

New affine minimal ruled hypersurfaces

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Abstract. In this paper, we study a new class of affine minimal hypersurfaces as higher dimensional analogues of affine minimal ruled surfaces.

1. Introduction.

The study of ruled surfaces in \mathbf{R}^3 is an important classical subject in affine differential geometry. They are visualized by means of wire models and the study can often be applied to architecture. For higher dimensions, an n -dimensional manifold immersed in \mathbf{R}^{n+1} is ruled if the manifold admits a continuous foliation of codimension one such that the immersion takes each leaf onto an open subset of an affine subspace of \mathbf{R}^{n+1} (see for instance [1], [3]).

In this paper, we generalize affine minimal ruled surfaces to higher dimensions in a manner different from those in [1] and [3]. Indeed, by introducing a new notion of ruled hypersurfaces, we study some large family of affine minimal hypersurfaces. Especially in the family, we find a class of minimal hypersurfaces with vanishing Pick invariant. Another class of ruled hypersurfaces given by formula (20) will also be considered. In particular, in Theorem 5.1 and 5.2, we give characterizations of affine hyperspheres and hypersurfaces with parallel shape operator. Finally in Theorem 5.3, we study ruled hypersurfaces in the class which are semiparallel, i.e., have shape operator with vanishing curvature tensor.

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2. Preliminaries.

Let M^n be a smooth manifold and $f: M \rightarrow \mathbf{R}^{n+1}$, a smooth immersion. We can choose a smooth vector field ξ along f which is transversal to M , i.e. for all $x \in M$,

$$T_{f(x)}\mathbf{R}^{n+1} = f_*(T_xM) \oplus \mathbf{R}\xi_x.$$

Let $\mathfrak{X}(M)$ be the set of all smooth vector fields on M . The canonical connection D on \mathbf{R}^{n+1} , induces the torsion free affine connection ∇ and the symmetric $(0, 2)$ -tensor field h on M with Gauss' formula:

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$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi,$$

and the (1, 1)-tensor field S and the 1-form τ on M with Weingarten’s formula:

$$D_X \xi = -f_*(SX) + \tau(X)\xi,$$

for arbitrary $X, Y \in \mathfrak{X}(M)$.

If h is non-degenerate, f is called a *non-degenerate immersion*. Then, there is a transversal vector field ξ , which is unique up to sign, and satisfies $\tau = 0$ and $\theta = \omega_h$, where ω_h is the volume form of h and θ is the volume form on M defined by:

$$\theta(X_1, \dots, X_n) := \det[f_*(X_1), \dots, f_*(X_n), \xi].$$

In this case, ξ is called an *affine normal vector field*, f with ξ is called a *Blaschke immersion*, h is called the *affine metric*, and (∇, h, S) is called the *Blaschke structure* of f . S is called the *affine shape operator* for ξ . In Blaschke structure, it is known that ∇S is always symmetric. And if S in Blaschke structure is a constant multiple of the identity, $f: M \rightarrow \mathbf{R}^{n+1}$ is called an *affine hypersphere*. On a Blaschke immersion, the function defined by $H := \frac{1}{n} \text{tr } S$ is called the *affine mean curvature*. Clearly, on an affine hypersphere, $S = HI$. And if $H \equiv 0$, $f: M \rightarrow \mathbf{R}^{n+1}$ is called an *affine minimal hypersurface*.

On a Blaschke immersion, the (0, 3)-tensor field C on M defined by $C(X, Y, Z) := (\nabla_X h)(Y, Z)$, called the *cubic form*, which is totally symmetric. It is known that if ∇C is totally symmetric then M is an affine hypersphere. And Pick-Berwald’s theorem (cf. [6, p. 53, Theorem 4.5]) says that $C \equiv 0$ (i.e. h is parallel with respect to ∇) if and only if the immersion $f: M \rightarrow \mathbf{R}^{n+1}$ is a quadratic hypersurface.

The function J on M defined by $J := \frac{1}{4n(n-1)}h(C, C)$ is called the *Pick invariant*. It is known that $\hat{\rho} = H + J$, where $\hat{\rho}$ is the scalar curvature of the affine metric h (cf. [6, p. 78, Proposition 9.3]). And in the case $n = 2$, the immersion $f: M \rightarrow \mathbf{R}^3$ is a ruled surface if and only if h is an indefinite metric and $J \equiv 0$ (cf. [6, pp. 89, 90, Definition 11.1, and Theorems 11.3, 11.4]). Hence the surface with $J \equiv 0$ is quadratic or ruled, because definiteness of h and $J \equiv 0$ imply $C \equiv 0$.

In [2], we obtained the following.

