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Combinatorial principles on ω_1, cardinal invariants of the meager ideal and destructible gaps

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Abstract. We show that (1) $\dagger$ plus $\mathsf{cov}(\mathscr{M}) > \aleph_1$ implies the existence of a destructible gap and (2) $\clubsuit$ plus $\mathsf{cof}(\mathscr{M}) = \aleph_1$ implies the existence of a destructible gap.

Introduction.

In this paper, we deal with a pregap in the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$. A pregap in $\mathscr{P}(\omega)/\mathrm{fin}$ is a pair $(\mathscr{A}, \mathscr{B})$ of subsets of $\mathscr{P}(\omega)$ such that for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$, the set $a \cap b$ is finite. For subsets a and b of ω, we say that a is almost contained in b (and denote $a \subseteq^* b$) if the set $a \smallsetminus l$ is a subset of b for some $l \in \omega$. For a pregap $(\mathscr{A}, \mathscr{B})$, both ordered sets $\langle \mathscr{A}, \subseteq^* \rangle$ and $\langle \mathscr{B}, \subseteq^* \rangle$ are well ordered and these order types are κ and λ respectively, then we say that a pregap $(\mathscr{A}, \mathscr{B})$ has the type (κ, λ) or is a (κ, λ)-pregap. Moreover if $\kappa = \lambda$, we say that the pregap is symmetric. For a pregap $(\mathscr{A}, \mathscr{B})$, we say that $(\mathscr{A}, \mathscr{B})$ is separated if for some $c \in \mathscr{P}(\omega)$, $a \subseteq^* c$ and the set $c \cap b$ is finite for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has the type (κ, λ), it is called a (κ, λ)-gap.

We note that being a pregap is absolute in any model having the pregap, but being a gap is not. In [**9**], Kunen has investigated an (ω_1, ω_1)-gap and has given a characterization of being a gap in the forcing extension and in [**18**, Chapter 9], Todorčević has introduced a notion of an open coloring and has given Ramsey theoretic characterization of being a gap in the forcing extension (see the theorem below). From their characterizations, we note that an (ω_1, ω_1)-gap constructed by Hausdorff is still a gap in any extension preserving cardinals. We say that such a gap is indestructible. If an (ω_1, ω_1)-gap is not indestructible, that is, it is not a gap in some forcing extension not collapsing cardinals, it is called destructible. (We note that every gap not having the type (ω_1, ω_1), it can be separated by a ccc-forcing extension.) Kunen has proved that under Martin's Axiom for $\aleph_1$ many dense sets of ccc-forcing notions, all (ω_1, ω_1)-gap are indestructible. In [**10**], Laver has implied that a destructible gap consistently exists. Therefore it is not decided from ZFC that there exists a destructible gap.

A notion of a destructible gap can be considered an analogy of one of a Suslin tree ([**1**]). A Suslin tree is an ω_1-tree having no uncountable chains and antichains. A

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destructible gap can be considered as a similar notion. For an (ω_1,ω_1)-pregap $(\mathscr{A},\mathscr{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$, we say here that α and β in ω_1 are compatible if

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing.$$

Then by the characterization due to Kunen and Todorčević, we notice that an (ω_1,ω_1)-pregap is a destructible gap iff it has no uncountable pairwise compatible and incompatible subsets of ω_1. (We must notice that from results of Farah and Hirschorn [**5**], [**7**], the existence of a destructible gap is independent with the existence of a Suslin tree.)

Jensen has proved that if $\boldsymbol{V} = \boldsymbol{L}$, then there exists a Suslin tree. After that, he has introduced a combinatorial principle $\diamondsuit$ and has constructed a Suslin tree from $\diamondsuit$. Shelah has proved that adding a Cohen real adds a Suslin tree. The same results for a destructible gap are also true and proved by Todorčević ([**4**, Proposition 2.5] and [**18**, Theorem 9.3]). In this paper, we construct a destructible gap by two ways.

One is a modification of the construction from adding a Cohen real. In [**19**], Velleman has modified a construction of a Suslin tree due to Shelah using a morass, and after that Miyamoto has modified a Velleman's construction using a connection of two models. The first version of Miyamoto's theorem also have a morass as a condition to build a Suslin tree, but in [**3**, §7], Brendle has modified again that situation and consequently, he constructed a Suslin tree from ⫯ plus the covering number $\mathsf{cov}(\mathscr{M})$ of the meager ideal is larger than $\aleph_1$. ⫯ is a combinatorial principle on ω_1, introduced in the paper [**2**], as follow: there is a sequence $\langle A_\alpha; \alpha \in \omega_1\rangle$ of countable subsets of ω_1 such that for any uncountable subset B of ω_1 there is $\alpha \in \omega_1$ so that A_α is a subset of B. A destructible gap can be constructed under the same situation, that is, ⫯ plus $\mathsf{cov}(\mathscr{M}) > \aleph_1$ implies the existence of a destructible gap (Theorem 1.1).

