ with $\sup_n\{|a_n|, |a'_n|\} \leq \lambda(\sigma)/2$, there is a sequence of functions $\{h_n\}_n$ in $B(H^\infty)$ satisfying $|h_n(z_n) - a_n| \to 0$ and $|h_n(z'_n) - a'_n| \to 0$ as $n \to \infty$.*

PROOF. Take a sequence of numbers $\{r_j\}_j$ with $0 < r_j < 1$ and $r_j \to 1$. By the assumption, for each j there is a positive integer N_j such that $r_j\lambda(\sigma)/2 < \lambda(\rho(z_n, z'_n))/2$ for every $n \geq N_j$. We may assume that $N_j < N_{j+1}$ for every j. We have $\max\{|r_j a_n|, |r_j a'_n|\} < \lambda(\rho(z_n, z'_n))/2$ for every $n \geq N_j$. For each n with $N_j \leq n < N_{j+1}$, there is $h_n \in B(H^\infty)$ satisfying $h_n(z_n) = r_j a_n$ and $h_n(z'_n) = r_j a'_n$. Since $j \to \infty$ as $n \to \infty$, we get the assertion. □

We denote by $\mathscr{A}$ the disk algebra, that is, $\mathscr{A}$ is the space of analytic functions on D which can be extended continuously on the closed unit disk $\overline{D}$. The main result in this section is the following. The idea for the proof is essentially the same as the one of Theorem 3.1 in [**8**].

THEOREM 2.2. *Let σ be a positive number with $\sigma \leq 1$ and $\{(w_n, w'_n)\}_n$ be a sequence in $D \times D$ with $|w_n| \to 1$ and $|w'_n| \to 1$. If $\rho(w_n, w'_n) \to \sigma$ as $n \to \infty$, then there is a subsequence $\{(z_k, z'_k)\}_k$ of $\{(w_n, w'_n)\}_n$ such that $\{z_k\}_k \cup \{z'_k\}_k$ is $\lambda(\sigma)/2$-asymptotic interpolating.*

PROOF. Assume that $w_n \to \alpha$ and $w'_n \to \alpha'$ as $n \to \infty$ for some $\alpha, \alpha' \in \partial D$. Then there are two functions f, g in $\mathscr{A}$ satisfying

$$f(\alpha) = f(\alpha') = 1, \quad 0 < |f| < 1 \text{ on } \overline{D} \setminus \{\alpha, \alpha'\} \tag{2.2}$$

and

$$g(\alpha) = g(\alpha') = 0, \quad 0 < |g| \leq 1 \text{ on } \overline{D} \setminus \{\alpha, \alpha'\}.$$

Write $g_n = g^{1/n}$ for every positive integer n. Then $g_n \in \mathscr{A}$, $\|g_n\|_\infty \leq 1$, $g_n(\alpha) = g_n(\alpha') = 0$, and

$$\lim_{n \to 1} |g_n(z)| = 1 \quad \text{for each } z \in D. \tag{2.3}$$

Write

$$c_k = 1 - (1/2)^k \tag{2.4}$$

for $k \geq 1$. By induction, we shall find two sequences of increasing positive integers $\{m_k\}_k, \{n_k\}_k$, and a subsequence $\{(z_k, z'_k)\}_k$ in $\{(w_n, w'_n)\}_n$ satisfying the following four conditions; for every $N \geq 1$,

$$\sup_{z \in \overline{D}} \sum_{k=1}^{N} |(c_k f^{m_k} g_{n_k})(z)| < 1, \tag{2.5}$$

$$\max\left\{ \sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z_N)|, \sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z'_N)| \right\} < 1 - c_N, \tag{2.6}$$

$$\min\left\{ |(f^{m_N} g_{n_N})(z_N)|, |(f^{m_N} g_{n_N})(z'_N)| \right\} > c_N, \tag{2.7}$$

and

$$\max\left\{ |f^{m_N}(z_j)|, |f^{m_N}(z'_j)| \right\} < 1 - c_N \quad \text{for } 1 \leq j < N. \tag{2.8}$$

First, take $m_1 = 1$. By (2.2), there is a point (z_1, z'_1) in $\{(w_n, w'_n)\}_n$ with

$\min\{|f(z_1)|, |f(z'_1)|\} > c_1$. By (2.3), there is a positive integer n_1 satisfying

$$\min\left\{|(f^{m_1} g_{n_1})(z_1)|, |(f^{m_1} g_{n_1})(z'_1)|\right\} > c_1.$$

Then we have (2.5) and (2.7) for $N = 1$.

Next, assume that $\{m_k\}_{k=1}^N, \{n_k\}_{k=1}^N$, and $\{(z_k, z'_k)\}_{k=1}^N$ are chosen to satisfy the conditions. Put

$$F_N = \sum_{k=1}^N |c_k f^{m_k} g_{n_k}| \quad \text{on } \overline{D}.$$

Noting that $g_n(\alpha) = g_n(\alpha') = 0$, we have that $F_N(\alpha) = F_N(\alpha') = 0$. Take an open subset U_N of $\overline{D}$ containing $\{\alpha, \alpha'\}$ such that

$$\{z_k, z'_k; 1 \le k \le N\} \cap U_N = \varnothing \tag{2.9}$$

and

$$F_N < 1 - c_{N+1} \quad \text{on } U_N. \tag{2.10}$$

By (2.4) and (2.5), there is a positive integer m_{N+1} such that $m_N < m_{N+1}$,

$$|f^{m_{N+1}}| < 1 - c_{N+1} \quad \text{on } \overline{D} \setminus U_N, \tag{2.11}$$

and

$$F_N + |f^{m_{N+1}}| < 1 \quad \text{on } \overline{D} \setminus U_N. \tag{2.12}$$

Combining (2.9) and (2.11), we have (2.8) for $N + 1$. By (2.2) again, there is a point (z_{N+1}, z'_{N+1}) in $\{(w_n, w'_n)\}_n \cap (U_N \times U_N)$ with

$$\min\left\{|f^{m_{N+1}}(z_{N+1})|, |f^{m_{N+1}}(z'_{N+1})|\right\} > c_{N+1}.$$

By (2.10), we have (2.6) for $N + 1$. By (2.3), there is a positive integer n_{N+1} satisfying $n_N < n_{N+1}$ and

$$\min\left\{|(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1})|, |(f^{m_{N+1}} g_{n_{N+1}})(z'_{N+1})|\right\} > c_{N+1}.$$

This leads (2.7) for $N + 1$. By (2.12), $F_N + |f^{m_{N+1}} g_{n_{N+1}}| < 1$ on $\overline{D} \setminus U_N$. Since $\|f^{m_{N+1}} g_{n_{N+1}}\|_\infty < 1$, by (2.10)

$$\sup_{z \in \overline{D}} \left(F_N(z) + c_{N+1} |(f^{m_{N+1}} g_{n_{N+1}})(z)|\right) < 1.$$

Thus we get (2.5) for $N+1$. This completes the induction.

