J. Math. Soc. Japan
Vol. 58, No. 3, 2006

Lou's fixed point theorem in a space of continuous mappings

By Tomonari SUZUKI

(Received Apr. 22, 2005)
(Revised Aug. 15, 2005)

Abstract. We present a very simple proof of Lou's fixed point theorem in a space of continuous mappings [Proc. Amer. Math. Soc., 127 (1999), 2259–2264]. We also discuss another similar fixed point theorem.

1. Introduction.

The following famous theorem is referred to as the *Banach contraction principle.*

THEOREM 1 (Banach [**1**]). *Let F be a nonempty closed subset of a Banach space $(X, \|\cdot\|)$. Let A be a contractive mapping from F into itself, i.e., there exists $r \in [0,1)$ such that*

$$\|Ax - Ay\| \leq r\,\|x - y\|$$

for all $x, y \in F$. Then A has a unique fixed point.

Put $I = [0,T]$ for some $T > 0$ and let $(E, \|\cdot\|_E)$ be a Banach space. Let $C(I,E)$ be the Banach space consisting of all continuous mappings from I into E with norm

$$\|u\|_C = \max\left\{\|u(t)\|_E : t \in I\right\}$$

for $u \in C(I,E)$. In 1999, Lou [**4**] proved the following fixed point theorem.

THEOREM 2 (Lou [**4**]). *Let F be a nonempty closed subset of $C(I,E)$ and let A be a mapping from F into itself. Assume that there exist $\alpha, \beta \in (0,1)$ and $K \geq 0$ such that*

$$\|Au(t) - Av(t)\|_E \leq \beta\,\|u(t) - v(t)\|_E + \frac{K}{t^\alpha}\int_0^t \|u(s) - v(s)\|_E\,ds$$

for all $u, v \in F$ and $t \in I \setminus \{0\}$. Then A has a unique fixed point.

Lou applied this theorem to integro-differential equations. Using the notion of K-normed spaces, de Pascale and de Pascale in [**2**] proved a fixed point theorem (Theorem 3) similar to Theorem 2. Very recently, de Pascale and Zabreiko generalized them in [**3**].

2000 *Mathematics Subject Classification.* 47H10.
Key Words and Phrases. Lou's fixed point theorem, the Banach contraction principle.
The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

We remark that the mapping A in Theorem 2 is not necessarily contractive relative to the original norm $\|\cdot\|_C$ (so that Theorem 1 cannot be directly applied). Nevertheless, it was shown in [**4**] that iterations $A^n u$ form a Cauchy sequence (so that the limit point gives rise to a fixed point as in the standard proof of Theorem 1). In this paper, we shall present a very simple proof of Theorem 2. Namely, we show that a modified norm equivalent to $\|\cdot\|_C$ can be introduced on $C(I,E)$ in such a way that A is contractive relative to this new norm. This obviously implies that Theorem 2 follows from Theorem 1. We will also present an alternative proof to [**2**] by a similar method, and our method has the advantage that the notion of K-normed spaces is not needed.

2. Proof of Theorem 2.

In this section, we present a very simple proof of Theorem 2. Compare it with the proof in [**4**].

Proof of Theorem 2. We choose $\tau \in (0,T)$ satisfying

$$\beta + K\,\tau^{1-\alpha} < 1$$

and define a nonincreasing function f from I into $(0,\infty)$ by

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t \le \tau, \\ \exp(1 - t/\tau), & \text{if } \tau \le t \le T \end{cases}$$

for $t \in I$. Define another norm $\|\cdot\|$ on $C(I,E)$ by

$$\|u\| = \max\big\{ f(t)\,\|u(t)\|_E : t \in I \big\}$$

for $u \in C(I,E)$. Since

$$f(T)\,\|u\|_C \le \|u\| \le \|u\|_C$$

for all $u \in C(I,E)$, the two norms $\|\cdot\|_C$ and $\|\cdot\|$ are equivalent. Thus, $(C(I,E), \|\cdot\|)$ is complete and F is also closed with respect to $\|\cdot\|$. We shall show

$$\|Au - Av\| \le (\beta + K\,\tau^{1-\alpha})\,\|u - v\| \tag{1}$$

for all $u, v \in F$. Fix $u, v \in F$. In the case of $0 < t \le \tau$, we note $\|u(t) - v(t)\|_E \le \|u - v\|$. We have

$$\begin{aligned} f(t)\,\|Au(t) - Av(t)\|_E &= \|Au(t) - Av(t)\|_E \\ &\le \beta\,\|u(t) - v(t)\|_E + \frac{K}{t^\alpha}\int_0^t \|u(s) - v(s)\|_E\,ds \end{aligned}$$

$$
\begin{aligned}
&\le \beta \, \|u - v\| + \frac{K}{t^\alpha} \int_0^t \|u - v\| \, ds \\
&= (\beta + K \, t^{1-\alpha}) \, \|u - v\| \\
&\le (\beta + K \, \tau^{1-\alpha}) \, \|u - v\|.
\end{aligned}
$$