THEOREM 2.1. (i) *Every affine surface with constant Pick invariant in \mathbf{R}^3 satisfying the condition $R(X, Y) \cdot S = 0$ for any vector fields X and Y on M is either an affine sphere with constant curvature metric or an affine minimal ruled surface.*

(ii) *Every affine minimal ruled surface can be written as $z = y\Psi(x) + \Phi(x)$, where $\Psi(x)$ is a non-constant smooth function in x and $\Phi(x)$ is any smooth function in x . Conversely, every surface which can be written as above is an affine minimal ruled surface.*

REMARK 2.2. The classification of affine spheres with constant curvature metrics was shown by M. A. Magid and P. J. Ryan [5] and U. Simon [7]. Any such a sphere is affinely congruent to one of the following surfaces.

$$xyz = 1 \tag{1}$$

$$(x^2 + y^2)z = 1 \tag{2}$$

$$z = x^2 + y^2 \tag{3}$$

$$z = xy + \Phi(x) \tag{4}$$

for some smooth function $\Phi(x)$ in x ,

$$x^2 + y^2 + z^2 = 1 \tag{5}$$

$$x^2 - y^2 - z^2 = 1 \tag{6}$$

$$f(u, v) = uA(v) + A'(v) \tag{7}$$

where $A(v)$ is an \mathbf{R}^3 -valued smooth function in v satisfying that $\det[A, A', A'']$ is a non-zero constant. Furthermore, S is parallel with respect to ∇ (i.e. $\alpha := \nabla S$ vanishes) if and only if the immersion $f: M \rightarrow \mathbf{R}^3$ is either an affine sphere, or can be written as following (8) or (9) (cf. [4]).

$$z = ye^x + \Phi(x) \tag{8}$$

$$z = y \tan x + \Phi(x) \tag{9}$$

for some smooth function $\Phi(x)$ in x .

3. Higher dimensional ruled hypersurfaces.

In this section, we introduce the notion of a higher dimensional ruled hypersurface. It is well known that all ruled surfaces immersed in \mathbf{R}^3 can be written of the form

$$f(u, v) = vA(u) + B(u),$$

where $A(u)$ and $B(u)$ are \mathbf{R}^3 -valued smooth functions in u . The curve $B(u)$ in \mathbf{R}^3 is called the *base line* of the surface. And this ruled surface can be regarded as the surface formed of the family of lines, called the *generators*, passing through each points on $B(u)$.

DEFINITION 3.1. We call a hypersurface in \mathbf{R}^{n+1} a *ruled hypersurface* if it consists of a family of $(n - m)$ -dimensional hyperplanes passing through a fixed m -dimensional hypersurface. Here, $1 \leq m < n$.

It is easy to see that any ruled hypersurface is written as

$$f(u, v) = A(u)v + B(u), \quad u = \begin{bmatrix} u^1 \\ \vdots \\ u^m \end{bmatrix}, \quad v = \begin{bmatrix} v^1 \\ \vdots \\ v^{n-m} \end{bmatrix}, \tag{10}$$

where $B(u)$ is an \mathbf{R}^{n+1} -valued smooth function in u and $A(u)$ is an $(n + 1) \times (n - m)$ -

matrix-valued smooth function in u .

We first show the necessary condition for our ruled hypersurfaces to be non-degenerate.

THEOREM 3.2. *Assume that a hypersurface (10) is non-degenerate. Then, n and m satisfy $n \leq 2m < 2n$ and the affine metric h is a non-degenerate indefinite metric.*

PROOF. For the hypersurface (10), we have

$$f_* \left(\frac{\partial}{\partial u^i} \right) = \frac{\partial f}{\partial u^i} \quad (1 \leq i \leq m), \quad f_* \left(\frac{\partial}{\partial v^j} \right) = A e_j \quad (1 \leq j \leq n - m), \tag{11}$$

where $e_j = {}^t[0, \dots, \overset{j}{1}, \dots, 0] \in \mathbf{R}^{n-m}$ for $j = 1, \dots, n - m$ and A is an $(n + 1) \times (n - m)$ -matrix-valued smooth function in u .

We designate $[\frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^m}]$ by $\frac{\partial f}{\partial u}$. We think about $[\frac{\partial f}{\partial u}, A, XYf]$ which is an $(n + 1) \times (n + 1)$ -matrix for $X, Y \in \mathfrak{X}(M)$. For a transversal vector field ξ , its determinant can be calculated as follows.

$$\begin{aligned} \det \left[\frac{\partial f}{\partial u}, A, XYf \right] &= \det \left[\frac{\partial f}{\partial u}, A, D_X f_*(Y) \right] \\ &= \det \left[f_* \left(\frac{\partial}{\partial u^1} \right), \dots, f_* \left(\frac{\partial}{\partial u^m} \right), f_* \left(\frac{\partial}{\partial v^1} \right), \dots, f_* \left(\frac{\partial}{\partial v^{n-m}} \right), \right. \\ &\quad \left. f_*(\nabla_X Y) + h(X, Y)\xi \right] \\ &= h(X, Y) \det \left[\frac{\partial f}{\partial u}, A, \xi \right]. \end{aligned}$$

Then, we have

$$h(X, Y) = \frac{\det \left[\frac{\partial f}{\partial u}, A, XYf \right]}{\det \left[\frac{\partial f}{\partial u}, A, \xi \right]}.$$

Hence the necessary and sufficient condition that the hypersurface (10) is non-degenerate is that

$$\det \left[\begin{array}{c|c} [h(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})]_{1 \leq i, j \leq m} & [h(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^j})]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n-m}} \\ \hline [h(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial u^j})]_{\substack{1 \leq i \leq n-m \\ 1 \leq j \leq m}} & [h(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j})]_{1 \leq i, j \leq n-m} \end{array} \right] \neq 0,$$

which is equivalent to that

$$\det \left[\begin{array}{c|c} [\det \left[\frac{\partial f}{\partial u}, A, \frac{\partial^2 f}{\partial u^i \partial u^j} \right]]_{1 \leq i, j \leq m} & [\det \left[\frac{\partial f}{\partial u}, A, \frac{\partial A}{\partial u^i} e_j \right]]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n-m}} \\ \hline [\det \left[\frac{\partial f}{\partial u}, A, \frac{\partial A}{\partial u^j} e_i \right]]_{\substack{1 \leq i \leq n-m \\ 1 \leq j \leq m}} & O_{n-m, n-m} \end{array} \right] \neq 0, \tag{12}$$

where $O_{n-m,n-m}$ is the $(n - m) \times (n - m)$ -zero matrix.

