J. Math. Soc. Japan
Vol. 58, No. 3, 2006

L^q estimates of weak solutions to the stationary Stokes equations around a rotating body

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(Received Feb. 3, 2005)
(Revised Aug. 10, 2005)

Abstract. We establish the existence, uniqueness and L^q estimates of weak solutions to the stationary Stokes equations with rotation effect both in the whole space and in exterior domains. The equation arises from the study of viscous incompressible flows around a body that is rotating with a constant angular velocity, and it involves an important drift operator with unbounded variable coefficient that causes some difficulties.

1. Introduction.

Consider the motion of a viscous incompressible fluid around a compact rigid body $\mathscr{B} = \boldsymbol{R}^3 \setminus D$ (with smooth boundary ∂D), that is formulated as the exterior problem for the Navier-Stokes equations. The case that the body $\mathscr{B}$ is rotating with a prescribed angular velocity ω is of particular interest. Assume that ω is a constant vector, say, $\omega = (0,0,1)^T$. The problem is then to solve the Navier-Stokes equations in the domain $D(t) = O(t)D$, that depends on the time-variable unless the body $\mathscr{B}$ is axisymmetric, subject to the inhomogeneous nonslip boundary condition, where $O(t)$ is the rotation matrix given below. It is reasonable to reduce the problem to an equivalent one in the exterior domain D by using the coordinate system attached to the body $\mathscr{B}$. The reduced problem is ([**13**, subsection 2.1]; see also [**1**], [**4**], [**5**], [**7**])

$$\begin{aligned} \partial_t u + u\cdot\nabla u &= \Delta u + (\omega\wedge x)\cdot\nabla u - \omega\wedge u - \nabla p, && \text{in } D\times(0,\infty),\\ \nabla\cdot u &= 0, && \text{in } D\times[0,\infty), \end{aligned}$$

subject to

$$u|_{\partial D} = \omega\wedge x, \qquad u\to 0 \ \text{ as } |x|\to\infty, \qquad u|_{t=0} = a,$$

where $u = (u_1,u_2,u_3)^T$ and p are unknown velocity and pressure, respectively, and $\wedge$ stands for the usual exterior product of three-dimensional vectors; so, $\omega\wedge x = (-x_2, x_1, 0)^T$.

The most interesting and difficult feature is that the drift term $(\omega\wedge x)\cdot\nabla u$ is not subordinate to the viscous term Δu and thus cannot be treated as a simple perturbation.

2000 *Mathematics Subject Classification.* Primary 35Q30; Secondary 76D07.

Key Words and Phrases. Stokes flow, rotating body, exterior domain, L^q estimate.

Supported in part by Grant-in-Aid for Scientific Research, No. 16540143, from the Japan Society for the Promotion of Science.

In fact, the fundamental solution of the linear operator

$$L = -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge \tag{1.1}$$

cannot be estimated from above by $C/|x-y|$ unlike the Laplace operator, see Proposition 4.1. Furthermore, the generated semigroup

$$e^{-tL} f(x) = O(t)^T \big(e^{t\Delta} f\big)(O(t)x)$$

on $L^2(\boldsymbol{R}^3)^3$ is not analytic unlike the heat semigroup $e^{t\Delta}$, see Proposition 3.7 of [**13**], where

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $O(t)^T = O(-t)$, although it possesses some smoothing properties (the related semigroup [**12**] for the exterior problem enjoys such properties as well, see [**13**], [**14**], [**15**] and also the recent work [**11**]).

There are some studies on the nonlinear problem above in exterior domains within the framework of L^2 space; weak solution [**1**], local unique solution [**13**], stationary solution (time-periodic solution of the original problem) [**1**], [**7**], [**8**], [**21**], local and global strong solutions [**9**]. Among them, Galdi [**8**] has derived some pointwise estimates at infinity such as $|u(x)| \le C/|x|$ for stationary solutions provided the angular velocity of the body is sufficiently small. Recently, a local unique solution has been constructed by [**11**] within the framework of L^q space.

In the present paper, toward further analysis of the problem above in general L^q spaces, we prove the fundamental estimate

$$\|\nabla u\|_q + \|p\|_q \le C\|f\|_{-1,q} \tag{1.2}$$

of weak solutions to the linearized stationary problem

$$Lu + \nabla p = f, \qquad \nabla \cdot u = 0. \tag{1.3}$$

See Theorem 2.1 (whole space problem) and Theorem 2.2 (exterior problem) in the next section. Note that, when one ignores the crucial term $(\omega \wedge x) \cdot \nabla u$, a multiplier theory leads to some L^q estimates; in fact, this was done by [**19**]. However, such a theory does not seem to work well for the operator L.

After preliminaries (section 3), we discuss in section 4 the whole space problem by real analytic method based on dyadic decomposition, square function and maximal function to derive the estimate (1.2) for $1 < q < \infty$. We make use of an explicit representation formula of the solution and consider the integral operator $F \mapsto \nabla u$, which does not seem to be of Calderón-Zygmund type, where $f = \nabla \cdot F$ with $F \in C_0^\infty(\boldsymbol{R}^3)^9$. The argument is a development of the previous study [**6**], in which the L^q estimate of

$\{\nabla^2 u, \nabla p\}$ for (1.3) in the whole space $\boldsymbol{R}^3$ was provided. See also Farwig [**5**], in which the translation of the body as well as its rotation has been taken into account and the L^q estimate of $\partial_{x_3} u$, that arises from the translation, as well as $\{\nabla^2 u, \nabla p\}$ has been derived. The method of the square function of Littlewood-Paley type (see Stein [**23**], [**24**]) enables us to reduce the study of L^q norms to that of some quadratic expressions. This method is related to the recent progress, due to Weis [**25**], of a characterization of maximal L^p-regularity (optimal regularity estimate in $L^p(0,T;X)$ for the Cauchy problem) in terms of R-boundedness, the notion of which is equivalent to a certain square function estimate in case $X = L^q, 1 < q < \infty$.

The final section is devoted to the analysis of the exterior problem by means of a localization procedure, which was developed in [**2**], [**16**] and [**17**]. Unlike the whole space problem, there is the restriction $n/(n-1) = 3/2 < q < 3 = n$ so that the existence, uniqueness and estimate (1.2) of solutions hold. For the usual Stokes problem (the case $\omega = 0$) in general space dimensions $n \geq 3$, Theorem 2.2 is due to Borchers and Miyakawa [**2**], Galdi and Simader [**10**], Kozono and Sohr [**16**], [**17**], where the restriction above is optimal; that is, $q > n/(n-1)$ is necessary for the solvability in the class $\{u, p\} \in \widehat{W}_0^{1,q}(D)^n \times L^q(D)$ for all $f \in \widehat{W}^{-1,q}(D)^n$, and so is $q < n$ for the uniqueness in that class. For the function spaces, see the next section. Indeed the behavior of the fundamental solution of (1.1) is a little worse than that of the Laplace operator, but Theorem 2.2 tells us that the same result as in the case $\omega = 0$ holds true as far as we are concerned with L^q theory.

We note, owing to the restriction $q > 3/2$ in Theorem 2.2, that our theory for the exterior problem is not sufficient to solve the stationary Navier-Stokes equations because $\|u \cdot \nabla u\|_{-1,q} \leq C\|\nabla u\|_q^2$ holds if and only if $q = 3/2 = n/2$. In the case of the usual Navier-Stokes problem, this difficulty was overcome by [**18**] and, later on, [**20**] with use of the Lorentz spaces, especially $L^{3/2}_w$ (weak $L^{3/2}$ space) that is larger than the usual $L^{3/2}$. A right space to find a nonlinear solution seems to be $L^{3/2}_w(\ni \nabla u)$ for our problem as well and this will be discussed elsewhere [**26**].

Acknowledgement. The author is grateful to the referee for helpful suggestions.

2. Results.

To begin with, we introduce some notations. Given a domain $\Omega\,(= \boldsymbol{R}^3, D, \ldots)$, the class $C_0^\infty(\Omega)$ consists of C^∞ functions with a compact support contained in Ω. By $L^q(\Omega)$ we denote the usual Lebesgue space with norm $\|\cdot\|_{q,\Omega}$. For $1 < q < \infty$ and $\Omega = \boldsymbol{R}^3$ or D, we need the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\boldsymbol{R}^3) = \overline{C_0^\infty(\boldsymbol{R}^3)}^{\|\nabla(\cdot)\|_{q,\boldsymbol{R}^3}} = \{v \in L^q_{loc}(\boldsymbol{R}^3); \nabla v \in L^q(\boldsymbol{R}^3)^3\}/\boldsymbol{R},$$

$$\widehat{W}_0^{1,q}(D) = \overline{C_0^\infty(D)}^{\|\nabla(\cdot)\|_{q,D}} = \begin{cases} \{v \in L^{3q/(3-q)}(D); \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} & \text{for } 1 < q < 3\,(= n), \\ \{v \in L^q_{loc}(\overline{D}); \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} & \text{for } 3 \leq q < \infty, \end{cases} \tag{2.1}$$

and their dual spaces

$$\widehat{W}^{-1,q}(\boldsymbol{R}^3) = \widehat{W}^{1,q/(q-1)}(\boldsymbol{R}^3)^*, \qquad \widehat{W}^{-1,q}(D) = \widehat{W}_0^{1,q/(q-1)}(D)^*,$$

with norms $\|\cdot\|_{-1,q,\boldsymbol{R}^3}$ and $\|\cdot\|_{-1,q,D}$, respectively. The characterization above of the space $\widehat{W}_0^{1,q}(D)$ is due to Galdi and Simader [**10**] (see also Kozono and Sohr [**17**]). For a bounded domain Ω, we use the usual Sobolev spaces $W_0^{1,q}(\Omega)$ and $W^{-1,q}(\Omega) = W_0^{1,q/(q-1)}(\Omega)^*$ with norm $\|\cdot\|_{-1,q,\Omega}$. For simplicity, we use the abbreviations $\|\cdot\|_q = \|\cdot\|_{q,D}$ and $\|\cdot\|_{-1,q} = \|\cdot\|_{-1,q,D}$ for the exterior domain D.

