J. Math. Soc. Japan
Vol. 58, No. 3, 2006

A Möbius characterization of submanifolds

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(Received Sep. 9, 2005)

Abstract. In this paper, we study Möbius characterizations of submanifolds without umbilical points in a unit sphere $S^{n+p}(1)$. First of all, we proved that, for an n-dimensional ($n \geq 2$) submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ without umbilical points and with vanishing Möbius form Φ, if $(n-2)\|\tilde{\boldsymbol{A}}\| \leq \sqrt{\frac{n-1}{n}}\{nR - \frac{1}{n}[(n-1)(2-\frac{1}{p}) - 1]\}$ is satisfied, then, $\mathbf{x}$ is Möbius equivalent to an open part of either the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or the image of the conformal diffeomorphism σ of the standard cylinder $S^{n-1}(1) \times \boldsymbol{R}$ in $\boldsymbol{R}^{n+1}$, or the image of the conformal diffeomorphism τ of the Riemannian product $S^{n-1}(r) \times \boldsymbol{H}^1(\sqrt{1+r^2})$ in $\boldsymbol{H}^{n+1}$, or $\mathbf{x}$ is locally Möbius equivalent to the Veronese surface in $S^4(1)$. When $p = 1$, our pinching condition is the same as in Main Theorem of Hu and Li [**6**], in which they assumed that M is compact and the Möbius scalar curvature $n(n-1)R$ is constant. Secondly, we consider the Möbius sectional curvature of the immersion $\mathbf{x}$. We obtained that, for an n-dimensional compact submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ without umbilical points and with vanishing form Φ, if the Möbius scalar curvature $n(n-1)R$ of the immersion $\mathbf{x}$ is constant and the Möbius sectional curvature K of the immersion $\mathbf{x}$ satisfies $K \geq 0$ when $p = 1$ and $K > 0$ when $p > 1$. Then, $\mathbf{x}$ is Möbius equivalent to either the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$, in $S^{n+1}(1)$; or $\mathbf{x}$ is Möbius equivalent to a compact minimal submanifold with constant scalar curvature in $S^{n+p}(1)$.

1. Introduction.

Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional immersed submanifold in an $(n+p)$-dimensional unit sphere $S^{n+p}(1)$. In [**11**], Wang introduced a Möbius metric, Möbius form and the Möbius second fundamental form of the immersion $\mathbf{x}$. By making use of these Möbius invariants, he founded the fundamental formulas on Möbius geometry of submanifolds in $S^{n+p}(1)$. By following these results of Wang, the Möbius geometry on submanifolds in $S^{n+p}(1)$ was researched by many mathematicians (see. [**6**], [**7**], [**8**] and [**9**]). In particular, Li, Wang and Wu [**8**] studied the Möbius characterization of Veronese surface. They proved that if $\mathbf{x} : S^2(1) \mapsto S^m(1)$ is an immersion without umbilical points of the 2-sphere with vanishing Möbius form, then there exists a Möbius transformation $\tau : S^m(1) \mapsto S^m(1)$ such that $\tau \circ \mathbf{x} : S^2(1) \mapsto S^{2k}(1)$ is the Veronese surface, where $S^{2k}(1) \subset S^m(1)$ with $2 \leq k \leq [m/2]$. Furthermore, a kind of pinching problems on Möbius geometry of submanifolds in $S^{n+p}(1)$ was studied by Akivis and Goldberg [**2**], Hu and Li [**6**] and so on.

2000 *Mathematics Subject Classification.* 53C42, 53C20.

Key Words and Phrases. submanifold, Möbius metric, Möbius scalar curvature, Möbius sectional curvature, Blaschke tensor and Möbius form.

*The first author's research was partially supported by a Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science.

**The second author's research was partially supported by the Natural Science Foundation of China and NSF of Shaanxi.

Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional immersed submanifold in $S^{n+p}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j,\alpha} h^\alpha_{ij}\theta_i\theta_j e_\alpha$ be the second fundamental form of the immersion $\mathbf{x}$ and $\vec{H} = \sum_\alpha H^\alpha e_\alpha$ the mean curvature vector of the immersion $\mathbf{x}$, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $\mathbf{x}$. By putting $\rho^2 = \frac{n}{n-1}\big\{\sum_{\alpha,i,j}(h^\alpha_{ij})^2 - n\|\vec{H}\|^2\big\}$, the Möbius metric of the immersion $\mathbf{x}$ is defined by $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$, which is a Möbius invariant. $\Phi = \Sigma_{i,\alpha} C^\alpha_i \theta_i e_\alpha$ and $\boldsymbol{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i\theta_j$ are Möbius form and Blaschke tensor of the immersion $\mathbf{x}$, respectively, where C^α_i and A_{ij} are defined by formulas (2.13) and (2.14) in section 2. It was proved that Φ and $\boldsymbol{A}$ are Möbius invariants (cf. [**11**]).

In particular, Akivis and Goldberg [**1**], [**2**] and Wang [**11**] proved that two hypersurfaces $\mathbf{x} : M \mapsto S^{n+1}(1)$ and $\tilde{\mathbf{x}} : \tilde{M} \mapsto S^{n+1}(1)$ are Möbius equivalent if and only if there exists a diffeomorphism $\sigma' : M \mapsto \tilde{M}$ which preserves the Möbius metric and the Möbius shape operator such that $\mathbf{x} = \sigma' \circ \tilde{\mathbf{x}}$.

Let $\boldsymbol{H}^{n+p}$ be an $(n+p)$-dimensional hyperbolic space defined by

$$\boldsymbol{H}^{n+p} = \big\{(y_0, y_1) \in \boldsymbol{R}^+ \times \boldsymbol{R}^{n+p} | -y_0^2 + y_1 \cdot y_1 = -1\big\}.$$

We denote the open hemisphere in $S^{n+p}(1)$ whose first coordinate is positive by $S^{n+p}_+(1)$. We consider conformal diffeomorphisms $\sigma_p : \boldsymbol{R}^{n+p} \mapsto S^{n+p}(1)\backslash\{(-1, 0)\}$ and $\tau_p : \boldsymbol{H}^{n+p} \mapsto S^{n+p}_+(1)$ defined by:

$$\sigma_p(u) = \left(\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2}\right), \quad u \in \boldsymbol{R}^{n+p}, \tag{1.1}$$

$$\tau_p(y_0, y_1) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad (y_0, y_1) \in \boldsymbol{H}^{n+p}, \tag{1.2}$$

respectively. The conformal diffeomorphisms σ_p and τ_p assign any submanifold in $\boldsymbol{R}^{n+p}$ or $\boldsymbol{H}^{n+p}$ to a submanifold in $S^{n+p}(1)$. If $p = 1$, we denote σ_1 and τ_1 by σ and τ. In [**7**], Li, Liu, Wang and Zhao classified Möbius isoparametric hypersurfaces with two distinct principal curvatures. They obtained the following:

THEOREM 1.1. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be a Möbius isoparametric hypersurface with two distinct principal curvatures. Then $\mathbf{x}$ is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in $S^{n+1}(1)$:*

1. *the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$,*
2. *the image of σ of the standard cylinder $S^k(1) \times \boldsymbol{R}^{n-k}$ in $\boldsymbol{R}^{n+1}$,*
3. *the image of τ of the Riemannian product $S^k(r) \times \boldsymbol{H}^{n-k}(\sqrt{1+r^2})$ in $\boldsymbol{H}^{n+1}$.*

A submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ is called Möbius isotropic if $\Phi \equiv 0$ and $\boldsymbol{A} = \lambda d\mathbf{x} \cdot d\mathbf{x}$ for some function λ. In [**9**], Liu, Wang and Zhao proved the following:

THEOREM 1.2. *Any Möbius isotropic submanifolds in $S^{n+p}(1)$ is Möbius equivalent to an open part of one of the following Möbius isotropic submanifolds*:

1. *a minimal submanifold with constant scalar curvature in* $S^{n+p}(1)$,
2. *the image of* σ_p *of a minimal submanifold with constant scalar curvature in* $\boldsymbol{R}^{n+p}$,
3. *the image of* τ_p *of a minimal submanifolds with constant scalar curvature in* $\boldsymbol{H}^{n+p}$.