Clearly, if $m < n - m$, then the left hand side of (12) must vanish. So we obtain that $m \geq n - m$ if (10) is non-degenerate. We have the first statement of Theorem 3.2.

The non-degeneracy of the hypersurface (10) means that there exists the affine metric on it. Since $h\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^i}\right) = 0$ for all $i = 1, \dots, n - m$, we have the second statement of Theorem 3.2. \square

REMARK 3.3. Generally, n -dimensional ruled hypersurface in the sense of [1] and [3] implies a hypersurface with a foliation by $(n - 1)$ -dimensional hyperplanes, namely, it is our ruled hypersurface (10) in the case of $m = 1$. But it has no Blaschke structure in the case $n \geq 3$ due to Theorem 3.2. This is the main reason why we consider our ruled hypersurface (10) is general.

4. New ruled hypersurfaces of the form of graphs.

In this section, we concentrate ourselves to consider special ruled hypersurfaces which can be written as

$$z = \sum_{k=1}^{n-m} y^k \Psi_k(x^1, \dots, x^m) + \Phi(x^1, \dots, x^m), \tag{13}$$

with smooth $(n - m + 1)$ functions $\Psi_1, \dots, \Psi_{n-m}, \Phi$ in (x^1, \dots, x^m) . If we want to write this hypersurface as the form of (10), it is the following:

$$f(x, y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} O_{m,n-m} & & \\ & I_{n-m} & \\ \Psi_1(x) & \dots & \Psi_{n-m}(x) \end{bmatrix} y + \begin{bmatrix} x \\ \mathbf{0} \\ \Phi(x) \end{bmatrix}, \quad x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix}, \quad y = \begin{bmatrix} y^1 \\ \vdots \\ y^{n-m} \end{bmatrix},$$

where I_{n-m} is the identity matrix of degree $n - m$, $O_{m,n-m}$ is the $m \times (n - m)$ -zero matrix and $\mathbf{0} = {}^t[0, \dots, 0] \in \mathbf{R}^{n-m}$.

REMARK 4.1. Clearly, the form of (13) is a natural generalization of the form of affine minimal ruled surfaces given by Theorem 2.1 (ii). Furthermore, there is an advantage of thinking the hypersurface (13), which is as follows. For calculation of Blaschke structure of a hypersurface, at least one vector field ξ along f which is transversal to M must be found. On the hypersurface (10), tangent vector spaces for each point of $f(M)$ is spanned by (11). So it is difficult in general to find a vector field ξ on \mathbf{R}^{n+1} which is linearly independent to $f_*\left(\frac{\partial}{\partial u^i}\right)$ and $f_*\left(\frac{\partial}{\partial v^j}\right)$ for all $i = 1, \dots, m$ and $j = 1, \dots, n - m$. But on the hypersurface (13), the tangent vector space of $f(M)$ at each point in it is spanned by

$$f_*\left(\frac{\partial}{\partial x^i}\right) = \begin{bmatrix} \tilde{e}_i \\ \mathbf{0} \\ \sum_{k=1}^{n-m} y^k \frac{\partial \Psi_k}{\partial x^i} + \frac{\partial \Phi}{\partial x^i} \end{bmatrix}, \quad i = 1, \dots, m,$$

$$f_*\left(\frac{\partial}{\partial y^j}\right) = \begin{bmatrix} \tilde{\mathbf{0}} \\ \mathbf{e}_j \\ \Psi_j \end{bmatrix}, \quad j = 1, \dots, n - m,$$

where $\tilde{\mathbf{0}} = {}^t[0, \dots, 0]$, $\tilde{\mathbf{e}}_i = {}^t[0, \dots, \overset{i}{1}, \dots, 0] \in \mathbf{R}^m$ for $i = 1, \dots, m$. Thus, the vector ${}^t[0, \dots, 0, 1] \in \mathbf{R}^{n+1}$ does not belong to the hyperplanes generated by $f_*\left(\frac{\partial}{\partial x^i}\right)$ and $f_*\left(\frac{\partial}{\partial y^j}\right)$ for all $i = 1, \dots, m$ and $j = 1, \dots, n - m$. Therefore, we can easily describe the Blaschke structure of the hypersurface (13).

We can easily show that the hypersurface (13) is non-degenerate if and only if

$$\det \left[\begin{array}{c|c} \left[\frac{\partial^2 z}{\partial x^i \partial x^j} \right]_{1 \leq i, j \leq m} & \left[\frac{\partial \Psi_j}{\partial x^i} \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n-m}} \\ \hline \left[\frac{\partial \Psi_i}{\partial x^j} \right]_{\substack{1 \leq i \leq n-m \\ 1 \leq j \leq m}} & 0 \end{array} \right] \neq 0.$$

Then, because of the second statement of Theorem 3.2, the affine metric h is indefinite. But both the affine mean curvature H and the Pick invariant J do not always vanish.

Now we give a simple non-trivial example of 3-dimensional ruled hypersurface in \mathbf{R}^4 .

