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Stickelberger ideals of conductor p and their application

By Humio Ichimura and Hiroki Sumida-Takahashi

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Abstract. Let p be an odd prime number and F a number field. Let $K = F(\zeta_p)$ and $\Delta = \mathrm{Gal}(K/F)$. Let $\mathscr{S}_\Delta$ be the Stickelberger ideal of the group ring $\boldsymbol{Z}[\Delta]$ defined in the previous paper [**8**]. As a consequence of a p-integer version of a theorem of McCulloh [**15**], [**16**], it follows that F has the Hilbert-Speiser type property for the rings of p-integers of elementary abelian extensions over F of exponent p if and only if the ideal $\mathscr{S}_\Delta$ annihilates the p-ideal class group of K. In this paper, we study some properties of the ideal $\mathscr{S}_\Delta$, and check whether or not a subfield of $\boldsymbol{Q}(\zeta_p)$ satisfies the above property.

1. Introduction.

Let $p \geq 3$ be a fixed odd prime number. Let $\boldsymbol{F}_{p^r}$ be the finite field with p^r elements, and let $\Gamma_r = \boldsymbol{F}_{p^r}^+$ and $G_r = \boldsymbol{F}_{p^r}^\times$ be the additive group and the multiplicative group of $\boldsymbol{F}_{p^r}$, respectively. For a number field F, denote by $Cl = Cl(\mathscr{O}_F[\Gamma_r])$ and $R = R(\mathscr{O}_F[\Gamma_r])$ the locally free class group of the group ring $\mathscr{O}_F[\Gamma_r]$ and the subset of classes realized by rings of integers of tame Γ_r-Galois extensions over F, respectively. Here, $\mathscr{O}_F$ is the ring of integers of F. As G_r naturally acts on Γ_r, the group ring $\boldsymbol{Z}[G_r]$ acts on Cl. McCulloh [**15**], [**16**] characterized the realizable classes R by the action on Cl of a naturally defined Stickelberger ideal $\mathscr{S}_r$ of $\boldsymbol{Z}[G_r]$. On the other hand, we defined in [**8**] another Stickelberger ideal $\mathscr{S}_H$ of $\boldsymbol{Z}[H]$ for each subgroup H of the multiplicative group $\boldsymbol{F}_p^\times$ in connection with a normal integral basis problem (for the definition, see Section 2). The Stickelberger ideal $\mathscr{S}_H$ is a "H-part" of McCulloh's $\mathscr{S}_1$, and when $H = \boldsymbol{F}_p^\times$, it equals $\mathscr{S}_1$ and the classical one for the extension $\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}$. For the ideal $\mathscr{S}_H$, the following assertion (Theorem 1) holds as a consequence of a p-integer version of the above theorem of McCulloh. For details, see Section 7. A direct and simpler proof is given in [**8**].

Let F be a number field, $\mathscr{O}_F$ the ring of integers, and $\mathscr{O}_F' = \mathscr{O}_F[1/p]$ the ring of p-integers. Let Cl_F and Cl_F' be the ideal class groups of the Dedekind domains $\mathscr{O}_F$ and $\mathscr{O}_F'$, respectively. Letting P be the subgroup of Cl_F generated by the classes containing a prime ideal of $\mathscr{O}_F$ over p, we naturally have $Cl_F' \cong Cl_F/P$. A finite Galois extension N/F with group Γ has a normal p-integral basis (p-NIB for short) when $\mathscr{O}_N'$ is cyclic over the group ring $\mathscr{O}_F'[\Gamma]$. We say that F satisfies the condition (H_p') when any cyclic extension N/F of degree p has a p-NIB, and that it satisfies $(H_{p,\infty}')$ when any abelian extension N/F of exponent p has a p-NIB. It is known that when $F = \boldsymbol{Q}$, these conditions

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are satisfied for any p. This is shown similarly to the classical theorem of Hilbert and Speiser. Let $K = F(\zeta_p)$ and $\Delta = \mathrm{Gal}(K/F)$. For an integer $i \in \boldsymbol{Z}$, let $\bar{i}$ denote the class in $\boldsymbol{F}_p = \boldsymbol{Z}/p\boldsymbol{Z}$ represented by i. We have a natural embedding

$$\iota : \Delta \to \boldsymbol{F}_p^\times, \quad \sigma \to \bar{i}$$

with $\zeta_p^\sigma = \zeta_p^i$, and we identify Δ with the image $H = H_F = \iota(\Delta)$. Then, the Stickelberger ideal $\mathscr{S}_\Delta = \mathscr{S}_H$ naturally acts on the class group Cl'_K.

Theorem 1. *Let F be a number field. Let $K = F(\zeta_p)$ and $\Delta = \mathrm{Gal}(K/F)$. Then, the following three conditions are equivalent.*

(I) *F satisfies (H'_p).*

(II) *F satisfies $(H'_{p,\infty})$.*

(III) *The Stickelberger ideal $\mathscr{S}_\Delta$ annihilates the class group Cl'_K.*

For $p \leq 19$, it is known that the class number of $\boldsymbol{Q}(\zeta_p)$ is one (cf. Washington [**19**, Theorem11.1]), and hence it follows from Theorem 1 that any subfield F of $\boldsymbol{Q}(\zeta_p)$ satisfies (H'_p).

The purposes of this paper are (a) to study some properties of the ideal $\mathscr{S}_H$, and as an application, (b) to check whether or not a subfield of $\boldsymbol{Q}(\zeta_p)$ satisfies the condition (H'_p) for $23 \leq p \leq 499$. As a consequence of our results, we propose the following conjecture in Section 3.

Conjecture. *Let p be a prime number with $p \geq 23$ and F a subfield of $\boldsymbol{Q}(\zeta_p)$ with $F \neq \boldsymbol{Q}$. If $[F : \boldsymbol{Q}] > 2$ or $p \equiv 1 \bmod 4$, then F does not satisfy (H'_p) except for the case where $p = 29$ and $[F : \boldsymbol{Q}] = 2$ or 7.*

When $23 \leq p \leq 499$, this assertion is valid for any F. It is also valid for any $p \geq 23$ if $[\boldsymbol{Q}(\zeta_p) : F] \leq 4$ or $[\boldsymbol{Q}(\zeta_p) : F] = 6$. When $p \equiv 3 \bmod 4$ and F is the quadratic subfield of $\boldsymbol{Q}(\zeta_p)$, the matters seem to be more complicated. For these, see Proposition 4 and Remark 2 in Section 3.

Remark 1. (1) A relation between Stickelberger ideals and Galois module structure of rings of integers was observed first by Hilbert [**6**, Theorem 136] in his alternative proof of the classical Stickelberger theorem for the ideal class group of $\boldsymbol{Q}(\zeta_p)$. After Hilbert, this connection was pursued by Fröhlich [**3**], McCulloh [**15**], [**16**], Childs [**1**], *etc.* For details, see Fröhlich [**4**, Chapter IV]. (2) For the rings of integers in the usual sense, a result corresponding to (but weaker than) Theorem 1 is given in [**9**, Theorem 5]. It is obtained from the above mentioned theorem of McCulloh.

This paper is organized as follows. In Section 2, we recall the definition of the ideal $\mathscr{S}_H$, and give several properties of $\mathscr{S}_H$. In Section 3, we derive corollaries on the property (H'_p) from Theorem 1 and the results in Section 2. In Sections 3-6, we prove the results in Section 2. In the final section, we give the p-integer version of McCulloh's theorem, and derive a part of Theorem 1 from this.

2. Results.

Let us first recall the definition of the Stickelberger ideal associated with a subgroup of $\boldsymbol{F}_p^\times$. Let H be a subgroup of $\boldsymbol{F}_p^\times$. For an integer $i \in \boldsymbol{Z}$ with $\bar{i} \in \boldsymbol{F}_p^\times$, we often write $\sigma_i = \bar{i}$. For an integer $r \in \boldsymbol{Z}$, let

$$\theta_r = \theta_{H,r} = \sum_i{}' \left[\frac{ri}{p}\right]\sigma_i^{-1} \in \boldsymbol{Z}[H].$$

Here, in the sum $\sum_i{}'$, i runs over the integers with $1 \leq i \leq p-1$ and $\bar{i} \in H$, and for a real number x, $[x]$ denotes the largest integer $\leq x$. Let $\mathscr{S}_H$ be the submodule of $\boldsymbol{Z}[H]$ generated by θ_r for all integers r over $\boldsymbol{Z}$:

$$\mathscr{S}_H = \langle \theta_r \mid r \in \boldsymbol{Z} \rangle_{\boldsymbol{Z}}.$$

This is an ideal of $\boldsymbol{Z}[H]$ as $\sigma_s\theta_r = \theta_{sr} - r\theta_s$ for $\bar{s} \in H$ ([**8**, Section 2]).