The other is the modification of the construction from $\diamondsuit$. $\clubsuit$ is a combinatorial principle on ω_1 introduced by Ostaszewski ([**12**]. See also [**14**, I.§7].): There exists a sequence $\langle A_\alpha; \alpha \in \omega_1\rangle$ of subsets of ω_1 such that for all $\alpha \in \omega_1$, A_α is a cofinal subset of α and for every uncountable subset A of ω_1, the set $\{\alpha \in \omega_1; A_\alpha \subseteq A\}$ is stationary. We note that $\diamondsuit$ implies $\clubsuit$ and $\clubsuit$ plus the Continuum Hypothesis implies $\diamondsuit$ ([**14**, Chapter 1, 7.4 Theorem]). From the result of Baumgartner [**8**, Theorem IV. 4] (or the result [**11**, Corollary 6.14]), it is consistent with ZFC that $\clubsuit$, the cofinality $\mathsf{cof}(\mathscr{M})$ of the meager ideal on the real line is equal to $\aleph_1$ and the continuum is larger than $\aleph_1$, hence in this model, $\diamondsuit$ does not hold. Brendle has proved that a Suslin tree exists in the model satisfying $\clubsuit$ plus $\mathsf{cof}(\mathscr{M}) = \aleph_1$ ([**3**, Theorem 6]). As same as a Suslin tree, we can show that $\clubsuit$ and $\mathsf{cof}(\mathscr{M}) = \aleph_1$ implies the existence of a destructible gap (Theorem 2.2).

Throughout this paper, we always deal with a symmetric pregap. For an ordinal α, if we say that $\langle a_\xi, b_\xi; \xi \in \alpha\rangle$ is a pregap, we always assume that if $\xi < \eta$ in α, $a_\xi \subseteq^* a_\eta$ and $b_\xi \subseteq^* b_\eta$, and for every $\xi \in \alpha$, the set $a_\xi \cap b_\xi$ is empty. We have the following characterizations of being a gap and indestructibility.

Theorem (E.g. [**9**], [**13**], [**16**], [**18**]). *Let $(\mathscr{A},\mathscr{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ be an (ω_1,ω_1)-pregap with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$.*

(1) *$(\mathscr{A},\mathscr{B})$ forms a gap iff for any $X \in [\omega_1]^{\omega_1}$, there are $\alpha \neq \beta$ in X such that*

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \varnothing.$$

(2) $(\mathscr{A}, \mathscr{B})$ *is destructible* (*may not be a gap*) *iff for any* $X \in [\omega_1]^{\omega_1}$, *there are* $\alpha \neq \beta$ *in* X *such that*

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing.$$

1. ⫯ plus $\mathsf{cov}(\mathscr{M}) > \aleph_1$.

Miyamoto has proved the existence of a Suslin tree under the assumptions in Theorem 1.1. This theorem says not only the existence of a Suslin tree but also the preservation of a Suslin tree constructed from a Cohen real between models with some properties. To prove Miyamoto's theorem, we use Todorčević's coding of Aronszajn trees. In the following proof, we just use a K_0-homogeneous gap (see in the proof). To prove Theorem 1.1, we note that $\mathsf{cov}(\mathscr{M})$ is equal to the smallest number κ such that there is a family of dense subsets of $\boldsymbol{C}$ of size κ so that there are no filters which meets all members of the family.

Theorem 1.1. (1) *Assume* $\boldsymbol{V} \subseteq \boldsymbol{W}$ *are models of* (*fragments of*) ZFC *so that*

(a) ${\aleph_1}^{\boldsymbol{V}} = {\aleph_1}^{\boldsymbol{W}}$,
(b) $\forall B \in [\omega_1]^{\omega_1} \cap \boldsymbol{W}\ \exists A \in [\omega_1]^{\omega} \cap \boldsymbol{V}\ (A \subseteq B)$, *and*
(c) *there exists* $c \in \boldsymbol{W}$ *which is Cohen over* $\boldsymbol{V}$.

Then there exists a destructible gap in $\boldsymbol{W}$.
(2) ⫯ *plus* $\mathsf{cov}(\mathscr{M}) > \aleph_1$ *implies the existence of a destructible gap.*

Proof. We will prove only (2). (1) can be proved by the same way.