EXAMPLE 4.2. A hypersurface in \mathbf{R}^4 written of the form

$$z = y(a(x^1)^2 + 2bx^1x^2 + c(x^2)^2)$$

with constants $a, b, c \in \mathbf{R}$ is the simplest higher dimensional hypersurface written of the form (13). We can calculate H and J of this hypersurface explicitly, and give the necessary and sufficient condition for the non-degeneracy.

The necessary and sufficient condition for this hypersurface to be non-degenerate is that $b^2 - ac \neq 0$ except only for the line $x^1 = x^2 = 0$ and the plane $y = 0$. Indeed, the condition for the non-degeneracy is that

$$\gamma := (8(b^2 - ac)y(a(x^1)^2 + 2bx^1x^2 + c(x^2)^2))^{1/5} \neq 0.$$

The affine normal vector field of this hypersurface is

$$\xi = {}^t \left[-\frac{4}{5}(b^2 - ac)x^1\gamma^{-4}, -\frac{4}{5}(b^2 - ac)x^2\gamma^{-4}, -\frac{4}{5}(b^2 - ac)y\gamma^{-4}, \frac{7}{10}\gamma \right].$$

Then we obtain $H = \frac{4}{25}(b^2 - ac)\gamma^{-4} \neq 0$ and $J = -\frac{4}{5}(b^2 - ac)\gamma^{-4} \neq 0$.

From now on, we consider only the case $n = 2m$.

THEOREM 4.3. Any ruled hypersurface which is written of the form

$$z = \sum_{k=1}^m y^k \Psi_k(x^1, \dots, x^m) + \Phi(x^1, \dots, x^m), \quad (14)$$

with $(m + 1)$ smooth functions $\Psi_1, \dots, \Psi_m, \Phi$ in (x^1, \dots, x^m) , is always affine minimal and the Pick invariant J always vanishes.

PROOF. The necessary and sufficient condition for this hypersurface (14) to be non-degenerate is

$$\gamma(x) := \left| \det \left[\frac{\partial \Psi_i}{\partial x^j} \right] \right|^{\frac{1}{m+1}} \neq 0.$$

Then, the affine normal vector field is

$$\xi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{bmatrix} - f_* \left(\varphi(x) \frac{\partial}{\partial y} \right),$$

where

$$\varphi(x) := \left[\frac{\partial \gamma}{\partial x^1}, \dots, \frac{\partial \gamma}{\partial x^m} \right] \left[\frac{\partial \Psi_i}{\partial x^j} \right]^{-1}, \quad \frac{\partial}{\partial y} := \begin{bmatrix} \frac{\partial}{\partial y^1} \\ \vdots \\ \frac{\partial}{\partial y^m} \end{bmatrix}.$$

Therefore we obtain following Blaschke structure:

$$\begin{cases} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{1}{\gamma(x)} \frac{\partial^2 z}{\partial x^i \partial x^j} \varphi(x) \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} = \frac{1}{\gamma(x)} \frac{\partial \Psi_j}{\partial x^i} \varphi(x) \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \end{cases} \quad (15)$$

$$h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \frac{1}{\gamma(x)} \frac{\partial^2 z}{\partial x^i \partial x^j}, \quad h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = \frac{1}{\gamma(x)} \frac{\partial \Psi_j}{\partial x^i}, \quad h \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0, \quad (16)$$

$$S \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial \varphi}{\partial x^i} \frac{\partial}{\partial y}, \quad S \left(\frac{\partial}{\partial y^i} \right) = 0, \quad (17)$$

for $i, j = 1, \dots, m$. Clearly, because of (17), we obtain

$$H = \frac{1}{2m} \operatorname{tr} S = 0.$$

To see that J vanishes, we first see with using (15) and (16),

$$\left\{ \begin{aligned} C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= \frac{1}{\gamma(x)} \left(\frac{\partial^3 z}{\partial x^i \partial x^j \partial x^k} - \frac{\partial \log \gamma}{\partial x^i} \frac{\partial^2 z}{\partial x^j \partial x^k} \right. \\ &\quad \left. - \frac{\partial \log \gamma}{\partial x^j} \frac{\partial^2 z}{\partial x^k \partial x^i} - \frac{\partial \log \gamma}{\partial x^k} \frac{\partial^2 z}{\partial x^i \partial x^j} \right), \\ C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) &= \frac{1}{\gamma(x)} \left(\frac{\partial^2 \Psi_k}{\partial x^i \partial x^j} - \frac{\partial \log \gamma}{\partial x^i} \frac{\partial \Psi_k}{\partial x^j} - \frac{\partial \log \gamma}{\partial x^j} \frac{\partial \Psi_k}{\partial x^i} \right), \\ C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) &= C\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = 0, \end{aligned} \right. \tag{18}$$

for $i, j, k = 1, \dots, m$ where C is the cubic form of (14). For calculation of the Pick invariants, we rewrite the local coordinate system as:

$$u^1 := x^1, \dots, u^m := x^m, u^{m+1} := y^1, \dots, u^{2m} := y^m.$$

Then we have

$$\begin{aligned} J &= \frac{1}{8m(2m-1)} h(C, C) \\ &= \frac{1}{8m(2m-1)} \sum_{i,j,k,p,q,r=1}^{2m} h^{ip} h^{jq} h^{kr} C\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right) C\left(\frac{\partial}{\partial u^p}, \frac{\partial}{\partial u^q}, \frac{\partial}{\partial u^r}\right), \end{aligned}$$