Let $B_r(x)$ be the open ball centered at x with radius $r > 0$. For sufficiently large $r > 0$, we set $D_r = D \cap B_r$ as well as $B_r = B_r(0)$.

Let us consider the boundary value problem for the linearized equation

$$\begin{cases} -\Delta u - (\omega \wedge x)\cdot \nabla u + \omega \wedge u + \nabla p = f & \text{in } D, \\ \nabla \cdot u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{2.2}$$

Let $1 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(D)^3$, we call $\{u,p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ *weak solution* to (2.2) if

1. $\nabla \cdot u = 0$ in $L^q(D)$;
2. $(\omega \wedge x)\cdot \nabla u - \omega \wedge u \in \widehat{W}^{-1,q}(D)^3$;
3. $\{u,p\}$ satisfies (2.2) in the sense of distributions, that is,

$$\langle \nabla u, \nabla \varphi\rangle - \langle (\omega \wedge x)\cdot\nabla u - \omega\wedge u, \varphi\rangle - \langle p, \nabla\cdot\varphi\rangle = \langle f, \varphi\rangle \tag{2.3}$$

holds for all $\varphi \in C_0^\infty(D)^3$, where $\langle\cdot,\cdot\rangle$ stands for various duality pairings; by continuity, $\{u,p\}$ satisfies (2.3) for all $\varphi \in \widehat{W}_0^{1,q/(q-1)}(D)^3$.

Since we make use of a cut-off technique, we first consider the whole space problem with the inhomogeneous divergence condition

$$-\Delta u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f, \quad \nabla\cdot u = g \qquad \text{in } \boldsymbol{R}^3, \tag{2.4}$$

a weak solution of which is defined in the same way as above.

The results on the existence, uniqueness and L^q estimates of weak solutions to (2.4) and to (2.2) are, respectively, as follows.

Theorem 2.1. *Let $1 < q < \infty$ and suppose that*

$$f \in \widehat{W}^{-1,q}(\boldsymbol{R}^3)^3, \quad g \in L^q(\boldsymbol{R}^3), \quad (\omega\wedge x)g \in \widehat{W}^{-1,q}(\boldsymbol{R}^3)^3.$$

Then the problem (2.4) *possesses a weak solution $\{u,p\} \in \widehat{W}^{1,q}(\boldsymbol{R}^3)^3 \times L^q(\boldsymbol{R}^3)$ subject to the estimate*

$$\begin{aligned} &\|\nabla u\|_{q,\boldsymbol{R}^3} + \|p\|_{q,\boldsymbol{R}^3} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,\boldsymbol{R}^3} \\ &\qquad \le C(\|f\|_{-1,q,\boldsymbol{R}^3} + \|g\|_{q,\boldsymbol{R}^3} + \|(\omega\wedge x)g\|_{-1,q,\boldsymbol{R}^3}), \end{aligned} \tag{2.5}$$

with some $C > 0$. The solution is unique in the class above up to a constant multiple of ω for u.

THEOREM 2.2. *Let $3/2 < q < 3$. For every $f \in \widehat{W}^{-1,q}(D)^3$, there exists a unique weak solution $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ of the problem (2.2) subject to the estimate*

$$\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q} \le C\|f\|_{-1,q}, \tag{2.6}$$

with some $C > 0$.

REMARK 2.1. Since $q < 3$ in Theorem 2.2, we have the embedding relation $\widehat{W}_0^{1,q}(D) \hookrightarrow L^{3q/(3-q)}(D)$ by (2.1). In this sense, u is small at infinity.

3. Preliminaries.

For convenience, we collect a few important lemmata in this section. We first introduce the square function S (see [**24**, Chapter I, 6.3]), which is the operator $f \mapsto Sf$ given by

$$Sf(x) = \left(\int_0^\infty |(\phi_s * f)(x)|^2 \frac{ds}{s} \right)^{1/2}, \tag{3.1}$$

where $\{\phi_s\}_{s>0} \subset \mathscr{S}(\boldsymbol{R}^n)$ is a fixed family of radially symmetric functions constructed in the following way: we take $\gamma \in C_0^\infty((1/2, 2); \boldsymbol{R})$ so that

$$\int_{1/2}^2 \gamma(\sigma)^2 \frac{d\sigma}{\sigma} = \frac{1}{2},$$

define $\phi(x)$ by

$$\widehat{\phi}(\xi) \equiv (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} \phi(x)dx = \gamma(|\xi|),$$

where $i = \sqrt{-1}$, and set

$$\phi_s(x) = s^{-n/2}\phi(x/\sqrt{s}) \qquad \big(\widehat{\phi_s}(\xi) = \gamma(\sqrt{s}|\xi|)\big),$$

for $s > 0$. Then we have

$$\int_{\boldsymbol{R}^n} \phi_s(x)dx = 0; \qquad \int_0^\infty \widehat{\phi_s}(\xi)^2 \frac{ds}{s} = 1 \qquad (\xi \in \boldsymbol{R}^n \setminus \{0\}), \tag{3.2}$$

and

$$\operatorname{supp} \widehat{\phi_s} \subset \left\{\xi;\ \frac{1}{2\sqrt{s}} < |\xi| < \frac{2}{\sqrt{s}}\right\}. \tag{3.3}$$

The following lemma plays a crucial role in the next section.

Lemma 3.1 (Stein [**24**, Chapter I, 8.23]). *Let* $1 < q < \infty$*. Then there is a constant* $C = C(q,n) \geq 1$ *such that*

$$\frac{1}{C}\|f\|_{q,\boldsymbol{R}^n} \leq \|Sf\|_{q,\boldsymbol{R}^n} \leq C\|f\|_{q,\boldsymbol{R}^n}$$

for all $f \in L^q(\boldsymbol{R}^n)$*.*

We next introduce the Hardy-Littlewood maximal function (see, for instance, [**23**, Chapter I])

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy \tag{3.4}$$

and need a variant of its L^q-boundedness.

Lemma 3.2. *Let* $1 < q \leq \infty$*.*

1. *There is a constant* $C = C(q,n) > 0$ *such that*

$$\|Mf\|_{q,\boldsymbol{R}^n} \leq C\|f\|_{q,\boldsymbol{R}^n}$$

for all $f \in L^q(\boldsymbol{R}^n)$*.*

2. *Let us consider* (3.4) *in one dimension. There is a constant* $C = C(q) > 0$ *such that*

$$\|Mf\|_{q,I} \leq C\|f\|_{q,I}$$

for all 2π*-periodic function* f *on* $\boldsymbol{R}$ *with* $f \in L^q(I)$*, where* $I = (0, 2\pi)$*.*

Proof. We briefly show the second part, although the proof is essentially the same as that of the well-known first part. We note that Mf is also 2π-periodic. Since

$$\|Mf\|_{\infty,I} = \|Mf\|_{\infty,\boldsymbol{R}} \leq C\|f\|_{\infty,\boldsymbol{R}} = C\|f\|_{\infty,I}$$

for all 2π-periodic $f \in L^\infty(I)$, it suffices to show the weak $(1,1)$ estimate. For 2π-periodic $f \in L^1(I)$ and $\lambda > 0$, we set $E_\lambda = \{\theta \in I;\, Mf(\theta) > \lambda\}$ and

$$A_\theta(r) = \frac{1}{2r}\int_{B_r(\theta)} |f(t)|dt, \qquad r > 0,$$

where $B_r(\theta) = (\theta - r, \theta + r)$. We then find $Mf(\theta) = \sup_{0<r<2\pi} A_\theta(r)$. Fix $\lambda > 0$ arbitrarily, and for $\theta \in E_\lambda$ we choose $r \in I$ so that $A_\theta(r) > \lambda$; then, we have

$$\int_{B_r(\theta)} |f(t)|dt > \lambda|B_r(\theta)| = 2\lambda r. \tag{3.5}$$

From the family $\{B_r(\theta)\}_{\theta\in E_\lambda}$, one can select at most countable sub-family $\{B^{(k)}\}_k$, whose members are disjoint each other, such that $\sum_k |B^{(k)}| \geq |E_\lambda|/5$. This combined with (3.5) yields

$$|E_\lambda| \leq 5\sum_k |B^{(k)}| < \frac{5}{\lambda}\sum_k \int_{B^{(k)}} |f(t)|dt \leq \frac{5}{\lambda}\int_{-2\pi}^{4\pi} |f(t)|dt = \frac{15}{\lambda}\|f\|_{1,I},$$

which is the desired weak $(1,1)$ boundedness. □

Finally, the following density property will be used both for the whole space problem and for the exterior one.