Let ρ be a generator of the cyclic group H. We put

$$N_H = 1 + \rho + \rho^2 + \cdots + \rho^{|H|-1},$$

and

$$\mathfrak{n}_H = \begin{cases} 1, & \text{if } |H| \text{ is odd} \\ 1 + \rho + \rho^2 + \cdots + \rho^{|H|/2-1}, & \text{if } |H| \text{ is even.} \end{cases}$$

For an element $x \in \boldsymbol{Z}[H]$, let $\langle x \rangle = x\boldsymbol{Z}[H]$ for simplicity. We see that the ideal $\langle \mathfrak{n}_H \rangle$ does not depend on the choice of ρ since for integers n, $k > 1$ with $(n, k) = 1$, we have

$$1 + X + \cdots + X^{n-1} \big| 1 + X^k + \cdots + (X^k)^{n-1}$$

in the polynomial ring $\boldsymbol{Z}[X]$.

Lemma 1. *We have* $\langle N_H \rangle \subseteq \mathscr{S}_H \subseteq \langle \mathfrak{n}_H \rangle$.

Let $h(F)$ be the class number of a number field F, and let h_p^- be the relative class number of $\boldsymbol{Q}(\zeta_p)$. For groups A and B, we write $A \leq B$ when A is a subgroup of B.

Theorem 2. *For any subgroup H of $\boldsymbol{F}_p^\times$, the quotient $\langle \mathfrak{n}_H \rangle / \mathscr{S}_H$ is a finite abelian group, and the following assertions hold.*

(I) *When* $H = \boldsymbol{F}_p^\times$, $|\langle \mathfrak{n}_H \rangle / \mathscr{S}_H| = h_p^-$.

(II) *Let A and B be subgroups of $\boldsymbol{F}_p^\times$ with $A \leq B$. Then, the finite abelian group $\langle \mathfrak{n}_A \rangle / \mathscr{S}_A$ is isomorphic to a subquotient of $\langle \mathfrak{n}_B \rangle / \mathscr{S}_B$. In particular, the order and the exponent of $\langle \mathfrak{n}_A \rangle / \mathscr{S}_A$ divide those of $\langle \mathfrak{n}_B \rangle / \mathscr{S}_B$, respectively.*

(III) *When* $|H| = 1, 2, 3, 4$ *or* 6, *we have* $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle$.

Theorem 3. *Let* $p \equiv 3 \bmod 4$, *and let H be the subgroup of $\boldsymbol{F}_p^\times$ of order* $(p-1)/2$.

A prime number q divides the order of $\boldsymbol{Z}[H]/\mathscr{S}_H = \langle \mathfrak{n}_H \rangle/\mathscr{S}_H$ if and only if one of the following conditions is satisfied:

(i) *q divides the quotient $h_p^-/h(\boldsymbol{Q}(\sqrt{-p}))$,*
(ii) *q divides both $p-1$ and $h(\boldsymbol{Q}(\sqrt{-p}))$.*

It is known that $h_p^- = 1$ if and only if $p \leq 19$. For this, confer Uchida [**17**] or [**19**, Corollary 11.18]. Hence, we obtain the following corollary from Theorem 2.

Corollary 1. *When $p \leq 19$, $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle$ for any $H \leq \boldsymbol{F}_p^\times$.*

We obtain the following numerical result from Theorem 3 using the table of Wada and Saito [**18**] on the class numbers of imaginary quadratic fields and the tables in [**19**, pp. 412–420] and Lehmer-Masley [**14**] on the values of h_p^-.

Proposition 1. *Let p be a prime number with $23 \leq p \leq 499$ and $p \equiv 3 \bmod 4$, and let H be the subgroup of $\boldsymbol{F}_p^\times$ of order $(p-1)/2$.*

(I) *For $p = 23$, $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle$.*

(II) *We have $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q \neq \{0\}$ for all prime numbers q dividing h_p^- when $p = 31$, 43, 67, 71, 131, 139, 163, 199, 211, 283, 307, 331, 367, 379, 463, 499.*

(III) *For any p not in* (I) *nor in* (II), *$(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q = \{0\}$ for some prime number q dividing h_p^-, and it is nontrivial for some other q.*

Using Theorem 3 and Proposition 1, we can show the following:

Proposition 2. *Let p and H be as in Theorem* 3. *Then, we have $\mathscr{S}_H \subsetneqq \langle \mathfrak{n}_H \rangle$ when $p \geq 31$.*

For those p (≤ 499) and H not dealt with in Proposition 1, we practiced some computer calculation on $\langle \mathfrak{n}_H \rangle/\mathscr{S}_H$, and obtain the following numerical result.

Proposition 3. *Let p be a prime number with $23 \leq p \leq 499$, and let H be a proper subgroup of $\boldsymbol{F}_p^\times$. Assume that $|H| < (p-1)/2$ or $p \equiv 1 \bmod 4$. Then $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q$ is nontrivial if and only if the triple $(p, (p-1)/|H|, q)$ is one of the following*:

(149, 2, 3), (277, 2, 2), (277, 4, 2), (293, 2, 3), (313, 2, 37), (337, 2, 17), (349, 2, 2), (349, 4, 2), (397, 2, 2), (397, 4, 2), (401, 2, 41), (409, 2, 5), (331, 5, 3), (331, 10, 3).

In particular, we have $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q = \{0\}$ for some odd prime factor q of h_p^- except for the case $p = 29$ where $h_p^- = 8$ and $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle$ for any H ($\neq \boldsymbol{F}_p^\times$). Further, we have $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle$ for p and H not contained in the above list.

From Proposition 3, it is natural to propose the following conjecture.

Conjecture A. *Let p be a prime number with $p \geq 23$ and H a proper subgroup of $\boldsymbol{F}_p^\times$. If $|H| < (p-1)/2$ or $p \equiv 1 \bmod 4$, then $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q = \{0\}$ for some odd prime number q dividing h_p^-, except for the case $p = 29$.*

We obtained Proposition 3 as follows. First, we calculated whether or not $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q$ is trivial for each prime number q up to 2^{16}, and observed that (1) for each prime p in Proposition 3, $(\langle \mathfrak{n}_H \rangle/\mathscr{S}_H) \otimes \boldsymbol{F}_q \neq \{0\}$ happens quite rarely (and hence $\mathscr{S}_H$ is very large in $\langle \mathfrak{n}_H \rangle$) and that (2) for primes p in Proposition 1, the opposite phenomenon occurs. A part of Theorem 2 and Theorem 3 were obtained after these computation and observation.

Let us briefly explain the computation. For simplicity, we restrict ourselves to the case where $h = |H|$ is odd. Then, $\boldsymbol{Z}[H]/\mathscr{S}_H$ is a finite abelian group by Theorem 2. Hence, as an abelian group, $\mathscr{S}_H$ is freely generated by h elements. Further, these h elements generate $\boldsymbol{Q}[H]$ over $\boldsymbol{Q}$. For a finite number of elements α, β, $***$ in $\boldsymbol{Z}[H]$, let $\langle \alpha, \beta, *** \rangle_{\boldsymbol{Z}}$ be the submodule of $\boldsymbol{Z}[H]$ generated by these elements over $\boldsymbol{Z}$. From the definition, we can show that

$$\begin{aligned} \mathscr{S}_H &= \langle \theta_r,\, N_H \bigm| 1 \le r \le p-1 \rangle_{\boldsymbol{Z}} \\ &= \langle \theta_r,\, N_H,\, h_p^- \bigm| 1 \le r \le p-1 \rangle_{\boldsymbol{Z}}. \end{aligned} \tag{1}$$

For the first equality, see Remark 3 in Section 4. The second equality holds by Theorem 2. Therefore, there exist polynomials $f_r \in \boldsymbol{Z}[T]$ $(1 \le r \le p)$ with indeterminate T such that $\deg f_r \le h-1$ and

$$\mathscr{S}_H = \langle f_r(\rho),\, h_p^- \bigm| 1 \le r \le p \rangle_{\boldsymbol{Z}}.$$

As $h_p^- \in \mathscr{S}_H$, the polynomials satisfying these two conditions are determined modulo h_p^-. Starting from these polynomials $f_r(T)$ (or the above expression for $\mathscr{S}_H$), we can inductively calculate a basis $\{e_n\}_{0 \le n \le h-1}$ of $\mathscr{S}_H$ over $\boldsymbol{Z}$ such that

$$e_n = \sum_{i=0}^{n} a_{i,n}\rho^i \quad \text{and} \quad a_{n,n} | a_{\ell,\ell}$$

for $n \ge \ell$. From this, it follows that

$$[\langle \mathfrak{n}_H \rangle : \mathscr{S}_H] = [\boldsymbol{Z}[H] : \mathscr{S}_H] = \prod_{n=0}^{h-1} a_{n,n}.$$

To calculate e_n, we used a version of the Gaussian elimination method over $\boldsymbol{Z}$ (cf. Knuth [**13**, 4.6]).