Let $\langle A_\alpha; \alpha \in \omega_1\rangle \in ([\omega_1]^{\omega})^{\omega_1}$ be a ⫯ sequence, i.e. for every $B \in [\omega_1]^{\omega_1}$ we can find $\alpha \in \omega_1$ so that $A_\alpha \subseteq B$. Let $\langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ be an (ω_1, ω_1)-pregap such that for all $\alpha \in \omega_1$ and $n \in \omega$, there exists $\xi \neq \eta$ in A_α such that

$$(a_\xi \cap b_\eta \cap n) \cup (a_\eta \cap b_\xi \cap n) = \varnothing \text{ and } \big((a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)\big) \smallsetminus n \neq \varnothing.$$

(We note that by ⫯, this pregap really forms a gap.) This pregap exists under ZFC. For example, let $\langle a_\xi, b_\xi; \xi \in \omega_1\rangle$ is K_0-homogeneous, i.e. the set $a_\xi \cap b_\xi$ is empty for every $\xi \in \omega_1$ and for every $\xi \neq \eta$ in ω_1,

$$(a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi) \neq \varnothing$$

(see [**13**, Lemma 12] or [**16**, 8.6.Theorem]). Then for any countable subset A of ω_1 and a natural number n, there are $\xi \neq \eta$ in A such that

$$(a_\xi \cap b_\eta \cap n) \cup (a_\eta \cap b_\xi \cap n) = \varnothing \text{ and } \big((a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)\big) \smallsetminus n \neq \varnothing.$$

In this proof, we let $\boldsymbol{C}$ be a partial order $\langle 2^{<\omega}, \supseteq\rangle$ and we identify a condition p in

$\boldsymbol{C}$ with a finite subset $\{i \in |p|\,; p(i) = 1\}$ of $|p|$. For each $\alpha \in \omega_1$, we define

$$D_\alpha = \big\{p \in \boldsymbol{C}; \exists \xi \neq \eta \in A_\alpha\ \big((a_\xi \cap b_\eta \cap p) \cup (a_\eta \cap b_\xi \cap p) \neq \varnothing\big)\big\}$$

and

$$E_\alpha = \big\{p \in \boldsymbol{C}; \exists \xi < \eta \in A_\alpha\ (a_\xi \setminus |p| \subseteq a_\eta \ \&\ b_\xi \setminus |p| \subseteq b_\eta \\ \&\ (a_\xi \cap b_\eta \cap p) \cup (a_\eta \cap b_\xi \cap p) = \varnothing)\big\}.$$

Claim. *All D_α and E_α are dense in $\boldsymbol{C}$.*

Proof of Claim. For each $\alpha \in \omega_1$ and $p \in \boldsymbol{C}$, there exist $\xi < \eta$ in A_α such that

$$(a_\xi \cap b_\eta \cap |p|) \cup (a_\eta \cap b_\xi \cap |p|) = \varnothing \text{ and } \big((a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)\big) \setminus |p| \neq \varnothing.$$

Let $k \geq |p|$ such that $k \in (a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)$ and let $q := p \cup (\mathbf{0} \restriction (k \setminus |p|)) \cup \{\langle k, 1\rangle\}$. ($\mathbf{0}$ is a constant function with the value 0.) Then

$$q \Vdash_{\boldsymbol{C}} \text{“} (\check{a}_\xi \cap \check{b}_\eta \cap \dot{c}) \cup (\check{a}_\eta \cap \check{b}_\xi \cap \dot{c}) \neq \varnothing \text{”},$$

i.e. q is in D_α.

Let $l \geq |p|$ be so that $a_\xi \setminus l \subseteq a_\eta$, $b_\xi \setminus l \subseteq b_\eta$ and let $r := p \cup \mathbf{0} \restriction (l \setminus |p|)$. Then

$$r \Vdash_{\boldsymbol{C}} \text{“} (\check{a}_\xi \cap \check{b}_\eta \cap \dot{c}) \cup (\check{a}_\eta \cap \check{b}_\xi \cap \dot{c}) = \varnothing \text{”},$$

i.e. r is in E_α. ⊣

Let $G \subseteq \boldsymbol{C}$ be a filter which meets all D_α and E_α, and the dense subsets $\{p \in \boldsymbol{C}; |p| \geq n\}$ of $\boldsymbol{C}$ for all $n \in \omega$. Let $c := \bigcup G$.

Since G meets all D_α, $\langle a_\alpha \cap c, b_\alpha \cap c; \alpha \in \omega_1\rangle$ forms a gap: Assume not, then there is an uncountable subset B of ω_1 such that for every $\xi \neq \eta$ in B, the set $(a_\xi \cap b_\eta \cap c) \cup (a_\eta \cap b_\xi \cap c)$ is empty, i.e. there is $p \in G$ such that

$$p \Vdash_{\boldsymbol{C}} \text{“} \forall \xi \neq \eta \in \check{B}\big((\check{a}_\xi \cap \check{b}_\eta \cap \dot{c}) \cup (\check{a}_\eta \cap \check{b}_\xi \cap \dot{c}) = \varnothing\big) \text{”}.$$

(By the countability of $\boldsymbol{C}$, we may assume that B is an object lying in the ground model by shrinking B if necessary.) We can find $\alpha \in \omega_1$ so that $A_\alpha \subseteq B$. Since G meets D_α, there is $q \in G \cap D_\alpha$, say ξ and η as witnesses for $q \in D_\alpha$. Then $p \cup q$ is in G (in fact, $p \cup q$ is just either p or q), both ξ and η are in B and

$$p \cup q \Vdash_{\boldsymbol{C}} \text{“} (\check{a}_\xi \cap \check{b}_\eta \cap \dot{c}) \cup (\check{a}_\eta \cap \check{b}_\xi \cap \dot{c}) \neq \varnothing \text{”},$$

which is a contradiction.