By (2.4) and (2.8), we have

$$\sum_{k=N+1}^{\infty} |(c_k f^{m_k} g_{n_k})(z_N)| < \sum_{k=N+1}^{\infty} (1/2)^k = 1/2^N. \tag{2.13}$$

Let $\{a_k\}_k$ and $\{a'_k\}_k$ be sequences of complex numbers with

$$\sup_k \{|a_k|, |a'_k|\} \leq \lambda(\sigma)/2.$$

Set

$$A_k = \frac{a_k |(f^{m_k} g_{n_k})(z_k)|}{(f^{m_k} g_{n_k})(z_k)}, \quad A'_k = \frac{a'_k |(f^{m_k} g_{n_k})(z'_k)|}{(f^{m_k} g_{n_k})(z'_k)}. \tag{2.14}$$

Then $\max_k \{|A_k|, |A'_k|\} \leq \lambda(\sigma)/2$. Since $\rho(z_k, z'_k) \to \sigma$, by Lemma 2.1 there is $h_k \in B(H^\infty)$ satisfying

$$\lim_{k\to\infty} |h_k(z_k) - A_k| = 0, \quad \lim_{k\to\infty} |h_k(z'_k) - A'_k| = 0. \tag{2.15}$$

Here we define a function $h(z)$ as

$$h(z) = \sum_{k=1}^{\infty} c_k h_k(z)(f^{m_k} g_{n_k})(z)$$

for $z \in D$. Then by (2.5), $h \in B(H^\infty)$. We have

$$\lim_{N\to\infty} |a_N - c_N h_N(z_N)(f^{m_N} g_{n_N})(z_N)| = 0. \tag{2.16}$$

For,

$$\begin{aligned}
&|a_N - c_N h_N(z_N)(f^{m_N} g_{n_N})(z_N)| \\
&\quad = |a_N - c_N(h_N(z_N) - A_N)(f^{m_N} g_{n_N})(z_N) - c_N A_N (f^{m_N} g_{n_N})(z_N)| \\
&\quad \leq |h_N(z_N) - A_N| + |a_N|(1 - c_N|(f^{m_N} g_{n_N})(z_N)|) \qquad \text{by (2.14)} \\
&\quad = |h_N(z_N) - A_N| + |a_N|(1 - c_N) + |a_N| c_N (1 - |(f^{m_N} g_{n_N})(z_N)|) \\
&\quad \leq |h_N(z_N) - A_N| + |a_N|(1 - c_N) + |a_N| c_N / 2^N \qquad \text{by (2.4) and (2.7)} \\
&\quad \to 0 \qquad \text{as } N \to \infty \ \text{ by (2.15)}.
\end{aligned}$$

Therefore by (2.6) and (2.13),

$$\begin{aligned}|h(z_N)-a_N| &\le |a_N - c_N h_N(z_N)(f^{m_N}g_{n_N})(z_N)| \\ &\quad + \sum_{k=1}^{N-1}|(c_k f^{m_k}g_{n_k})(z_N)| + \sum_{k=N+1}^{\infty}|c_k(f^{m_k}g_{n_k})(z_N)| \\ &< |a_N - c_N h_N(z_N)(f^{m_N}g_{n_N})(z_N)| + 2(1/2)^N \\ &\to 0 \qquad \text{as } N\to\infty \text{ by (2.16).}\end{aligned}$$

Similarly, we can prove that $|h(z'_N) - a'_N| \to 0$ as $N \to \infty$. □

3. Lower bounds.

In this section, we will obtain lower bounds for $\|C_\varphi - C_\psi\|_e$. The main idea in this section comes from Lemma 4.2 of [**8**]. The following theorem is a key for lower bounds.

Theorem 3.1. *Let T be a bounded linear operator on H^∞ and σ be a positive number. Suppose that there exist a sequence $\{h_k\}_k$ in $B(H^\infty)$ and a sequence $\{x_n\}_n$ in $\mathscr{M}(H^\infty)$ satisfying the following conditions;*

(i) $\lim_{n\to\infty}(Th_k)(x_n) = \sigma$ *for each fixed* k,
(ii) $\lim_{n\to\infty}\big(\lim_{k\to\infty}(Th_k)(x_n)\big) = -\sigma$.

Then $\|T\|_e \ge \sigma$.

Proof. Let $\{h_k\}_k$ be a sequence in $B(H^\infty)$ and $\{x_n\}_n$ in $\mathscr{M}(H^\infty)$ satisfying conditions (i) and (ii). Let K be an arbitrary compact operator on H^∞. Then we have

$$|(Th_k)(x_n) + (Kh_k)(x_n)| \le \|T+K\|. \tag{3.1}$$

Let x_0 be a cluster point of $\{x_n\}_n$ in $\mathscr{M}(H^\infty)$. By (i),

$$|\sigma + (Kh_k)(x_0)| \le \|T+K\|$$

for every k. Since K is compact, considering a subsequence of $\{h_k\}_k$ we may assume that $\|Kh_k - h\|_\infty \to 0$ as $k\to\infty$ for some $h \in H^\infty$. Then

$$|\sigma + h(x_0)| \le \|T+K\|. \tag{3.2}$$

On the other hand, letting $k\to\infty$ in (3.1), we have

$$\Big|\lim_{k\to\infty}(Th_k)(x_n) + h(x_n)\Big| \le \|T+K\|$$

for each n, and by (ii), $|-\sigma + h(x_0)| \le \|T+K\|$. Combining with (3.2), we get

$$2\sigma \le |\sigma - h(x_0)| + |\sigma + h(x_0)| \le 2\|T+K\|.$$

This completes the proof. □

THEOREM 3.2. *Let φ, ψ be functions in $\mathscr{S}(D)$. If $\|\varphi\psi\|_\infty = 1$, then $\|C_\varphi - C_\psi\|_e \geq \lambda(\sigma_\infty(\varphi, \psi))$.*