From the continuity of Au and Av, we obtain

$$
f(0) \, \|Au(0) - Av(0)\|_E \le (\beta + K \, \tau^{1-\alpha}) \, \|u - v\|.
$$

In the case of $\tau < t \le T$, we note

$$
\|u(t) - v(t)\|_E \le \exp(-1 + t/\tau) \, \|u - v\|.
$$

We have

$$
\begin{aligned}
\int_0^t \|u(s) - v(s)\|_E \, ds &= \int_0^\tau \|u(s) - v(s)\|_E \, ds + \int_\tau^t \|u(s) - v(s)\|_E \, ds \\
&\le \int_0^\tau \|u - v\| \, ds + \int_\tau^t \exp(-1 + s/\tau) \, \|u - v\| \, ds \\
&= \tau \, \exp(-1 + t/\tau) \, \|u - v\|
\end{aligned}
$$

and hence

$$
\begin{aligned}
&f(t) \, \|Au(t) - Av(t)\|_E \\
&\quad = \exp(1 - t/\tau) \, \|Au(t) - Av(t)\|_E \\
&\quad \le \exp(1 - t/\tau) \Big(\beta \, \|u(t) - v(t)\|_E + \frac{K}{t^\alpha} \int_0^t \|u(s) - v(s)\|_E \, ds \Big) \\
&\quad \le \exp(1 - t/\tau) \Big(\beta \, \exp(-1 + t/\tau) \, \|u - v\| + \frac{K}{\tau^\alpha} \, \tau \, \exp(-1 + t/\tau) \, \|u - v\| \Big) \\
&\quad = (\beta + K \, \tau^{1-\alpha}) \, \|u - v\|.
\end{aligned}
$$

Therefore

$$
f(t) \, \|Au(t) - Av(t)\|_E \le (\beta + K \, \tau^{1-\alpha}) \, \|u - v\|
$$

for all $t \in I$. This implies (1). By Theorem 1, A has a unique fixed point. □

3. De Pascale and De Pascale's Theorem.

In this section, we present an alternative proof of de Pascale and de Pascale's theorem in [**2**] without using the notion of K-normed spaces. We put $I = [1, \infty)$ and let $BC(I, E)$

be the Banach space consisting of all bounded continuous mappings from I into E with norm

$$\|u\|_B = \sup \left\{ \|u(t)\|_E : t \in I \right\}$$

for $u \in BC(I, E)$. De Pascale and de Pascale in [**2**] proved the following.

THEOREM 3 (de Pascale and de Pascale [**2**]). *Let F be a nonempty closed subset of $BC(I, E)$ and let A be a mapping from F into itself. Assume that there exist $\alpha \in (1, \infty)$, $\beta \in (0, 1)$ and $K \geq 0$ such that*

$$\|Au(t) - Av(t)\|_E \leq \beta \|u(t) - v(t)\|_E + \frac{K}{t^\alpha} \int_1^t \|u(s) - v(s)\|_E \, ds$$

for all $u, v \in F$ and $t \in I$. Then A has a unique fixed point.

PROOF. We choose $c > 0$ and $\tau \in (1, \infty)$ satisfying

$$\beta + \frac{K}{c} + \frac{K}{\tau^{\alpha - 1}} < 1.$$

Define a nonincreasing function f from I into $(0, \infty)$ by

$$f(t) = \begin{cases} \exp(-c\,t), & \text{if } 1 \leq t \leq \tau, \\ \exp(-c\,\tau), & \text{if } \tau \leq t \end{cases}$$

for $t \in I$. Define another norm $\|\cdot\|$ on $BC(I, E)$ by

$$\|u\| = \sup \left\{ f(t) \, \|u(t)\|_E : t \in I \right\}$$

for $u \in BC(I, E)$. Then we have

$$f(\tau) \, \|u\|_B \leq \|u\| \leq f(1) \, \|u\|_B$$

for all $u \in BC(I, E)$. So the two norms $\|\cdot\|_B$ and $\|\cdot\|$ are equivalent. Hence, $(BC(I, E), \|\cdot\|)$ is complete and F is also closed with respect to $\|\cdot\|$. We shall show