On the other hand, Hu and Li [**6**] studied a pinching problem on the squared norm of the Blaschke tensor of the immersion $\mathbf{x}$ and obtained the following:

THEOREM 1.3. *Let* $\mathbf{x} : M \to S^{n+p}(1)$ *be an* n*-dimensional* $(n \geq 3)$ *compact submanifold without umbilical points and with vanishing Möbius form* Φ *in* $S^{n+p}(1)$. *If the Möbius scalar curvature* $n(n-1)R \geq \frac{(n-1)(n-2)}{n}$ *is constant and if*

$$\|\tilde{\boldsymbol{A}}\| \leq \sqrt{\frac{n-1}{n}}\left(\frac{n}{n-2}R - \frac{1}{n}\right),$$

then, either $\mathbf{x}$ *is Möbius equivalent to a minimal submanifold with constant scalar curvature in* $S^{n+p}(1)$ *or* $\mathbf{x}$ *is Möbius equivalent to* $S^1(r) \times S^{n-1}\big(\sqrt{\frac{1}{1+c^2} - r^2}\big)$ *in* $S^{n+1}(1/\sqrt{1+c^2})$ *for some constant* $c \geq 0, r = \sqrt{\frac{nR}{(n-2)(1+c^2)}}$, *where* $\tilde{\boldsymbol{A}} = \rho^2 \sum_{ij} \tilde{A}_{ij}\theta_i\theta_j$ *with* $\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\sum_k A_{kk}\delta_{ij}$.

REMARK 1.4. In the original statement of the theorem 1.3 of Hu and Li [**6**], they did not write out the condition that M has no umbilical points. But this condition is necessary for their proof. Further, We should note that these assumptions that M is compact and the Möbius scalar curvature $n(n-1)R$ is constant plays an important role in the proof of Theorem 1.3 of Hu and Li [**6**].

In this paper, first of all, we prove the following:

MAIN THEOREM 1. *Let* $\mathbf{x} : M \to S^{n+p}(1)$ *be an* n*-dimensional* $(n \geq 2)$ *submanifold without umbilical points and with vanishing Möbius form* Φ, *if*

$$(n-2)\|\tilde{\boldsymbol{A}}\| \leq \sqrt{\frac{n-1}{n}}\left\{nR - \frac{1}{n}\left[(n-1)\left(2 - \frac{1}{p}\right) - 1\right]\right\}, \tag{1.3}$$

then $\mathbf{x}$ *is locally Möbius equivalent to either the Veronese surface in* $S^4(1)$, *or* $\mathbf{x}$ *is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in* $S^{n+1}(1)$:

1. *the Riemannian product* $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ *in* $S^{n+1}(1)$,
2. *the image of* σ *of the standard cylinder* $S^{n-1}(1) \times \boldsymbol{R}$ *in* $\boldsymbol{R}^{n+1}$,
3. *the image of* τ *of the Riemannian product* $S^{n-1}(r) \times \boldsymbol{H}^1(\sqrt{1+r^2})$ *in* $\boldsymbol{H}^{n+1}$,

where $n(n-1)R$ *denotes the Möbius scalar curvature of the immersion* $\mathbf{x}$ *and* $\tilde{\boldsymbol{A}} = \rho^2\sum_{ij}\tilde{A}_{ij}\theta_i\theta_j$ *with* $\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\sum_k A_{kk}\delta_{ij}$.

REMARK 1.5. In our Main Theorem 1, we do not assume the global condition that M is compact and we do not need to assume that the Möbius scalar curvature is constant. Further, when $p = 1$ and $(n \geq 3)$ our pinching condition is the same as in Hu and Li [**6**]. Since Hu and Li [**6**] assumed that M is compact, the cases of 2 and 3 above in Main Theorem 1 do not appear in their theorem. If $n = 2$, since the Möbius metric g is flat, we know that $R \equiv 0$. Main Theorem 1 reduces to the Theorem 5.1 in [**11**].

Since Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$, have nonnegative Möbius sectional curvature and they do not satisfy the inequality in Theorem 1.3 of Hu and Li [**6**] except $k = 1$ or $k = n-1$ (see Proposition 3.2 and Remark 3.3 in section 3), we will consider the immersion $\mathbf{x}$ with nonnegative Möbius sectional curvature and prove the following:

MAIN THEOREM 2. *Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional compact submanifold without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+p}(1)$. If the Möbius sectional curvature K of M satisfies*

$$\begin{cases} K \geq 0, & \textit{if } p = 1 \\ K > 0, & \textit{if } p > 1, \end{cases}$$

then, $\mathbf{x}$ is Möbius equivalent to the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$, in $S^{n+1}(1)$; or $\mathbf{x}$ is Möbius equivalent to an n-dimensional compact minimal submanifold with constant scalar curvature in $S^{n+p}(1)$.

2. Preliminaries and fundamental formulas on Möbius geometry.

In this section, we review the definitions of Möbius invariants and give the fundamental formulas on Möbius geometry of submanifolds in $S^{n+p}(1)$, which can be found in [**11**].

Let $\boldsymbol{R}_1^{n+p+2}$ be the Lorentzian space with inner product

$$\langle x, w \rangle = -x_0 w_0 + x_1 w_1 + \cdots + x_{n+p+1} w_{n+p+1}, \tag{2.1}$$

where $x = (x_0, x_1, \cdots, x_{n+p+1})$ and $w = (w_0, w_1, \cdots, w_{n+p+1})$. Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional submanifold of $S^{n+p}(1)$ without umbilical points. Putting

$$Y = \rho(1, \mathbf{x}), \quad \rho^2 = \frac{n}{n-1}\big(\|II\|^2 - n\|\vec{H}\|^2\big) > 0, \tag{2.2}$$

then, $Y : M \mapsto \boldsymbol{R}_1^{n+p+2}$ is called *Möbius position vector* of $\mathbf{x}$. It is easy to prove that

$$g = \langle dY, dY \rangle = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$$

is a Möbius invariant which is recalled *Möbius metric* of the immersion $\mathbf{x}$. Let Δ denote the Laplacian on M with respect to the Möbius metric g. Defining

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}(1 + n^2 R)Y, \tag{2.3}$$

we can infer

$$\langle \Delta Y, Y\rangle = -n, \quad \langle \Delta Y, dY\rangle = 0, \quad \langle \Delta Y, \Delta Y\rangle = 1 + n^2 R, \tag{2.4}$$

$$\langle Y, Y\rangle = 0, \quad \langle N, Y\rangle = 1, \quad \langle N, N\rangle = 0, \tag{2.5}$$

where $n(n-1)R$ denotes the Möbius scalar curvature of the immersion $\mathbf{x}$. Let $\{E_1, \cdots, E_n\}$ denote a local orthonormal frame on (M, g) with dual frame $\{\omega_1, \cdots, \omega_n\}$. Putting $Y_i = E_i(Y)$, then we have, from (2.2), (2.4) and (2.5),

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \quad \langle Y_i, Y_j\rangle = \delta_{ij}, \quad 1 \le i, j \le n. \tag{2.6}$$

Let V be the orthogonal complement to the subspace $\mathrm{Span}\{Y, N, Y_1, \cdots, Y_n\}$ in $\boldsymbol{R}_1^{n+p+2}$. Along M, we have the following orthogonal decomposition:

$$\boldsymbol{R}_1^{n+p+2} = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \cdots, Y_n\} \oplus V, \tag{2.7}$$

where V is called *Möbius normal bundle* of the immersion $\mathbf{x}$. It is not difficult to prove that

$$E_\alpha = (H^\alpha, H^\alpha \mathbf{x} + e_\alpha), \quad n+1 \le \alpha \le n+p, \tag{2.8}$$

is a local orthonormal frame of V. Then $\{Y, N, Y_1, \cdots, Y_n, E_{n+1}, \cdots, E_{n+p}\}$ forms a moving frame in $\boldsymbol{R}_1^{n+p+2}$ along M. We use the following range of indices throughout this paper:

$$1 \le i, j, k, l, m \le n, \quad n+1 \le \alpha, \beta \le n+p.$$

The structure equations on M with respect to the Möbius metric g can be written as follows:

$$dY = \sum_i Y_i \omega_i, \tag{2.9}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_{i,\alpha} C_i^\alpha \omega_i E_\alpha, \tag{2.10}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_{j,\alpha} B_{ij}^\alpha \omega_j E_\alpha, \tag{2.11}$$

$$dE_\alpha = -\sum_i C_i^\alpha \omega_i Y - \sum_{i,j} B_{ij}^\alpha \omega_j Y_i + \sum_\beta \omega_{\alpha\beta} E_\beta, \tag{2.12}$$

where ω_{ij} is the connection form with respect to the Möbius metric g, $\omega_{\alpha\beta}$ is the normal connection form of $\mathbf{x} : M \to S^{n+p}(1)$, which is a Möbius invariant. $\boldsymbol{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$ and $\Phi = \sum_{i,\alpha} C_i^\alpha \omega_i (\rho^{-1} e_\alpha)$ are called *Blaschke tensor* and *Möbius form* of the immersion $\mathbf{x}$, respectively, where

$$C_i^\alpha = -\rho^{-2}\Big\{H^\alpha_{,i} + \sum_j \big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big)e_j(\log\rho)\Big\}, \tag{2.13}$$

$$\begin{aligned} A_{ij} = & -\rho^{-2}\Big\{\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h^\alpha_{ij}\Big\} \\ & - \frac{1}{2}\rho^{-2}\big(\|\nabla(\log\rho)\|^2 - 1 + \|\vec{H}\|^2\big)\delta_{ij}. \end{aligned} \tag{2.14}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $d\mathbf{x}\cdot d\mathbf{x}$. It was proved that $\Phi = \Sigma_{i,\alpha} C_i^\alpha \theta_i e_\alpha$ and $\boldsymbol{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i\theta_j$ are Möbius invariants. $\boldsymbol{B} = \sum_{i,j,\alpha} B^\alpha_{ij}\omega_i\omega_j(\rho^{-1}e_\alpha)$ is called *Möbius second fundamental form* of the immersion $\mathbf{x}$, where

$$B^\alpha_{ij} = \rho^{-1}\big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big). \tag{2.15}$$

