J. Math. Soc. Japan
Vol. 57, No. 4, 2005

On the nilpotency of rational *H*-spaces

By Shizuo KAJI

(Received Jan. 24, 2004)
(Revised Dec. 18, 2004)

Abstract. In [**BG**], it is proved that the Whitehead length of a space Z is less than or equal to the nilpotency of ΩZ. As for rational spaces, those two invariants are equal. We show this for a 1-connected rational space Z by giving a way to calculate those invariants from a minimal model for Z. This also gives a way to calculate the nilpotency of an homotopy associative rational H-space.

1. Introduction.

We assume that all spaces in this paper are connected based spaces with the homotopy types of CW-complexes and all maps are based maps.

In [**Ark**], the generalized Whitehead product $[f,g] : \Sigma(X \wedge Y) \to Z$ was defined, where $f : \Sigma X \to Z, g : \Sigma Y \to Z$. Moreover Arkowitz showed that for given space Z, the following three conditions are equivalent.

(i) ΩZ is homotopy commutative.
(ii) For any spaces X, Y, all the generalized Whitehead products vanish.
(iii) For any spaces X, Y and any maps f, g, there exists a map H which gives the following homotopy commutative diagram:

$$\begin{array}{ccc} \Sigma X \vee \Sigma Y & \xrightarrow{f \vee g} & Z \\ {\scriptstyle incl.}\downarrow & \nearrow_{H} & \\ \Sigma X \times \Sigma Y. & & \end{array}$$

As for rational spaces, suspension spaces decompose to wedges of spheres. Therefore the third is equivalent to the condition that all (ordinary) Whitehead products of Z vanish. In other words, for a rational space Z, $\mathbf{WL}(Z) = 0$ if and only if $\mathbf{nil}(\Omega Z) = 0$. Here $\mathbf{WL}(Z)$ and $\mathbf{nil}(\Omega Z)$ stand for the Whitehead length of Z and the nilpotency of ΩZ, respectively (see Definitions 4.2 and 4.10).

In this paper, we prove that $\mathbf{WL}(Z)$ is equal to $\mathbf{nil}(\Omega Z)$ for a simply connected rational space Z by comparing these invariants with another numerical one, which is called the d_1-depth of a space. We note that the fact $\mathbf{WL}(Z)$ is equal to $\mathbf{nil}(\Omega Z)$ is proved in [**Sal**] without assuming the 1-connectedness of Z.

In the rest of this paper, we assume that all spaces are nilpotent connected based spaces with the homotopy types of rational CW-complexes whose homologies are of finite type, and all maps are based maps. We also assume that all vector spaces and algebras

---

2000 *Mathematics Subject Classification.* 55P45, 55P62.
Key Words and Phrases. H-Spaces, nilpotency, rational homotopy theory.

are defined over the rational field $\boldsymbol{Q}$.

An outline for the paper is as follows. We prove some facts on H-spaces in §2 using the correspondence between homotopy types of rational H-spaces and isomorphism classes of the Sullivan models whose differentials vanish. In §3, we construct a minimal KS-model for a path space fibration and investigate some properties of it for the following sections. In §4, we investigate the nilpotency of the loop space ΩX for a space X. To this end, we define a rational homotopy invariant d_1-depth(X) for a minimal model for X. We prove that this invariant is equal to the Whitehead length of X and the nilpotency of ΩX. Note that d_1-depth$(X) = \mathbf{WL}(X)$ is also proved in [**KY**, Appendix]. The nilpotency of homotopy associative H-spaces is given in §5.

2. Definitions and basic results.

Definition 2.1. A Sullivan model $(\bigwedge V, d)$ is a differential graded algebra(DGA) with the following properties [**FHT**].

- $\bigwedge V$ is the free graded commutative algebra on a graded vector space $V = \{V^i\}_{i \geq 1}$.
- V admits a filtration $V = \bigcup_{i=0}^{\infty} V_i$, where $0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots$ such that $d : V_i \to \bigwedge V_{i-1}$.