Since h_p^- is contained in $\mathscr{S}_H$ by virtue of Theorem 2, all the polynomials which appear in the calculation (such as f_r) are determined modulo h_p^-. Hence, we can choose them so that their coefficients are non-negative and less than h_p^-. Namely, their coefficients do not become too large. This is a reason that we were able to complete the calculation.

For example, we obtained when $(p,\, |H|) = (331,\, 33)$,

$$\mathscr{S}_H = \langle \rho^i(\rho+2),\, 3 \bigm| 0 \le i \le 31 \rangle_{\boldsymbol{Z}}$$

with $\rho = \sigma_{283}$ $(= \sigma_3^{(1-p)/|H|})$, and when $(p,\, |H|) = (349,\, 87)$,

$$\mathscr{S}_H = \langle \rho^i(\rho^2+\rho+1),\, 2\rho,\, 2 \bigm| 0 \le i \le 84\rangle_{\boldsymbol{Z}}$$

with $\rho = \sigma_{240}$ ($= \sigma_2^{(1-p)/|H|}$). Here, 3 (resp. 2) is a primitive root modulo 331 (resp. 349).

3. Corollaries.

Let F, K and Δ be as in Theorem 1. As in Section 1, we identify Δ with a subgroup $H = H_F$ of $\boldsymbol{F}_p^\times$ through the Galois action on ζ_p. As the conditions (H'_p) and $(H'_{p,\infty})$ are equivalent, we refer only to (H'_p) in what follows. The following assertion is an immediate consequence of Theorems 1 and 2, and contains [**8**, Corollaries 1, 2].

Corollary 2. *Under the above setting, the following conditions are equivalent if* $[K:F] \le 3$.

(i) *F satisfies (H'_p).*
(ii) *K satisfies (H'_p).*
(iii) $h'_K = 1$.

When $[K:F]$ is even, let $J \in \Delta$ be the automorphism of order 2. For an odd prime number q, let $Cl'_K(q)^- = Cl'_K(q)^{J-1}$ be the odd part of the Sylow q-subgroup $Cl'_K(q)$.

Corollary 3. *Let the notation be as above. When $[K:F]$ is odd, F does not satisfy (H'_p) if there exists a prime number q with $q|h'_K$ and $q \nmid h_p^-$. When $[K:F]$ is even, F does not satisfy (H'_p) if there exists an odd prime number q with $Cl'_K(q)^- \ne \{0\}$ and $q \nmid h_p^-$.*

Proof. Because of Theorem 2, the condition $q \nmid h_p^-$ implies that $\mathscr{S}_\Delta \otimes \boldsymbol{F}_q = \mathfrak{n}_\Delta \boldsymbol{F}_q[\Delta]$. Therefore, the first assertion follows from Theorem 1 as $\mathfrak{n}_\Delta = 1$. Let us deal with the case where $[K:F]$ is even, assuming the existence of an odd prime number q with $Cl'_K(q)^- \ne \{0\}$ and $q \nmid h_p^-$. Let c be a nontrivial class in $Cl'_K(q)^-$ of order q. Then, $c^J = c^{-1}$. On the other hand, $J-1$ is an element of $\mathscr{S}_\Delta \otimes \boldsymbol{F}_q = \mathfrak{n}_\Delta \boldsymbol{F}_q[\Delta]$ as $J-1$ is a multiple of $\mathfrak{n}_\Delta$. Therefore, if F satisfies (H'_p), then $c^J = c$ by Theorem 1, and hence $c^2 = 1$. This is a contradiction as c is of order q. $\square$

In the following, let $K = \boldsymbol{Q}(\zeta_p)$ and let F be a subfield of K. In this case, we have $Cl'_F = Cl_F$ as the unique prime ideal of F over p is principal. As we mentioned in Section 1, the condition (H'_p) is satisfied for $F = \boldsymbol{Q}$. So, we deal with the case $F \ne \boldsymbol{Q}$ in what follows. Let $\Delta = H = \mathrm{Gal}(K/F)$. The following is shown similarly to Corollary 3.

Corollary 4. *Let the notation be as above. When $[K:F]$ is odd, F does not satisfy (H'_p) if there exists a prime number q with $q|h_p$ and $\mathscr{S}_\Delta \otimes \boldsymbol{F}_q = \boldsymbol{F}_q[\Delta]$. When $[K:F]$ is even, F does not satisfy (H'_p) if there exists an odd prime number q with $q|h_p^-$ and $\mathscr{S}_\Delta \otimes \boldsymbol{F}_q = \mathfrak{n}_\Delta \boldsymbol{F}_q[\Delta]$.*

Let $K^+ = \boldsymbol{Q}(\cos(2\pi/p))$ and let Cl_K^- be the kernel of the norm map $Cl_K \to Cl_{K^+}$. Let $h_p = |Cl_K|$ and $h_p^+ = |Cl_{K^+}|$. Then, we have $h_p = h_p^+ h_p^-$.

Corollary 5. *Let the notation be as above, and let* $G = \mathrm{Gal}(K/\boldsymbol{Q}) = \boldsymbol{F}_p^\times$.

Assume that $h_p^+ = 1$ and that h_p^- is odd and square free. If the exponents of the abelian groups $\langle \mathfrak{n}_\Delta \rangle / \mathscr{S}_\Delta$ and $\langle \mathfrak{n}_G \rangle / \mathscr{S}_G$ are equal, then F satisfies (H'_p).

PROOF. By the assumptions and Lemma 5 (in Section 5), we see that

$$\mathscr{S}_\Delta \boldsymbol{Z}[G] \cap \langle \mathfrak{n}_G \rangle = \mathscr{S}_G.$$

Further, we have $Cl_K = Cl_K^-$ as $h_p^+ = 1$. By the classical Stickelberger theorem (cf. [**19**, Theorem 6.10]), $\mathscr{S}_G$ annihilates Cl_K. Let J be the complex conjugation in G. We have $2\mathscr{S}_\Delta \subset (1+J)\mathscr{S}_\Delta + (1-J)\mathscr{S}_\Delta$ in $\boldsymbol{Z}[G]$. Clearly, $(1+J)\mathscr{S}_\Delta$ annihilates $Cl_K^- = Cl_K$. On the other hand, $(1-J)\mathscr{S}_\Delta$ annihilates Cl_K since $(1-J)\mathscr{S}_\Delta \subseteq \mathscr{S}_\Delta \boldsymbol{Z}[G] \cap \langle \mathfrak{n}_G \rangle$. Therefore, $2\mathscr{S}_\Delta$ annihilates Cl_K. As h_p is odd, it follows that $\mathscr{S}_\Delta$ annihilates Cl_K. Hence, F satisfies (H'_p) by Theorem 1. □

From the corollaries and Propositions 1 and 3, we obtain the following:

PROPOSITION 4. (I) *Let p be a prime number with $23 \leq p \leq 499$ and let F be a subfield of $K = \boldsymbol{Q}(\zeta_p)$ with $F \neq \boldsymbol{Q}$. If $[F : \boldsymbol{Q}] > 2$ or $p \equiv 1 \bmod 4$, then F does not satisfy (H'_p) except for the case where $p = 29$ and $[F : \boldsymbol{Q}] = 2$ or 7.*

(II) *When $p = 29$ and $[F : \boldsymbol{Q}] = 2$ or 7, F satisfies (H'_p).*

(III) *For any $p \geq 23$ and any subfield F of $K = \boldsymbol{Q}(\zeta_p)$ with $[K : F] = 1, 2, 3, 4$ or 6, F does not satisfy (H'_p) except for the case where $p = 29$ and $[K : F] = 4$.*

(VI) *Let F be the quadratic subfield of $\boldsymbol{Q}(\zeta_p)$. For $p = 23$ and any prime number p in the third assertion of Proposition 1, F does not satisfy (H'_p).*

PROOF. First, we show the assertion (I). When $[K : F] \leq 2$, it is an immediate consequence of Corollary 2 as $h_p > 1$. When $p \neq 29$ (and $[K : F] > 2$), the assertion follows from Proposition 3 and Corollary 4. When $p = 29$ and $[F : \boldsymbol{Q}] = 4$, we have $h_p^- = 8$ and $\mathscr{S}_H = \langle \mathfrak{n}_H \rangle = \boldsymbol{Z}[H]$ by Proposition 3 where $H = \mathrm{Gal}(K/F)$. Hence, the condition (H'_p) is not satisfied for this case by Corollary 4. Thus, the assertion (I) holds in all cases. The assertion (III) follows from Theorem 2 (III), Corollaries 2, 4 and the assertion (I) for the case $p = 29$. This is because h_p^- is a power of 2 if and only if $p \leq 19$ or $p = 29$ by Horie [**7**]. The assertion (VI) follows from Corollary 4.