LEMMA 3.3 (Kozono and Sohr [**16**, Lemma 2.2, Corollary 2.3]). *Let $\Omega \subset \boldsymbol{R}^n (n \geq 2)$ be any domain and let $1 < q < \infty$. For all $f \in \widehat{W}^{-1,q}(\Omega)$, there is $F \in L^q(\Omega)^n$ such that*

$$\nabla \cdot F = f, \qquad \|F\|_{q,\Omega} \leq C\|f\|_{-1,q,\Omega}$$

with some $C > 0$. As a result, the space $\{\nabla \cdot F; F \in C_0^\infty(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

4. Whole space problem.

This section is devoted to the analysis of the whole space problem (2.4). Theorem 2.1 is implied by the following.

THEOREM 4.1. *Let $1 < q < \infty$ and $f \in \widehat{W}^{-1,q}(\boldsymbol{R}^3)^3$. Then the equation*

$$Lu \equiv -\Delta u - (\omega \wedge x)\cdot \nabla u + \omega \wedge u = f \quad \text{in } \boldsymbol{R}^3 \tag{4.1}$$

possesses a weak solution $u \in \widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ subject to the estimate

$$\|\nabla u\|_{q,\boldsymbol{R}^3} + \|(\omega \wedge x)\cdot\nabla u - \omega \wedge u\|_{-1,q,\boldsymbol{R}^3} \leq C\|f\|_{-1,q,\boldsymbol{R}^3}, \tag{4.2}$$

with some $C > 0$. The solution is unique in $\widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ up to a constant multiple of ω for u.

REMARK 4.1. For the equation (4.1) with angular velocity $\omega = (0,0,|\omega|)^T$, we have only to replace $O(t)$ by $O(|\omega|t)$ in every formula below. Thus, one can easily find that Theorem 4.1 holds for such angular velocity ω as well and that the constant $C > 0$ in (4.2) is independent of $|\omega|$. As a consequence, Theorem 2.1 also holds with a constant $C > 0$ independent of $|\omega|$.

The outline of the proof of Theorem 4.1 is as follows. Thanks to Lemma 3.3 on the density property, the essential step is to show

$$\|\nabla u\|_{q,\boldsymbol{R}^3} \leq C\|F\|_{q,\boldsymbol{R}^3}, \tag{4.3}$$

for the force term of the form $f = \nabla \cdot F$ with $F \in C_0^\infty(\boldsymbol{R}^3)^9$. We set the operator $T : F \mapsto \nabla u$, see its integral expression (4.7) with (4.5) below. In order to prove the L^q-boundedness of the operator T, we decompose it on the Fourier side by use of the Littlewood-Paley dyadic decomposition. Next we derive the L^q estimates of the decomposed operators for $2 < q < \infty$ by the method of the square function (see Lemma 3.1), and then we sum up them. For $1 < q < 2$ we employ the duality argument by use of the adjoint operator T^*.

For $f \in \mathscr{S}(\boldsymbol{R}^3)^3$, the equation (4.1) admits a solution of the form ([**6**])

$$u(x) = \int_{\boldsymbol{R}^3} \Gamma(x,y) f(y) dy = \int_0^\infty O(t)^T \big(e^{t\Delta} f\big)(O(t)x) dt \tag{4.4}$$

with the non-symmetric kernel

$$\Gamma(x,y) = \int_0^\infty O(t)^T E_t(O(t)x - y) dt, \tag{4.5}$$

where $e^{t\Delta} = E_t *$ is the heat semigroup and

$$E_t(x) = t^{-3/2} E(x/\sqrt{t}), \qquad E(x) = (4\pi)^{-3/2} e^{-|x|^2/4}.$$

On the Fourier side, the solution (4.4) is written as

$$\widehat{u}(\xi) = \int_0^\infty O(t)^T e^{-|\xi|^2 t} \widehat{f}(O(t)\xi) dt. \tag{4.6}$$

As we mentioned in section 1, we have the following negative assertion on a pointwise estimate of $\Gamma(x,y)$, which shows that the operator $(\omega \wedge x) \cdot \nabla$ is not subordinate to the Laplacian.

Proposition 4.1. *There is* no *constant* $C > 0$ *such that*

$$|x - y||\Gamma(x,y)| \le C, \quad \forall (x,y) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3.$$

Proof. This was shown in [**6**], but we give the proof for completeness. We intend to estimate the right-hand side of

$$|\Gamma(x,y)| > \Gamma_{33}(x,y) = \int_0^\infty E_t(O(t)x - y) dt.$$

We take, for instance, $x_\rho = (\rho, 0, 0)^T$ and $y_\rho = (0, \rho, 0)^T$ to show

$$\Gamma_{33}(x_\rho, y_\rho) = \int_0^\infty (4\pi t)^{-3/2} e^{-\rho^2(1-\sin t)/2t} dt \ge \frac{C \log \rho}{\rho},$$

for all $\rho > 1$ with $C > 0$ independent of ρ. In fact, we have

$$\Gamma_{33}(x_\rho, y_\rho) \geq \sum_{k=0}^{\infty} J_k(\rho) \geq \sum_{k=1}^{[\rho^2]} J_k(\rho)$$

with

$$J_k(\rho) = \int_{-\pi/6}^{\pi/6} \{4\pi(t+\pi/2+2k\pi)\}^{-3/2} e^{-\rho^2(1-\cos t)/2(t+\pi/2+2k\pi)} dt,$$

which is estimated from below as

$$\begin{aligned} J_k(\rho) &\geq (12k\pi^2)^{-3/2} \int_{-\pi/6}^{\pi/6} e^{-\rho^2(1-\cos t)/4k\pi} dt \\ &\geq 2(12k\pi^2)^{-3/2} \int_0^{\pi/6} e^{-\rho^2 t^2/8k\pi} dt = \frac{C}{k\rho} \int_0^{\sqrt{\pi}\rho/12\sqrt{2k}} e^{-t^2} dt \end{aligned}$$

for $k \geq 1$. If in particular $k \leq \rho^2$, we then find

$$J_k(\rho) \geq \frac{C}{k\rho} \int_0^{\sqrt{\pi}/12\sqrt{2}} e^{-t^2} dt = \frac{C}{k\rho}.$$

As a consequence,

$$\Gamma_{33}(x_\rho, y_\rho) \geq \frac{C}{\rho} \sum_{k=1}^{[\rho^2]} \frac{1}{k} \geq \frac{C}{\rho} \int_1^{\rho^2} \frac{ds}{s} = \frac{C \log \rho}{\rho},$$

which completes the proof. □

Let $f = \nabla \cdot F$ with $F = (F_{\mu\nu}) \in C_0^\infty(\boldsymbol{R}^3)^9$ and consider the L^q estimate of the operator T defined by

$$TF(x) = \nabla u(x) = -\int_{\boldsymbol{R}^3} \nabla_x \nabla_y \Gamma(x,y) : F(y) dy \tag{4.7}$$

to show (4.3), where

$$(\nabla_y \Gamma(x,y) : F(y))_\ell = \sum_{1 \leq \mu,\nu \leq 3} \partial_{y_\nu} \Gamma_{\ell\mu}(x,y) F_{\mu\nu}(y) \qquad (1 \leq \ell \leq 3).$$

As in Proposition 4.1, the kernel of (4.7) does *not* seem to enjoy the pointwise estimate $|\nabla_x \nabla_y \Gamma(x,y)| \leq C/|x-y|^3$; that is, the operator T does *not* seem to be of Calderón-Zygmund type. However the L^2 estimate can be easily obtained.

PROPOSITION 4.2. *The operator T enjoys*

$$\|TF\|_{2,\boldsymbol{R}^3} \leq \|F\|_{2,\boldsymbol{R}^3},$$

for all $F \in C_0^\infty(\boldsymbol{R}^3)^9$.