Hence, we have

$$\sum_i B^\alpha_{ii} = 0, \quad \sum_{i,j,\alpha}\big(B^\alpha_{ij}\big)^2 = \frac{n-1}{n}. \tag{2.16}$$

We define the covariant derivative of $C_i^\alpha, A_{ij}, B^\alpha_{ij}$ by

$$\sum_j C^\alpha_{i,j}\omega_j = dC_i^\alpha + \sum_j C_j^\alpha\omega_{ji} + \sum_\beta C_i^\beta\omega_{\beta\alpha},$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.17}$$

$$\sum_k B^\alpha_{ij,k}\omega_k = dB^\alpha_{ij} + \sum_k B^\alpha_{ik}\omega_{kj} + \sum_k B^\alpha_{kj}\omega_{ki} + \sum_\beta B^\beta_{ij}\omega_{\beta\alpha}. \tag{2.18}$$

From the structure equations (2.9), (2.10), (2.11) and (2.12), we can infer

$$A_{ij,k} - A_{ik,j} = \sum_\alpha \big(B^\alpha_{ik}C^\alpha_j - B^\alpha_{ij}C^\alpha_k\big), \tag{2.19}$$

$$C^\alpha_{i,j} - C^\alpha_{j,i} = \sum_k \big(B^\alpha_{ik}A_{kj} - B^\alpha_{kj}A_{ki}\big), \tag{2.20}$$

$$B^\alpha_{ij,k} - B^\alpha_{ik,j} = \delta_{ij}C^\alpha_k - \delta_{ik}C^\alpha_j, \tag{2.21}$$

$$R_{ijkl} = \sum_\alpha \big(B^\alpha_{ik}B^\alpha_{jl} - B^\alpha_{il}B^\alpha_{jk}\big) + \big(\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}\big), \tag{2.22}$$

$$R_{\alpha\beta ij} = \sum_k \big(B^\alpha_{ik}B^\beta_{kj} - B^\beta_{ik}B^\alpha_{kj}\big), \tag{2.23}$$

where R_{ijkl} and $R_{\alpha\beta ij}$ denote the curvature tensor with respect to the Möbius metric g

on M and the normal curvature tensor of the normal connection. $n(n-1)R = \sum_{i,j} R_{ijij}$ is the Möbius scalar curvature of the immersion $\mathbf{x} : M \to S^{n+p}(1)$. From (2.3) and the structure equation (2.11), we have, (cf. [**11**]),

$$\mathrm{tr}\boldsymbol{A} = \frac{1}{2n}(1 + n^2 R). \tag{2.24}$$

By taking exterior differentiation of (2.17) and (2.18), and defining

$$\sum_l A_{ij,kl}\omega_l = dA_{ij,k} + \sum_l A_{lj,k}\omega_{li} + \sum_l A_{il,k}\omega_{lj} + \sum_l A_{ij,l}\omega_{lk},$$

$$\sum_l B^\alpha_{ij,kl}\omega_l = dB^\alpha_{ij,k} + \sum_l B^\alpha_{lj,k}\omega_{li} + \sum_l B^\alpha_{il,k}\omega_{lj} + \sum_l B^\alpha_{ij,l}\omega_{lk} + \sum_\beta B^\beta_{ij,k}\omega_{\beta\alpha},$$

we have the following Ricci identities

$$A_{ij,kl} - A_{ij,lk} = \sum_m A_{mj}R_{mikl} + \sum_m A_{im}R_{mjkl}, \tag{2.25}$$

$$B^\alpha_{ij,kl} - B^\alpha_{ij,lk} = \sum_m B^\alpha_{mj}R_{mikl} + \sum_m B^\alpha_{im}R_{mjkl} + \sum_\beta B^\beta_{ij}R_{\beta\alpha kl}. \tag{2.26}$$

For a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \mathrm{tr}(AA^t) = \sum_{i,j}(a_{ij})^2,$$

where A^t denotes the transposed matrix of A. It is obvious that $N(A) = N(T^tAT)$ holds for any orthogonal matrix T.

The following algebraic lemmas will be used in order to prove our Main Theorems.

Lemma 2.1 ([**5**]). *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \le 2N(A) \cdot N(B) \tag{2.27}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$, respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and satisfy

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)\cdot N(A_\beta), \qquad 1 \le \alpha, \beta \le 3,$$

then at least one of the matrices A_α must be zero.

Lemma 2.2 (Cheng [**4**] and Santos [**10**]). *Let A and B be $n\times n$-symmetric matrices satisfying* $\mathrm{tr}A = 0, \mathrm{tr}B = 0$ *and* $AB - BA = 0$. *Then,*

$$\mathrm{tr}(B^2A) \ge -\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}B^2)(\mathrm{tr}A^2)^{1/2}, \tag{2.28}$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = \frac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}}$, $x_ix_j \ge 0$, $y_i = \frac{(\mathrm{tr}A^2)^{1/2}}{\sqrt{n(n-1)}}$.

3. Möbius invariants on typical examples.

In this section, we shall study Möbius invariants on typical examples. These results in this section will be used in the proof of Main Theorem 1 and the results in the following proposition 3.2 will support our assumption in Main Theorem 2. Throughout this section, we shall make the following convention on the ranges of indices:

$$1 \le i, j \le n, \quad 1 \le a, b \le k, \quad k+1 \le s, t \le n.$$

The following Lemma 3.1 due to Li, Liu, Wang and Zhao [**7**] will be used.

Lemma 3.1. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n-dimensional hypersurface with two distinct principal curvatures with multiplicities k and $n-k$, respectively. Then the principal curvatures of the Möbius second fundamental form $\boldsymbol{B}$ of $\mathbf{x}$ are constant, which are given by*

$$\mu_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \quad \mu_2 = -\frac{1}{n}\sqrt{\frac{(n-1)k}{(n-k)}}.$$

Proposition 3.2. *Let $\mathbf{x}_1 : S^k(1) \mapsto \boldsymbol{R}^{k+1}$ and $\mathbf{x}_2 : S^{n-k}(1) \mapsto \boldsymbol{R}^{n-k+1}$ be the standard embeddings of the unit spheres. Then, for Riemannian product $\mathbf{x} : S^k(r)\times S^{n-k}(\sqrt{1-r^2}) \mapsto S^{n+1}(1)$ defined by $\mathbf{x} = (r\mathbf{x}_1, \sqrt{1-r^2}\mathbf{x}_2)$, for any $1 \le k \le n-1$ and any $0 < r < 1$, we have*

$$\Phi = 0, \tag{3.1}$$

$$R = \frac{k-1}{n(n-k)} + \frac{(n-1)(n-2k)}{nk(n-k)}r^2, \tag{3.2}$$

$$(n-2k)^2\|\tilde{\boldsymbol{A}}\|^2 = \frac{k(n-k)}{n}\left(nR - \frac{n-2}{n}\right)^2, \tag{3.3}$$

$$R_{abab} = \frac{n-1}{k(n-k)}(1-r^2), \quad R_{asas} = 0, \quad R_{stst} = \frac{n-1}{k(n-k)}r^2, \tag{3.4}$$

where R_{ijij} *denotes the Möbius sectional curvature of the plane section spanned by* $\{E_i, E_j\}$.