Let us show the assertion (II). Let $p = 29$, $K = \boldsymbol{Q}(\zeta_p)$ and $G = \mathrm{Gal}(K/\boldsymbol{Q}) = \langle \rho \rangle$. For each positive divisor i of $p-1$, let F_i be the subfield of K with $[F_i : \boldsymbol{Q}] = i$, and let $H_i = \mathrm{Gal}(K/F_i) = \langle \rho^i \rangle$. It is known that $h_p = 8$ and $h_p^+ = 1$. In particular, $Cl_K = Cl_K^-$. Further, it is known that

$$Cl_K = (\boldsymbol{Z}/2)^{\oplus 3} \tag{2}$$

(see Iwasawa [**11**, page 244] or [**19**, page 412]). First, let us show the assertion for $F = F_7$. We have $\mathscr{S}_{H_7} = \langle \mathfrak{n}_{H_7} \rangle$ by Theorem 2 (III) or Proposition 3. We show that $\mathfrak{n}_{H_7}$ annihilates Cl_K. For this purpose, we first regard Cl_K as a module over $\boldsymbol{Z}_2[H_4]$. There are six nontrivial $\bar{\boldsymbol{Q}}_2$-valued characters of the cyclic group H_4 of order 7, and they are divided into two $\boldsymbol{Q}_2$-equivalent classes. Here, $\boldsymbol{Q}_2$ is the field of 2-adic rationals, and $\bar{\boldsymbol{Q}}_2$ is an algebraic closure of $\boldsymbol{Q}_2$. Let χ_1 and χ_2 be representatives of the two classes, respectively. Let χ_0 be the trivial character of H_4. We can canonically decompose the

$\boldsymbol{Z}_2[H_4]$-module Cl_K as

$$Cl_K = Cl_K^- = Cl_K(\chi_0) \oplus Cl_K(\chi_1) \oplus Cl_K(\chi_2).$$

Here, $Cl_K(\chi)$ is the χ-part of the $\boldsymbol{Z}_2[H_4]$-module Cl_K. We have $Cl_K(\chi_0) = \{0\}$ as the class number of the subfield F_4 of K corresponding to H_4 is one (cf. Hasse [**5**, Tafel II]). For a nontrivial character χ of H_4, let $\mathscr{O}_\chi = \boldsymbol{Z}_2[\chi]$ be the subring of $\bar{\boldsymbol{Q}}_2$ generated by the values of χ over $\boldsymbol{Z}_2$, where $\boldsymbol{Z}_2$ is the ring of 2-adic integers. We can naturally regard $Cl_K(\chi)$ as a module over $\mathscr{O}_\chi$. Then, since $|\mathscr{O}_\chi/2| = 8 = h_p$, we see that

$$Cl_K = Cl_K^- = Cl_K(\chi) \cong \mathscr{O}_\chi/2 \ (\cong (\boldsymbol{Z}/2)^{\oplus 3})$$

for $\chi = \chi_1$ or χ_2. (This assertion is essentially contained in [**11**]. Actually, Iwasawa obtained (2) in a similar way.) From this, it follows that H_7 acts trivially on the $(\mathscr{O}_\chi/2)[H_7]$-module $Cl_K = Cl_K(\chi)$. Therefore, $\mathfrak{n}_{H_7} = 1 + \rho^7$ annihilates $Cl_K = (\boldsymbol{Z}/2)^{\oplus 3}$. Hence, $\mathscr{S}_{H_7}$ annihilates Cl_K, and F_7 satisfies (H_p') by Theorem 1.

Next, we show the assertion (II) for $F = F_2$. We have $\mathscr{S}_{H_2} = \langle \mathfrak{n}_{H_2} \rangle$ by Proposition 3. The elements N_{H_4} and $N_{H_{14}}$ of $\boldsymbol{Z}[G]$ annihilate Cl_K since the class groups of F_4 and $F_{14} = K^+$ are trivial. We see however that

$$\mathfrak{n}_{H_2} = N_{H_4} + (\rho^2 + \rho^6 + \rho^{10})(2 - N_{H_{14}}).$$

Hence, $\mathscr{S}_{H_2}$ annihilates Cl_K, and F_2 satisfies (H_p') by Theorem 1. □

In view of Conjecture A and Proposition 4, we can propose the following:

Conjecture B. *Let p be a prime number with $p \geq 23$, and let F be a subfield of $\boldsymbol{Q}(\zeta_p)$ with $F \neq \boldsymbol{Q}$. If $[F : \boldsymbol{Q}] > 2$ or $p \equiv 1 \bmod 4$, then F does not satisfy (H_p') except for the case where $p = 29$ and $[F : \boldsymbol{Q}] = 2$ or 7.*

Remark 2. For the primes in Proposition 1 (II), h_p^- is square free only when $p = 43, 67$ (see the tables in [**19**, pp. 412–420] and [**14**], or the table of Yamamura [**20**]). For $p = 43$, 67, $h_p^+ = 1$ and h_p^- is square free and odd. Therefore, we see that $F = \boldsymbol{Q}(\sqrt{-p})$ satisfies (H_p') for $p = 43, 67$ by Proposition 1 (II) and Corollary 5. For the other primes p in Proposition 1 (II), we did not check whether or not the quadratic subfield satisfy (H_p') mainly because we have, at present, no exact data for the class group of K^+ (cf. [**19**, pp. 420–421]).

4. Proof of Theorem 2 (I).

For $x \in \boldsymbol{Z}$ and $\alpha \in \boldsymbol{Q}$, we easily see that

$$[x + \alpha] = x + [\alpha], \tag{3}$$

and

$$[x-\alpha] = \begin{cases} x-1-[\alpha], & \text{if } \alpha \notin \boldsymbol{Z} \\ x-[\alpha], & \text{if } \alpha \in \boldsymbol{Z}. \end{cases} \tag{4}$$

For $x \in \boldsymbol{Z}$, let $(x)_p$ be the unique integer satisfying $0 \le (x)_p \le p-1$ and $(x)_p \equiv x \bmod p$. Clearly, we have

$$x = \left[\frac{x}{p}\right]p + (x)_p.$$

Using this and (3), we easily show the following simple formulas.

$$(-x)_p = p-(x)_p \quad \text{when } p \nmid x. \tag{5}$$

$$\left[\frac{xy(z)_p}{p}\right] = \left[\frac{x(yz)_p}{p}\right] + x\left[\frac{y(z)_p}{p}\right] \quad \text{for } y,\ z \in \boldsymbol{Z}. \tag{6}$$

Let $H = \langle \bar{g} \rangle$ be a subgroup of $\boldsymbol{F}_p^\times$ of order h, and let $\rho = \sigma_g$. By definition,

$$\theta_r = \theta_{H,r} = \sum_{i=0}^{h-1}\left[\frac{r(g^i)_p}{p}\right]\rho^{-i}. \tag{7}$$

When $|H| = 2\ell$ is even, let

$$X_{H,r} = (\rho-1)\sum_{i=0}^{\ell-1}\left[\frac{r(g^{\ell-1-i})_p}{p}\right]\rho^i$$

and put

$$\tilde{\theta}_r = \tilde{\theta}_{H,r} = \begin{cases} X_{H,r} + (r-1), & \text{if } p \nmid r \\ X_{H,r} + r, & \text{if } p | r. \end{cases}$$

We see that $N_H = -\theta_{-1} \in \mathscr{S}_H$. Therefore, Lemma 1 is obtained immediately from the following:

Lemma 2. *When $|H|$ is even, we have $\theta_r = \rho \mathfrak{n}_H \tilde{\theta}_r$.*

Proof. By (7), we see that

$$\begin{aligned}\theta_r &= \sum_{i=0}^{\ell-1}\left[\frac{r(g^i)_p}{p}\right]\rho^{2\ell-i} + \sum_{i=\ell}^{2\ell-1}\left[\frac{r(g^i)_p}{p}\right]\rho^{2\ell-i} \\ &= \rho^\ell \sum_{j=1}^{\ell}\left[\frac{r(g^{\ell-j})_p}{p}\right]\rho^j + \sum_{j=1}^{\ell}\left[\frac{r(g^{2\ell-j})_p}{p}\right]\rho^j.\end{aligned}$$

Noting that $g^\ell \equiv -1 \bmod p$ in the last term, we obtain the assertion using (4) and (5). $\square$

Proof of Theorem 2 (I). Let $\ell = (p-1)/2$, $H = \boldsymbol{F}_p^\times = \langle \rho \rangle$, and $J = \rho^\ell$. Let $R = \boldsymbol{Z}[H]$, $\mathscr{S} = \mathscr{S}_H$, $R^- = (J-1)R$, and $\mathscr{S}^- = \mathscr{S} \cap R^-$. In [**10**], Iwasawa proved that

$$|R^-/\mathscr{S}^-| = h_p^-$$

(cf. [**19**, Theorem 6.19]). Let $\mathfrak{n} = \mathfrak{n}_H$ and $A = \langle \mathfrak{n} \rangle$. We see that $R^- \subseteq A$ as $J-1 = (\rho-1)\mathfrak{n}$. We show that there exists a submodule R' of A with $R' \cap R^- = \{0\}$ such that

$$A = \theta_2 \boldsymbol{Z} + (R' \oplus R^-) \quad \text{and} \quad \mathscr{S} \supseteqq R'. \tag{8}$$

Using this, we easily see that $R^-/\mathscr{S}^- \cong A/\mathscr{S}$ considering the natural homomorphism $R^- \to A/\mathscr{S}$, and we obtain Theorem 2 (I).