Proof. By the solution formula (4.6) we have

$$(\widehat{TF})(\xi) = -\xi \otimes \int_0^\infty O(t)^T e^{-|\xi|^2 t}(O(t)\xi) \cdot \widehat{F}(O(t)\xi)dt.$$

The Planchrel theorem thus leads us to

$$\begin{aligned}\|TF\|_{2,\boldsymbol{R}^3}^2 = \|\widehat{TF}\|_{2,\boldsymbol{R}^3}^2 &\le \int_{\boldsymbol{R}^3} |\xi|^4 \left\{ \int_0^\infty e^{-|\xi|^2 t}|\widehat{F}(O(t)\xi)|dt \right\}^2 d\xi \\ &\le \int_{\boldsymbol{R}^3} |\xi|^2 \int_0^\infty e^{-|\xi|^2 t}|\widehat{F}(O(t)\xi)|^2 dt d\xi \\ &= \int_0^\infty \int_{\boldsymbol{R}^3} |\xi|^2 e^{-|\xi|^2 t}|\widehat{F}(\xi)|^2 d\xi dt = \|\widehat{F}\|_{2,\boldsymbol{R}^3}^2 = \|F\|_{2,\boldsymbol{R}^3}^2,\end{aligned}$$

which completes the proof. □

We rewrite (4.7) as the form

$$TF = (T_{\ell m}F)_{1 \le \ell, m \le 3} \quad \text{for} \quad F = (F_{\mu\nu})_{1 \le \mu,\nu \le 3}$$

with

$$T_{\ell m}F(x) = \partial_{x_m} u_\ell(x) = \sum_{\mu,\nu,k} \int_0^\infty O(t)_{\ell\mu}^T O(t)_{km}(H_{k\nu,t} * F_{\mu\nu})(O(t)x)\frac{dt}{t}, \tag{4.8}$$

where $H = (H_{k\nu})_{1 \le k,\nu \le 3}$ is the Hessian matrix of E, that is,

$$H_{k\nu}(x) = \partial_{x_\nu}\partial_{x_k}E(x), \qquad H_{k\nu,t}(x) = t^{-3/2}H_{k\nu}(x/\sqrt{t}). \tag{4.9}$$

We need also the adjoint operator

$$T^*G = \left(T_{\mu\nu}^*G\right)_{1 \le \mu,\nu \le 3} \quad \text{for} \quad G = (G_{\ell m})_{1 \le \ell, m \le 3}$$

with

$$T_{\mu\nu}^*G(y) = \sum_{k,\ell,m} \int_0^\infty O(t)_{\ell\mu}^T O(t)_{km} \int_{\boldsymbol{R}^3} H_{k\nu,t}(O(t)x - y)G_{\ell m}(x)dx\frac{dt}{t}, \tag{4.10}$$

for which the argument will be parallel to that for the operator T.

We now introduce the Littlewood-Paley dyadic decomposition

$$\sum_{j=-\infty}^\infty \widehat{\eta_j}(\xi) = 1 \qquad (\xi \in \boldsymbol{R}^3 \setminus \{0\})$$

with

$$\widehat{\eta_j}(\xi) = \beta(2^{-j}|\xi|) - \beta(2^{-j+1}|\xi|),$$

where $\beta \in C^\infty((0,\infty);[0,1])$ is a fixed function so that $\beta \equiv 1$ on $(0,1]$ and $\beta \equiv 0$ on $[2,\infty)$. Note that

$$\operatorname{supp} \widehat{\eta_j} \subset \{\xi;\ 2^{j-1} \leq |\xi| \leq 2^{j+1}\}. \tag{4.11}$$

By use of η_j, we decompose the function H in (4.9) as

$$H_{k\nu} = \sum_{j=-\infty}^{\infty} H_{k\nu}^{(j)}, \qquad H_{k\nu}^{(j)} = (2\pi)^{-3/2}\eta_j * H_{k\nu} \quad \left(\widehat{H_{k\nu}^{(j)}} = \widehat{\eta_j}\,\widehat{H_{k\nu}}\right).$$

In (4.8) and (4.10), respectively, we replace H by $H^{(j)} = \big(H_{k\nu}^{(j)}\big)_{1\leq k,\nu\leq 3}$ to define the decomposed operators

$$T^{(j)} = \big(T_{\ell m}^{(j)}\big)_{1\leq \ell,m\leq 3}, \qquad T^{*(j)} = \big(T_{\mu\nu}^{*(j)}\big)_{1\leq \mu,\nu\leq 3},$$

with

$$T_{\ell m}^{(j)}F(x) = \sum_{\mu,\nu,k} \int_0^\infty O(t)_{\ell\mu}^T O(t)_{km}\big(H_{k\nu,t}^{(j)} * F_{\mu\nu}\big)(O(t)x)\frac{dt}{t}, \tag{4.12}$$

$$T_{\mu\nu}^{*(j)}G(y) = \sum_{k,\ell,m} \int_0^\infty O(t)_{\ell\mu}^T O(t)_{km} \int_{\boldsymbol{R}^3} H_{k\nu,t}^{(j)}(O(t)x - y)G_{\ell m}(x)dx\frac{dt}{t}, \tag{4.13}$$

where

$$H_{k\nu,t}^{(j)}(x) = t^{-3/2}H_{k\nu}^{(j)}(x/\sqrt{t}),$$

namely,

$$\widehat{H_{k\nu,t}^{(j)}}(\xi) = \widehat{H_{k\nu}^{(j)}}(\sqrt{t}\xi) = \widehat{\eta_j}(\sqrt{t}\xi)\widehat{H_{k\nu}}(\sqrt{t}\xi),$$

so that (4.11) leads to

$$\operatorname{supp} \widehat{H_{k\nu,t}^{(j)}} \subset \left\{\xi;\ \frac{2^{j-1}}{\sqrt{t}} \leq |\xi| \leq \frac{2^{j+1}}{\sqrt{t}}\right\}. \tag{4.14}$$

In order to estimate $T_{\ell m}^{(j)}F$ and $T_{\mu\nu}^{*(j)}G$ defined by (4.12) and (4.13), respectively, we make use of the square function (3.1). From Lemma 3.1 it follows that

$$\big\|T_{\ell m}^{(j)}F\big\|_{q,\boldsymbol{R}^3}^2 \leq C\big\|ST_{\ell m}^{(j)}F\big\|_{q,\boldsymbol{R}^3}^2 = C\big\|(ST_{\ell m}^{(j)}F)^2\big\|_{q/2,\boldsymbol{R}^3}. \tag{4.15}$$

Assume now that $1 < q/2 < \infty$. Then we will estimate

$$\langle \big(ST^{(j)}_{\ell m}F\big)^2, w\rangle \equiv \int_{\boldsymbol{R}^3} w(x)\int_0^\infty \big|\big(\phi_s * T^{(j)}_{\ell m}F\big)(x)\big|^2 \frac{ds}{s}dx \tag{4.16}$$

for $w \in L^{q/(q-2)}(\boldsymbol{R}^3)$. By (4.12) we see

$$\big(\phi_s * T^{(j)}_{\ell m}F\big)(x) = \sum_{\mu,\nu,k}\int_{I(s,j)} O(t)^T_{\ell\mu}O(t)_{km}\big(\phi_s * H^{(j)}_{k\nu,t} * F_{\mu\nu}\big)(O(t)x)\frac{dt}{t}$$

with

$$I(s,j) = \big[2^{2j-4}s,\ 2^{2j+4}s\big]$$

because (4.14) and (3.3) imply

$$\widehat{\phi_s}(\xi)\widehat{H^{(j)}_{k\nu,t}}(\xi) \equiv 0, \qquad t \notin I(s,j)$$

and because ϕ_s is radially symmetric. We use the Schwarz inequality twice to obtain

$$\begin{aligned}\big|\big(\phi_s * T^{(j)}_{\ell m}F\big)(x)\big|^2 &\le C\sum_{\mu,\nu,k}\int_{I(s,j)}\left\{\int_{\boldsymbol{R}^3}\big|H^{(j)}_{k\nu,t}(O(t)x-y)\big|\big|(\phi_s * F_{\mu\nu})(y)\big|dy\right\}^2\frac{dt}{t}\\ &\le C\sum_{\mu,\nu,k}\big\|H^{(j)}_{k\nu}\big\|_{1,\boldsymbol{R}^3}\int_{I(s,j)}\big(\big|H^{(j)}_{k\nu,t}\big| * |\phi_s * F_{\mu\nu}|^2\big)(O(t)x)\frac{dt}{t}.\end{aligned}$$

Therefore, (4.16) is estimated as

$$\begin{aligned}&\big|\langle \big(ST^{(j)}_{\ell m}F\big)^2, w\rangle\big|\\ &\quad\le C\sum_{\mu,\nu,k}\big\|H^{(j)}_{k\nu}\big\|_{1,\boldsymbol{R}^3}\int_0^\infty\int_{I(s,j)}\int_{\boldsymbol{R}^3}|w(O(t)^Tx)|\big(\big|H^{(j)}_{k\nu,t}\big| * |\phi_s * F_{\mu\nu}|^2\big)(x)dx\frac{dt}{t}\frac{ds}{s}\\ &\quad= C\sum_{\mu,\nu,k}\big\|H^{(j)}_{k\nu}\big\|_{1,\boldsymbol{R}^3}\int_0^\infty\int_{\boldsymbol{R}^3}|(\phi_s * F_{\mu\nu})(x)|^2\\ &\qquad\qquad\int_{I(s,j)}\Big(\big|\widetilde{H^{(j)}_{k\nu,t}}\big| * |w(O(t)^T\,\cdot)|\Big)(x)\frac{dt}{t}dx\frac{ds}{s},\end{aligned}$$

where $\widetilde{H^{(j)}_{k\nu,t}}$ is the reflection of $H^{(j)}_{k\nu,t}$, that is, $\widetilde{H^{(j)}_{k\nu,t}}(x) = H^{(j)}_{k\nu,t}(-x)$. Set

$$M^{(j)}_{k\nu}w(x) = \sup_{r>0}\int_{2^{-4}r}^{2^4 r}\Big(\big|\widetilde{H^{(j)}_{k\nu,t}}\big| * |w(O(t)^T\,\cdot)|\Big)(x)\frac{dt}{t}. \tag{4.17}$$