Proof. Since Riemannian product $\mathbf{x} : S^k(r) \times S^{n-k}(\sqrt{1-r^2}) \mapsto S^{n+1}(1)$ is the standard embedding, we know that the second fundamental form of $\mathbf{x}$ has two distinct principal curvatures $\frac{\sqrt{1-r^2}}{r}$ and $-\frac{r}{\sqrt{1-r^2}}$ with multiplicities k and $n-k$, respectively. Putting $c = \frac{\sqrt{1-r^2}}{r}$, we have

$$h_{ab} = c\delta_{ab}, \quad h_{as} = 0, \quad h_{st} = -\frac{1}{c}\delta_{st}, \tag{3.5}$$

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii} = \frac{1}{n}\left\{kc - (n-k)\frac{1}{c}\right\}, \tag{3.6}$$

$$\|II\|^2 = kc^2 + (n-k)\frac{1}{c^2}, \tag{3.7}$$

$$\rho^2 = \frac{n}{n-1}(\|II\|^2 - nH^2) = \frac{k(n-k)}{n-1}\frac{(c^2+1)^2}{c^2}. \tag{3.8}$$

Hence, the Möbius metric g of $\mathbf{x}$ is given by

$$g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}.$$

Since ρ^2 is constant, from (2.13) and (2.14), we have $C_i = 0$ and $A_{ij} = -\frac{1}{2}\rho^{-2}\{(H^2-1)\delta_{ij} - 2Hh_{ij}\}$, where C_i and A_{ij} denote components of Möbius form Φ and components of the Blaschke tensor $\boldsymbol{A}$. Hence, we infer $\Phi = 0$ and

$$A_{ab} = \frac{n-1}{2k(n-k)n^2}\{k(2n-k) - n^2r^2\}\delta_{ab}, \tag{3.9}$$

$$A_{as} = 0, \tag{3.10}$$

$$A_{st} = \frac{n-1}{2k(n-k)n^2}\{n^2r^2 - k^2\}\delta_{st}. \tag{3.11}$$

Thus, we have

$$\mathrm{tr}\boldsymbol{A} = \frac{n-1}{2k(n-k)n}\{k^2 + n(n-2k)r^2\}. \tag{3.12}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)} + \frac{(n-1)(n-2k)}{k(n-k)n}r^2. \tag{3.13}$$

According to

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}(\mathrm{tr}\boldsymbol{A})\delta_{ij},$$

we have

$$\tilde{A}_{ab} = \frac{n-1}{kn^2}\{k - nr^2\}\delta_{ab}, \tag{3.14}$$

$$\tilde{A}_{as} = 0, \tag{3.15}$$

$$\tilde{A}_{st} = \frac{n-1}{(n-k)n^2}\{nr^2 - k\}\delta_{st}. \tag{3.16}$$

Therefore, we infer

$$\|\tilde{\boldsymbol{A}}\|^2 = \frac{(n-1)^2}{k(n-k)n}\left(r^2 - \frac{k}{n}\right)^2. \tag{3.17}$$

From (3.13) and (3.17), we obtain

$$(n-2k)^2\|\tilde{\boldsymbol{A}}\|^2 = \frac{k(n-k)}{n}\left(nR - \frac{n-2}{n}\right)^2. \tag{3.18}$$

From Lemma 3.1, (2.22), (3.9), (3.10) and (3.11), we have

$$R_{abab} = B_{aa}B_{bb} + A_{aa} + A_{bb} = \frac{n-1}{k(n-k)}(1-r^2), \tag{3.19}$$

$$R_{asas} = B_{aa}B_{ss} + A_{aa} + A_{ss} = 0, \tag{3.20}$$

$$R_{stst} = B_{ss}B_{tt} + A_{ss} + A_{tt} = \frac{n-1}{k(n-k)}r^2. \tag{3.21}$$

This completes the proof of Proposition 3.2. □

REMARK 3.3. From (3.3), we know that $(n-2)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{n-1}{n}}\left(nR - \frac{n-2}{n}\right)$ if and only if $k = 1$ or $k = n-1$.

PROPOSITION 3.4. *Let $\hat{\mathbf{x}} : S^k(1) \times \boldsymbol{R}^{n-k} \mapsto \boldsymbol{R}^{n+1}$ be the standard cylinder. Then, the hypersurface $\mathbf{x} = \sigma \circ \hat{\mathbf{x}} : S^k(1) \times \boldsymbol{R}^{n-k} \mapsto S^{n+1}(1)$ satisfies*

$$\Phi = 0, \tag{3.22}$$

$$R = \frac{k-1}{n(n-k)}, \tag{3.23}$$

$$\|\tilde{A}\|^2 = \frac{k(n-1)^2}{n^3(n-k)}, \tag{3.24}$$

$$R_{abab} = \frac{n-1}{k(n-k)}, \quad R_{asas} = 0, \quad R_{stst} = 0, \tag{3.25}$$

where σ is the conformal diffeomorphism defined by (1.1) with $p = 1$.

PROOF. Since $\hat{\mathbf{x}} : S^k(1) \times \boldsymbol{R}^{n-k} \mapsto \boldsymbol{R}^{n+1}$ is the standard cylinder, we know that the second fundamental form of $\hat{\mathbf{x}}$ has two distinct principal curvatures 1 and 0 with multiplicities k and $n-k$, respectively. Let $\hat{h}_{ij}$ and $\hat{H}$ denote components of the second fundamental form $\hat{II}$ and the mean curvature of $\hat{\mathbf{x}}$, respectively. Then, we have

$$\hat{h}_{ab} = \delta_{ab}, \quad \hat{h}_{as} = 0, \quad \hat{h}_{st} = 0, \tag{3.26}$$

$$\hat{H} = \frac{k}{n}, \quad \|\hat{II}\|^2 = k. \tag{3.27}$$

By defining

$$\hat{\rho}^2 = \frac{n}{n-1}\big(\|\hat{II}\|^2 - n\hat{H}^2\big) = \frac{k(n-k)}{n-1},$$

then, the Möbius metric $\hat{g}$ of $\hat{\mathbf{x}}$ is given by

$$\hat{g} = \hat{\rho}^2 d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}.$$

Let $\{\hat{e}_i\}$ be an orthonormal basis for the first fundamental form $\hat{I} = d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}$ with the dual basis $\{\hat{\theta}_i\}$. Define

$$\hat{C}_i = -\hat{\rho}^{-2}\Big\{\hat{H}_{,i} + \sum_j \big(\hat{h}_{ij} - \hat{H}\delta_{ij}\big)\hat{e}_j(\log\hat{\rho})\Big\}, \tag{3.28}$$

$$\begin{aligned}\hat{A}_{ij} = &-\hat{\rho}^{-2}\big\{\mathrm{Hess}_{ij}(\log\hat{\rho}) - \hat{e}_i(\log\hat{\rho})\hat{e}_j(\log\hat{\rho}) - \hat{H}\hat{h}_{ij}\big\} \\ &-\frac{1}{2}\hat{\rho}^{-2}\big(\|\nabla(\log\hat{\rho})\|^2 + \hat{H}^2\big)\delta_{ij},\end{aligned} \tag{3.29}$$

$$\hat{B}_{ij} = \hat{\rho}^{-1}\big(\hat{h}_{ij} - \hat{H}\delta_{ij}\big). \tag{3.30}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $\hat{I} = d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}$. $\hat{\Phi} = \sum_i \hat{C}_i\hat{\theta}_i\hat{e}_{n+1}$, $\hat{\boldsymbol{A}} = \hat{\rho}^2\sum_{i,j}\hat{A}_{ij}\hat{\theta}_i\hat{\theta}_j$ and $\hat{\boldsymbol{B}} = \sum_{i,j}\hat{B}_{ij}\hat{\theta}_i\hat{\theta}_j$ $(\hat{\rho}^{-1}\hat{e}_{n+1})$ is called *Möbius form*, *Blaschke tensor* and *Möbius second fundamental form* of the immersion $\hat{\mathbf{x}}$, respectively (cf. [**9**]).