Let us show the assertion (8). Let $\boldsymbol{Z}[T]$ be the polynomial ring with indeterminate T. An element α of A can be written in the form $\alpha = \mathfrak{n} f(\rho)$ for some $f \in \boldsymbol{Z}[T]$. Using the relation $\mathfrak{n}(\rho-1)(\rho^\ell+1) = 0$, we see that the polynomial f is uniquely determined by α modulo $(T-1)(T^\ell+1)$ and that $\alpha = \mathfrak{n} f(\rho) = 0$ if and only if f is a multiple of $(T-1)(T^\ell+1)$. Thus, the map

$$\mathfrak{n} f(\rho) \to f(T) \text{ modulo } (T-1)(T^\ell+1)$$

is a well defined isomorphism between the $\boldsymbol{Z}[H]$-module A and the $\boldsymbol{Z}[T]$-module $\boldsymbol{Z}[T]/((T-1)(T^\ell+1))$. We identify these two modules by this isomorphism. Consider the following homomorphism over $\boldsymbol{Z}[T]$.

$$\varphi : A \longrightarrow B := \frac{\boldsymbol{Z}[T]}{(T-1)} \oplus \frac{\boldsymbol{Z}[T]}{(T^\ell+1)},$$
$$\mathfrak{n} f(\rho) \to (f \bmod (T-1),\ f \bmod (T^\ell+1)).$$

We easily see that φ is injective. Define submodules R_1 and R_2 of B by

$$R_1 = \varphi(\langle (\rho^\ell+1)\mathfrak{n} \rangle) = (2,\, T-1)/(T-1) \oplus \{0\}$$
$$R_2 = \varphi(R^-) = \varphi(\langle (\rho-1)\mathfrak{n} \rangle) = \{0\} \oplus (T-1,\, 2,\, T^\ell+1)/(T^\ell+1).$$

Then, it follows that

$$\varphi(A) \supseteq R_1 \oplus R_2 \quad \text{and} \quad B/(R_1 \oplus R_2) \cong \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2.$$

By Lemma 2 and the definition of $\tilde{\theta}_r$, we see that

$$\varphi(\theta_2) = (1,\, *) \notin R_1 \oplus R_2 \quad \text{and} \quad \varphi((\rho^\ell+1)\theta_2) = (2,\, 0).$$

The latter implies that $R_1 \subseteq \varphi(\mathscr{S})$. On the other hand, we see that $\varphi(A) \neq B$ since A is cyclic over $\boldsymbol{Z}[H]$ but B is not cyclic over $\boldsymbol{Z}[T]$. From the above, we see that

$$\varphi(A) = \varphi(\theta_2)\boldsymbol{Z} + (R_1 \oplus R_2) \quad \text{and} \quad R_1 \subseteq \varphi(\mathscr{S}).$$

We obtain the assertion (8) from this. □

REMARK 3. We can show the first equality in (1) using (3) and $\theta_{-1} = \theta_{H,-1} = -N_H$.

5. Proofs of Theorem 2 (II) and (III).

In this section, we prove the finiteness of $\langle \mathfrak{n}_H \rangle / \mathscr{S}_H$ for general H and Theorem 2 (II), (III). In the following, A and B are subgroups of $\boldsymbol{F}_p^\times$ with $A \leq B$.

LEMMA 3. *$\mathscr{S}_B \subseteq \mathscr{S}_A \boldsymbol{Z}[B] \cap \langle \mathfrak{n}_B \rangle$.*

PROOF. In view of Lemma 1, it suffices to show that $\mathscr{S}_B \subseteq \mathscr{S}_A \boldsymbol{Z}[B]$. Let $|A| = a$, $|B| = at$, $B = \langle \bar{g} \rangle$, and $\rho = \sigma_g$. By (6) and (7), we see that

$$\begin{aligned}
\theta_{B,r} &= \sum_{\lambda=0}^{t-1} \rho^{-\lambda} \sum_{i=0}^{a-1} \left[\frac{r(g^{ti+\lambda})_p}{p} \right] \rho^{-ti} \\
&= \sum_{\lambda=0}^{t-1} \rho^{-\lambda} \sum_{i=0}^{a-1} \left\{ \left[\frac{rg^\lambda (g^{ti})_p}{p} \right] - r \left[\frac{g^\lambda (g^{ti})_p}{p} \right] \right\} \rho^{-ti} \\
&= \sum_{\lambda=0}^{t-1} \rho^{-\lambda} \big(\theta_{A,rg^\lambda} - r\theta_{A,g^\lambda} \big). \qquad (9)
\end{aligned}$$

The assertion follows immediately from this. □

LEMMA 4. *There is a natural injective homomorphism*

$$\bar{\varphi} : \langle \mathfrak{n}_A \rangle / \mathscr{S}_A \longrightarrow \frac{\langle \mathfrak{n}_B \rangle}{\mathscr{S}_A \boldsymbol{Z}[B] \cap \langle \mathfrak{n}_B \rangle}.$$

PROOF. Let $B = \langle \rho \rangle$ and $t = |B/A|$. Then, as $A = \langle \rho^t \rangle$, an element of $\langle \mathfrak{n}_A \rangle = \mathfrak{n}_A \boldsymbol{Z}[A]$ is of the form $\mathfrak{n}_A f(\rho^t)$ for some polynomial $f(T) \in \boldsymbol{Z}[T]$. Consider the homomorphism

$$\varphi : \langle \mathfrak{n}_A \rangle \longrightarrow \frac{\langle \mathfrak{n}_B \rangle}{\mathscr{S}_A \boldsymbol{Z}[B] \cap \langle \mathfrak{n}_B \rangle};\ \mathfrak{n}_A f(\rho^t) \to [\mathfrak{n}_B f(\rho^t)].$$

Here, $[\mathfrak{n}_B f(\rho^t)]$ is the class containing $\mathfrak{n}_B f(\rho^t)$. As $\mathfrak{n}_A | \mathfrak{n}_B$ in $\boldsymbol{Z}[B]$, it is clear that φ is well defined and that $\mathscr{S}_A \subseteq \ker \varphi$. Let us show that $\ker \varphi \subseteq \mathscr{S}_A$. There are three cases; (i) $|B|$ is odd, (ii) $|A|$ is even, and (iii) $|A|$ is odd and $|B|$ is even.

The case (i). In this case, $\mathfrak{n}_A = \mathfrak{n}_B = 1$. Assume that $f(\rho^t) \in \mathscr{S}_A \boldsymbol{Z}[B]$. Then, it follows that

$$f(\rho^t) = \sum_{\lambda=0}^{t-1} \alpha_\lambda \rho^\lambda$$

with some $\alpha_\lambda \in \mathscr{S}_A$ for $0 \le \lambda \le t-1$. This implies that $f(\rho^t) = \alpha_0 \in \mathscr{S}_A$.

The case (ii). In this case, we have $\mathfrak{n}_B = (1+\rho+\cdots+\rho^{t-1})\mathfrak{n}_A$. Assume that $f(\rho^t)\mathfrak{n}_B \in \mathscr{S}_A \boldsymbol{Z}[B]$. Then, it follows that

$$f(\rho^t)\mathfrak{n}_B = f(\rho^t)\mathfrak{n}_A(1+\rho+\cdots+\rho^{t-1}) = \sum_{\lambda=0}^{t-1} \alpha_\lambda \rho^\lambda$$

with some $\alpha_\lambda \in \mathscr{S}_A$ for $0 \le \lambda \le t-1$. This implies that $f(\rho^t)\mathfrak{n}_A = \alpha_0 \in \mathscr{S}_A$.