PROOF. By the definition of $\sigma_\infty(\varphi, \psi)$, there is a sequence $\{z_n\}_n$ in D satisfying $|z_n| \to 1$, $|\varphi(z_n)| \to 1$, $|\psi(z_n)| \to 1$, and $\rho(\varphi(z_n), \psi(z_n)) \to \sigma_\infty(\varphi, \psi)$. By Theorem 2.2, we may assume that $\{\varphi(z_n)\}_n \cup \{\psi(z_n)\}_n$ is $\lambda(\sigma_\infty(\varphi, \psi))/2$-asymptotic interpolating. Write $E = \{z_n\}_n$ and take a sequence of subsets $\{E_k\}_k$ in E such that $E_{k+1} \subset E_k$ and $E_k \setminus E_{k+1}$ is an infinite set for every k. Then for each positive integer k, there exists $h_k \in B(H^\infty)$ satisfying the following four conditions;

$$|h_k(\varphi(z_n)) - \lambda(\sigma_\infty(\varphi, \psi))/2| \to 0, \tag{3.3}$$

$$|h_k(\psi(z_n)) + \lambda(\sigma_\infty(\varphi, \psi))/2| \to 0 \tag{3.4}$$

as $|z_n| \to 1$ and $z_n \in E_k$, and

$$|h_k(\varphi(z_n)) + \lambda(\sigma_\infty(\varphi, \psi))/2| \to 0, \tag{3.5}$$

$$|h_k(\psi(z_n)) - \lambda(\sigma_\infty(\varphi, \psi))/2| \to 0 \tag{3.6}$$

as $|z_n| \to 1$ and $z_n \notin E_k$. Take a point x_k in $\overline{E_k \setminus E_{k+1}} \setminus (E_k \setminus E_{k+1})$. Since $E_n \subset E_k$ for $n \geq k$, $x_n \in \overline{E_k} \setminus E_k$ for every $n \geq k$. By (3.3) and (3.4),

$$((C_\varphi - C_\psi)h_k)(x_n) = \lambda(\sigma_\infty(\varphi, \psi)) \quad \text{for every } n \geq k.$$

Noting that $E_n \setminus E_{n+1} \subset E \setminus E_k$ for $n < k$, we have that $x_n \in \overline{E \setminus E_k} \setminus (E \setminus E_k)$ for $n < k$. Thus (3.5) and (3.6) give

$$((C_\varphi - C_\psi)h_k)(x_n) = -\lambda(\sigma_\infty(\varphi, \psi)) \quad \text{for every } n < k.$$

By Theorem 3.1, we get the assertion. □

COROLLARY 3.3. *Let φ, ψ be functions in $\mathscr{S}(D)$ with $\|\varphi\psi\|_\infty = 1$. If $\varphi \neq \psi$ on $E_{H^\infty}(|\varphi\psi|)$, then $\|C_\varphi - C_\psi\|_e = 2$.*

PROOF. Let x be a point in $\mathscr{M}(H^\infty)$ satisfying $|(\varphi\psi)(x)| = 1$ and $\varphi(x) \neq \psi(x)$. By the corona theorem, there exists a net $\{z_\alpha\}_\alpha$ in D with $z_\alpha \to x$. Then $\varphi(z_\alpha) \to \varphi(x)$ and $\psi(z_\alpha) \to \psi(x)$. Since $|\varphi(x)| = |\psi(x)| = 1$ and $\varphi(x) \neq \psi(x)$, $\rho(\varphi(z_\alpha), \psi(z_\alpha)) \to 1$. Hence by Theorem 3.2, we have the assertion. □

COROLLARY 3.4. *Let E be a measurable subset in ∂D with $d\theta(E) > 0$. Let φ, ψ be functions in $\mathscr{S}(D)$ satisfying $\varphi \neq \psi$ and $|\varphi| = |\psi| = 1$ for almost all points in E. Then $\|C_\varphi - C_\psi\|_e = 2$.*

COROLLARY 3.5. *Let φ, ψ be functions in $\mathscr{S}(D)$ with $\|\varphi\psi\|_\infty = 1$. If $E_{H^\infty}(|\varphi\psi|) \neq E_{H^\infty}(|\varphi|)$, and $E_{H^\infty}(|\varphi\psi|)$ is not an open and closed subset of $E_{H^\infty}(|\varphi|)$, then*

$\sigma_\infty(\varphi,\psi)=1$ *and* $\|C_\varphi - C_\psi\|_e = 2$.

Proof. By the assumption, there exists a sequence $\{x_n\}_n$ in $\mathscr{M}(H^\infty)\setminus D$ such that $|\varphi(x_n)|=1$, $|\psi(x_n)|<1$ for every n, and $|\psi(x_n)|\to 1$. By the corona theorem, for each fixed n there exists a net $\{z_{n,\alpha}\}_\alpha$ in D satisfying $\lim_{\alpha\to\infty}|\varphi(z_{z_n,\alpha})|=1$ and $\lim_{\alpha\to\infty}|\psi(z_{z_n,\alpha})|=|\psi(x_n)|<1$. Then $\rho(\varphi(z_{z_n,\alpha}),\psi(z_{z_n,\alpha}))\to 1$ as $\alpha\to\infty$, so that there exists α_n such that $|\varphi(z_{n,\alpha_n})|>1-1/n$, $|\psi(z_{n,\alpha_n})|>|\psi(x_n)|-1/n$, and $\rho(\varphi(z_{n,\alpha_n}),\psi(z_{n,\alpha_n}))>1-1/n$. Hence $|\psi(z_{n,\alpha_n})|\to 1$ as $n\to\infty$, and by Theorem 3.2 we get the assertion. □

The following corollary is one of the main results in [**13**].

Corollary 3.6. *Let* φ *be a function in* $\mathscr{S}(D)$. *If* C_φ *is not a compact operator on* H^∞, *then* $\|C_\varphi\|_e = 1$.

Proof. We have $\|C_\varphi\|=1$ and $\|C_\varphi\|_e\le 1$. Since C_φ is not compact, $\|\varphi\|_\infty=1$. Put $\psi=-\varphi$. Then φ and ψ satisfy the assumption of Corollary 3.3, so that $2=\|C_\varphi-C_\psi\|_e\le\|C_\varphi\|_e+\|C_\psi\|_e\le 2$. Thus we get $\|C_\varphi\|_e=1$. □

4. Upper bounds.

In this section we will obtain upper bounds for the essential norm of $C_\varphi - C_\psi$. For each $g\in H^\infty$, define the multiplication operator M_g on H^∞ by $M_g f = gf$ for every $f\in H^\infty$. Clearly, M_g is a bounded linear operator on H^∞. For a function f in H^∞ and a subset E of D, write

$$\|f\|_E = \sup_{z\in E}|f(z)|.$$

The following lemma characterizes the compactness of $M_g C_\varphi$.

Lemma 4.1. *Let* φ *be a function in* $\mathscr{S}(D)$ *with* $\|\varphi\|_\infty = 1$ *and* $g\in H^\infty$. *Then* $M_g C_\varphi$ *is a compact operator on* H^∞ *if and only if* $\lim_{|\varphi(z)|\to 1} g(z)=0$, *that is,* $g=0$ *on* $E_{H^\infty}(|\varphi|)$.