Then we have

$$\big|\big\langle \big(ST_{\ell m}^{(j)}F\big)^2, w\big\rangle\big| \le C \sum_{\mu,\nu,k} \|H_{k\nu}^{(j)}\|_{1,\boldsymbol{R}^3} \int_{\boldsymbol{R}^3} M_{k\nu}^{(j)} w(x) SF_{\mu\nu}(x)^2 dx. \tag{4.18}$$

Similarly, for the adjoint operator we find

$$\big|\big\langle \big(ST_{\mu\nu}^{*(j)}G\big)^2, w\big\rangle\big| \le C \sum_{k,\ell,m} \|H_{k\nu}^{(j)}\|_{1,\boldsymbol{R}^3} \int_{\boldsymbol{R}^3} \mathscr{M}_{k\nu}^{(j)} w(y) SG_{\ell m}(y)^2 dy, \tag{4.19}$$

where

$$\mathscr{M}_{k\nu}^{(j)} w(y) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} \big(\big|H_{k\nu,t}^{(j)}\big| * |w|\big)(O(t)y)\frac{dt}{t}. \tag{4.20}$$

To proceed with the estimates, it is necessary to find the behavior of the following for $j \to \pm\infty$: $M_{k\nu}^{(j)}w$ and $\mathscr{M}_{k\nu}^{(j)}w$ as well as $\big\|H_{k\nu}^{(j)}\big\|_{1,\boldsymbol{R}^3}$. For this aim, the following lemma on a pointwise estimate of $H_{k\nu}^{(j)}(x)$, independently of (k,ν), plays an important role. The proof may be omitted since it is the same as in [**6**, Lemma 3.1].

Lemma 4.1. *Let $\psi(x) = (1+|x|^2)^{-2}$. Then there is a constant $C > 0$, independent of $x \in \boldsymbol{R}^3$, $j \in \boldsymbol{Z}$ and $1 \le k,\nu \le 3$, such that*

$$\big|H_{k\nu}^{(j)}(x)\big| \le C2^{-2|j|}\psi_{2^{-2j}}(x), \tag{4.21}$$

where $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t})$.

Proposition 4.3. *Let $1 < p < \infty$. Then the sublinear operators defined by* (4.17) *and* (4.20), *respectively, enjoy*

$$\big\|M_{k\nu}^{(j)}w\big\|_{p,\boldsymbol{R}^3} \le C2^{-2|j|}\|w\|_{p,\boldsymbol{R}^3}, \qquad \big\|\mathscr{M}_{k\nu}^{(j)}w\big\|_{p,\boldsymbol{R}^3} \le C2^{-2|j|}\|w\|_{p,\boldsymbol{R}^3},$$

with some $C = C(p) > 0$ independent of $w \in L^p(\boldsymbol{R}^3), j \in \boldsymbol{Z}$ and $1 \le k,\nu \le 3$.

Proof. The reflection $\widetilde{H_{k\nu}^{(j)}}(x)$ also satisfies (4.21) on account of $\psi(-x) = \psi(x)$. Note that $\psi_{2^{-2j}t}(x) \le C\psi_{2^{-2j}r}(x)$ for $2^{-4}r \le t \le 2^4 r$. Thus, we have

$$\begin{aligned} 0 \le M_{k\nu}^{(j)}w(x) &\le C2^{-2|j|}\sup_{r>0}\int_{2^{-4}r}^{2^4 r}\int_{\boldsymbol{R}^3}\psi_{2^{-2j}t}(x-y)|w(O(t)^T y)|dy\frac{dt}{t} \\ &\le C2^{-2|j|}\sup_{r>0}\int_{\boldsymbol{R}^3}\psi_{2^{-2j}r}(x-y)\int_{2^{-4}r}^{2^4 r}|w(O(t)^T y)|\frac{dt}{t}dy. \end{aligned}$$

Set

$$Rw(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} |w(O(t)^T x)| \frac{dt}{t}. \tag{4.22}$$

By use of this together with the maximal function (3.4), we obtain

$$\begin{aligned} M_{k\nu}^{(j)} w(x) &\le C 2^{-2|j|} \sup_{t>0} (\psi_t * Rw)(x) \\ &\le C 2^{-2|j|} (MRw)(x) \int_{\boldsymbol{R}^3} \psi(y) dy, \end{aligned}$$

see [**24**, Chapter II, 2.1]. Lemma 3.2 thus implies

$$\left\| M_{k\nu}^{(j)} w \right\|_{p,\boldsymbol{R}^3} \le C 2^{-2|j|} \|Rw\|_{p,\boldsymbol{R}^3},$$

as long as $Rw \in L^p(\boldsymbol{R}^3)$. It remains to show that the sublinear operator R is bounded in $L^p(\boldsymbol{R}^3)$. Using the cylindrical coordinate $x_1 = \rho \cos\theta$, $x_2 = \rho \sin\theta$, $x_3 = z$, we set

$$w_{(\rho,z)}(\theta) = w(\rho\cos\theta, \rho\sin\theta, z).$$

Then we have

$$Rw(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} |w_{(\rho,z)}(\theta - t)| \frac{dt}{t} \le 2^9 (M w_{(\rho,z)})(\theta).$$

By Lemma 3.2 we find

$$\|M w_{(\rho,z)}\|_{p,I} \le C \|w_{(\rho,z)}\|_{p,I}, \qquad I = (0, 2\pi).$$

Hence,

$$\begin{aligned} \|Rw\|_{p,\boldsymbol{R}^3}^p &\le C \int_{\boldsymbol{R}} \int_0^\infty \rho \int_0^{2\pi} (M w_{(\rho,z)})(\theta)^p d\theta d\rho dz \\ &\le C \int_{\boldsymbol{R}} \int_0^\infty \rho \int_0^{2\pi} w_{(\rho,z)}(\theta)^p d\theta d\rho dz = C \|w\|_{p,\boldsymbol{R}^3}^p, \end{aligned}$$

which implies the estimate for $M_{k\nu}^{(j)}$. By use of

$$\mathscr{R} w(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} |w(O(t)x)| \frac{dt}{t}$$

instead of (4.22), the second estimate on $\mathscr{M}_{k\nu}^{(j)}$ can be proved similarly. □

PROOF OF THEOREM 4.1. In view of (4.18), we use Proposition 4.3, Lemma 3.1 as well as

$$\|H_{k\nu}^{(j)}\|_{1,\boldsymbol{R}^3} \le C2^{-2|j|} \int_{\boldsymbol{R}^3} \psi(x)dx,$$

which follows from (4.21), to see that

$$\begin{aligned}\left|\left\langle \left(ST_{\ell m}^{(j)}F\right)^2, w\right\rangle\right| &\le C \sum_{\mu,\nu,k} \|H_{k\nu}^{(j)}\|_{1,\boldsymbol{R}^3} \left\|M_{k\nu}^{(j)} w\right\|_{q/(q-2),\boldsymbol{R}^3} \|SF_{\mu\nu}\|_{q,\boldsymbol{R}^3}^2 \\ &\le C(2^{-2|j|})^2 \|w\|_{q/(q-2),\boldsymbol{R}^3} \sum_{\mu,\nu} \|F_{\mu\nu}\|_{q,\boldsymbol{R}^3}^2,\end{aligned}$$

for all $w \in L^{q/(q-2)}(\boldsymbol{R}^3)$. By duality and by (4.15) we arrive at

$$\left\|T_{\ell m}^{(j)} F\right\|_{q,\boldsymbol{R}^3} \le C2^{-2|j|} \|F\|_{q,\boldsymbol{R}^3}, \tag{4.23}$$

with some $C > 0$ independent of $F \in C_0^\infty(\boldsymbol{R}^3)^9$, $j \in \boldsymbol{Z}$ and $1 \le \ell, m \le 3$. Hence, as long as $2 < q < \infty$,

$$T = (T_{\ell m})_{1 \le \ell, m \le 3} \quad \text{with} \quad T_{\ell m} = \sum_{j=-\infty}^{\infty} T_{\ell m}^{(j)}$$

is well-defined as a bounded operator on $L^q(\boldsymbol{R}^3)^9$. For $1 < q < 2$, we use the adjoint operator T^* given by (4.10). The same argument as above implies that T^* is also a bounded operator on $L^{q/(q-1)}(\boldsymbol{R}^3)^9$; so, T is L^q-bounded for $1 < q < 2$ as well. We have thus proved (4.3) for $1 < q < \infty$.