By the similar argument, we can show that $\langle a_\alpha \cap c, b_\alpha \cap c; \alpha \in \omega_1 \rangle$ is destructible using the dense sets E_α instead of D_α. □

2. ♣ plus $\mathsf{cof}(\mathscr{M}) = \aleph_1$.

In [**4**, Proposition 2.5], a destructible gap is constructed from $\diamondsuit$. This proof uses the enumeration of the reals of length ω_1 to show the pregap constructed by recursion is really a gap. The following proof says that we do not need the enumeration to construct a destructible gap from $\diamondsuit$ also.

The following condition is a useful notion to construct a destructible gap. This is used in the proof of [**4**, Proposition 2.5]. (But we slightly modify the original one.)

Definition 2.1 ([**20**]). We say that a pregap $(\mathscr{A}, \mathscr{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ admits finite changes if for all $\alpha \in \omega_1$, the set $a_\alpha \cap b_\alpha$ is empty and the set $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite, and for any $\beta < \alpha$ with $\beta = \eta + k$ for some $\eta \in \mathsf{Lim} \cap \alpha$ and $k \in \omega$, $H, J \in [\omega]^{<\omega}$ with $H \cap J = \varnothing$ and $i > \max(H \cup J)$ there exists $n \in \omega$ so that

$$a_{\eta+n} \cap i = H,\ a_{\eta+n} \setminus i = a_\beta \setminus i,\ b_{\eta+n} \cap i = J, \text{ and } b_{\eta+n} \setminus i = b_\beta \setminus i.$$

Theorem 2.2. *♣ and $\mathsf{cof}(\mathscr{M}) = \aleph_1$ implies the existence of a destructible gap.*

Proof. At first, we give some notation in the proof to avoid using many symbols in formulae.

For each $\alpha \in \omega_1$ and a pregap $\langle a_\xi, b_\xi; \xi < \alpha \rangle$, let $g \in 2^{\alpha \times \omega \times 2}$ be a function such that for all $\xi \in \alpha$, $a_\xi = \{n \in \omega; g(\xi, n, 0) = 1\}$ and $b_\xi = \{n \in \omega; g(\xi, n, 1) = 1\}$, that is, g is a code of this pregap. Assume that α is a countable ordinal and g is a code of an (α, α)-pregap $\langle a_\xi, b_\xi; \xi \in \alpha \rangle$ which admits finite changes, and $a_\xi \cap b_\xi = \varnothing$ and $\omega \setminus (a_\xi \cup b_\xi)$ is infinite for all $\xi \in \alpha$. Then we define a subset $\mathscr{X}(g)$ of α^ω which is a collection of members x in α^ω such that

$$\bigcup_{\xi \in \mathrm{ran}(x)} a_\xi \cap \bigcup_{\xi \in \mathrm{ran}(x)} b_\xi = \varnothing.$$

We can identify $\mathscr{X}(g)$ as the Baire space ω^ω. (By the admission of finite changes of g, any node in $\mathscr{X}(g)$ has infinitely many successors.) For each $s \in \alpha^{<\omega}$, we let $[s] := \{x \in \mathscr{X}(g); s \subseteq x\}$ and denote $\mathscr{X}^{<\omega}(g)$ as the set of $s \in \alpha^{<\omega}$ such that $[s]$ is a basic open set in $\mathscr{X}(g)$, i.e.

$$\bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \cap \bigcup_{\xi \in \mathrm{ran}(s)} b_\xi = \varnothing.$$

Let O be a dense open subset of ω^ω. O is a union of countably many basic open sets, that is, O has a code as a countable sequence of members of $\omega^{<\omega}$. In this proof, we can consider O as a dense open subset of $\mathscr{X}(g)$ using its code. Moreover we define a space $\mathscr{Y}(g)$ such that

$$\mathscr{Y}(g) := \{y \in (\alpha \times \omega)^\omega; \text{the sequence of the first coordinates of } y \text{ is in } \mathscr{X}(g) \\ \text{and the second coordinates are strictly increasing}\}.$$

$\mathscr{Y}(g)$ is also considered as the Baire space. For $y \in (\alpha \times \omega)^{\leq\omega}$ and $l < |y|$, we denote $y(l) = \langle y(l)(0), y(l)(1)\rangle$ and $\mathrm{ran}_0(y) := \{y(l)(0); l < |y|\}$. As in the definition of $\mathscr{X}^{<\omega}(g)$, we denote $\mathscr{Y}^{<\omega}(g)$ as the set of $t \in (\alpha \times \omega)^{<\omega}$ such that $[t]$ is a basic open set in $\mathscr{Y}(g)$.