Proof. Write $K=M_gC_\varphi$. Suppose that $\lim_{|\varphi(z)|\to 1} g(z)=0$. We will show that K is compact. Let $\{f_j\}_j$ be a sequence of functions in $B(H^\infty)$ satisfying $f_j\to 0$ uniformly on each compact subset in D. It is sufficient to show that $\|Kf_j\|_\infty\to 0$ as $j\to\infty$, see [**3**, Proposition 3.11]. For every $\varepsilon>0$, there exists a positive number r with $r<1$ satisfying $|g|<\varepsilon$ on U_r, where $U_r=\{z\in D;|\varphi(z)|>r\}$. Then we have $\|Kf_j\|_{U_r}\le\varepsilon$ for every j. Since $\|\varphi\|_{D\setminus U_r}\le r<1$,

$$\lim_{j\to\infty}\|Kf_j\|_{D\setminus U_r}=\lim_{j\to\infty}\|g(f_j\circ\varphi)\|_{D\setminus U_r}=0.$$

Thus we obtain $\|Kf_j\|_\infty\to 0$ as $j\to\infty$.

Next, suppose that $\lim_{|\varphi(z)|\to 1} g(z)\ne 0$. Then there is a sequence $\{z_k\}_k$ in D satisfying $|\varphi(z_k)|\to 1$ and $|g(z_k)|\to c$ for some $c>0$. We may assume that $\varphi(z_k)\to$

$a \in \partial D$. Write $f_n = (z+a)^n/(2a)^n$. Then $\|f_n\|_\infty = 1$, $f_n(a) = 1$, and $f_n \to 0$ uniformly on every compact subset in D. We have

$$\|Kf_n\|_\infty \geq \lim_{k\to\infty} |g(z_k)|\,|(f_n \circ \varphi)(z_k)| = c > 0.$$

Therefore K is not compact. □

Let f be a continuous function on $\mathscr{M}(L^\infty)$. Recall that

$$E_{L^\infty}(f) = \{m \in \mathscr{M}(L^\infty); f(m) = 1\}$$

and for $\varphi, \psi \in \mathscr{S}(D)$,

$$\sigma_\infty(\varphi,\psi) = \limsup_{|(\varphi\psi)(z)|\to 1} \rho(\varphi(z),\psi(z)).$$

For each $m \in \mathscr{M}(H^\infty)$, there is a representing measure μ_m on $\mathscr{M}(L^\infty)$, that is, $f(m) = \int_{\mathscr{M}(L^\infty)} f\,d\mu_m$ for every $f \in H^\infty$. We denote by $\operatorname{supp}\mu_m$ the closed support set of μ_m.

THEOREM 4.2. *Let φ, ψ be functions in $\mathscr{S}(D)$ with $\|\varphi\psi\|_\infty = 1$. Suppose that the following conditions hold*;

(i) $E_{L^\infty}(|\varphi|) = E_{L^\infty}(|\psi|)$,
(ii) $E_{L^\infty}(|\varphi|)$ *is a peak set for* H^∞.

Then $\|C_\varphi - C_\psi\|_e \leq \lambda(\sigma_\infty(\varphi,\psi))$.

PROOF. We may assume that $\varphi \neq \psi$. If $\sigma_\infty(\varphi,\psi) = 0$, by [**10**, Theorem 3] $C_\varphi - C_\psi$ is compact. Hence $\|C_\varphi - C_\psi\|_e = 0 = \lambda(0)$.

If $\sigma_\infty(\varphi,\psi) = 1$, then $\lambda(\sigma_\infty(\varphi,\psi)) = 2$. Since $\|C_\varphi\|_e \leq \|C_\varphi\| \leq 1$,

$$\|C_\varphi - C_\psi\|_e \leq \|C_\varphi\|_e + \|C_\psi\|_e \leq 2.$$

Thus $\|C_\varphi - C_\psi\|_e \leq 2 = \lambda(\sigma_\infty(\varphi,\psi))$.

So, we assume that $0 < \sigma_\infty(\varphi,\psi) < 1$. Then we have that $\varphi = \psi$ on $E_{H^\infty}(|\varphi|)$, especially

$$\varphi = \psi \quad \text{on } E_{L^\infty}(|\varphi|). \tag{4.1}$$

Since $\varphi \neq \psi$, $E_{L^\infty}(|\varphi|) \neq \mathscr{M}(L^\infty)$. Let F be a peaking function in H^∞ for $E_{L^\infty}(|\varphi|)$. For each positive integer k, define

$$U_k = \{z \in D; |F(z) - 1| < 1/k\}$$

and

$$\sigma_k = \sup_{z\in U_k} \rho(\varphi(z),\psi(z)). \tag{4.2}$$

We shall prove that

$$\lim_{k\to\infty} \sigma_k = \sigma_\infty(\varphi, \psi). \tag{4.3}$$

Let $\{z_n\}_n$ be a sequence in D with $|(\varphi\psi)(z_n)| \to 1$. Let m be a cluster point of $\{z_n\}_n$ in $\mathscr{M}(H^\infty)$. Then $|\varphi(m)| = |\psi(m)| = 1$, so that $\operatorname{supp}\mu_m \subset E_{L^\infty}(|\varphi|)$. Since F is a peaking function for $E_{L^\infty}(|\varphi|)$, $F = 1$ on $\operatorname{supp}\mu_m$, so that $F(m) = 1$. This implies that $F(z_n) \to 1$ as $n \to \infty$. Therefore by (4.2), we get $\sigma_\infty(\varphi, \psi) \le \sigma_k$ for every k. It is clear that $\{\sigma_k\}_k$ is decreasing. To prove (4.3) by contradiction, we assume that there exists $\delta > 0$ such that $\sigma_\infty(\varphi, \psi) + \delta < \sigma_k$ for every k. Then for each k, there exists $w_k \in U_k$ satisfying

$$\sigma_\infty(\varphi, \psi) + \delta < \rho(\varphi(w_k), \psi(w_k)). \tag{4.4}$$

Let ζ be a cluster point of $\{w_k\}_k$ in $\mathscr{M}(H^\infty)$. Then $F(\zeta) = 1$, so that $\operatorname{supp}\mu_\zeta \subset E_{L^\infty}(|\varphi|)$. By (4.1), we have $\varphi(\zeta) = \psi(\zeta)$. Let $\{w_{k_\alpha}\}_\alpha$ be a net in $\{w_k\}_k$ with $w_{k_\alpha} \to \zeta$. If $|\varphi(\zeta)| = 1$, then $|(\varphi\psi)(w_{k_\alpha})| \to 1$. Hence

$$\limsup_{\alpha\to\infty} \rho(\varphi(w_{k_\alpha}), \psi(w_{k_\alpha})) \le \sigma_\infty(\varphi, \psi).$$

But this contradicts (4.4). So, we have $|\varphi(\zeta)| < 1$. In this case, since $\varphi(\zeta) = \psi(\zeta)$, we have

$$\lim_{\alpha\to\infty} \rho(\varphi(w_{k_\alpha}), \psi(w_{k_\alpha})) = 0.$$

This also contradicts (4.4). Thus we get (4.3).