Let $f \in \widehat{W}^{-1,q}(\boldsymbol{R}^3)^3$. By Lemma 3.3 there is $F \in L^q(\boldsymbol{R}^3)^9$ such that

$$\nabla \cdot F = f, \qquad \|F\|_{q,\boldsymbol{R}^3} \le C\|f\|_{-1,q,\boldsymbol{R}^3}. \tag{4.24}$$

We take $F_k \in C_0^\infty(\boldsymbol{R}^3)^9$ so that $\|F_k - F\|_{q,\boldsymbol{R}^3} \to 0$ as $k \to \infty$. Let u_k be the solution given by (4.4) with $f = \nabla \cdot F_k$. For each k and $m \in \boldsymbol{N}$, we take a constant vector $b_k^{(m)} \in \boldsymbol{R}^3$ satisfying

$$\int_{B_m} \left(u_k(x) + b_k^{(m)}\right) dx = 0$$

so that

$$\left\|u_k + b_k^{(m)}\right\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,\boldsymbol{R}^3} \le C_m \|F_k\|_{q,\boldsymbol{R}^3}$$

by the Poincaré inequality and by (4.3). Therefore, there exist $u^{(m)} \in W^{1,q}(B_m)^3$ and $V \in L^q(\boldsymbol{R}^3)^9$ such that

$$\left\|u_k + b_k^{(m)} - u^{(m)}\right\|_{q,B_m} \to 0, \qquad \|\nabla u_k - V\|_{q,\boldsymbol{R}^3} \to 0 \qquad (k \to \infty)$$

with $\nabla u^{(m)}(x) = V(x)$ (a.a. $x \in B_m$). We first set

$$\widetilde{u} = u^{(1)} \quad \text{on } B_1; \quad b_k = b_k^{(1)}.$$

Next we consider the case $m = 2$; since $\nabla u^{(2)}(x) = V(x) = \nabla u^{(1)}(x) = \nabla \widetilde{u}(x)$ for a.a. $x \in B_1 \subset B_2$, the difference $u^{(2)}(x) - \widetilde{u}(x) =: a$ is a constant vector and

$$\begin{aligned} |B_1|^{1/q} |b_k^{(2)} - b_k - a| &= \|b_k^{(2)} - b_k - a\|_{q,B_1} \\ &\le \|u_k + b_k - \widetilde{u}\|_{q,B_1} + \left\|u_k + b_k^{(2)} - u^{(2)}\right\|_{q,B_2} \to 0 \end{aligned} \tag{4.25}$$

as $k \to \infty$. One extends $\widetilde{u}$ by

$$\widetilde{u} = u^{(2)} - a \quad \text{on } B_2.$$

Then (4.25) implies

$$\|u_k + b_k - \widetilde{u}\|_{q,B_2} \le \left\|u_k + b_k^{(2)} - u^{(2)}\right\|_{q,B_2} + |B_2|^{1/q} |b_k^{(2)} - b_k - a| \to 0$$

as $k \to \infty$. We repeat this procedure for $m = 3, 4, \cdots$. By induction there is a function $\widetilde{u} \in \widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ so that

$$\|u_k + b_k - \widetilde{u}\|_{q,B_m} + \|\nabla u_k - \nabla \widetilde{u}\|_{q,\boldsymbol{R}^3} \to 0 \tag{4.26}$$

as $k \to \infty$ for all $m \in \boldsymbol{N}$. By use of the operator L, see (1.1), it follows from (4.26) together with $Lu_k = \nabla \cdot F_k$ that

$$Lb_k = \omega \wedge b_k = L(u_k + b_k) - \nabla \cdot F_k \to L\widetilde{u} - \nabla \cdot F \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^3)^3$$

as $k \to \infty$. But then, there is a constant vector $b \in \boldsymbol{R}^3$ such that

$$\omega \wedge b_k \to \omega \wedge b = Lb$$

as $k \to \infty$. Consequently, we get

$$L(\widetilde{u} - b) = \nabla \cdot F \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^3)^3$$

and $u = \widetilde{u} - b$ is the desired solution. By (4.26) we have $\|\nabla u_k - \nabla u\|_{q,\boldsymbol{R}^3} \to 0$ and, therefore, the estimate (4.3) holds true for the obtained solution u as well (we note that $\nabla u = TF$, where T is the extended operator on $L^q(\boldsymbol{R}^3)^9$, since $\nabla u_k = TF_k$). This together with (4.24) implies

$$\|\nabla u\|_{q,\boldsymbol{R}^3} \le C\|f\|_{-1,q,\boldsymbol{R}^3},$$

which combined with

$$\|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q,\boldsymbol{R}^3} = \|f + \Delta u\|_{-1,q,\boldsymbol{R}^3} \leq \|f\|_{-1,q,\boldsymbol{R}^3} + \|\nabla u\|_{q,\boldsymbol{R}^3}$$

yields (4.2).

It remains to prove the uniqueness. We use the duality method. Let us consider the adjoint equation

$$L^* v \equiv -\Delta v + (\omega \wedge x) \cdot \nabla v - \omega \wedge v = \nabla \cdot F \tag{4.27}$$

with $F \in C_0^\infty(\boldsymbol{R}^3)^9$. This admits the solution

$$v(x) = \int_0^\infty O(t)(e^{t\Delta} \nabla \cdot F)(O(t)^T x) dt,$$

where one has only to replace $O(t)$ by $O(t)^T$ in the formula (4.4). By the same argument as for (4.4) we have $v \in \widehat{W}^{1,r}(\boldsymbol{R}^3)^3$ for all $r \in (1, \infty)$ with $\|\nabla v\|_{r,\boldsymbol{R}^3} \leq C\|F\|_{r,\boldsymbol{R}^3}$. We now let $u \in \widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ be a weak solution of $Lu = 0$ in $\widehat{W}^{-1,q}(\boldsymbol{R}^3)^3$. One can take v as a test function to get

$$\langle Lu, v \rangle = 0.$$

Similarly, one takes u as a test function for (4.27) in $\widehat{W}^{-1,q/(q-1)}(\boldsymbol{R}^3)^3$ to obtain

$$\langle u, L^* v \rangle = \langle u, \nabla \cdot F \rangle.$$

Therefore,

$$\langle u, \nabla \cdot F \rangle = 0.$$

Since $F \in C_0^\infty(\boldsymbol{R}^3)^9$ is arbitrary, we obtain $u = 0$ in $\widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ by Lemma 3.3. Namely, u is a constant vector; but, it should be a constant multiple of ω because $\omega \wedge u = 0$. □

To complete the proof of Theorem 2.1, we need

LEMMA 4.2. *Let $v \in \mathscr{S}'(\boldsymbol{R}^3)$ be the solution of*

$$-\Delta v - (\omega \wedge x) \cdot \nabla v = 0 \qquad \text{in } \boldsymbol{R}^3.$$

Then $\operatorname{supp} \widehat{v} \subset \{0\}$. *Especially, if $v \in L^q(\boldsymbol{R}^3)$ for $1 \leq q < \infty$, then $v = 0$.*

PROOF. This was shown in [**6**], but we give the proof for completeness. We first see that

$$|\xi|^2 \widehat{v} - (\omega \wedge \xi) \cdot \nabla_\xi \widehat{v} = 0 \qquad \text{in } \boldsymbol{R}^3_\xi.$$

For any $\varphi \in C_0^\infty(\boldsymbol{R}^3_\xi \setminus \{0\})$, the adjoint equation

$$|\xi|^2\chi + (\omega \wedge \xi) \cdot \nabla_\xi \chi = \varphi \qquad \text{in } \boldsymbol{R}^3_\xi$$

is solvable; in fact,

$$\chi(\xi) = \int_0^\infty e^{-|\xi|^2 t}\varphi(O(t)^T\xi)dt \in C_0^\infty\big(\boldsymbol{R}^3_\xi \setminus \{0\}\big)$$

is a solution. Hence, we have

$$\langle \widehat{v}, \varphi\rangle = \langle \widehat{v}, |\xi|^2\chi + (\omega \wedge \xi) \cdot \nabla_\xi \chi\rangle = \langle |\xi|^2\widehat{v} - (\omega \wedge \xi) \cdot \nabla_\xi \widehat{v}, \chi\rangle = 0,$$

which completes the proof. □

Proof of Theorem 2.1. To complete the proof, it suffices to obtain the estimate involving the pressure term p. The relation

$$\nabla \cdot [(\omega \wedge x) \cdot \nabla u - \omega \wedge u] = (\omega \wedge x) \cdot \nabla(\nabla \cdot u) = \nabla \cdot [(\omega \wedge x)\nabla \cdot u]$$

gives the pressure term by f and g as

$$p = -\nabla \cdot (-\Delta)^{-1}[f + \nabla g + (\omega \wedge x)g].$$

Since $(-\Delta)^{-1}$ can be justified as a bounded operator from $\widehat{W}^{-1,q}(\boldsymbol{R}^3)$ to $\widehat{W}^{1,q}(\boldsymbol{R}^3)$ ([**10**], [**17**]), we get

$$\|p\|_{q,\boldsymbol{R}^3} \le C\|f + \nabla g + (\omega \wedge x)g\|_{-1,q,\boldsymbol{R}^3}, \tag{4.28}$$

which implies

$$\|f - \nabla p\|_{-1,q,\boldsymbol{R}^3} \le C\big(\|f\|_{-1,q,\boldsymbol{R}^3} + \|\nabla g + (\omega \wedge x)g\|_{-1,q,\boldsymbol{R}^3}\big). \tag{4.29}$$