where

$$\begin{aligned} [h^{pq}]_{1 \leq p,q \leq 2m} &:= \left[h\left(\frac{\partial}{\partial u^p}, \frac{\partial}{\partial u^q}\right) \right]_{1 \leq p,q \leq 2m}^{-1} \\ &= \begin{bmatrix} 0 & \gamma \left[\frac{\partial \Psi_i}{\partial x^j} \right]^{-1} \\ \gamma \left[\frac{\partial \Psi_j}{\partial x^i} \right]^{-1} & -\gamma \left[\frac{\partial \Psi_j}{\partial x^i} \right]^{-1} \left[\frac{\partial^2 z}{\partial x^i \partial x^j} \right] \left[\frac{\partial \Psi_i}{\partial x^j} \right]^{-1} \end{bmatrix}. \end{aligned} \tag{19}$$

Because of (18), if $C\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right)$ does not vanish, then at least two of $\{i, j, k\}$ are smaller than $m+1$. Therefore, if $J_{i,j,k,p,q,r} := h^{ip} h^{jq} h^{kr} C\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right) C\left(\frac{\partial}{\partial u^p}, \frac{\partial}{\partial u^q}, \frac{\partial}{\partial u^r}\right)$ does not vanish for some i, j, k, p, q and r which belong to the set $\{1, \dots, 2m\}$, then at least two of $\{i, j, k\}$ are smaller than $m+1$ and at least two of $\{p, q, r\}$ are smaller than $m+1$, i.e. there exists at least one pair of $\{i, p\}$, $\{j, q\}$ and $\{k, r\}$ satisfy the condition that both number in the pair are smaller than $m+1$. But because of (19), if i and p are smaller than $m+1$, then h^{ip} vanishes, and so on. Therefore $J_{i,j,k,p,q,r}$ vanishes for all i, j, k, p, q and r . Hence J always vanishes. \square

5. Examples of affine hyperspheres.

In this section, we consider the hypersurfaces (14) with $\Psi_i(x^i)$ which is a smooth function in only one variable x^i for each $i = 1, \dots, m$, i.e.

$$z = \sum_{k=1}^m y^k \Psi_k(x^k) + \Phi(x^1, \dots, x^m). \tag{20}$$

We have following Theorems 5.1, 5.2 and 5.3.

THEOREM 5.1. *The hypersurface (20) in \mathbf{R}^{2m+1} is an affine hypersphere if and only if it can be written as (21) or (22).*

$$z = \sum_{k=1}^m y^k x^k + \Phi(x^1, \dots, x^m). \tag{21}$$

$$z = y^1(x^1)^{-\frac{1}{m}} + \sum_{k=2}^m y^k x^k + \Phi(x^1, \dots, x^m). \tag{22}$$

THEOREM 5.2. *Assume that the hypersurface (20) satisfies the condition that $\nabla S = 0$. Then, it can be written as:*

$$z = y^1 \Psi_1(x^1) + \sum_{k=2}^m y^k x^k + \Phi(x^1, \dots, x^m), \tag{23}$$

where Ψ_1 satisfies $|\Psi_1'(x^1)| = \mu(x^1)^{-\frac{m+1}{m}}$, where $\mu(x^1)$ is a smooth function in x^1 satisfying that $\mu''(x^1) = c\mu(x^1)^{\frac{1}{m}}$ for some constant $c \in \mathbf{R}$. And the converse is also true.

THEOREM 5.3. *Assume that the hypersurface (20) is semiparallel, that is, it satisfies the condition that $R(X, Y) \cdot S = 0$ for any $X, Y \in \mathfrak{X}(M)$. Then, it can be written as one of the following four cases:*

$$z = \sum_{k=1}^s y^k \log x^k + \sum_{k=s+1}^m y^k x^k + \Phi(x^1, \dots, x^m) \tag{24}$$

with some $s = 0, \dots, m$,

$$z = y^1 \Psi_1(x^1) + \sum_{k=2}^m y^k x^k + \Phi(x^1, \dots, x^m) \tag{25}$$

where $\Psi_1(x^1)$ is not a constant,

$$z = y^1(x^1)^{\frac{c+1}{c-m}} + y^2(x^2)^{\frac{c+1}{1-cm}} + \sum_{k=3}^m y^k x^k + \Phi(x^1, \dots, x^m) \tag{26}$$

where $c \in \mathbf{R}$ is a constant which is not either $0, -1, m$ or $\frac{1}{m}$,

$$z = y^1 e^{x^1} + y^2 (x^2)^{-\frac{1}{m-1}} + \sum_{k=3}^m y^k x^k + \Phi(x^1, \dots, x^m). \tag{27}$$

And the converse is also true.

REMARK 5.4. (i) In 2-dimensional case, i.e. $m = 1$, any surface which is an affine minimal ruled surface and an affine sphere is (4). Since we can find (21) as a natural generalization of (4), it may be called as a trivial affine minimal ruled affine hypersphere. And (22) in 2-dimensional case is that

$$y^1 = zx^1 - x^1 \Phi(x^1),$$

so it is same as (4).

(ii) In 2-dimensional case, any affine minimal ruled surface which satisfies $\nabla S = 0$ is (4), (8) or (9) (cf. [4]). And (23) in 2-dimensional case is same as (4) if $c = 0$, (8) if $c > 0$, and (9) if $c < 0$.