A Sullivan model is called a minimal model if its differential maps into decomposables. We say that an element $x \in \bigwedge V$ has the *word length* n if $x \in \bigwedge^n V$, and that an element $x \in \bigwedge V$ has the *degree* i if $x \in (\bigwedge V)^i$. We denote by $|x|$ the degree of x.

Definition 2.2. Let $(\bigwedge V, d)$ be a Sullivan model with $d = d_0 + d_1 + \cdots$ where $d_i : V \to \bigwedge^{i+1} V$. We call d_0 the *linear part of* d, and d_1 the *quadratic part of* d. We say that $(\bigwedge V, d)$ is *coformal* if $d = d_1$.

Definition 2.3. An H-space (X, μ) is a based space X with a homotopy class of map $\mu : X \times X \to X$ which is homotopic to the identity when restricted to each factor. We call μ a *multiplication*.

Let X be a connected rational H-space. It is known that X has a minimal Sullivan model whose differential vanishes. Since $H^*(X, \boldsymbol{Q})$ is free, its minimal model is isomorphic to $H^*(X, \boldsymbol{Q})$. Hence the Sullivan representative of a map f between connected rational H-spaces is uniquely determined. We denote the Sullivan representative of f by f^*. Note that $f^* \cong H^*(f)$. Let $(\bigwedge V, 0)$ be a minimal model for X and $x_1, x_2, \ldots$ be a basis of V such that $0 < |x_1| \leq |x_2| \leq \cdots$. Homotopy classes of multiplications correspond bijectively to maps of graded algebras $\mu^* : \bigwedge V \to \bigwedge V \otimes \bigwedge V$ of the form

$$\mu^*(x_i) = x_i \otimes 1 + 1 \otimes x_i + \sum_j P_{ij} \otimes Q_{ij}, \quad \mu^*(1) = 1 \otimes 1,$$

where P_{ij}, Q_{ij} are polynomials in $x_k (k < i)$ having positive degrees. For a Sullivan model, a map in the above form is also called a *multiplication*. We call x_i is *primitive* when $\mu^*(x_i) = x_i \otimes 1 + 1 \otimes x_i$.

We derive bijective correspondence between the homotopy category of connected rational H-spaces and isomorphism classes of connected augmented graded commutative

Hopf algebras with finite generators in each degree. In the rest of this section, we prove some properties on inverses of H-spaces using this correspondence.

Definition 2.4. A *left inverse* $\lambda : X \to X$ and a *right inverse* $\rho : X \to X$ of an H-space (X, μ) are maps such that the compositions

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\lambda \times 1} X \times X \xrightarrow{\mu} X,$$

and

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times \rho} X \times X \xrightarrow{\mu} X$$

are null homotopic, where $\Delta : X \to X \times X$ is the diagonal map.

Theorem 2.5 ([**Jam**]). *An H-space (X, μ) has a left inverse λ and a right inverse ρ unique up to homotopy.*

Proof. A proof of general case is found in [**Jam**]. In rational case, we can calculate a Sullivan representative of inverses from a Sullivan representative of the multiplication.

By the definition of the left inverse, we have

$$\Delta^*(\lambda^* \otimes 1)\mu^*(x_i) = \lambda^*(x_i) + x_i + \sum_j \lambda^*(P_{ij})Q_{ij} = 0.$$

Since P_{ij} is a polynomial in $x_k (k < i)$, by induction on i we have $\lambda^*(x_i) = -x_i - \sum_j \lambda^*(P_{ij})Q_{ij}$ and $\rho^*(x_i) = -x_i - \sum_j P_{ij}\rho^*(Q_{ij})$. □

Corollary 2.6. *$\lambda\rho$ and $\rho\lambda$ are homotopic to the identity.*

Proof. By induction on i, we have

$$\rho^*\lambda^*(x_i) = x_i + \sum_j P_{ij}\rho^*(Q_{ij}) - \sum_j \rho^*\lambda^*(P_{ij})\rho^*(Q_{ij}) = x_i.$$ □

Corollary 2.7. *The following three conditions are equivalent.*

(i) $\lambda^2 \simeq 1$.
(ii) $\lambda \simeq \rho$.
(iii) $\rho^2 \simeq 1$.