The case (iii). Let $t = 2s$ and $|A| = a$. Assume that $f(\rho^{2s})\mathfrak{n}_B \in \mathscr{S}_A \boldsymbol{Z}[B]$. Then, it follows that

$$f(\rho^{2s})\mathfrak{n}_B = f(\rho^{2s})(1+\rho+\cdots+\rho^{as-1}) = \sum_{\lambda=0}^{2s-1} \alpha_\lambda \rho^\lambda$$

with some $\alpha_\lambda \in \mathscr{S}_A$ for $0 \le \lambda \le 2s-1$. Let $\ell = (a-1)/2+1$ and $\tau = \rho^{2s} \in A$. From the above, we see that

$$f(\rho^{2s})(1+\tau+\cdots+\tau^{\ell-1}) = f(\rho^{2s}) \cdot \frac{1-\tau^\ell}{1-\tau} = \alpha_0 \in \mathscr{S}_A.$$

Let k be the least integer with $\ell^k \equiv 1 \bmod a$, and write $\ell^k = 1 + aX$ for some $X \in \boldsymbol{Z}$. It follows that

$$f(\rho^{2s}) \cdot \frac{1-\tau^\ell}{1-\tau} \times \cdots \times \frac{1-\tau^{\ell^k}}{1-\tau^{\ell^{k-1}}} \in \mathscr{S}_A.$$

The left hand side equals

$$\begin{aligned} f(\rho^{2s}) \cdot (1+\tau+\tau^2+\cdots+\tau^{aX}) &= f(\rho^{2s}) \cdot \left\{\tau^{aX} + N_A(1+\tau^a+\cdots+\tau^{a(X-1)})\right\} \\ &\equiv f(\rho^{2s}) \bmod \mathscr{S}_A. \end{aligned}$$

The last congruence holds as $N_A \in \mathscr{S}_A$ (Lemma 1). Therefore, we obtain $f(\rho^{2s}) = f(\rho^{2s})\mathfrak{n}_A \in \mathscr{S}_A$. $\square$

Proof of the finiteness of $\langle \mathfrak{n}_H \rangle / \mathscr{S}_H$ and Theorem 2 (II). The assertions follow from Theorem 2 (I) and Lemmas 3 and 4. $\square$

LEMMA 5. *Assume that h_p^- is square free. If the exponents of the abelian groups $\langle \mathfrak{n}_A\rangle/\mathscr{S}_A$ and $\langle \mathfrak{n}_B\rangle/\mathscr{S}_B$ are equal, then $\mathscr{S}_B = \mathscr{S}_A \boldsymbol{Z}[B] \cap \langle \mathfrak{n}_B\rangle$.*

PROOF. This assertion follows immediately from Lemmas 3 and 4. $\square$

PROOF OF THEOREM 2 (III). By Theorem 2 (II), it suffices to deal with the cases where $|H| = 4$ or 6. Let $H = \langle \bar{g}\rangle$ and $\rho = \sigma_g$.

The case $|H| = 4$. Let $r = (g)_p$. As $r^2 \equiv -1 \bmod p$, we see that $(g^3)_p = (-g)_p = p - r$. Hence, it follows that $2(g)_p < p \Leftrightarrow 2(g^3)_p > p$. Therefore, we may as well assume that $(g)_p < p/2$ replacing g with g^3 if necessary. Then, it follows that $\tilde{\theta}_2 = 1$, and hence $\mathscr{S}_H = \langle \mathfrak{n}_H\rangle$ by Lemmas 1 and 2.

The case $|H| = 6$. Let $r = (g)_p$. We show that if $2r < p$, then $2(g^2)_p < p$, and that if $2r > p$, then $2(g^5)_p < p$. As $\bar{r}$ is a primitive 6-th root of unity in $\boldsymbol{F}_p^\times$, we have $r^2 \equiv r - 1 \bmod p$. From this, we see that $2r \not\equiv 1 \bmod p$. It also follows that $(g^2)_p = r - 1$. From this, the first assertion follows. Next, assume that $2r > p$. Then, as $2r \geq p + 1$,

$$2(g^2)_p = 2(r-1) \geq p - 1.$$

However, the last equality does not hold as $2r \not\equiv 1 \bmod p$. Hence, we obtain $2(g^2)_p > p$. As $g^5 \equiv -g^2 \bmod p$, it follows that $(g^5)_p = p - (g^2)_p < p/2$.

When $2r < p$, it follows from the above that $\tilde{\theta}_2 = 1$, and hence $\mathscr{S}_H = \langle \mathfrak{n}_H\rangle$. When $2r > p$, we see from the above that $\mathscr{S}_H = \langle \mathfrak{n}_H\rangle$ replacing g with g^5. $\square$

6. Proofs of Theorem 3 and Proposition 2.

Let p be a prime number with $p \equiv 3 \bmod 4$. Let $G = \boldsymbol{F}_p^\times$, and let H be the subgroup of G of order $(p-1)/2$. Let $G = \langle \bar{g}\rangle$ and $\rho = \sigma_g$. Let χ be an odd character of G. Namely, $\chi(\rho^{(p-1)/2}) = -1$. We naturally regard χ as a homomorphism $\boldsymbol{Z}[G] \to \boldsymbol{Z}[\mu_{p-1}]$. Let χ_0 be the trivial character of G. Let $\delta_r = 0$ or 1 according to whether $p|r$ or $p \nmid r$.

LEMMA 6. *Let χ be an odd character of G. For any $r \in \boldsymbol{Z}$, we have*

$$\chi(\theta_{G,r}) = \begin{cases} 2\chi(\theta_{H,r}), & \text{if } \chi^2 \neq \chi_0, \\ 2\chi(\theta_{H,r}) - (r - \delta_r)(p-1)/2, & \text{if } \chi^2 = \chi_0. \end{cases}$$

PROOF. Let $\ell = (p-1)/2$. From (7), it follows that

$$\chi(\theta_{G,r}) = \sum_{i=0}^{\ell-1} \left[\frac{r(g^{2i})_p}{p}\right] \chi(\rho^{-2i}) + \sum_{i=0}^{\ell-1} \left[\frac{r(g^{2i+1})_p}{p}\right] \chi(\rho^{-(2i+1)}).$$

By (7) and $H = \langle \rho^2\rangle$, the first term of the right hand side equals $\chi(\theta_{H,r})$. Since $\ell = (p-1)/2$ is odd and χ is odd, the second term of the right hand side equals

$$\sum_{i=0}^{\ell-1} \left[\frac{r(g^{\ell+2i})_p}{p}\right] \chi(\rho^{-(\ell+2i)}) = \sum_{i=0}^{\ell-1} \left[\frac{r(-g^{2i})_p}{p}\right] \chi(\rho^{-2i})(-1).$$

We see from (4) and (5) that the last term equals

$$-\sum_{i=0}^{\ell-1}\left(r-\delta_r-\left[\frac{r(g^{2i})_p}{p}\right]\right)\chi(\rho^{-2i})=\chi(\theta_{H,r})-(r-\delta_r)\sum_{i=0}^{\ell-1}\chi(\rho^{-2i}).$$

Now, the assertion follows from the above. □

Proof of Theorem 3. For a character χ of G, we easily observe that

$$\begin{aligned}\chi(\theta_{G,r})&=\sum_{i=1}^{p-1}\left[\frac{ri}{p}\right]\chi(i)^{-1}=\sum_{i=1}^{p-1}\frac{1}{p}(ri-(ri)_p)\chi(i)^{-1}\\&=(r-\chi(r))B_{1,\chi^{-1}},\end{aligned}\tag{10}$$

where

$$B_{1,\chi^{-1}}=\frac{1}{p}\sum_{i=1}^{p-1}i\chi(i)^{-1}$$

is the first Bernoulli number. For a prime number q, let $\boldsymbol{Q}_q$ be the field of q-adic rationals, $\boldsymbol{Z}_q$ the ring of q-adic integers, and $\bar{\boldsymbol{Q}}_q$ an algebraic closure of $\boldsymbol{Q}_q$. For a $\bar{\boldsymbol{Q}}_q$-valued character χ of G or H, let $\mathfrak{Q}_\chi$ be the maximal ideal of the integer ring of the subfield of $\bar{\boldsymbol{Q}}_q$ generated by the values of χ over $\boldsymbol{Q}_q$.