(iii) In 2-dimensional case, all affine minimal ruled surfaces satisfy $R(X, Y) \cdot S = 0$ (cf. Theorem 2.1 (i)). Because any 2-dimensional affine minimal ruled surface is written as (25).

REMARK 5.5. The hypersurface (23) is an affine hypersphere if and only if $c = 0$. In fact, μ'' vanishes if and only if μ is either a constant or a linear expression. If $\mu = |\Psi_1'|^{-\frac{m}{m+1}}$ is a constant then $\Psi_1(x^1)$ is a linear expression, so the hypersurface is (21). The converse is true. And if $\mu = |\Psi_1'|^{-\frac{m}{m+1}}$ is a linear expression, then $\Psi_1(x^1)$ can be written as $\Psi_1(x^1) = (x^1)^{-\frac{1}{m}}$ with affine transformation, so the hypersurface is (22). The converse is true.

PROOF OF THEOREM 5.1. The necessary and sufficient condition for this hypersurface (20) to be non-degenerate is

$$\gamma(x) := \prod_{k=1}^m |\Psi_k'|^{\frac{1}{m+1}} \neq 0,$$

where $\Psi_k' := \frac{d\Psi_k}{dx^k}$. It implies that all Ψ_k are non-constant functions. From now on, if we write \pm or \mp then we choose $+$ or $-$ such that $\pm\Psi_k' > 0$, i.e. $|\Psi_k'| = \pm\Psi_k'$.

Then, the affine normal vector field is given by

$$\xi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{bmatrix} - f_* \left(\varphi(x) \frac{\partial}{\partial y} \right),$$

where

$$\varphi(x) = -\frac{\gamma(x)}{m+1} \left[\left(\frac{1}{\Psi_1'} \right)', \dots, \left(\frac{1}{\Psi_m'} \right)' \right].$$

Because of (17), the necessary and sufficient condition for the surface to be an affine

sphere is that

$$\frac{\partial \varphi}{\partial x^i} = \left[-\frac{\gamma}{m+1} g_{1i}, \dots, -\frac{\gamma}{m+1} g_{mi} \right] = 0$$

for all $i = 1, \dots, m$, i.e. $g_{pq} = 0$ for all $p, q = 1, \dots, m$, where

$$g_{pq} := \delta_{pq} \left(\frac{1}{\Psi_p'} \right)'' + \frac{1}{m+1} \left(\frac{1}{\Psi_p'} \right)' (\log |\Psi_q'|)'. \tag{28}$$

If $p \neq q$, because of (28), either Ψ_p' or Ψ_q' is a constant. Therefore, Ψ_k is a linear expression, i.e. it can be written as $\Psi_k(x^k) = x^k$ with affine transformation, except for at most one of $k = 1, \dots, m$. Then we assume that

$$\Psi_2(x^2) = x^2, \dots, \Psi_m(x^m) = x^m.$$

Clearly, the cases $p = q = 2, \dots, m$ always satisfy (28). So we consider only the case $p = q = 1$. Then, we have

$$0 = g_{11} = \pm \frac{m+1}{m} \left((\pm \Psi_1')^{-\frac{m}{m+1}} \right)'' (\pm \Psi_1')^{-\frac{1}{m+1}},$$

so we have the condition $(|\Psi_1'|^{-\frac{m}{m+1}})'' = 0$. If $(|\Psi_1'|^{-\frac{m}{m+1}})'$ vanishes, then we have $\Psi_1(x^1) = x^1$, so we obtain (21). And if it does not vanish, then we have $\Psi_1(x^1) = (x^1)^{-\frac{1}{m}}$ with affine transformation, so we obtain (22). Hence we obtain Theorem 5.1. \square

PROOF OF THEOREM 5.2. Since (15) and (17), we obtain

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial x^p}} S) \frac{\partial}{\partial y^q} &= (\nabla_{\frac{\partial}{\partial y^p}} S) \frac{\partial}{\partial x^q} = (\nabla_{\frac{\partial}{\partial y^p}} S) \frac{\partial}{\partial y^q} = 0, \\ (\nabla_{\frac{\partial}{\partial x^p}} S) \frac{\partial}{\partial x^q} &= -\frac{\gamma}{m+1} \sum_{r=1}^m \alpha_{pq}^r \frac{\partial}{\partial y^r} \end{aligned}$$

for $p, q = 1, \dots, m$, where

$$\begin{aligned} \alpha_{pq}^r &= \delta_p^r \delta_q^r \left(\frac{1}{\Psi_r'} \right)''' + \frac{1}{m+1} (\delta_p^r (\log |\Psi_q'|)' + \delta_q^r (\log |\Psi_p'|)') \left(\frac{1}{\Psi_r'} \right)'' \\ &+ \frac{1}{m+1} \delta_{pq} \left(2 \frac{\Psi_p'''}{\Psi_p'} - 3 ((\log |\Psi_p'|)')^2 \right) \left(\frac{1}{\Psi_r'} \right)' \\ &+ \frac{2}{(m+1)^2} (\log |\Psi_p'|)' (\log |\Psi_q'|)' \left(\frac{1}{\Psi_r'} \right)'. \end{aligned} \tag{29}$$

Then we should find the condition $\alpha_{pq}^r = 0$ for all $p, q, r = 1, \dots, m$.