Let $\langle A_\alpha; \alpha \in \omega_1\rangle$ be a $\clubsuit$-sequence. Since $\mathsf{cof}(\mathscr{M})$ is equal to the cofinality of the collection of closed nowhere dense sets (e.g. [**15**, Lemma 3.7]) and now $\mathsf{cof}(\mathscr{M}) = \aleph_1$, there exists a family $\mathscr{O}$ of open dense subsets of ω^ω of size $\aleph_1$ such that for any dense open subset O of ω^ω, there exists a member of $\mathscr{O}$ which is a subset of O. We write Lim as a class of limit ordinals. Let $\langle P_\beta; \beta \in \omega_1 \cap \mathsf{Lim}\rangle$ be a partition and f a function from ω_1 onto $\mathscr{O}$ such that for all $\beta \in \omega_1 \cap \mathsf{Lim}$,

- P_β is uncountable,
- the set $P_\beta \cap \beta$ is empty, and
- $f \restriction P_\beta$ is surjective.

We construct a pregap $\langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ with the following properties:

(1) $a_0 = b_0 = \varnothing$, $a_\alpha \cap b_\alpha = \varnothing$ and the set $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite for all $\alpha \in \omega_1$.
(2) If $\beta \leq \alpha < \omega_1$, then both $a_\beta \subseteq^* a_\alpha$ and $b_\beta \subseteq^* b_\alpha$.
(3) $\langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ admits finite changes.
(4) For each $\alpha \in \omega_1 \cap \mathsf{Lim}$, if for any $\gamma, \delta \in A_\alpha$ with $\gamma < \delta$, there is $\beta > \gamma$ such that $\delta \in P_\beta$, then there exists a strictly increasing sequence $\langle j^\alpha_k; k \in \omega\rangle$ of natural numbers such that for each $\beta \in \alpha \cap \mathsf{Lim}$ and $\gamma \in P_\beta \cap A_\alpha$, there is an infinite subset S of ω so that for any $j \in \{j^\alpha_k; k \in S\}$ and $K \subseteq j$, there exists $s \in \mathscr{X}^{<\omega}(g_\beta)$ such that $[s]$ is a subset of the dense open subset $f(\gamma)$ in $\mathscr{X}(g_\beta)$, and
$$\bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \cap K = \varnothing, \qquad \bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \setminus j \subseteq a_\alpha,$$
$$\bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \cap j \subseteq K \text{ and } \bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \setminus j \subseteq b_\alpha.$$
(5) For each $\alpha \in \omega_1 \cap \mathsf{Lim}$, if for any $\gamma, \delta \in A_\alpha$ with $\gamma < \delta$, there is $\beta > \gamma$ such that $\delta \in P_\beta$, then there exists a strictly increasing sequence $\langle i^\alpha_k : k \in \omega\rangle$ of natural numbers such that for each $\beta \in \alpha \cap \mathsf{Lim}$ and $\gamma \in P_\beta \cap A_\alpha$, there is an infinite subset T of ω so that for any $i \in \{i^\alpha_k; k \in T\}$, there exists $t \in \mathscr{Y}^{<\omega}(g_\beta)$ such that $t(0)(1) \geq i$, $[t]$ is a subset of the dense open subset $f(\gamma)$ in $\mathscr{Y}(g_\beta)$, and
$$\bigcup_{\xi \in \mathrm{ran}_0(t)} a_\xi \cap \big[i,\ t(|t|-1)(1)\big) \subseteq a_\alpha \text{ and } \bigcup_{\xi \in \mathrm{ran}_0(t)} b_\xi \cap \big[i,\ t(|t|-1)(1)\big) \subseteq b_\alpha.$$

The construction at successor stages are trivial by the property 3. Assume that α is a limit ordinal. We enumerate the set $\{\langle\beta,\gamma\rangle; \beta \in \alpha \cap \mathsf{Lim} \text{ and } \gamma \in P_\beta \cap A_\alpha\}$ by $\{\langle\beta_k,\gamma_k\rangle; k \in \omega\}$ such that each pair $\langle\beta,\gamma\rangle$ appears infinitely many often. (These sets may be empty. If so, we let all $\langle\beta_k,\gamma_k\rangle$ not be defined.) In order to construct a_α and b_α, we construct an increasing cofinal sequence $\langle\zeta_k; k\in\omega\rangle$ of α and natural numbers $i^\alpha_k = i_k$, $j^\alpha_k = j_k$, with properties that

- $\langle\zeta_k; k\in\omega\rangle \in \mathscr{X}(g_\alpha)$,
- $\beta_k < \zeta_{k-1}$ and $i_k < j_k < i_{k+1}$ for every $k\in\omega$, and
- $a_{\zeta_{k-1}} \cap j_{k-1} = a_{\zeta_k}\cap j_{k-1}$ and $b_{\zeta_{k-1}}\cap j_{k-1} = b_{\zeta_k}\cap j_{k-1}$ for every $k\in\omega$

as follows; then we define $a_\alpha := \bigcup_{k\in\omega} a_{\zeta_k}$ and $b_\alpha := \bigcup_{k\in\omega} b_{\zeta_k}$.