Since $\hat{\rho}^2$ is constant, from (3.28) and (3.29), we have $\hat{C}_i = 0$ and $\hat{A}_{ij} = -\frac{1}{2}\hat{\rho}^{-2}$ $\{\hat{H}^2\delta_{ij} - 2\hat{H}\hat{h}_{ij}\}$. Hence, we infer $\hat{\Phi} = 0$ and

$$\hat{A}_{ab} = -\frac{(n-1)(k-2n)}{2(n-k)n^2}\delta_{ab},$$

$$\hat{A}_{as} = 0,$$

$$\hat{A}_{st} = -\frac{(n-1)k}{2(n-k)n^2}\delta_{st}.$$

Thus, from Theorem 4.1 of Liu, Wang and Zhao [**9**], we know $\Phi = \hat{\Phi} = 0$ and

$$A_{ab} = \hat{A}_{ab} = -\frac{(n-1)(k-2n)}{2(n-k)n^2}\delta_{ab}, \tag{3.31}$$

$$A_{as} = \hat{A}_{as} = 0, \tag{3.32}$$

$$A_{st} = \hat{A}_{st} = -\frac{(n-1)k}{2(n-k)n^2}\delta_{st}. \tag{3.33}$$

Thus, we infer

$$\mathrm{tr}\boldsymbol{A} = \frac{(n-1)k}{2(n-k)n}. \tag{3.34}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)}. \tag{3.35}$$

From

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\mathrm{tr}\boldsymbol{A}\delta_{ij},$$

we have

$$\tilde{A}_{ab} = \frac{n-1}{n^2}\delta_{ab},$$

$$\tilde{A}_{as} = 0,$$

$$\tilde{A}_{st} = -\frac{(n-1)k}{(n-k)n^2}\delta_{st}.$$

Therefore, we infer

$$\|\tilde{\boldsymbol{A}}\|^2 = \frac{(n-1)^2k}{(n-k)n^3}. \tag{3.36}$$

From (3.35) and (3.36), we obtain

$$(n-2k)^2\|\tilde{\boldsymbol{A}}\|^2 = \frac{k(n-k)}{n}\left(nR - \frac{n-2}{n}\right)^2. \tag{3.37}$$

From Lemma 3.1, (2.22), (3.31), (3.32) and (3.33), we have

$$R_{abab} = B_{aa}B_{bb} + A_{aa} + A_{bb} = \frac{n-1}{k(n-k)},$$

$$R_{asas} = B_{aa}B_{ss} + A_{aa} + A_{ss} = 0,$$

$$R_{stst} = B_{ss}B_{tt} + A_{ss} + A_{tt} = 0.$$

This completes the proof of Proposition 3.4. $\square$

REMARK 3.5. From (3.23) and (3.24), we know that $(n-2)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{n-1}{n}}\left(nR - \frac{n-2}{n}\right)$ if and only if $k = n-1$.

PROPOSITION 3.6. *Let* $\bar{\mathbf{x}} : S^k(r) \times \boldsymbol{H}^{n-k}(\sqrt{1+r^2}) \mapsto \boldsymbol{H}^{n+1}$ *be the standard embedding. Then, the hypersurface* $\mathbf{x} = \tau \circ \bar{\mathbf{x}} : S^k(r) \times \boldsymbol{H}^{n-k}(\sqrt{1+r^2}) \mapsto S^{n+1}(1)$ *satisfies*

$$\Phi = 0, \tag{3.38}$$

$$R = \frac{k-1}{n(n-k)} - \frac{(n-1)(n-2k)}{nk(n-k)} r^2, \tag{3.39}$$

$$(2k-n)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{k(n-k)}{n}}\left(nR - \frac{n-2}{n}\right), \tag{3.40}$$

$$R_{abab} = \frac{n-1}{k(n-k)}(1+r^2), \quad R_{asas} = 0, \quad R_{stst} = -\frac{n-1}{k(n-k)} r^2, \tag{3.41}$$

where τ *is the conformal diffeomorphism defined by* (1.2) *with* $p = 1$.

PROOF. Since $\bar{\mathbf{x}} : S^k(1) \times \boldsymbol{H}^{n-k}(\sqrt{1+r^2}) \mapsto \boldsymbol{H}^{n+1}$ is the standard embedding, we know that the second fundamental form of $\bar{\mathbf{x}}$ has two distinct principal curvatures $\frac{\sqrt{1+r^2}}{r} = d$ and $\frac{r}{\sqrt{1+r^2}}$ with multiplicities k and $n-k$, respectively. Let $\bar{h}_{ij}$ and $\bar{H}$ denote the components of the second fundamental form $\bar{II}$ and the mean curvature of $\bar{\mathbf{x}}$, respectively. Then, we have

$$\bar{h}_{ab} = d\delta_{ab}, \quad \bar{h}_{as} = 0, \quad \bar{h}_{st} = \frac{1}{d}\delta_{st},$$
$$\bar{H} = \frac{1}{n}\{kd + (n-k)d\}, \quad \|\bar{II}\|^2 = kd^2 + (n-k)\frac{1}{d^2}.$$

By defining

$$\bar{\rho}^2 = \frac{n}{n-1}\left(\|\bar{II}\|^2 - n\bar{H}^2\right) = \frac{k(n-k)}{n-1}\frac{(d^2-1)^2}{d^2},$$

then, the Möbius metric $\bar{g}$ of $\bar{\mathbf{x}}$ is given by

$$\bar{g} = \bar{\rho}^2 d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}.$$

Let $\{\bar{e}_i\}$ be an orthonormal basis for the first fundamental form $\bar{I} = d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}$ with the dual basis $\{\bar{\theta}_i\}$. Define

$$\bar{C}_i = -(\bar{\rho})^{-2}\left\{\bar{H}_{,i} + \sum_j \left(\bar{h}_{ij} - \bar{H}\delta_{ij}\right)\bar{e}_j(\log\bar{\rho})\right\}, \tag{3.42}$$

$$\begin{aligned}\bar{A}_{ij} = & -(\bar{\rho})^{-2}\big\{\mathrm{Hess}_{ij}(\log\bar{\rho}) - \bar{e}_i(\log\bar{\rho})\bar{e}_j(\log\bar{\rho}) - \bar{H}\bar{h}_{ij}\big\}\\ & -\frac{1}{2}(\bar{\rho})^{-2}\big(\|\nabla(\log\bar{\rho})\|^2 + 1 + \bar{H}^2\big)\delta_{ij},\end{aligned} \tag{3.43}$$

$$\bar{B}_{ij} = (\bar{\rho})^{-1}\big(\bar{h}_{ij} - \bar{H}\delta_{ij}\big). \tag{3.44}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $\bar{I} = d\bar{\mathbf{x}}\cdot d\bar{\mathbf{x}}$. $\bar{\Phi} = \Sigma_i\bar{C}_i\bar{\theta}_i\bar{e}_{n+1}$, $\bar{\boldsymbol{A}} = \bar{\rho}^2\sum_{ij}\bar{A}_{ij}\bar{\theta}_i\bar{\theta}_j$ and $\bar{\boldsymbol{B}} = \sum_{i,j}\bar{B}_{ij}\bar{\theta}_i\bar{\theta}_j$ $((\bar{\rho})^{-1}\bar{e}_{n+1})$ is called *Möbius form*, *Blaschke tensor* and *Möbius second fundamental form* of the immersion $\bar{\mathbf{x}}$, respectively (cf. [**9**]).

Since $\bar{\rho}^2$ is constant, from (3.42) and (3.43), we have $\bar{C}_i = 0$ and $\bar{A}_{ij} = -\frac{1}{2}(\bar{\rho})^{-2}\{(1+\bar{H}^2)\delta_{ij} - 2\bar{H}\bar{h}_{ij}\}$. Hence, we infer $\bar{\Phi} = 0$ and

$$\begin{aligned}\bar{A}_{ab} &= \frac{n-1}{2k(n-k)n^2}\{k(2n-k) + n^2r^2\}\delta_{ab},\\ \bar{A}_{as} &= 0,\\ \bar{A}_{st} &= -\frac{n-1}{2k(n-k)n^2}\{k^2 + n^2r^2\}\delta_{st}.\end{aligned}$$

Thus, from Theorem 4.4 of Liu, Wang and Zhao [**9**], we know $\Phi = \bar{\Phi} = 0$ and

$$A_{ab} = \bar{A}_{ab} = \frac{n-1}{2k(n-k)n^2}\{k(2n-k) + n^2r^2\}\delta_{ab}, \tag{3.45}$$

$$A_{as} = \bar{A}_{as} = 0, \tag{3.46}$$

$$A_{st} = \bar{A}_{st} = -\frac{n-1}{2k(n-k)n^2}\{k^2 + n^2r^2\}\delta_{st}. \tag{3.47}$$

Thus, we infer

$$\mathrm{tr}\boldsymbol{A} = \frac{n-1}{2k(n-k)n}\{k^2 - n(n-2k)r^2\}. \tag{3.48}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)} - \frac{(n-1)(n-2k)}{nk(n-k)}r^2. \tag{3.49}$$

From

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\mathrm{tr}\boldsymbol{A}\delta_{ij},$$

we have

$$\begin{aligned}\tilde{A}_{ab} &= \frac{n-1}{kn^2}(k+nr^2)\delta_{ab},\\ \tilde{A}_{as} &= 0,\\ \tilde{A}_{st} &= -\frac{n-1}{(n-k)n^2}(k+nr^2)\delta_{st}.\end{aligned}$$

Therefore, we infer

$$\|\tilde{\boldsymbol{A}}\|^2 = \frac{(n-1)^2}{k(n-k)n}\left(r^2+\frac{k}{n}\right)^2.$$

From (3.49) and (3.50), we obtain

$$(2k-n)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{k(n-k)}{n}}\left(nR-\frac{n-2}{n}\right).$$

From Lemma 3.1, (2.22), (3.45), (3.46) and (3.47), we have

$$\begin{aligned}R_{abab} &= B_{aa}B_{bb}+A_{aa}+A_{bb} = \frac{n-1}{k(n-k)}(1+r^2),\\ R_{asas} &= B_{aa}B_{ss}+A_{aa}+A_{ss} = 0,\\ R_{stst} &= B_{ss}B_{tt}+A_{ss}+A_{tt} = -\frac{n-1}{k(n-k)}r^2.\end{aligned}$$

This completes the proof of Proposition 3.6. □

Remark 3.7. From (3.40), we know that $(n-2)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{n-1}{n}}\left(nR-\frac{n-2}{n}\right)$ if and only if $k=n-1$.