$$\|Au - Av\| \leq \left(\beta + \frac{K}{c} + \frac{K}{\tau^{\alpha - 1}} \right) \|u - v\| \tag{2}$$

for all $u, v \in F$. Fix $u, v \in F$. In the case of $1 \leq t \leq \tau$, we note

$$\|u(t) - v(t)\|_E \leq \exp(c\,t) \, \|u - v\|.$$

We have

$$\begin{aligned}\int_1^t \|u(s)-v(s)\|_E\,ds &\le \int_1^t \exp(c\,s)\,\|u-v\|\,ds\\ &\le \frac{\exp(c\,t)}{c}\,\|u-v\|\end{aligned}$$

and hence

$$\begin{aligned}f(t)\,\|Au(t)-Av(t)\|_E &= \exp(-c\,t)\,\|Au(t)-Av(t)\|_E\\ &\le \exp(-c\,t)\Big(\beta\,\|u(t)-v(t)\|_E + \frac{K}{t^\alpha}\int_1^t \|u(s)-v(s)\|_E\,ds\Big)\\ &\le \exp(-c\,t)\Big(\beta\,\exp(c\,t)\,\|u-v\| + \frac{K}{t^\alpha}\,\frac{\exp(c\,t)}{c}\,\|u-v\|\Big)\\ &= \Big(\beta+\frac{K}{t^\alpha}\,\frac{1}{c}\Big)\|u-v\|\\ &\le \Big(\beta+\frac{K}{c}\Big)\|u-v\|\\ &\le \Big(\beta+\frac{K}{c}+\frac{K}{\tau^{\alpha-1}}\Big)\|u-v\|.\end{aligned}$$

In the case of $\tau < t$, we note

$$\|u(t)-v(t)\|_E \le \exp(c\,\tau)\,\|u-v\|.$$

We have

$$\begin{aligned}\int_1^t \|u(s)-v(s)\|_E\,ds &= \int_1^\tau \|u(s)-v(s)\|_E\,ds + \int_\tau^t \|u(s)-v(s)\|_E\,ds\\ &\le \frac{\exp(c\,\tau)}{c}\,\|u-v\| + \int_\tau^t \exp(c\,\tau)\,\|u-v\|\,ds\\ &= \Big(\frac{1}{c}+t-\tau\Big)\exp(c\,\tau)\,\|u-v\|\\ &\le \Big(\frac{1}{c}+t\Big)\exp(c\,\tau)\,\|u-v\|\end{aligned}$$

and hence

$$\begin{aligned}f(t)\,\|Au(t)-Av(t)\|_E &= \exp(-c\,\tau)\,\|Au(t)-Av(t)\|_E\\ &\le \exp(-c\,\tau)\Big(\beta\,\|u(t)-v(t)\|_E + \frac{K}{t^\alpha}\int_1^t \|u(s)-v(s)\|_E\,ds\Big)\end{aligned}$$

$$
\begin{aligned}
&\le \exp(-c\,\tau)\left(\beta \exp(c\,\tau)\,\|u-v\| + \frac{K}{t^\alpha}\left(\frac{1}{c}+t\right)\exp(c\,\tau)\,\|u-v\|\right)\\
&= \left(\beta + \frac{K}{t^\alpha}\left(\frac{1}{c}+t\right)\right)\|u-v\|\\
&\le \left(\beta + \frac{K}{c} + \frac{K}{\tau^{\alpha-1}}\right)\|u-v\|.
\end{aligned}
$$

Therefore

$$
f(t)\,\|Au(t)-Av(t)\|_E \le \left(\beta + \frac{K}{c} + \frac{K}{\tau^{\alpha-1}}\right)\|u-v\|
$$

for all $t \in I$. This implies (2). By Theorem 1, A has a unique fixed point. □

Acknowledgment. The author wishes to express his hearty thanks to the referee for his/her helpful advice on Section 1.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3** (1922), 133–181.
[2] E. de Pascale and L. de Pascale, Fixed points for some non-obviously contractive operators, Proc. Amer. Math. Soc., **130** (2002), 3249–3254.
[3] E. de Pascale and P. P. Zabreiko, Fixed point theorems for operators in spaces of continuous functions, Fixed Point Theory, **5** (2004), 117–129.
[4] B. Lou, Fixed points for operators in a space of continuous functions and applications, Proc. Amer. Math. Soc., **127** (1999), 2259–2264.

Tomonari Suzuki
Department of Mathematics
Kyushu Institute of Technology
Sensuicho, Tobata, Kitakyushu
804-8550, Japan
E-mail: suzuki-t@mns.kyutech.ac.jp