For each positive integer n, define

$$\tau_n(z) = 1 - F^n(z), \quad z \in D \tag{4.5}$$

and

$$K_n = M_{\tau_n}(C_\varphi - C_\psi) \quad \text{on } H^\infty. \tag{4.6}$$

By the above argument, we note that $E_{H^\infty}(|\varphi|) \cup E_{H^\infty}(|\psi|) \subset E_{H^\infty}(F)$. By the assumption and Lemma 4.1, (4.5) gives that K_n is a compact operator on H^∞.

We need to prove that

$$\liminf_{n\to\infty} \|C_\varphi - C_\psi - K_n\| \le \lambda(\sigma_\infty(\varphi, \psi)). \tag{4.7}$$

Let ε be a positive number and $f \in B(H^\infty)$. Combining (4.5) with (4.6), we have

$$(C_\varphi - C_\psi - K_n)f = F^n(f \circ \varphi - f \circ \psi). \tag{4.8}$$

By (4.3), there exists a positive integer k_1 with $\lambda(\sigma_{k_1}) < \lambda(\sigma_\infty(\varphi,\psi)) + \varepsilon$. Thus (2.1) and (4.2) yield

$$|f \circ \varphi - f \circ \psi| \le \lambda(\sigma_{k_1}) < \lambda(\sigma_\infty(\varphi,\psi)) + \varepsilon \quad \text{on } U_{k_1}.$$

Hence by (4.8),

$$\|(C_\varphi - C_\psi - K_n)f\|_{U_{k_1}} < \lambda(\sigma_\infty(\varphi,\psi)) + \varepsilon$$

for every n. Since $\|F\|_{D\setminus U_{k_1}} < 1$, there exists a positive integer n_0 with $\|F^{n_0}\|_{D\setminus U_{k_1}} < \varepsilon/2$. Then by (4.8) again,

$$\|(C_\varphi - C_\psi - K_{n_0})f\|_{D\setminus U_{k_1}} < \varepsilon.$$

Therefore

$$\|C_\varphi - C_\psi - K_{n_0}\| < \lambda(\sigma_\infty(\varphi,\psi)) + \varepsilon.$$

Thus we get (4.7). This completes the proof. □

Using the same idea as in the proof of Theorem 4.2, we have the following.

THEOREM 4.3. *Let φ,ψ be functions in $\mathscr{S}(D)$ satisfying $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ and $\|\varphi\psi\|_\infty < 1$. If there exist peak sets E_1 and E_2 for H^∞ such that $E_{L^\infty}(|\varphi|) \subset E_1$, $E_{L^\infty}(|\psi|) \subset E_2$, and $E_1 \cap E_2 = \varnothing$, then $\|C_\varphi - C_\psi\|_e = 1$.*

PROOF. By the assumption, $\sup_{z\in D} \rho(\varphi(z),\psi(z)) = 1$. Then by [**8**, Lemma 4.2] and [**10**, Theorem 1], $\|C_\varphi - C_\psi\|_e \ge 1$. Let F_1 and F_2 be peaking functions in H^∞ for the peak sets E_1 and E_2, respectively. For each positive number r with $r < 1$, we write

$$U = \{z \in D; |F_1(z)| > r\}, \quad V = \{z \in D; |F_2(z)| > r\}.$$

Since $E_1 \cap E_2 = \varnothing$, we may assume that $U \cap V = \varnothing$. Take a positive number ε arbitrary. Then there exists a positive integer n satisfying $|F_1^n| < \varepsilon$ on $D\setminus U$ and $|F_2^n| < \varepsilon$ on $D\setminus V$. Define an operator K by

$$K = M_{(1-F_1^n)}C_\varphi - M_{(1-F_2^n)}C_\psi.$$

Then by Lemma 4.1, K is compact. For a function f in $B(H^\infty)$, we have

$$\|(C_\varphi - C_\psi - K)f\|_{D\setminus V} = \|(M_{F_1^n}C_\varphi - M_{F_2^n}C_\psi)f\|_{D\setminus V} \le 1 + \varepsilon$$

and $\|(C_\varphi - C_\psi - K)f\|_{D\setminus U} \le 1+\varepsilon$. Since $U \cap V = \varnothing$, we get $\|(C_\varphi - C_\psi - K)f\|_\infty \le 1+\varepsilon$ for every $f \in B(H^\infty)$. Thus $\|C_\varphi - C_\psi - K\| \le 1 + \varepsilon$, so that $\|C_\varphi - C_\psi\|_e \le 1$. □

A typical example satisfying conditions of Theorem 4.3 is $\varphi(z) = (z + 1)/2$ and

$\psi(z) = (z-1)/2$.

Theorem 4.4. *Let φ, ψ be functions in $\mathscr{S}(D)$ with $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. Let δ be a positive number with $\delta < 1$, and U, V be open subsets in D satisfying the following conditions*;

(i) $|\psi(z)| \le \delta$ *for* $z \in U$,
(ii) $|\varphi(z)| \le \delta$ *for* $z \in V$,
(iii) $\sup_{z \in D \setminus (U \cup V)} \rho(\varphi(z), \psi(z)) < 1$,

Then $\|C_\varphi - C_\psi\|_e < 2$.

Proof. Let a be a small positive number with $a < 1$. Define the operator K_a by

$$(K_a h)(z) = a h(0) \quad \text{for } h \in H^\infty. \tag{4.9}$$

Then K_a is a compact operator on H^∞ and $\|K_a\| = a$. We shall prove that $\|C_\varphi - C_\psi + K_a\| < 2$ for a sufficiently small $a > 0$.

Write

$$A = \sup_{z \in D \setminus (U \cup V)} \rho(\varphi(z), \psi(z)).$$

Then by (iii), $A < 1$. Hence by (2.1),

$$\sup_{h \in B(H^\infty)} \|(C_\varphi - C_\psi)h\|_{D \setminus (U \cup V)} \le \lambda(A) < 2. \tag{4.10}$$

Since $(1+3\delta)/(2+\delta+\delta^2) < 1$, we may further assume that a satisfies

$$0 < a < \min\left\{1 - \frac{1+3\delta}{2+\delta+\delta^2}, \frac{1}{2+\delta}, \frac{2-\lambda(A)}{2}\right\}. \tag{4.11}$$

Then by (4.9)–(4.11),

$$\sup_{h \in B(H^\infty)} \|(C_\varphi - C_\psi + K_a)h\|_{D \setminus (U \cup V)} \le \frac{2+\lambda(A)}{2} < 2. \tag{4.12}$$

Next, we study the estimate on U. Let h be a function in $B(H^\infty)$ and $z \in U$. By (i), $|\psi(z)| \le \delta$. Then by Schwarz and Pick's lemma,

$$h(\psi(z)) \in \Delta(h(0), \delta). \tag{4.13}$$

By [**4**, p. 3],

$$\frac{|h(0)| - \delta}{1 - \delta|h(0)|} \le |w| \le \frac{|h(0)| + \delta}{1 + \delta|h(0)|} \quad \text{for } w \in \Delta(h(0), \delta). \tag{4.14}$$