4. Proofs of Main Theorems.

In this section, we will prove our Main Theorems.

Proof of Main Theorem 1. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$, we have, by (2.19), (2.20) and (2.21), that

$$A_{ij,k} = A_{ik,j}, \quad B^\alpha_{ij,k} = B^\alpha_{ik,j}, \quad \sum_k B^\alpha_{ik}A_{kj} = \sum_k B^\alpha_{kj}A_{ki}, \text{ for any } \alpha. \tag{4.1}$$

From the definition $\Delta B^\alpha_{ij} = \sum_k B^\alpha_{ij,kk}$ of the Laplacian of the Möbius second fundamental form of the immersion $\mathbf{x}$, we have

$$\frac{1}{2}\Delta\sum_{i,j,\alpha}\left(B^\alpha_{ij}\right)^2 = \sum_{i,j,k,\alpha}\left(B^\alpha_{ij,k}\right)^2 + \sum_{i,j,\alpha}B^\alpha_{ij}\Delta B^\alpha_{ij}. \tag{4.2}$$

From (2.16), we have

$$\sum_{i,j,k,\alpha}\left(B^{\alpha}_{ij,k}\right)^2+\sum_{i,j,\alpha}B^{\alpha}_{ij}\Delta B^{\alpha}_{ij}=0. \tag{4.3}$$

From (2.16), (2.22), (2.23), (2.26) and (4.1), we have, by a direct calculation, that

$$\begin{aligned}\sum_{i,j,\alpha}B^{\alpha}_{ij}\Delta B^{\alpha}_{ij}=&-2\sum_{\alpha,\beta}\left[\mathrm{tr}\big(B^2_{\alpha}B^2_{\beta}\big)-\mathrm{tr}\big\{(B_{\alpha}B_{\beta})^2\big\}\right]\\&-\sum_{\alpha,\beta}\big\{\mathrm{tr}(B_{\alpha}B_{\beta})\big\}^2+n\sum_{\alpha}\mathrm{tr}\big(AB^2_{\alpha}\big)+\frac{n-1}{n}\mathrm{tr}A,\end{aligned} \tag{4.4}$$

where B_α and A denote the $n\times n$-symmetric matrices (B^{α}_{ij}) and (A_{ij}) respectively. Putting $\tilde{A}=(\tilde{A}_{ij})$ with

$$\tilde{A}_{ij}=A_{ij}-\frac{1}{n}(\mathrm{tr}A)\delta_{ij}, \tag{4.5}$$

we have

$$\|A\|^2=\sum_{i,j}(A_{ij})^2=\sum_{i,j}\big(\tilde{A}_{ij}\big)^2+\frac{1}{n}(\mathrm{tr}A)^2=\|\tilde{A}\|^2+\frac{1}{n}(\mathrm{tr}A)^2, \tag{4.6}$$

From (4.5), we have

$$\mathrm{tr}\tilde{A}=0,\quad \mathrm{tr}\big(\tilde{A}B^2_{\alpha}\big)=\mathrm{tr}\big(AB^2_{\alpha}\big)-\frac{1}{n}(\mathrm{tr}A)\big(\mathrm{tr}B^2_{\alpha}\big). \tag{4.7}$$

From (4.1), we know that $B_\alpha A=AB_\alpha$. Therefore $B_\alpha\tilde{A}=\tilde{A}B_\alpha$ holds. From Lemma 2.2, we have

$$\mathrm{tr}\big(AB^2_{\alpha}\big)\geq-\frac{n-2}{\sqrt{n(n-1)}}\mathrm{tr}B^2_{\alpha}\|\tilde{A}\|+\frac{1}{n}(\mathrm{tr}A)\big(\mathrm{tr}B^2_{\alpha}\big). \tag{4.8}$$

CASE (i) where $p=1$. Put $B^{n+1}_{ij}=B_{ij}$ and $B_{n+1}=B$. Since $BA=AB$ holds from (4.1), we can choose a local orthonormal basis $\{E_1,E_2,\cdots,E_n\}$ such that $B_{ij}=\mu_i\delta_{ij}$ and $A_{ij}=\lambda_i\delta_{ij}$. Thus, we have from (4.4), (2.16) and (4.8)

$$\begin{aligned}\sum_{i,j}B_{ij}\Delta B_{ij}&=-(\mathrm{tr}B^2)^2+n\mathrm{tr}(AB^2)+\frac{n-1}{n}\mathrm{tr}A\\&\geq-\left(\frac{n-1}{n}\right)^2-\sqrt{\frac{n-1}{n}}(n-2)\|\tilde{A}\|+2\frac{n-1}{n}\mathrm{tr}A\\&=\sqrt{\frac{n-1}{n}}\left[\sqrt{\frac{n-1}{n}}\left(nR-\frac{n-2}{n}\right)-(n-2)\|\tilde{\boldsymbol{A}}\|\right],\end{aligned} \tag{4.9}$$

where $\|\tilde{\boldsymbol{A}}\|^2 = \|\tilde{A}\|^2$ and $\mathrm{tr}\boldsymbol{A} = \mathrm{tr}A$ are used. From the assumption (1.3) in Main Theorem 1, we know that the right hand side of formula (4.9) is nonnegative. Therefore, from (4.3) and (4.9), we obtain

$$B_{ij,k} = 0, \quad \text{for all } \ i, j, k \ \text{ and } \ \sum_{i,j} B_{ij} \Delta B_{ij} = 0. \tag{4.10}$$

Hence the equality in (4.9) holds. We have

$$\sqrt{\frac{n-1}{n}}\left(nR - \frac{n-2}{n}\right) - (n-2)\|\tilde{\boldsymbol{A}}\| = 0. \tag{4.11}$$

Further, the inequality (4.8) becomes equality. From Lemma 2.2, we know that $(n-1)$ of the eigenvalues μ_i of B satisfy $|\mu_i| = \frac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}} = \frac{1}{n}$ and $\mu_i\mu_j \geq 0$, which yields that the $(n-1)$ of μ_i's are equal and constant. Since $\mathrm{tr}B = 0$ and $\sum_{i,j} B_{ij}^2 = \frac{n-1}{n}$ hold, we know that B has two distinct principal curvatures, which are all constant. Therefore, we obtain $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isoparametric hypersurface with two distinct principal curvatures. By the result of Li, Liu, Wang and Zhao [**7**], we have that $\mathbf{x}$ is Möbius equivalent to an open part of the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or an open part of the image of σ of the standard cylinder $S^k(1) \times \boldsymbol{R}^{n-k}$ in $\boldsymbol{R}^{n+1}$ or an open part of the image of τ of $S^k(r) \times \boldsymbol{H}^{n-k}(\sqrt{1+r^2})$ in $\boldsymbol{H}^{n+1}$, for $k = 1, 2, \cdots, n-1$. From Remark 3.3, Remark 3.5 and Remark 3.7, we know that formula (4.11) holds if and only if $k = n-1$. Hence, Main Theorem 1 is true in this case.