Theorem 4.1 thus provides a solution $u \in \widehat{W}^{1,q}(\boldsymbol{R}^3)^3$ of

$$-\Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u = f - \nabla p.$$

Since

$$-\Delta(\nabla \cdot u) - (\omega \wedge x) \cdot \nabla(\nabla \cdot u) = \nabla \cdot f - \Delta p = -\Delta g - (\omega \wedge x) \cdot \nabla g,$$

Lemma 4.2 yields $\nabla \cdot u = g$ in $L^q(\boldsymbol{R}^3)$. The estimate (4.2) together with (4.29) and (4.28) implies (2.5). This completes the proof of Theorem 2.1. □

5. Exterior problem.

In this section we will prove Theorem 2.2 for the exterior problem (2.2) by means of a localization procedure. We combine Theorem 2.1 for the whole space problem with the following lemma on the interior one. Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and consider the usual Stokes problem with the inhomogeneous divergence condition

$$-\Delta u + \nabla p = f, \quad \nabla \cdot u = g \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0. \tag{5.1}$$

LEMMA 5.1 (Cattabriga [**3**], Solonnikov [**22**], Kozono and Sohr [**16**]). *Let Ω be as above and let $1 < q < \infty$. Suppose that*

$$f \in W^{-1,q}(\Omega)^3, \quad g \in L^q(\Omega), \quad \int_\Omega g(x)dx = 0.$$

Then the problem (5.1) *possesses a unique* (*up to an additive constant for* p) *weak solution* $\{u,p\} \in W_0^{1,q}(\Omega)^3 \times L^q(\Omega)$ *subject to the estimate*

$$\|\nabla u\|_{q,\Omega} + \|p - \overline{p}\|_{q,\Omega} \le C(\|f\|_{-1,q,\Omega} + \|g\|_{q,\Omega}), \tag{5.2}$$

where $\overline{p} = \frac{1}{|\Omega|}\int_\Omega p(x)dx$.

To begin with, we derive the following a priori estimate, which will be refined later, see Proposition 5.2.

LEMMA 5.2. *Let $3/2 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(D)^3$, let*

$$\{u,p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$$

be a weak solution to the problem (2.2). *Choose $\rho > \rho_0 > 0$ so large that $\boldsymbol{R}^3 \setminus D \subset B_{\rho_0}$, and take $\psi \in C_0^\infty(B_\rho;[0,1])$ such that $\psi = 1$ on B_{ρ_0}. Then*

$$\begin{aligned} &\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x)\cdot\nabla u - \omega \wedge u\|_{-1,q} \\ &\quad \le C\left(\|f\|_{-1,q} + \|u\|_{q,D_\rho} + \|p\|_{-1,q,D_\rho} + \left|\int_{D_\rho}\psi(x)p(x)dx\right|\right), \end{aligned} \tag{5.3}$$

with some $C > 0$, where $D_\rho = D \cap B_\rho$.

PROOF. By use of the cut-off function ψ, we decompose the solution $\{u,p\}$ as

$$\begin{cases} u = U + V, & U = (1-\psi)u, & V = \psi u, \\ p = \sigma + \tau, & \sigma = (1-\psi)p, & \tau = \psi p. \end{cases} \tag{5.4}$$

Then $\{U,\sigma\}$ is a weak solution of

$$-\Delta U - (\omega \wedge x) \cdot \nabla U + \omega \wedge U + \nabla \sigma = Z_1, \quad \nabla \cdot U = -u \cdot \nabla \psi \qquad \text{in } \boldsymbol{R}^3,$$

where

$$Z_1 = (1-\psi)f + 2\nabla\psi \cdot \nabla u + [\Delta\psi + (\omega \wedge x) \cdot \nabla\psi]u - (\nabla\psi)p;$$

in fact, for all $\phi \in C_0^\infty(\boldsymbol{R}^3)^3$, we see that

$$\langle \nabla U, \nabla\phi \rangle - \langle (\omega \wedge x) \cdot \nabla U - \omega \wedge U, \phi \rangle - \langle \sigma, \nabla \cdot \phi \rangle = \langle Z_1, \phi \rangle,$$

since $\{u, p\}$ satisfies (2.3) with $\varphi = (1-\psi)\phi$. Similarly, $\{V, \tau\}$ is a weak solution of

$$-\Delta V + \nabla \tau = Z_2, \quad \nabla \cdot V = u \cdot \nabla \psi \quad \text{in } D_\rho; \quad V|_{\partial D_\rho} = 0,$$

where

$$Z_2 = \psi[f + (\omega \wedge x) \cdot \nabla u - \omega \wedge u] - 2\nabla\psi \cdot \nabla u - (\Delta\psi)u + (\nabla\psi)p.$$

It therefore follows from Theorem 2.1 and Lemma 5.1, respectively, that

$$\begin{aligned} &\|\nabla U\|_{q,\boldsymbol{R}^3} + \|\sigma\|_{q,\boldsymbol{R}^3} \\ &\quad \le C\|Z_1\|_{-1,q,\boldsymbol{R}^3} + C\|u \cdot \nabla\psi\|_{q,\boldsymbol{R}^3} + C\|(\omega \wedge x)(u \cdot \nabla\psi)\|_{-1,q,\boldsymbol{R}^3}, \end{aligned} \tag{5.5}$$

and that

$$\begin{aligned} &\|\nabla V\|_{q,D_\rho} + \|\tau\|_{q,D_\rho} \\ &\quad \le C\|Z_2\|_{-1,q,D_\rho} + C\|u \cdot \nabla\psi\|_{q,D_\rho} + \frac{1}{|D_\rho|^{1-1/q}} \left| \int_{D_\rho} \tau(x)dx \right|, \end{aligned} \tag{5.6}$$

as long as the right-hand sides are finite. Let $\phi \in C_0^\infty(\boldsymbol{R}^3)^3$. We then have

$$\begin{aligned} |\langle (1-\psi)f, \phi \rangle| &\le \|f\|_{-1,q} \|\nabla[(1-\psi)\phi]\|_{q/(q-1)} \\ &\le \|f\|_{-1,q} \big(\|\nabla\phi\|_{q/(q-1)} + C\|\phi\|_{q/(q-1),D_\rho} \big) \\ &\le C\|f\|_{-1,q} \|\nabla\phi\|_{q/(q-1),\boldsymbol{R}^3}. \end{aligned}$$

Here, we have used the condition $q > 3/2$, so that $q/(q-1) < 3$, to apply the Sobolev inequality

$$\|\phi\|_{q/(q-1),D_\rho} \le |D_\rho|^{1/3} \|\phi\|_{r,D_\rho} \le C\|\phi\|_{r,\boldsymbol{R}^3} \le C\|\nabla\phi\|_{q/(q-1),\boldsymbol{R}^3},$$

where $1/r = (q-1)/q - 1/3$. Similarly, we obtain

$$\begin{aligned}&|\langle 2\nabla\psi\cdot\nabla u+[\Delta\psi+(\omega\wedge x)\cdot\nabla\psi]u,\phi\rangle|\\&\quad\le C\|u\|_{q,D_\rho}\big(\|\nabla\phi\|_{q/(q-1),D_\rho}+\|\phi\|_{q/(q-1),D_\rho}\big)\\&\quad\le C\|u\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\boldsymbol{R}^3},\end{aligned}$$

and

$$\begin{aligned}|\langle(\nabla\psi)p,\phi\rangle|&\le\|p\|_{-1,q,D_\rho}\|\nabla[(\nabla\psi)\phi]\|_{q/(q-1),D_\rho}\\&\le C\|p\|_{-1,q,D_\rho}\|\nabla\phi\|_{q/(q-1),\boldsymbol{R}^3},\end{aligned}$$

as well as

$$\begin{aligned}|\langle(\omega\wedge x)(u\cdot\nabla\psi),\phi\rangle|&\le C\|u\|_{q,D_\rho}\|\phi\|_{q/(q-1),D_\rho}\\&\le C\|u\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\boldsymbol{R}^3}.\end{aligned}$$

In view of (5.5), we collect the estimates above to find

$$\|\nabla U\|_{q,\boldsymbol{R}^3}+\|\sigma\|_{q,\boldsymbol{R}^3}\le C\big(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}\big). \tag{5.7}$$

In the same way, we see that

$$|\langle Z_2,\phi\rangle|\le C\big(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}\big)\|\nabla\phi\|_{q/(q-1),D_\rho}$$

for all $\phi\in C_0^\infty(D_\rho)^3$; here, we have used the Poincaré inequality and so the condition $q>3/2$ is not necessary. This combined with (5.6) implies that

$$\begin{aligned}&\|\nabla V\|_{q,D_\rho}+\|\tau\|_{q,D_\rho}\\&\quad\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}+\left|\int_{D_\rho}\psi(x)p(x)dx\right|\right).\end{aligned} \tag{5.8}$$

By (5.7) and (5.8) we obtain

$$\|\nabla u\|_q+\|p\|_q\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}+\left|\int_{D_\rho}\psi(x)p(x)dx\right|\right),$$

which together with

$$\|(\omega\wedge x)\cdot\nabla u-\omega\wedge u\|_{-1,q}=\|f+\Delta u-\nabla p\|_{-1,q}\le\|f\|_{-1,q}+\|\nabla u\|_q+\|p\|_q$$

yields (5.3). □

We next show the existence and summability of weak solutions to the problem (2.2) and of those to the adjoint one

$$\begin{cases} -\Delta v + (\omega \wedge x) \cdot \nabla v - \omega \wedge v - \nabla \pi = f & \text{in } D, \\ -\nabla \cdot v = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{5.9}$$

for nice force terms f, being in a dense subspace of $\widehat{W}^{-1,q}(D)^3$; see Lemma 3.3.