Assume that we have already constructed ζ_h, i_h and j_h, $h<k$, for some $k\in\omega$. (We put $i_{-1} = j_{-1} = 0$. If $\langle\beta_k,\gamma_k\rangle$'s are not defined, then we ignore the following construction and define a_α and b_α satisfying the properties 1 and 2 and for all $\mu\in\alpha$, both sets $a_\alpha\smallsetminus a_\mu$ and $b_\alpha\smallsetminus b_\mu$ are infinite.) Let $\left\{K_m; m<2^{j_{k-1}}\right\}$ enumerate $\mathscr{P}(j_{k-1})$. By the inductive hypothesis of the property 3, we pick $\eta_m\in\beta_k$ for each $m\le 2^{j_{k-1}}$ and $s_m\in\mathscr{X}^{<\omega}(g_{\beta_k})$ for each $m<2^{j_{k-1}}$ such that

- $a_{\eta_m}\cap j_{k-1} = j_{k-1}\smallsetminus K_m$ and $b_{\eta_m}\cap j_{k-1} = K_m$,
- $\langle\eta_m\rangle\subseteq s_m$ (i.e. $s_m(0)=\eta_m$),
- $[s_m]$ is a subset of the dense open subset $f(\gamma_k)$ in $\mathscr{X}(g_{\beta_k})$,
- $\max(\eta_{m+1}\cap\mathsf{Lim}) = \max\{\max(\xi\cap\mathsf{Lim}); \xi\in\operatorname{ran}(s_m)\}$, and
- $\displaystyle\bigcup_{\xi\in\operatorname{ran}(s_m)} a_\xi\smallsetminus j_{k-1} = a_{\eta_{m+1}}\smallsetminus j_{k-1}$ and $\displaystyle\bigcup_{\xi\in\operatorname{ran}(s_m)} b_\xi\smallsetminus j_{k-1} = b_{\eta_{m+1}}\smallsetminus j_{k-1}$.

(This can be done by the property 3.) Let $i_k > j_{k-1}$ be such that

$$a_{\eta_{2^{j_{k-1}}}}\smallsetminus i_k\subseteq a_{\zeta_{k-1}} \text{ and } b_{\eta_{2^{j_{k-1}}}}\smallsetminus i_k\subseteq b_{\zeta_{k-1}},$$

and then we take $\zeta'_{k-1}\in\alpha$ (by the inductive hypothesis of the property 3) so that

$$\begin{aligned}
&a_{\zeta'_{k-1}}\cap j_{k-1} = a_{\zeta_{k-1}}\cap j_{k-1},\quad a_{\zeta'_{k-1}}\cap\left[j_{k-1},\ i_k\right) = a_{\eta_{2^{j_{k-1}}}}\cap\left[j_{k-1},\ i_k\right)\\
&a_{\zeta'_{k-1}}\smallsetminus i_k = a_{\zeta_{k-1}}\smallsetminus i_k,\quad b_{\zeta'_{k-1}}\cap j_{k-1} = b_{\zeta_{k-1}}\cap j_{k-1},\\
&b_{\zeta'_{k-1}}\cap\left[j_{k-1},\ i_k\right) = b_{\eta_{2^{j_{k-1}}}}\cap\left[j_{k-1},\ i_k\right) \text{ and } b_{\zeta'_{k-1}}\smallsetminus i_k = b_{\zeta_{k-1}}\smallsetminus i_k.
\end{aligned}$$

The construction up to here is for the property 4. For the property 5, we pick $t\in\mathscr{Y}^{<\omega}(g_{\beta_k})$ such that $t(0)(1)\ge i_k$, $[t]$ is a subset of the dense open subset $f(\gamma_k)$ in $\mathscr{Y}(g_{\beta_k})$. (This can be done by the density of $f(\gamma_k)$. For the sequence $\langle\langle 0,i\rangle\rangle\in\mathscr{Y}(g_{\beta_k})^{<\omega}$, there is $t\in\mathscr{Y}(g_{\beta_k})^{<\omega}$ so that $\langle\langle 0,i\rangle\rangle\subseteq t$ and $[t]$ is a subset of $f(\gamma_k)$.) We let

$$\zeta''_{k-1} > \max\left(\operatorname{ran}_0(t)\cup\left\{\zeta'_{k-1}\right\}\right)$$

be a large enough ordinal less than α and $j_k > t(|t|-1)(1)(\ge i_k)$ be such that for all $\xi\in\operatorname{ran}_0(t)\cup\left\{\zeta'_{k-1}\right\}$,