Case (ii) where $p \geq 2$. Define $\sigma_{\alpha\beta} = \sum_{i,j} B_{ij}^\alpha B_{ij}^\beta$. Since the $(p \times p)$-matrix $(\sigma_{\alpha\beta})$ is symmetric, we can choose $E_{n+1}, \cdots, E_{n+p}$ such that $(\sigma_{\alpha\beta})$ is diagonal, that is,

$$\sigma_{\alpha\beta} = \sigma_\alpha \delta_{\alpha\beta}. \tag{4.12}$$

From Lemma 2.1, we have

$$\begin{aligned} -\sum_{\alpha,\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) - \sum_{\alpha,\beta} \left\{\mathrm{tr}(B_\alpha B_\beta)\right\}^2 &\geq -2\sum_{\alpha\neq\beta} \sigma_\alpha\sigma_\beta - \sum_\alpha \sigma_\alpha^2 \\ &= -2\left(\sum_\alpha \sigma_\alpha\right)^2 + \sum_\alpha \sigma_\alpha^2 \\ &\geq -2\left(\frac{n-1}{n}\right)^2 + \frac{1}{p}\left(\sum_\alpha \sigma_\alpha\right)^2 \\ &= -\left(2 - \frac{1}{p}\right)\left(\frac{n-1}{n}\right)^2. \end{aligned} \tag{4.13}$$

From (4.4), (4.8), (4.13), we have

$$\begin{aligned}\sum_{i,j,\alpha} B_{ij}^{\alpha}\Delta B_{ij}^{\alpha} \geq & -\left(2-\frac{1}{p}\right)\left(\frac{n-1}{n}\right)^2 + n\sum_{\alpha}\mathrm{tr}\big(AB_{\alpha}^2\big) + \frac{n-1}{n}\mathrm{tr}A \\ = & -\left(2-\frac{1}{p}\right)\left(\frac{n-1}{n}\right)^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\mathrm{tr}B_{\alpha}^2\|\tilde{A}\| \\ & + \mathrm{tr}A\sum_{\alpha}\mathrm{tr}B_{\alpha}^2 + \frac{n-1}{n}\mathrm{tr}A \\ = & -\left(2-\frac{1}{p}\right)\left(\frac{n-1}{n}\right)^2 - \sqrt{\frac{n-1}{n}}(n-2)\|\tilde{A}\| + 2\frac{n-1}{n}\mathrm{tr}A \\ = & \sqrt{\frac{n-1}{n}}\left\{\sqrt{\frac{n-1}{n}}\left(nR - \frac{1}{n}\left[(n-1)\left(2-\frac{1}{p}\right)-1\right]\right) - (n-2)\|\tilde{\boldsymbol{A}}\|\right\},\end{aligned} \tag{4.14}$$

where $\|\tilde{\boldsymbol{A}}\|^2 = \|\tilde{A}\|^2$ and $\mathrm{tr}\boldsymbol{A} = \mathrm{tr}A$ are used. From the assumption (1.3) in Main Theorem 1, we know that the right hand side of (4.14) is nonnegative. Therefore, from (4.3) and (4.14), we obtain $B_{ij,k}^{\alpha} = 0$, for all i, j, k, α, and $\sum_{i,j,\alpha} B_{ij}^{\alpha}\Delta B_{ij}^{\alpha} = 0$. Hence, the above inequalities become equalities. Thus, we have

$$(n-2)\|\tilde{\boldsymbol{A}}\| = \sqrt{\frac{n-1}{n}}\left\{nR - \frac{1}{n}\left[(n-1)\left(2-\frac{1}{p}\right)-1\right]\right\}, \tag{4.15}$$

and

$$\sigma_{n+1} = \sigma_{n+2} = \cdots = \sigma_{n+p} \tag{4.16}$$

because of $\frac{1}{p}(\sum_{\alpha}\sigma_{\alpha})^2 = \sum_{\alpha}\sigma_{\alpha}^2$. From Lemma 2.1, we know that at most two of the matrices $B_{\alpha} = (B_{ij}^{\alpha})$ are nonzero. From (2.16), we have $\sum_{\alpha}\sigma_{\alpha} = \frac{n-1}{n}$. Hence, (4.16) yields $p = 2$ and we may assume that

$$B_{n+1} = \lambda\widetilde{A}, \quad B_{n+2} = \mu\widetilde{B}, \quad \lambda, \mu \neq 0, \tag{4.17}$$

where $\widetilde{A}$ and $\widetilde{B}$ are defined in Lemma 2.1. Therefore, we have

$$B_{12}^{n+1} = B_{21}^{n+1} = \lambda, \quad B_{ij}^{n+1} = 0, \qquad (i,j) \notin \{(1,2),(2,1)\}, \tag{4.18}$$

$$B_{11}^{n+2} = \mu, \quad B_{22}^{n+2} = -\mu, \quad B_{ij}^{n+2} = 0, \qquad (i,j) \notin \{(1,1),(2,2)\}. \tag{4.19}$$

Since the inequality (4.8) becomes equality, from Lemma 2.2, we know that, for each α, $(n-1)$ of the eigenvalues μ_i^{α} of $B_{\alpha} = (B_{ij}^{\alpha})$ satisfy $|\mu_i^{\alpha}| = \frac{(\mathrm{tr}B_{\alpha}^2)^{1/2}}{\sqrt{n(n-1)}}$ and $\mu_i^{\alpha}\mu_j^{\alpha} \geq 0$, which infer that the $(n-1)$ of μ_i^{α} are equal. From (4.19), we have the eigenvalues of $B_{n+2} = (B_{ij}^{n+2})$ are $\mu, -\mu, 0, 0, \cdots 0$. Since $\mu \neq 0$, we infer $n = 2$. From (4.18), we can infer, by an algebraic method, that the eigenvalues of B_{n+1} are $\lambda, -\lambda$. Since $n = p = 2$

holds, from (4.1), (4.18) and (4.19), we have

$$A_{11} = A_{22}, \quad A_{12} = A_{21} = 0. \tag{4.20}$$

Therefore, $\mathbf{x} : M^2 \mapsto S^4(1)$ is a Möbius isotropic submanifold in S^4. Thus, we have $\|\tilde{\boldsymbol{A}}\| = 0$. Since $n = 2$ and $p = 2$ hold, from (4.15), we have $R = \frac{1}{8}$. We obtain $\mathrm{tr}A = \frac{3}{8}$. Hence $A_{11} = A_{22} = \frac{3}{16}$. From Liu, Wang and Zhao [**9**], we obtain that $\mathbf{x} : M^2 \mapsto S^4(1)$ is Möbius equivalent to an open part of either a minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$, or the image of σ_2 of a minimal surface with constant scalar curvature in $\boldsymbol{R}^4$ or the image of τ_2 of a minimal surface with constant scalar curvature in $\boldsymbol{H}^4$. For a surface, Gaussian curvature is constant if and only if the scalar curvature is constant. From the Proposition 4.1 and Theorem 4.2 of Bryant [**3**], we know that a minimal surface with constant scalar curvature in $\boldsymbol{R}^4$ is totally geodesic and a minimal surface with constant scalar curvature in $\boldsymbol{H}^4$ is also totally geodesic. Since $\mathbf{x} : M^2 \mapsto S^4(1)$ has no umbilical points, we infer that $\mathbf{x} : M^2 \mapsto S^4(1)$ is Möbius equivalent to an open part of a minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$. From the Gauss equation of the minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$, we know that the squared norm of the second fundamental form of this minimal surface is constant. According to the definition (2.2) of ρ, ρ^2 is constant. From (2.14), we have $\rho^2 = \frac{8}{3}$. Thus, the squared norm of the second fundamental form of $\tilde{\mathbf{x}}$ must be $\frac{4}{3}$, i.e. $\|II\|^2 = \frac{4}{3}$. Therefore, from the result of Chern, do Carmo and Kobayashi [**5**], we obtain that $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ is locally a Veronese surface in $S^4(1)$. This finishes the proof of Main Theorem 1.

PROOF OF MAIN THEOREM 2. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$ holds, we have

$$A_{ij,k} = A_{ik,j}, \quad B^\alpha_{ij,k} = B^\alpha_{ik,j}, \quad \sum_k B^\alpha_{ik} A_{kj} = \sum_k B^\alpha_{kj} A_{ki}. \tag{4.21}$$

Hence, for any α, $B_\alpha A = AB_\alpha$, where $A = (A_{ij})$ and $B_\alpha = (B^\alpha_{ik})$. For any fixed α, we can choose the basis $\{E_i\}$ such that $A = (A_{ij})$ and $B_\alpha = (B^\alpha_{ik})$ are diagonal, that is,

$$A_{ij} = \lambda_i \delta_{ij}, \quad B^\alpha_{ij} = \mu^\alpha_i \delta_{ij}. \tag{4.22}$$

Since $n(n-1)R$ is constant, from (2.24), we have that $\mathrm{tr}\boldsymbol{A} = \mathrm{tr}A = \sum_i A_{ii}$ is constant. From (2.25), (4.21), (4.22), we infer

$$\begin{aligned}\frac{1}{2}\Delta\|\boldsymbol{A}\|^2 &= \sum_{i,j,k}(A_{ij,k})^2 + \sum_{i,j,k} A_{ij}A_{ij,kk}\\ &= \sum_{i,j,k}(A_{ij,k})^2 + \sum_{i,j,k} A_{ij}A_{kk,ij} + \sum_{i,j,k,l} A_{ij}A_{li}R_{lkjk} + \sum_{i,j,k,l} A_{ij}A_{kl}R_{lijk}\\ &= \sum_{i,j,k}(A_{ij,k})^2 + \frac{1}{2}\sum_{i,k} R_{ikik}(\lambda_i - \lambda_k)^2. \end{aligned}\tag{4.23}$$

When $p > 1$, from the assumption $K > 0$ in Main Theorem 2, by integrating (4.23), we have

$$R_{ikik}(\lambda_i - \lambda_k)^2 = 0.$$

Therefore, we know that $\lambda_i = \lambda_k$, that is, $\mathbf{x} : M \mapsto S^{n+p}(1)$ is a Möbius isotropic submanifold in $S^{n+p}(1)$ with positive Möbius sectional curvature. From the result in [**9**], we know that $\mathbf{x}$ is Möbius equivalent to the compact minimal submanifolds with constant scalar curvature in $S^{n+p}(1)$.