Lemma 5.3. *Let $F \in C_0^\infty(D)^9$. Then the problem* (2.2) *with $f = \nabla \cdot F$ has a weak solution $\{u, p\}$ of class*

$$u \in \widehat{W}_0^{1,r}(D)^3, \qquad p \in L^r(D) \qquad \text{for } 3/2 < \forall r < \infty. \tag{5.10}$$

The same assertion for the adjoint problem (5.9) *holds true as well.*

Proof. We first employ the standard L^2 technique to show that there is a distribution solution

$$u \in \widehat{W}_0^{1,2}(D)^3 \hookrightarrow L^6(D)^3, \qquad p \in L^2_{loc}(\overline{D}).$$

When $q = 2$, one can take $\varphi = u$ in (2.3) to get

$$\|\nabla u\|_2^2 = \langle \nabla \cdot F, u \rangle$$

since

$$\int_{D_R} [(\omega \wedge x) \cdot \nabla u] \cdot u dx = \frac{1}{2} \int_{\partial D_R} n \cdot (\omega \wedge x) |u|^2 d\sigma = 0$$

for every sufficiently large $R > 0$, where $D_R = D \cap B_R$ and n is the unit exterior normal vector to the boundary ∂D_R; here, note that $x \cdot (\omega \wedge x) = 0$ on ∂B_R. We thus have the a priori estimate

$$\|\nabla u\|_2 \le \|F\|_2.$$

The Lax-Milgram theorem (together with a result of de Rham) provides a distribution solution $\{u_R, p_R\} \in W_{0,\sigma}^{1,2}(D_R) \times L^2(D_R)$ to the same equations in each bounded domain D_R, where $W_{0,\sigma}^{1,2}(D_R)$ denotes the completion of the class of solenoidal vector fields, whose components are in $C_0^\infty(D_R)$, under the norm $\|\nabla(\cdot)\|_{2,D_R}$. In fact, the bilinear form

$$(u, \varphi) \mapsto \langle \nabla u, \nabla \varphi \rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, \varphi \rangle$$

on $W_{0,\sigma}^{1,2}(D_R) \times W_{0,\sigma}^{1,2}(D_R)$ is not only continuous but also coercive. Since the a priori estimate $\|\nabla u_R\|_2 \le \|F\|_2$ is available, where u_R is understood as its extension by setting zero in $D \setminus D_R$, there exist $u \in \widehat{W}_0^{1,2}(D)^3$ and a sequence $\{R_n\} \nearrow \infty$ so that $u_{R_n} \rightharpoonup u$ weakly in $\widehat{W}_0^{1,2}(D)^3$ as $n \to \infty$. We then find $\langle Lu - \nabla \cdot F, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(D)^3$ with

$\nabla \cdot \varphi = 0$. By a result of de Rham, there is a distribution p such that $-\nabla p = Lu - \nabla \cdot F$ in $\mathscr{D}'(D)^3$. Since the right-hand side belongs to $W^{-1,2}(D_R)$ for every large $R > 0$, we see that $p \in L^2(D_R)$ and thus $p \in L^2_{loc}(\overline{D})$.

Now, as in (5.4), we use the cut-off technique to split the solution $\{u, p\}$ into flows $\{U, \sigma\}$ for the whole space problem and $\{V, \tau\}$ for the interior one. Along the same line as in the proof of Lemma 5.2, Theorem 2.1 and Lemma 5.1 lead to

$$u \in \widehat{W}_0^{1,r}(D)^3, \qquad p \in L^r(D) \qquad \text{for } 3/2 < \forall r \leq 6.$$

By the same localization argument once more, we obtain (5.10) for the problem (2.2). The problem (5.9) is nothing but (2.2) with $\{p, \omega\}$ replaced by $\{-\pi, -\omega\}$, and so the same assertion holds. □

As a corollary, we have the following uniqueness assertion.

PROPOSITION 5.1 (Uniqueness). *Let* $1 < q < 3$. *Suppose that* $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ *is a weak solution to the problem* (2.2) *with* $f = 0$. *Then* $\{u, p\} = \{0, 0\}$.

PROOF. Consider the adjoint problem (5.9) with $f = \nabla \cdot F$, where $F \in C_0^\infty(D)^9$. By Lemma 5.3 there is a weak solution $\{v, \pi\}$ of class (5.10). Since $q/(q-1) > 3/2$, one can put $\varphi = v$ in (2.3) with $f = 0$:

$$\langle \nabla u, \nabla v \rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, v \rangle = 0.$$

Similarly, one can take u as a test function for (5.9) to get

$$\langle \nabla v, \nabla u \rangle + \langle (\omega \wedge x) \cdot \nabla v - \omega \wedge v, u \rangle = \langle \nabla \cdot F, u \rangle.$$

From the equalities above it follows that $\langle \nabla \cdot F, u \rangle = 0$ for all $F \in C_0^\infty(D)^9$. By Lemma 3.3 we get $\langle f, u \rangle = 0$ for all $f \in \widehat{W}^{-1,q/(q-1)}(D)^3$, which yields $u = 0$ in $\widehat{W}_0^{1,q}(D)^3$, and thus $p = 0$ in $L^q(D)$. This completes the proof. □

By Lemma 5.2 together with Proposition 5.1 we find the following a priori estimate.

PROPOSITION 5.2 (A priori estimate). *Let* $3/2 < q < 3$. *Suppose that* $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ *is a weak solution to the problem* (2.2) *with* $f \in \widehat{W}^{-1,q}(D)^3$. *Then the estimate* (2.6) *holds.*

PROOF. Suppose on the contrary. Then there exist sequences $f_k \in \widehat{W}^{-1,q}(D)^3$ and $\{u_k, p_k\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$, the corresponding weak solution, so that

$$\|\nabla u_k\|_q + \|p_k\|_q + \|(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k\|_{-1,q} = 1,$$

while

$$\|f_k\|_{-1,q} \to 0$$

as $k \to \infty$. Then we have

$$\|u_k\|_{1,q,D_\rho} \le \|\nabla u_k\|_{q,D_\rho} + C\|u_k\|_{q_*,D_\rho} \le C\|\nabla u_k\|_q \le C$$

as well as $\|p_k\|_{q,D_\rho} \le \|p_k\|_q \le 1$, where $1/q_* = 1/q - 1/3$, $D_\rho = D \cap B_\rho$ (as in Lemma 5.2) and $\|\cdot\|_{1,q,D_\rho}$ is the norm of $W^{1,q}(D_\rho)$. There are subsequences, which we denote by u_k and p_k again, so that they weakly converge in $W^{1,q}(D_\rho)$ and $L^q(D_\rho)$, and by the Rellich compactness theorem, they strongly converge in $L^q(D_\rho)$ and $W^{-1,q}(D_\rho)$, respectively. From (5.3) it follows that $\{u_k, p_k\}$ and $\{(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k\}$ are the Cauchy sequences, respectively, in $\widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ and in $\widehat{W}^{-1,q}(D)^3$; hence, there exists $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ so that

$$\begin{cases} \|\nabla u_k - \nabla u\|_q + \|p_k - p\|_q \to 0, \\ \|[(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k] - [(\omega \wedge x) \cdot \nabla u - \omega \wedge u]\|_{-1,q} \to 0, \end{cases} \tag{5.11}$$

as $k \to \infty$. It easily turns out that the pair $\{u, p\}$ is a weak solution to (2.2) with $f = 0$. Since $q < 3$, Proposition 5.1 implies that $\{u, p\} = \{0, 0\}$, which contradicts

$$\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q} = 1.$$

This completes the proof. □

Proof of Theorem 2.2. The uniqueness part follows from Proposition 5.1. By Lemma 3.3, given $f \in \widehat{W}^{-1,q}(D)^3$, we take $F_k \in C_0^\infty(D)^9$ so that $\|\nabla \cdot F_k - f\|_{-1,q} \to 0$ as $k \to \infty$. By Lemma 5.3 there is a solution $\{u_k, p_k\}$ of class (5.10) to the problem (2.2) with the force $\nabla \cdot F_k$. One can take $r = q$ in (5.10) since $q > 3/2$. By Proposition 5.2 one can use (2.6) to show that there exists $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ so that the same convergence properties as in (5.11) hold. This pair $\{u, p\}$ is a weak solution to (2.2) with the estimate (2.6). We have completed the proof. □

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