First we assume that $|h(0)| \le (1+\delta)/2$. Then by (4.13) and (4.14),

$$|h(\psi(z))| \le \frac{1+3\delta}{2+\delta+\delta^2},$$

so that

$$|h(\varphi(z)) - h(\psi(z)) + ah(0)| \le 1 + \frac{1+3\delta}{2+\delta+\delta^2} + a.$$

Hence by (4.11),

$$\sup_{\{h \in B(H^\infty); |h(0)| \le (1+\delta)/2\}} \|(C_\varphi - C_\psi + K_a)h\|_U < 2. \tag{4.15}$$

Now, we assume that

$$(1+\delta)/2 < |h(0)| < 1. \tag{4.16}$$

By (4.14),

$$|w| \ge 1/(2+\delta) \quad \text{for } w \in \varDelta(h(0), \delta). \tag{4.17}$$

By (4.11), $ah(0) \notin \varDelta(h(0), \delta)$. Hence by (4.13),

$$|h(\psi(z)) - ah(0)| \le 1 - a|h(0)| < 1 - \frac{a(1+\delta)}{2}.$$

Therefore

$$\sup_{\{h \in B(H^\infty); (1+\delta)/2 < |h(0)| < 1\}} \|(C_\varphi - C_\psi + K_a)h\|_U < 2. \tag{4.18}$$

If $|h(0)| = 1$, then h is constant. Hence

$$\sup_{\{h \in B(H^\infty); |h(0)| = 1\}} \|(C_\varphi - C_\psi + K_a)h\|_U = a < 2.$$

Combined with (4.15) and (4.18), we get

$$\sup_{h \in B(H^\infty)} \|(C_\varphi - C_\psi + K_a)h\|_U < 2. \tag{4.19}$$

Similarly, we get

$$\sup_{h \in B(H^\infty)} \|(C_\varphi - C_\psi + K_a)h\|_V < 2$$

for a sufficiently small $a > 0$. Combined with (4.12) and (4.19), we obtain $\|C_\varphi - C_\psi + K_a\| < 2$. This completes the proof. □

Corollary 4.5. *Let φ, ψ be functions in $\mathscr{S}(D)$ with $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. If $\|\varphi\psi\|_\infty < 1$, then $\|C_\varphi - C_\psi\|_e < 2$.*

Proof. For each positive number δ with $\delta < 1$, we write

$$U = \{z \in D; |\varphi(z)| > \delta\}, \quad V = \{z \in D; |\psi(z)| > \delta\}.$$

Since $\|\varphi\psi\|_\infty < 1$, we may assume that $U \cap V = \varnothing$. Hence $|\psi(z)| \le \delta$ for $z \in U$ and $|\varphi(z)| \le \delta$ for $z \in V$. Let w be a complex number in $D \setminus (U \cup V)$. Then $|\psi(w)| \le \delta$ and $|\varphi(w)| \le \delta$. Hence

$$\sup_{w \in D \setminus (U \cup V)} \rho(\varphi(w), \psi(w)) < 1.$$

As an application of Theorem 4.4, we get the assertion. □

Theorem 4.6. *Let φ, ψ be functions in $\mathscr{S}(D)$. Then $\|C_\varphi - C_\psi\|_e = 2$ if and only if $\|\varphi\psi\|_\infty = 1$ and $\sigma_\infty(\varphi, \psi) = 1$.*

Proof. If either $\|\varphi\|_\infty < 1$ or $\|\psi\|_\infty < 1$, then by Corollary 3.6 $\|C_\varphi - C_\psi\|_e = 0$ or 1, because either C_φ or C_ψ is compact. So we may assume that $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. If $\|\varphi\psi\|_\infty < 1$, then by Corollary 4.5 $\|C_\varphi - C_\psi\|_e < 2$. So moreover we may assume that $\|\varphi\psi\|_\infty = 1$. By Theorem 3.2, if $\sigma_\infty(\varphi, \psi) = 1$, then $\|C_\varphi - C_\psi\|_e = 2$.

Next, suppose that

$$\sigma_\infty(\varphi, \psi) < 1. \tag{4.20}$$

We shall prove that $\|C_\varphi - C_\psi\|_e < 2$. By (4.20), we have $\varphi = \psi$ on $E_{H^\infty}(|\varphi\psi|)$. If $E_{H^\infty}(|\varphi|) = E_{H^\infty}(|\psi|)$, by [**10**, Theorem 1], C_φ and C_ψ are contained in the same connected component of $\mathscr{C}(H^\infty)$ and $\|C_\varphi - C_\psi\|_e \le \|C_\varphi - C_\psi\| < 2$. So we may assume that $E_{H^\infty}(|\varphi|) \ne E_{H^\infty}(|\psi|)$. If $E_{H^\infty}(|\varphi\psi|)$ is not an open and closed subset in $E_{H^\infty}(|\varphi|)$ (or $E_{H^\infty}(|\psi|)$), then by Corollary 3.5, $\sigma_\infty(\varphi, \psi) = 1$. This contradicts (4.20). Hence $E_{H^\infty}(|\varphi\psi|)$ is an open and closed subset in both sets $E_{H^\infty}(|\varphi|)$ and $E_{H^\infty}(|\psi|)$. We may take open subsets W_1, W_2, W_3 in $\mathscr{M}(H^\infty)$ such that

$$W_1 \supset E_{H^\infty}(|\varphi|) \setminus E_{H^\infty}(|\varphi\psi|),$$

$$W_2 \supset E_{H^\infty}(|\psi|) \setminus E_{H^\infty}(|\varphi\psi|), \tag{4.21}$$

$$W_3 \supset E_{H^\infty}(|\varphi\psi|), \tag{4.22}$$

and

$$W_1 \cap W_2 = \varnothing, \quad W_1 \cap W_3 = \varnothing, \quad W_2 \cap W_3 = \varnothing. \tag{4.23}$$

Then there exists $\delta, 0 < \delta < 1$, satisfying $|\varphi| < \delta$ on W_2 and $|\psi| < \delta$ on W_1. Write $U = W_1 \cap D$ and $V = W_2 \cap D$. Then

$$|\psi(z)| < \delta \text{ for } z \in U, \quad |\varphi(z)| < \delta \text{ for } z \in V.$$

We shall prove that

$$\sup_{z \in D \setminus (U \cup V)} \rho(\varphi(z), \psi(z)) < 1. \tag{4.24}$$

Assume that (4.24) does not hold. Then there exists a sequence $\{z_n\}_n$ in D satisfying

$$\lim_{n \to \infty} \rho(\varphi(z_n), \psi(z_n)) = 1 \tag{4.25}$$

and

$$z_n \notin U \cup V \tag{4.26}$$

for every n. By (4.25), $\max\{|\varphi(z_n)|, |\psi(z_n)|\} \to 1$. We may assume that $\delta < |\varphi(z_n)| \to 1$. By (4.20) and (4.25), we have

$$\limsup_{n \to \infty} |\psi(z_n)| < 1.$$

Hence by (4.21)–(4.23), $z_n \in W_1$ for large n. This contradicts (4.26). Hence (4.24) holds. By Theorem 4.4, we get the assertion. □

For $\varphi, \psi \in \mathscr{S}(D)$, we write $\varphi \sim \psi$ if $\|C_\varphi - C_\psi\| < 2$. Then by [**10**], the relation $\sim$ in $\mathscr{S}(D)$ is an equivalence one. But we note that $\|C_\varphi - C_\psi\|_e < 2$ does not induce an equivalence relation in $\mathscr{S}(D)$.