Next, we consider the case where $p = 1$. In this case, we know that the Möbius sectional curvature of the immersion $\mathbf{x}$ is nonnegative. By integrating (4.23), we infer

$$A_{ij,k} = 0, \text{ for any } i, j, k, \quad R_{ikik}(\lambda_i - \lambda_k)^2 = 0. \tag{4.24}$$

From (2.22) and (4.22), we have $R_{ikik} = \mu_i\mu_k + \lambda_i + \lambda_k$ for $i \neq k$. Hence, we infer

$$(\mu_i\mu_k + \lambda_i + \lambda_k)(\lambda_i - \lambda_k)^2 = 0. \tag{4.25}$$

Form (4.24) and (2.17), we have

$$0 = d\lambda_i\delta_{ij} + (\lambda_i - \lambda_j)\omega_{ij}, \quad 1 \leq i, j \leq n. \tag{4.26}$$

Setting $i = j$ in (4.26), we obtain $d\lambda_i = 0$, that is, eigenvalues of (A_{ij}) are all constant. From (4.26), we infer that for $\lambda_i \neq \lambda_j$,

$$\omega_{ij} = 0. \tag{4.27}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_l$ are these distinct eigenvalues of $A = (A_{ij})$. We can assume $\lambda_1 < \lambda_2 < \cdots < \lambda_l$. From (4.25), we have

$$\lambda_i = \lambda_k, \quad \text{or} \quad \mu_i\mu_k + \lambda_i + \lambda_k = 0. \tag{4.28}$$

In the second case, we will prove that $A = (A_{ij})$ has at most three distinct eigenvalues. In fact, if we assume $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_l$ are these distinct eigenvalues of $A = (A_{ij})$. Let $\lambda_1, \lambda_2, \lambda_i$ are the three distinct eigenvalues of $A = (A_{ij})$, we have

$$\mu_i\mu_1 + \lambda_i + \lambda_1 = 0.$$
$$\mu_i\mu_2 + \lambda_i + \lambda_2 = 0.$$

Hence, we have

$$\mu_i = -\frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2}, \tag{4.29}$$

$$\lambda_i = -\lambda_1 + \mu_1\frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2}. \tag{4.30}$$

Hence, for $r = 3, 4, \cdots, l$, we have $\lambda_r = \lambda_i$. This is a contradiction. Therefore, $A = (A_{ij})$ has at most three distinct eigenvalues.

(1). In the first case, we consider the case that (A_{ij}) only has one distinct eigenvalues. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$, we know $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isotropic hypersurface in $S^{n+1}(1)$ with nonnegative Möbius sectional curvature. By the result in [**9**], we know that $\mathbf{x}$ is Möbius equivalent to a minimal hypersurface with constant scalar curvature in $S^{n+1}(1)$.

(2). We consider the second case that (A_{ij}) has two or three distinct eigenvalues. From (4.29), we know that at most three of the principal curvatures of (B_{ij}) are distinct. Since $\mathbf{x}$ has no umbilical points, we know that the distinct principal curvatures of (B_{ij}) is two or three.

(i) If two of the principal curvatures of (B_{ij}) are distinct, without loss of generality, we may assume $\mu_1 < \mu_2$. From (2.16), we know that μ_1 and μ_2 are constant, that is, $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isoparametric hypersurface with two distinct principal curvatures in $S^{n+1}(1)$. Since $\mathbf{x}$ is compact, from Theorem 1.1 in the introduction, we infer that $\mathbf{x}$ is Möbius equivalent to the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$.

(ii) If three of the principal curvatures of (B_{ij}) are distinct, without loss of generality, we may assume $\mu_1 < \mu_2 < \mu_3$. From (2.16) and (4.29), we know that μ_1, μ_2, μ_3 are constant. From the proof of Main Theorem 1, we infer

$$\begin{aligned}\frac{1}{2}\Delta\sum_{i,j} B_{ij}^2 &= \sum_{i,j,k} B_{ij,k}^2 + \sum_{i,j} B_{ij}\Delta B_{ij} \\ &= \sum_{i,j,k} B_{ij,k}^2 - (\mathrm{tr}B^2)^2 + n\mathrm{tr}(AB^2) + \frac{n-1}{n}\mathrm{tr}A \\ &= \sum_{i,j,k} B_{ij,k}^2 + \frac{1}{2}\sum_{i,j}(\mu_i - \mu_j)^2 R_{ijij} \geq 0.\end{aligned}$$

Since $\sum_{i,j} B_{ij}^2$ is constant, we obtain $B_{ij,k} = 0$ for any i, j, k. From (2.18), we have, for each $\mu_i \neq \mu_j$,

$$\omega_{ij} = 0. \tag{4.31}$$

Hence, we know that the distributions of the eigenspaces with respect to μ_i are integrable. Since the distinct principal curvatures of M is three, we can write $M = M_1 \times M_2 \times M_3$, where M_i $(1 \leq i \leq 3)$ is the integrable manifold corresponding to the principal curvature μ_i. Since μ_i's are constant, we know that M_i, $i = 1, 2, 3$, are closed. Thus, they are compact because M is compact. From (2.22), we have, for $j, k, l \in [i]$,

$$R_{ijkl} = \big(\mu_i^2 + 2\lambda_i\big)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{4.32}$$

that is, M_i are constant curvature space with respect to the Möbius metric g. Putting $k_i = \mu_i^2 + 2\lambda_i$, $1 \leq i \leq 3$, then, we have

$$k_1 = (\mu_1 - \mu_2)(\mu_1 - \mu_3) > 0,$$
$$k_2 = (\mu_2 - \mu_1)(\mu_2 - \mu_3) < 0,$$
$$k_3 = (\mu_3 - \mu_1)(\mu_3 - \mu_2) > 0. \tag{4.33}$$

Therefore, we may infer dim $M_2 = 1$. In fact, if dim $M_2 \geq 2$ holds, by the assumption that the Möbius sectional curvature of M is nonnegative, we have $k_2 \geq 0$. This is a contradiction.

Let (u, v, w) be a coordinate system for M such that $u \in M_1$, $v \in M_2$, $w \in M_3$ and $E_l = \frac{\partial}{\partial v}$, where $l = \dim M_1 + 1$. Then, from structure equations (2.9), (2.10), (2.11) and (2.12) and (4.31), by a direct and simple calculation, we obtain

$$N_v = \lambda_2 Y_v, \tag{4.34}$$

$$Y_{vv} = -\lambda_2 Y - N + \mu_2 E, \quad Y_{vj} = 0, \text{ for } j \neq l, \tag{4.35}$$

$$E_v = -\mu_2 Y_v, \tag{4.36}$$

where we denote E_{n+1} by E. From (4.35), we can write $Y = f(v) + F(u, w)$. Then, by (4.34), (4.35) and (4.36), we have

$$f'''(v) + k_2 f'(v) = 0, \tag{4.37}$$

where $k_2 = \mu_2^2 + 2\lambda_2 < 0$. The solution of (4.37) can be easily written as

$$f(v) = C_1 \frac{1}{\sqrt{-k_2}} \cosh\left(\sqrt{-k_2} v\right) + C_2 \frac{1}{\sqrt{-k_2}} \sinh\left(\sqrt{-k_2} v\right), \tag{4.38}$$

where $C_1, C_2 \in \boldsymbol{R}_1^{n+3}$ are constant vectors. From (4.38), we know that M_2 must be a hyperbola. This is a contradiction because M_2 is compact. Hence, the case (ii) does not occur, that is, M is a Möbius isoparametric hypersurface with two distinct principal curvatures. This completes the proof of Main Theorem 2.

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