If p, q and r differ each other, because of (29), at least one of Ψ_p', Ψ_q' and Ψ_r' is a

constant. Therefore, Ψ_k is a linear expression, i.e. it can be written as $\Psi_k(x^k) = x^k$ with affine transformations, except for at most two of $k = 1, \dots, m$. Then we assume that

$$\Psi_3(x^3) = x^3, \dots, \Psi_m(x^m) = x^m.$$

And we have

$$\begin{aligned} \alpha_{22}^1 &= \frac{1}{(m+1)^2} \left(\frac{1}{\Psi_1'} \right)' \left(2(m+1) \frac{\Psi_2'''}{\Psi_2'} - (3m+1) ((\log |\Psi_2'|)')^2 \right), \\ \alpha_{12}^2 &= \frac{1}{(m+1)^2} (\log |\Psi_1'|)' \left(2m ((\log |\Psi_2'|)')^2 - (m+1) \frac{\Psi_2'''}{\Psi_2'} \right) \frac{1}{\Psi_2'}. \end{aligned}$$

So at least one of Ψ_1' and Ψ_2' is a constant. Therefore, we may assume that $\Psi_2(x^2) = x^2$. Clearly, if at least one of p, q and r is larger than 1, then α_{pq}^r vanishes. So we consider only the case that $p = q = r = 1$. Then, we have

$$\alpha_{11}^1 = \pm \frac{m+1}{m} (\pm \Psi_1')^{-\frac{2}{m+1}} \left(((\pm \Psi_1')^{-\frac{m}{m+1}})'' (\pm \Psi_1')^{\frac{1}{m+1}} \right)'$$

So $\alpha_{11}^1 = 0$ if and only if $(|\Psi_1'|^{-\frac{m}{m+1}})'' |\Psi_1'|^{\frac{1}{m+1}}$ is a constant. By taking a function $\mu(x^1)$ in x^1 as $\mu(x^1) = |\Psi_1'|^{-\frac{m}{m+1}}$, we obtain (23). Hence we obtain Theorem 5.2. \square

PROOF OF THEOREM 5.3. To see the condition $R(X, Y) \cdot S = 0$, by (15) and (17), we obtain

$$\begin{aligned} R\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial y^q}\right) \cdot S &= R\left(\frac{\partial}{\partial y^p}, \frac{\partial}{\partial y^q}\right) \cdot S = 0, \\ \left(R\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) \cdot S\right) \frac{\partial}{\partial y^r} &= 0, \\ \left(R\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) \cdot S\right) \frac{\partial}{\partial x^r} &= \frac{\gamma}{(m+1)^3} \sum_{s=1}^m \beta_{pq,r}^s \frac{\partial}{\partial y^s} \end{aligned}$$

for $p, q, r = 1, \dots, m$, where

$$\begin{aligned} \beta_{pq,r}^s &= (\delta_{qr} (\log |\Psi_p'|)' - \delta_{pr} (\log |\Psi_q'|)') \Psi_r' \left(\frac{1}{\Psi_r'} \right)'' \left(\frac{1}{\Psi_s'} \right)', \quad (s \neq p, q, r) \\ \beta_{pq,r}^p &= -\beta_{qp,r}^p = -(1 - \delta_{pr}) \left(\frac{1}{\Psi_p'} \right)'' (\log |\Psi_q'|)' (\log |\Psi_r'|)' \\ &\quad + \delta_{qr} \left((2m+1) \frac{(\Psi_p'')^2}{(\Psi_p')^3} - (m+1) \frac{\Psi_p'''}{(\Psi_p')^2} \right) \Psi_r' \left(\frac{1}{\Psi_r'} \right)'' \\ &\quad + \delta_{pq} \left(\frac{1}{\Psi_p'} \right)'' \left((\log |\Psi_p'|)' (\log |\Psi_r'|)' - (m+1) \delta_{pr} \Psi_r' \left(\frac{1}{\Psi_r'} \right)'' \right), \\ \beta_{pq,r}^r &= 0. \end{aligned} \tag{30}$$

Then we should find the condition $\beta_{pq,r}^s = 0$ for all $p, q, r, s = 1, \dots, m$.

Clearly in (30), $\beta_{pp,r}^s$ vanishes and $\beta_{qp,r}^s$ equals to $-\beta_{pq,r}^s$ for all $p, q, r, s = 1, \dots, m$, and $\beta_{pq,r}^s$ vanishes for p, q, r, s which differ each other. And because of

$$\beta_{pq,q}^r = (\log |\Psi_p'|)'\Psi_q' \left(\frac{1}{\Psi_q'}\right)'' \left(\frac{1}{\Psi_r'}\right)',$$

$$\beta_{pq,r}^p = -\left(\frac{1}{\Psi_p'}\right)'' (\log |\Psi_q'|)'(\log |\Psi_r'|)'$$

for p, q, r which differ each other, we should find only the condition for

$$B_{p,q,r} := \left(\frac{1}{\Psi_p'}\right)'' \Psi_q'' \Psi_r'' = 0$$

and

$$\beta_{pq,q}^p = \left((2m+1) \frac{(\Psi_p'')^2}{(\Psi_p')^3} - (m+1) \frac{\Psi_p'''}{(\Psi_p')^2} \right) \Psi_q' \left(\frac{1}{\Psi_q'}\right)'' - \left(\frac{1}{\Psi_p'}\right)'' ((\log |\Psi_q'|)')^2 = 0$$

for all p, q, r which differ each other.