EXAMPLE 4.7. Let φ, ψ be peaking functions in the disk algebra $\mathscr{A}$ for a point $z = 1$. Moreover we may assume that φ is an extreme point in $B(H^\infty)$ but ψ is not. By [**8**], [**10**], $\sigma_\infty(\varphi, \psi) = 1$. Then by Theorem 4.6, $\|C_\varphi - C_\psi\|_e = 2$. Let q be a peaking function in $\mathscr{A}$ for $z = -1$. Then by Theorem 4.3, $\|C_\varphi - C_q\|_e = \|C_\psi - C_q\|_e = 1$. Thus $\|C_\varphi - C_\psi\|_e < 2$ does not induce an equivalence relation in $\mathscr{S}(D)$.

We show that $\|C_\varphi - C_\psi\|_e < 2$ induces an equivalence relation in some part of $\mathscr{S}(D)$.

PROPOSITION 4.8. *Let φ be a function in $\mathscr{S}(D)$ such that $E_{L^\infty}(|\varphi|)$ is a peak set for H^∞. Let $\Omega_\varphi = \{\psi \in \mathscr{S}(D); E_{L^\infty}(|\psi|) = E_{L^\infty}(|\varphi|)\}$. Then $\|C_{\psi_1} - C_{\psi_2}\|_e = \lambda(\sigma_\infty(\psi_1, \psi_2))$, and $\|C_{\psi_1} - C_{\psi_2}\|_e < 2$ induces an equivalence relation in Ω_φ.*

PROOF. Let ψ_1, ψ_2 be functions in Ω_φ with $\psi_1 \neq \psi_2$. By Theorems 3.2 and 4.2, $\|C_{\psi_1} - C_{\psi_2}\|_e = \lambda(\sigma_\infty(\psi_1, \psi_2))$. It is not difficult to see that $\sigma_\infty(\psi_1, \psi_2) < 1$ induces an equivalence relation in Ω_φ, see [**10**]. Thus we get the assertion. □

By [**8**], [**10**], it is known that C_φ and C_ψ are in the same connected component in

$\mathscr{C}(H^\infty)$ with the operator norm (which is the same as the essential semi-norm) if and only if $\sup_{z\in D}\rho(\varphi(z),\psi(z))<1$. If $\|\varphi\|_\infty=1$ and $\sup_{z\in D}\rho(\varphi(z),\psi(z))<1$, it is not difficult to see that $E_{L^\infty}(|\varphi|)=E_{L^\infty}(|\psi|)$. So under the assumption that $\|\varphi\|_\infty=1$ and $E_{L^\infty}(|\varphi|)$ is a peak set for H^∞, if C_φ and C_ψ are in the same connected component in $\mathscr{C}(H^\infty)$, then $\|C_\varphi-C_\psi\|_e=\lambda(\sigma_\infty(\varphi,\psi))$.

5. Limits on z^nH^∞.

In this section, we prove the following. An idea for the proof is the same as the one of Theorem 4.4.

Theorem 5.1. *Let φ,ψ be functions in $\mathscr{S}(D)$. Then we have the following.*

(i) *If $\|\varphi\|_\infty<1$ and $\|\psi\|_\infty<1$, then $\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|=0$.*
(ii) *If $\|\varphi\|_\infty=1$ and $\|\varphi\psi\|_\infty<1$, then $\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|=1$.*

Moreover suppose that $\|\varphi\psi\|_\infty=1$. Then we have the following.

(iii) *If $\lambda(\sigma_\infty(\varphi,\psi))>1$, then $\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|=\lambda(\sigma_\infty(\varphi,\psi))$.*
(iv) *If $\lambda(\sigma_\infty(\varphi,\psi))\le 1$ and $E_{H^\infty}(|\varphi|)=E_{H^\infty}(|\psi|)$, then $\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|=\lambda(\sigma_\infty(\varphi,\psi))$.*
(v) *If $\lambda(\sigma_\infty(\varphi,\psi))\le 1$ and $E_{H^\infty}(|\varphi|)\ne E_{H^\infty}(|\psi|)$, then $\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|=1$.*

Proof. (i) is clear.

(ii) If $\|\psi\|_\infty<1$, then the assertion is also clear. So, we assume that $\|\psi\|_\infty=1$. For each positive number δ with $\delta<1$, we write

$$U_\varphi=\{z\in D;|\varphi(z)|>\delta\},\quad U_\psi=\{z\in D;|\psi(z)|>\delta\}.$$

Since $\|\varphi\psi\|_\infty<1$ we may assume that $U_\varphi\cap U_\psi=\varnothing$. Then $|\psi|\le\delta$ on U_φ. Let $\{\zeta_j\}_j$ be a sequence in U_φ with $\varphi(\zeta_j)\to\alpha$ for some $|\alpha|=1$. We have

$$\begin{aligned}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\| &\ge \|(C_\varphi-C_\psi)z^n(z+\alpha)/2\|_\infty\\ &\ge |(\varphi^n(\varphi+\alpha)/2-\psi^n(\psi+\alpha)/2)(z_j)|\\ &\ge |\varphi^n(z_j)(\varphi(z_j)+\alpha)/2|-\delta^n\\ &\to 1-\delta^n \qquad \text{as } j\to\infty.\end{aligned}$$

Since $0<\delta<1$, we get

$$\lim_{n\to\infty}\|(C_\varphi-C_\psi)_{|z^nH^\infty}\|\ge 1.$$

Let h be a function in H^∞ with $\|h\|_\infty\le 1$. Then

$$|\varphi^n(h\circ\varphi)-\psi^n(h\circ\psi)|\le 1+\delta^n \quad \text{on } U_\varphi\cup U_\psi$$

and

$$|\varphi^n(h \circ \varphi) - \psi^n(h \circ \psi)| \le 2\delta^n \quad \text{on } D \setminus (U_\varphi \cup U_\psi).$$

Hence $\|(C_\varphi - C_\psi)z^n h\|_\infty \le \max\{1 + \delta^n, 2\delta^n\}$, so that

$$\|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le \max\{1 + \delta^n, 2\delta^n\}.$$

Since $0 < \delta < 1$, we get

$$\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le 1.$$

Thus we obtain (ii).