Let us show the "if part" of the assertion. Let q be a prime number satisfying the condition (i) of Theorem 3. By the classical class number formula, we have

$$h_p^-/h(\boldsymbol{Q}(\sqrt{-p}))=p\cdot\prod_{\chi^2\neq\chi_0}\left(-\frac{1}{2}B_{1,\chi^{-1}}\right),$$

where χ runs over the odd characters of G with $\chi^2\neq\chi_0$ (cf. [**19**, Theorem 4.17]). Hence, we see that $B_{1,\chi^{-1}}\equiv 0 \bmod 2\mathfrak{Q}_\chi$ for some odd $\bar{\boldsymbol{Q}}_q$-valued character χ of G with $\chi^2\neq\chi_0$. Then, it follows from (10) and Lemma 6 that $\chi(\theta_{H,r})\equiv 0 \bmod \mathfrak{Q}_\chi$ for all r. Hence, we obtain the assertion. Let q be a prime number satisfying the condition (ii). Then, q is odd as $p\equiv 3 \bmod 4$. By the class number formula, we have $B_{1,\chi^{-1}}\equiv 0 \bmod q$ for the quadratic character χ associated with $\boldsymbol{Q}(\sqrt{-p})$. Hence, noting that q is odd and $q|p-1$, we obtain the assertion from (10) and Lemma 6 similarly to the above.

Let us show the "only if part". Assume that a prime number q divides the order of $\boldsymbol{Z}[H]/\mathscr{S}_H$. First, we deal with the case $q\nmid p-1$. In this case, we have the direct decomposition

$$(\boldsymbol{Z}[H]/\mathscr{S}_H)\otimes\boldsymbol{Z}_q=\bigoplus_\psi((\boldsymbol{Z}[H]/\mathscr{S}_H)\otimes\boldsymbol{Z}_q)(\psi).$$

Here, ψ runs over a complete set of representatives of the $\boldsymbol{Q}_q$-equivalent classes of the

$\bar{\boldsymbol{Q}}_q$-valued characters of H, and $(*)(\psi)$ denotes the ψ-component. Therefore, by the assumption, there exists a $\bar{\boldsymbol{Q}}_q$-valued character ψ of H such that $\psi(\theta_{H,r}) \equiv 0 \bmod \mathfrak{Q}_\psi$ for all r. Let χ be an odd character of G with $\chi_{|H} = \psi$. Then, from Lemma 3 it follows that $\chi(\theta_{G,r}) \equiv 0$ modulo $\mathfrak{Q}_\psi = \mathfrak{Q}_\chi$ for all r, and hence $B_{1,\chi^{-1}} \equiv 0 \bmod \mathfrak{Q}_\chi$ by (10). We see from Lemma 6 that $\chi^2 \neq \chi_0$ since $q \nmid p-1$ and $\chi(\theta_{G,r}) \equiv \psi(\theta_{H,r}) \equiv 0 \bmod \mathfrak{Q}_\chi$. Therefore, we see that q divides $h_p^-/h(\boldsymbol{Q}(\sqrt{-p}))$ by the class number formula. Next, we deal with the case $q|p-1$. From the assumption, we have $q|h_p^-$ by Theorem 2. Hence, q divides either $h_p^-/h(\boldsymbol{Q}(\sqrt{-p}))$ or $h(\boldsymbol{Q}(\sqrt{-p}))$. The assertion follows from this. □

Proof of Proposition 2. Let p be a prime number with $p \equiv 3 \bmod 4$. By Theorem 3 and Proposition 1, it suffices to show that $h_p^-/h(\boldsymbol{Q}(\sqrt{-p})) > 1$ for $p > 500$. It is known that

$$\log h_p^- \geq \frac{1}{4}(p-2)\log p - 1.08 \times (p-1)$$

for $p \geq 221$ (cf. [**19**, Proposition 11.16]). On the other hand, it is classically known that $h(\boldsymbol{Q}(\sqrt{-p})) < p$. This is an immediate consequence of the class number formula for imaginary quadratic fields (cf. [**19**, Theorem 4.17] or [**6**, Theorem 114]). Hence, it follows that

$$\log\left(h_p^-/h(\boldsymbol{Q}(\sqrt{-p}))\right) > g(p)$$

with the function

$$g(x) = \frac{1}{4}x\log x - \frac{3}{2}\log x - 1.08 \times (x-1).$$

We easily see that $g(x) > 1$ for all real numbers $x > 500$. The assertion follows from this. □

7. Appendix.

In this section, we give the p-integer version of McCulloh's theorem mentioned in Section 1, and derive a part of Theorem 1 from this. We add this appendix for the convenience of the reader following a suggestion of the referee.

Let p be a prime number and F a number field. Let $K = F(\zeta_p)$. Let $G = \boldsymbol{F}_p^\times$ and $\Gamma = \boldsymbol{F}_p^+$ be the multiplicative group and the additive group of the finite field $\boldsymbol{F}_p$, respectively. We write elements of G as $\sigma_i = \bar{i}$. We naturally regard $H = \mathrm{Gal}(K/F)$ as a subgroup of G through its Galois action on ζ_p. In this section, we simply write $\mathscr{O}'_F\Gamma$ (resp. $F\Gamma$) for the group ring $\mathscr{O}'_F[\Gamma]$ (resp. $F[\Gamma]$). Denote by $Cl(\mathscr{O}'_F\Gamma)$ and $R(\mathscr{O}'_F\Gamma)$ the locally free class group of the group ring $\mathscr{O}'_F\Gamma$ and the subset of classes realized by rings of p-integers of Γ-extensions over F, respectively. For the precise definition of $Cl(\mathscr{O}'_F\Gamma)$, see [**4**]. Later, we give a convenient description of $Cl(\mathscr{O}'_F\Gamma)$ following McCulloh's paper. Let $Cl^0(\mathscr{O}'_F\Gamma)$ be the kernel of the projection $Cl(\mathscr{O}'_F\Gamma) \to Cl'_F$. It is known and easily shown that $R(\mathscr{O}'_F\Gamma)$ is contained in $Cl^0(\mathscr{O}'_F\Gamma)$. The multiplicative group G naturally acts on Γ by

$$\bar{a}^{\sigma_i} = \overline{ia} \tag{11}$$

for $\sigma_i \in G$ and $\bar{a} \in \Gamma$. Through this action, the group ring $\boldsymbol{Z}[G]$ acts on the class group $Cl(\mathscr{O}'_F\Gamma)$. The following is the p-integer version of the main theorem of [**16**].

Theorem 4 (McCulloh). *Under the above setting, we have*

$$R(\mathscr{O}'_F\Gamma) = Cl^0(\mathscr{O}'_F\Gamma)^{\mathscr{S}_G}.$$

To prove this theorem, all one has to do is to replace $\mathscr{O}_F$ with $\mathscr{O}'_F$ in McCulloh's argument. From Theorem 4, it follows that $R(\mathscr{O}'_F\Gamma)$ is a subgroup of $Cl(\mathscr{O}'_F\Gamma)$. A number field F satisfies the condition (H'_p) if and only if the group $R(\mathscr{O}'_F\Gamma)$ is trivial because of the cancellation theorem (Jacobinski [**12**], Fröhlich [**2**, page 117]).

In the following, we derive the equivalence (I) $\Leftrightarrow$ (III) in Theorem 1 from Theorem 4. (For the other equivalences, see [**8**].) For this purpose, we give a convenient description of the class group $Cl(\mathscr{O}'_F\Gamma)$ following [**16**, page 113]. Let $I(\mathscr{O}'_F\Gamma)$ be the group of fractional ideals of $\mathscr{O}'_F\Gamma$ in $F\Gamma$, and let $P_{F,\Gamma}$ be the subgroup consisting of principal ideals $\alpha\mathscr{O}'_F\Gamma$ for units α of $F\Gamma$. The group G acts on $I(\mathscr{O}'_F\Gamma)$ and the quotient $I(\mathscr{O}'_F\Gamma)/P_{F,\Gamma}$ through its action (11) on Γ. Then, we have the following natural isomorphism compatible with the G-action.

$$\iota : Cl(\mathscr{O}'_F\Gamma) \cong I(\mathscr{O}'_F\Gamma)/P_{F,\Gamma}. \tag{12}$$

Let N/F be a Γ-extension. As is well known, we have $N = F\Gamma \cdot v$ for some element $v \in N$. We see that $\mathscr{O}'_N = A_N \cdot v$ for some fractional ideal A_N of $\mathscr{O}'_F\Gamma$. The class $[A_N]$ in $I(\mathscr{O}'_F\Gamma)/P_{F,\Gamma}$ represented by A_N depends only on the Γ-extension N/F. The image $\iota(R(\mathscr{O}'_F\Gamma))$ is the subset of classes $[A_N]$ for all Γ-extensions N/F.