Ψ_k'' vanishes if and only if Ψ_k can be written as $\Psi_k(x^k) = x^k$ with some affine transformations. And $\left(\frac{1}{\Psi_k'}\right)''$ vanishes if and only if Ψ_k can be written as $\Psi_k(x^k) = \log x^k$ or x^k with some affine transformations. Because of $B_{p,q,r} = 0$ for any p, q, r which differ each other, the number of $k = 1, \dots, m$ satisfying $\left(\frac{1}{\Psi_k'}\right)'' \neq 0$ is less than three.

(i) The case that the number of k satisfying $\left(\frac{1}{\Psi_k'}\right)'' \neq 0$ is zero.

In this case, $\left(\frac{1}{\Psi_k'}\right)''$ vanishes for all $k = 1, \dots, m$. Therefore, always $B_{p,q,r}$ and $\beta_{pq,q}^p$ vanish for any p, q, r . Then, by taking s as the number of k which satisfies $\Psi_k(x^k) = \log x^k$, we obtain (24).

(ii) The case that the number of k satisfying $\left(\frac{1}{\Psi_k'}\right)'' \neq 0$ is one.

We assume that $\left(\frac{1}{\Psi_1'}\right)'' \neq 0$. Then, because of

$$B_{1,q,r} = \left(\frac{1}{\Psi_1'}\right)'' \Psi_q'' \Psi_r'' = 0$$

for all $q, r = 2, \dots, m$, the number of k which satisfies $\Psi_k(x^k) = \log x^k$ is at most one.

Now, we assume that $\Psi_2(x^2) = \log x^2$. Then, we have

$$0 = \beta_{12,2}^1 = -\left(\frac{1}{\Psi_1'}\right)'' \frac{1}{(x^2)^2}.$$

But it contradicts the assumption $\left(\frac{1}{\Psi_1'}\right)'' \neq 0$. Therefore we have $\Psi_2(x^2) = x^2$. Then we obtain (25).

(iii) **The case that the number of k satisfying $(\frac{1}{\Psi_k})'' \neq 0$ is two.**

We assume that $(\frac{1}{\Psi_1})'' \neq 0$ and $(\frac{1}{\Psi_2})'' \neq 0$. Then, because of

$$B_{1,2r} = \left(\frac{1}{\Psi_1}\right)'' \Psi_2'' \Psi_r'' = 0,$$

we have $\Psi_r(x^r) = x^r$ for all $r = 3, \dots, m$. And because of

$$\begin{aligned} 0 &= \beta_{12,2}^1 \\ &= \frac{1}{(m+1)\Psi_1'} ((\log |\Psi_1'|)')^2 ((\log |\Psi_2'|)')^2 \\ &\quad \times \left(\left(\left(\frac{m+1}{(\log |\Psi_1'|)'} \right)' + m \right) \left(\left(\frac{m+1}{(\log |\Psi_2'|)'} \right)' + m \right) - 1 \right), \end{aligned}$$

we have $\left(\left(\frac{m+1}{(\log |\Psi_1'|)'}\right)' + m\right) \left(\left(\frac{m+1}{(\log |\Psi_2'|)'}\right)' + m\right) = 1$. Now, we take a non-zero function c defined by

$$c = \left(\frac{m+1}{(\log |\Psi_1'|)'}\right)' + m \tag{31}$$

and

$$\frac{1}{c} = \left(\frac{m+1}{(\log |\Psi_2'|)'}\right)' + m. \tag{32}$$

Since the right hand side of (31) is a function in x^1 and the right hand side of the (32) is a function in x^2 , c is a constant. Now, we should solve (31) and (32). We have

$$\frac{c-m}{m+1} = \left(\frac{1}{(\log |\Psi_1'|)'}\right)' \quad \text{and} \quad \frac{\frac{1}{c}-m}{m+1} = \left(\frac{1}{(\log |\Psi_2'|)'}\right)'.$$

If $c \neq m, \frac{1}{m}$, we may assume the condition that

$$(\log |\Psi_1'|)' = \frac{m+1}{(c-m)x^1} \quad \text{and} \quad (\log |\Psi_2'|)' = \frac{c(m+1)}{(1-cm)x^2}$$

with some affine transformations. Then we have $\Psi_1' = (x^1)^{\frac{m+1}{c-m}}$ and $\Psi_2' = (x^2)^{\frac{cm+c}{1-cm}}$.

If $c = -1$, we have $\Psi_1(x^1) = \log x^1$ and $\Psi_2(x^2) = \log x^2$. So we obtain (24).

If $c \neq -1$, with some affine transformations, we have $\Psi_1(x^1) = (x^1)^{\frac{c+1}{c-m}}$ and $\Psi_2(x^2) = (x^2)^{\frac{c+1}{1-cm}}$. So we obtain (26).

And if $c = m$, we may assume the condition that

$$(\log |\Psi_1'|)' = 1.$$

Then we have $\Psi_1(x^1) = e^{x^1}$. And with calculation same as above, we have $\Psi_2(x^2) = (x^2)^{-\frac{1}{m-1}}$. So we obtain (27). Obviously, if $c = \frac{1}{m}$ then we have $\Psi_1(x^1) = (x^1)^{-\frac{1}{m-1}}$ and $\Psi_2(x^2) = e^{x^2}$, so we obtain (27) by exchanging x^1 and x^2 . Hence we obtain Theorem 5.3. \square

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