Hereafter, we assume that $\|\varphi\psi\|_\infty = 1$.

Claim 1. $\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \ge \lambda(\sigma_\infty(\varphi,\psi))$.

Let $\{\zeta_j\}_j$ be a sequence in D with $|(\varphi\psi)(\zeta_j)| \to 1$ and $\rho(\varphi(\zeta_j), \psi(\zeta_j)) \to \sigma_\infty(\varphi,\psi)$. We may assume that $\lim_{j\to\infty} \varphi(\zeta_j) = \alpha$, $\lim_{j\to\infty} \psi(\zeta_j) = \beta$, and $|\alpha| = |\beta| = 1$. By Theorem 2.2, we may further assume that $\{\varphi(\zeta_j), \psi(\zeta_j)\}_j$ is $\lambda(\sigma_\infty(\varphi,\psi))/2$-asymptotic interpolating. Then there exists $h_n \in B(H^\infty)$ satisfying

$$\lim_{j\to\infty} h_n(\varphi(\zeta_j)) = \overline{\alpha}^n \lambda(\sigma_\infty(\varphi,\psi))/2$$

and

$$\lim_{j\to\infty} h_n(\psi(\zeta_j)) = -\overline{\beta}^n \lambda(\sigma_\infty(\varphi,\psi))/2.$$

Hence

$$\lim_{j\to\infty} |[(C_\varphi - C_\psi)z^n h_n](\zeta_j)| = \lambda(\sigma_\infty(\varphi,\psi)).$$

This shows Claim 1.

Claim 2. If $E_{H^\infty}(|\varphi|) \ne E_{H^\infty}(|\psi|)$, then $\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \ge 1$.

We may assume that $E_{H^\infty}(|\varphi|) \not\subset E_{H^\infty}(|\psi|)$. Then there is a sequence $\{\zeta_j\}_j$ in D satisfying $\varphi(\zeta_j) \to \alpha$ for some $|\alpha| = 1$ and $\sup_j |\psi(\zeta_j)| < 1$. In the same way as in the first paragraph of the proof of (ii), we get Claim 2.

Claim 3. If $\sigma_\infty(\varphi,\psi) < 1$ and $E_{H^\infty}(|\varphi|) \ne E_{H^\infty}(|\psi|)$, then $\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le \max\{\lambda(\sigma_\infty(\varphi,\psi)), 1\}$.

Since $\sigma_\infty(\varphi,\psi) < 1$, we have

$$\lim_{|(\varphi\psi)(z)|\to 1} |\varphi(z) - \psi(z)| = 0, \tag{5.1}$$

and by Corollary 3.5, $E_{H^\infty}(|\varphi\psi|)$ is an open and closed subset in the both sets $E_{H^\infty}(|\varphi|)$ and $E_{H^\infty}(|\psi|)$. For each positive number δ with $\delta < 1$, we write

$$U_\delta = \{z \in D; |(\varphi\psi)(z)| > \delta\}. \tag{5.2}$$

Let ε be a positive number with $\varepsilon < 1$ and n be a positive integer. By (5.1), we may assume that $|\varphi^n(z) - \psi^n(z)| < \varepsilon$ for every $z \in U_\delta$.

Let h be a function in $B(H^\infty)$ and ζ be a point in U_δ. Then

$$\begin{aligned}|\varphi^n(\zeta)h(\varphi(\zeta)) - \psi^n(\zeta)h(\psi(\zeta))| &\le |\psi^n(\zeta)||h(\varphi(\zeta)) - h(\psi(\zeta))| + \varepsilon \\ &\le \lambda(\rho(\varphi(\zeta), \psi(\zeta))) + \varepsilon.\end{aligned}$$

Hence

$$\|(C_\varphi - C_\psi)z^n h\|_{U_\delta} \le \lambda\left(\sup_{\zeta\in U_\delta} \rho(\varphi(\zeta), \psi(\zeta))\right) + \varepsilon$$

for every $h \in B(H^\infty)$.

Let ξ be a point in $D \setminus U_\delta$. Then by (5.2), either $|\varphi(\xi)| \le \delta^{1/2}$ or $|\psi(\xi)| \le \delta^{1/2}$. Then similarly as above we get

$$\|(C_\varphi - C_\psi)z^n h\|_{D\setminus U_\delta} \le 1 + \delta^{n/2} \tag{5.3}$$

for every $h \in B(H^\infty)$. As a consequence, we obtain

$$\|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le \max\left\{\lambda\left(\sup_{\zeta\in U_\delta} \rho(\varphi(\zeta), \psi(\zeta))\right) + \varepsilon, 1 + \delta^{n/2}\right\}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le \max\left\{\lambda\left(\sup_{\zeta\in U_\delta} \rho(\varphi(\zeta), \psi(\zeta))\right) + \varepsilon, 1\right\}.$$

Letting $\varepsilon \to 0$, we have

$$\lim_{n\to\infty} \|(C_\varphi - C_\psi)_{|z^n H^\infty}\| \le \max\left\{\lambda\left(\sup_{\zeta\in U_\delta} \rho(\varphi(\zeta), \psi(\zeta))\right), 1\right\}. \tag{5.4}$$

Here we note that by (5.2),

$$\sigma_\infty(\varphi, \psi) = \lim_{\delta\to 1} \sup_{\zeta\in U_\delta} \rho(\varphi(\zeta), \psi(\zeta)).$$

So, letting $\delta \to 1$ in (5.4) we obtain Claim 3.

Claim 4. If $\sigma_\infty(\varphi,\psi) \le 1$ and $E_{H^\infty}(|\varphi|) = E_{H^\infty}(|\psi|)$, then $\lim_{n\to\infty}\|(C_\varphi - C_\psi)_{|z^nH^\infty}\| \le \lambda(\sigma_\infty(\varphi,\psi))$.

To show this, we follow the proof of Claim 3. In this case we may assume that

$$\sup_{z\in D\setminus U_\delta} \max\{|\varphi(z)|, |\psi(z)|\} = \delta_1 < 1.$$

Then instead of (5.3), we get

$$\|(C_\varphi - C_\psi)z^n h\|_{D\setminus U_\delta} \le 2\delta_1^n$$

for every $h \in B(H^\infty)$. The rest is the same.

Now, we prove (iii). If $\sigma_\infty(\varphi,\psi) = 1$, by Claim 1 we get

$$\lim_{n\to\infty}\|(C_\varphi - C_\psi)_{|z^nH^\infty}\| = 2 = \lambda(\sigma_\infty(\varphi,\psi)).$$

If $1 < \lambda(\sigma_\infty(\varphi,\psi)) < 2$, by Claims 1, 3, and 4 we get the assertion.

(iv) follows from Claims 1 and 4.

(v) follows from Claims 2 and 3. □

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