Let us look at the group $I(\mathscr{O}'_F\Gamma)$ more explicitly. Let χ_0 be the trivial character of Γ. We fix a nontrivial character χ of Γ with values in $K = F(\zeta_p)$. Let $\rho = \sigma_g$ be a generator of G, where g is a primitive root modulo p. Let $t = [G : H]$. Then, ρ^t is a generator of $H = \mathrm{Gal}(K/F)$ sending ζ_p to $\zeta_p^{g^t}$. For a character ψ of Γ and an element $\alpha = \sum_\gamma a_\gamma\gamma$ of $F\Gamma$, let

$$\psi(\alpha) = \sum_\gamma a_\gamma\psi(\gamma),$$

where γ runs over Γ. We easily see that $\chi, \chi^g, \cdots, \chi^{g^{t-1}}$ form a complete set of representatives of the F-equivalent classes of nontrivial K-valued characters of Γ. From this, we see that the homomorphism

$$\varphi : F\Gamma \to F \oplus K \oplus K \oplus \cdots \oplus K$$

with

$$\varphi(\alpha) = \big(\chi_0(\alpha), \chi(\alpha), \chi^g(\alpha), \cdots, \chi^{g^{t-1}}(\alpha)\big)$$

is an isomorphism of F-algebras. We easily see that

$$\varphi(\mathscr{O}'_F\Gamma) = \mathscr{O}'_F \oplus \mathscr{O}'_K \oplus \mathscr{O}'_K \oplus \cdots \oplus \mathscr{O}'_K.$$

Via the isomorphism φ, a fractional ideal of $\mathscr{O}'_F\Gamma$ corresponds to the direct sum of fractional ideals of the components of $\varphi(\mathscr{O}'_F\Gamma)$. The image $\iota(Cl^0(\mathscr{O}'_F\Gamma))$ equals the subgroup of $I(\mathscr{O}'_F\Gamma)/P_{F,\Gamma}$ consisting of classes containing fractional ideals A of $\mathscr{O}'_F\Gamma$ for which the first component of $\varphi(A)$ is $\mathscr{O}'_F$. From the definition of $\psi(\alpha)$, we easily see that

$$\varphi(\alpha^{\rho^\lambda}) = \big(\chi_0(\alpha),\, \chi^{g^\lambda}(\alpha), \cdots, \chi^{g^{t-1}}(\alpha), \chi(\alpha)^{\rho^t}, \cdots, \chi^{g^{\lambda-1}}(\alpha)^{\rho^t}\big) \tag{13}$$

for $0 \leq \lambda \leq t-1$, and that

$$\varphi(\alpha^\delta) = \big(\chi_0(\alpha),\, \chi(\alpha)^\delta,\, \chi^g(\alpha)^\delta, \cdots, \chi^{g^{t-1}}(\alpha)^\delta\big) \tag{14}$$

for $\delta \in H$. Here, $\chi^{g^\lambda}(\alpha)^\delta$ denotes the Galois action of $\delta \in H$ on the element $\chi^{g^\lambda}(\alpha)$ of K. Namely, for $0 \leq \lambda \leq t-1$, the element ρ^λ acts on the components of $\varphi(\alpha)$ as a "cyclic permutation", and $\delta \in H$ acts on them by Galois action.

Proof of (I) $\Leftrightarrow$ (III) in Theorem 1. First, assume that F satisfies (H'_p). Then, by Theorem 4, the Stickelberger ideal $\mathscr{S}_G$ annihilates the class group $Cl^0(\mathscr{O}'_F\Gamma)$. Let $r \in \boldsymbol{Z}$ be an arbitrary integer. By (9), we see that

$$\theta_{G,r} = \theta_{H,r} + \sum_{\lambda=1}^{t-1} \rho^\lambda s_\lambda \tag{15}$$

with some $s_\lambda \in \mathscr{S}_H$ for $1 \leq \lambda \leq t-1$. Let $\mathfrak{A}$ be an arbitrary ideal of $\mathscr{O}'_K$, and let A be the ideal of $\mathscr{O}'_F\Gamma$ such that

$$\varphi(A) = \mathscr{O}'_F \oplus \mathfrak{A} \oplus \mathscr{O}'_K \oplus \cdots \oplus \mathscr{O}'_K.$$

From (13), (14) and (15), we see that

$$\varphi(A^{\theta_{G,r}}) = \mathscr{O}'_F \oplus \mathfrak{A}^{\theta_{H,r}} \oplus \cdots. \tag{16}$$

On the other hand, it follows from the assumption and the isomorphism (12) that

$$A^{\theta_{G,r}} = \alpha\mathscr{O}'_F\Gamma$$

for some unit $\alpha \in (F\Gamma)^\times$. From this and (16), we see that $\mathfrak{A}^{\theta_{H,r}} = \chi(\alpha)\mathscr{O}'_K$. Therefore, the Stickelberger ideal $\mathscr{S}_H$ annihilates the class group Cl'_K.

Conversely, assume that $\mathscr{S}_H$ annihilates Cl'_K. Then, we see from (13), (14) and (15) that $\theta_{G,r}$ annihilates the ideal of $\mathscr{O}'_F\Gamma$ corresponding to

$$\mathscr{O}'_F \oplus \mathfrak{A}_0 \oplus \cdots \oplus \mathfrak{A}_{t-1}$$

via φ. Here, $\mathfrak{A}_i$ denotes an arbitrary ideal of $\mathscr{O}'_K$. Therefore, $R(\mathscr{O}'_F\Gamma) = \{0\}$ by Theorem 4 and (12), and hence F satisfies (H'_p). $\square$

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References

[1] L. N. Childs, Tame Kummer extensions and Stickelberger ideals, Illinois J. Math., **25** (1981), 258–266.
[2] A. Fröhlich, Locally free modules over arithmetic orders, J. Reine Angew. Math., **274/275** (1975), 112–138.
[3] A. Fröhlich, Stickelberger without Gauss Sums, Algebraic Number Fields, Durham Symposium, 1975, (ed. A. Fröhlich), 587–607, Academic Press, London, 1977.
[4] A. Fröhlich, Galois Module Structure of Rings of Integers, Springer, Berlin-Heidelberg-New York, 1983.
[5] H. Hasse, Über die Klassenzahl Abelscher Zahlkörper, Akademia-Verlag, Berlin, 1952.
[6] D. Hilbert, The Theory of Algebraic Integers, Springer, Berlin-Heidelberg-New York, 1997.
[7] K. Horie, On the class numbers of cyclotomic fields, Manuscripta Math., **65** (1989), 465–477.
[8] H. Ichimura, Stickelberger ideals and normal bases of rings of p-integers, Math. J. Okayama Univ., in press.
[9] H. Ichimura, Normal integral bases and ray class groups, II, Yokohama Math. J., in press.
[10] K. Iwasawa, A class number formula for cyclotomic fields, Ann. of Math., **76** (1962), 171–179.
[11] K. Iwasawa, A note on ideal class groups, Nagoya Math. J., **27** (1966), 239–247.
[12] H. Jacobinski, Genera and decompositions of lattices over orders, Acta Math., **121** (1968), 1–29.
[13] D. E. Knuth, The Art of Computer Programming, **2**: Seminumerical Algorithm (2nd. ed.), Addison-Wesley, Massachusetts, 1981.
[14] D. H. Lehmer and J. Masley, Table of the cyclotomic class numbers $h^*(p)$ and their factors for $200 < p < 521$, Math. Comp., **32** (1978), 577–582.
[15] L. R. McCulloh, A Stickelberger condition on Galois module structure for Kummer extensions of prime degree, Algebraic Number Fields, Durham Symposium, 1975, (ed. A. Fröhlich), 190–204, Academic Press, London, 1977.
[16] L. R. McCulloh, Galois module structure of elementary abelian extensions, J. Algebra, **82** (1983), 102–134.
[17] K. Uchida, Class numbers of imaginary abelian number fields, III, Tohoku Math. J., **23** (1971), 573–580.
[18] H. Wada and M. Saito, A Table of Ideal Class Groups of Imaginary Quadratic Fields, Sophia Kokyuroku, **28**, Sophia Univ., Tokyo, 1988.
[19] L. C. Washington, Introduction to Cyclotomic Fields (2nd ed.), Springer, Berlin-Heidelberg-New York, 1996.
[20] K. Yamamura, Table of relative class numbers of imaginary abelian number fields of prime power conductor $\leq 2^{10} = 1024$, available at ftp://tnt.math.metro-u.ac.jp/pub/table/rcn/.

Humio ICHIMURA
Faculty of Science
Ibaraki University
Bunkyo 2-1-1, Mito
Ibaraki 310-8512
Japan

Hiroki SUMIDA-TAKAHASHI
Faculty and School of Engineering
The University of Tokushima
2-1, Minamijosanjima-cho
Tokushima, 770-8506
Japan