$$a_\xi\smallsetminus j_k\subseteq a_{\zeta''_{k-1}},\quad b_\xi\smallsetminus j_k\subseteq b_{\zeta''_{k-1}} \text{ and } \left|j_k\smallsetminus\left(a_{\zeta''_{k-1}}\cup b_{\zeta''_{k-1}}\right)\right|\ge k$$

and find $\zeta_k < \alpha$ (by the inductive hypothesis of the property 3) so that

$$a_{\zeta_k} \cap i_k = a_{\zeta'_{k-1}} \cap i_k, \quad a_{\zeta_k} \cap [i_k,\ j_k) = \Big(\bigcup_{\xi \in \mathrm{ran}_0(t)} a_\xi \cup a_{\zeta'_{k-1}}\Big) \cap [i_k,\ j_k),$$
$$a_{\zeta_k} \smallsetminus j_k = a_{\zeta''_{k-1}} \smallsetminus j_k, \quad b_{\zeta_k} \cap i_k = b_{\zeta'_{k-1}} \cap i_k,$$
$$b_{\zeta_k} \cap [i_k,\ j_k) = \Big(\bigcup_{\xi \in \mathrm{ran}_0(t)} b_\xi \cup b_{\zeta'_{k-1}}\Big) \cap [i_k,\ j_k) \text{ and } b_{\zeta_k} \smallsetminus j_k = b_{\zeta''_{k-1}} \smallsetminus j_k,$$

which completes the construction.

We check that $\langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ is a destructible gap, i.e. we will prove the following two statements:

(a) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing)$.
(b) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \varnothing)$.

(We recall that (a) means that the pregap is destructible, and (b) means that the pregap is a gap.)

For a proof of (a), assume that there exists an uncountable subset X of ω_1 such that for all $\gamma \neq \delta \in X$,

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \varnothing.$$

Without loss of generality, we may moreover assume that for all $\gamma \in \omega_1$, there exists $\delta \in X$ such that

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \varnothing.$$

We note that the set

$$C := \{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha\ \exists \delta \in X \cap \alpha\ ((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \varnothing)\}$$

is club on ω_1. We construct an uncountable subset A of ω_1 as follows. Assume that we have already constructed A up to δ for some countable ordinal δ. Then there is $\beta \in C \smallsetminus (\delta + 1)$. We notice that the set

$$D_\beta := \{x \in \mathscr{X}(g_\beta); \mathrm{ran}(x) \cap X \neq \varnothing\}$$

is dense open in $\mathscr{X}(g_\beta)$. So there exists $\gamma \in P_\beta$ such that $f(\gamma)$ is contained in D_β and let $A \cap (\gamma + 1) := (A \cap \delta) \cup \{\gamma\}$ which completes the construction of A.

By the $\clubsuit$-sequence, we can find $\alpha \in C$ such that $A_\alpha \subseteq A$. By the construction of A, A_α satisfies the first assumption of the property 4. We take any $\eta \in X \smallsetminus \alpha$. Then there is a natural number m such that

$$a_\alpha \setminus m \subseteq a_\eta \text{ and } b_\alpha \setminus m \subseteq b_\eta.$$

We fix any $\gamma \in A_\alpha$. Then by the construction of A, for some $\beta \in \alpha$, $\gamma \in P_\beta$ and $f(\gamma)$ is a subset of D_β. Applying the property 4 for $\langle \alpha, \beta, \gamma \rangle$, we can find $j \geq m$ which satisfies the conclusion of the property 4. Then we can find $s \in \mathscr{X}^{<\omega}(g_\beta)$ such that $[s]$ is a subset of $f(\gamma)$ and

$$\bigcup_{\xi \in \operatorname{ran}(s)} a_\xi \cap b_\eta \cap j = \varnothing, \qquad \bigcup_{\xi \in \operatorname{ran}(s)} a_\xi \setminus j \subseteq a_\alpha,$$

$$\bigcup_{\xi \in \operatorname{ran}(s)} b_\xi \cap j \subseteq b_\eta \cap j \text{ and } \bigcup_{\xi \in \operatorname{ran}(s)} b_\xi \setminus j \subseteq b_\alpha.$$

By the definition of D_β, there exists $\xi \in \operatorname{ran}(s) \cap X$. (Because if $\operatorname{ran}(s) \cap X = \varnothing$, then let $\zeta \in \operatorname{ran}(s)$ and $x \in \beta^\omega$ such that $s \subseteq x$ and $x(i) = \zeta$ for all $i \geq |s|$, and then $x \in ([s] \cap \mathscr{X}(g_\beta)) \setminus D_\beta$, which contradicts an assumption of s. The point is that for any $s_0, s_1 \in \alpha^{<\omega}$, the intersection $[s_0] \cap [s_1]$ is empty if s_0 and s_1 are incomparable, otherwise $[s_0] \cap [s_1]$ is either $[s_0]$ or $[s_1]$.) But then

$$(a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi) = \varnothing$$

which is a contradiction and completes the proof of (a).

A proof of (b) is similar to one of (a), but we will use the property 5 instead of 4. We assume that there exists an uncountable subset Y of ω_1 such that for all $\gamma \neq \delta \in Y$,

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \varnothing.$$

Without loss of generality, we may moreover assume that for all $\gamma \in \omega_1$, there exists $\delta \in Y$ such that

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \varnothing.$$

We note again that the set

$$C' := \{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha\ \exists \delta \in Y \cap \alpha\ ((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \varnothing)\}$$

is club on ω_1. We construct an uncountable subset B of ω_1 as follows. Assume that we have already constructed B up to δ for some countable ordinal δ. Then there is $\beta \in C' \setminus (\delta + 1)$. We define the subset E_β of $\mathscr{Y}(g_\beta)$ such that $y \in E_\beta$ if there exists $\xi \in Y$ so that for some $l \in \omega$, either

$$a_\xi \cap \Bigl(\bigcup_{\zeta \in \operatorname{ran}_0(y)} b_\zeta \Bigr) \cap \bigl[y(l), y(l+1)\bigr) \neq \varnothing$$

or

$$\Big(\bigcup_{\zeta\in\mathrm{ran}_0(y)} a_\zeta\Big)\cap b_\xi\cap\big[y(l),y(l+1)\big)\neq\varnothing.$$

We note that E_β is dense open in $\mathscr{Y}(g_\beta)$, hence there exists $\gamma\in P_\beta$ such that $f(\gamma)$ is contained in E_β and let

$$B\cap(\gamma+1):=(B\cap\delta)\cup\{\gamma\}$$

which completes the construction of B.

By the ♣-sequence, we can find $\alpha\in C'$ such that $A_\alpha\subseteq B$. By the construction of B, A_α satisfies the first assumption of the property 4. We take any $\eta\in Y\setminus\alpha$. Then there is a natural number m such that

$$a_\alpha\setminus m\subseteq a_\eta \text{ and } b_\alpha\setminus m\subseteq b_\eta.$$

We take any $\gamma\in A_\alpha$, then by the construction of B, for some $\beta\in\alpha$, $\gamma\in P_\beta$ and $f(\gamma)$ is a subset of E_β. Applying the property 5 for $\langle\alpha,\beta,\gamma\rangle$, we can find $i\geq m$ which satisfies the conclusion of the property 5. Then we can find $t\in\mathscr{Y}^{<\omega}(g_\beta)$ such that $t(0)(1)\geq i$, $[t]$ is a subset of $f(\gamma)$ and

$$\Big(\bigcup_{\zeta\in\mathrm{ran}_0(t)} a_\zeta\Big)\cap\big[i,\ t(|t|-1)(1)\big)\subseteq a_\alpha$$

and

$$\Big(\bigcup_{\zeta\in\mathrm{ran}_0(t)} b_\zeta\Big)\cap\big[i,\ t(|t|-1)(1)\big)\subseteq b_\alpha.$$

By the definition of E_β, there exists $\xi\in Y$ such that for some $l<|t|-1$, either

$$a_\xi\cap\Big(\bigcup_{\zeta\in\mathrm{ran}_0(t)} b_\zeta\Big)\cap\big[t(l)(1),\ t(l+1)(1)\big)\neq\varnothing$$

or

$$\Big(\bigcup_{\zeta\in\mathrm{ran}_0(t)} a_\zeta\Big)\cap b_\xi\cap\big[t(l)(1),\ t(l+1)(1)\big)\neq\varnothing.$$

But then, since $t(l)(1)\geq i$,

$$(a_\xi\cap b_\eta)\cup(a_\eta\cap b_\xi)\neq\varnothing$$

which is a contradiction and completes the proof of (b). □

3. Remarks.

Since we can show that the Lévy collapse of ω_1 to ω adds a destructible gap, there exists a destructible gap in Shelah' model of $\clubsuit + \neg\mathsf{CH}$ [**14**, Chapter 1, 7.4 Theorem]. As a corollary of Theorem 1.1, in the model of [**6**, Theorem 3.8], there exists a destructible gap as for a Suslin tree. Moreover as a corollary of Theorem 2.2, in the extension with the countable support iteration of Sacks forcing of length ω_2 ([**11**, Corollary 6.14]), and in the extension with the countable support product of Sacks forcing over $\diamondsuit$ ([**8**, Theorem IV. 4]), $\clubsuit$ holds and there exist a Suslin tree and a destructible gap. So it seems to be the following question still open.

QUESTION 3.1. *Is it consistent with* ZFC *that* $\clubsuit$ *holds and there are no destructible gaps?*

As a corollary of two theorems, if $\clubsuit$ holds and all gaps are indestructible, then the inequality

$$\mathsf{cov}(\mathscr{M}) = \aleph_1 < \mathsf{cof}(\mathscr{M})$$

holds. This is as same as the case of the existence of a Suslin tree.

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