Proof. It is clear from the previous Corollary that $\lambda \simeq \rho$ when $\lambda^2 \simeq 1$.

We show $\lambda^2 \simeq 1$ when $\lambda \simeq \rho$ by induction on i.

Applying λ^* to both sides of the equality

$$\lambda^*(x_i) = -x_i - \sum_j \lambda^*(P_{ij})Q_{ij},$$

we have

$$\begin{aligned}(\lambda^*)^2(x_i) &= -\lambda^*(x_i) - \sum_j (\lambda^*)^2(P_{ij})\lambda^*(Q_{ij}) \\ &= -\rho^*(x_i) - \sum_j P_{ij}\rho^*(Q_{ij}) \\ &= x_i.\end{aligned}$$

This completes the proof. □

In [**AOS**], an H-space with the left inverse having finite order other than two is given. Next Proposition states there is no such a rational H-space.

Proposition 2.8. *For any positive integer n, $\lambda^n \not\simeq 1$ when $\lambda \not\simeq \rho$.*

Proof. When n is odd, the term of $\lambda^*(x)$ having word length one is $-x$. Hence $\lambda^n \not\simeq 1$.

We assume n is even. Let i be the least number such that $(\lambda^*)^2(x_i) \neq x_i$. We write

$$(\lambda^*)^2(x_i) = x_i + P,$$

where P is a polynomial in $x_k (k < i)$. Then we have

$$(\lambda^*)^4(x_i) = x_i + P + (\lambda^*)^2(P) = x_i + 2P,$$

and

$$(\lambda^*)^{2n}(x_i) = x_i + \frac{n}{2}P.$$ □

Definition 2.9. An H-space (X, μ) is *homotopy associative* if $\mu(\mu\times 1) = \mu(1\times\mu) \in [X\times X\times X, X]$.

An Hopf algebra $(\bigwedge V, \mu^*)$ is *associative* if $(\mu^*\times 1)\mu^* = (1\times\mu^*)\mu^*$. We use the term "associative" after the manner in [**AOS**] so that homotopy associativity of H-spaces corresponds to associativity of Hopf algebras.

Proposition 2.10. *Homotopy associativity implies $\lambda \simeq \rho$.*

Proof. Since $\Delta^*(\lambda^*\otimes 1)\mu^*(P_{ij}) = 0$ and $\Delta^*(\lambda^*\otimes 1)\mu^*(x_i) = 0$, it follows that

$$\begin{aligned}&\Delta^*(\Delta^*\otimes 1)(\lambda^*\otimes 1\otimes\rho^*)(\mu^*\otimes 1)\mu^*(x_i) \\ &\quad = \Delta^*(\Delta^*\otimes 1)(\lambda^*\otimes 1\otimes\rho^*)(\mu^*\otimes 1)\Big(x_i\otimes 1 + \sum P_{ij}\otimes Q_{ij} + 1\otimes x_i\Big) \\ &\quad = \Delta^*(\Delta^*\otimes 1)(\lambda^*\otimes 1\otimes\rho^*)(1\otimes 1\otimes x_i) \\ &\quad = \rho^*(x_i).\end{aligned}$$

On the other hand, we have

$$\Delta^*(\Delta^* \otimes 1)(\lambda^* \otimes 1 \otimes \rho^*)(1 \otimes \mu^*)\mu^*(x_i) = \lambda^*(x_i).$$

Since $(1 \otimes \mu^*)\mu^* = (\mu^* \otimes 1)\mu^*$, it follows that $\lambda^*(x_i) = \rho^*(x_i)$. □

REMARK 2.11. The converse of Proposition 2.10 doesn't hold. We give a finite H-space that $\lambda \simeq \rho$ while it is not homotopy associative.

We consider the following Hopf algebra which is not associative:

$$\bigwedge(x, y, z), \quad |x| = 11, |y| = 3, |z| = 5,$$

where the elements y and z are primitive and $\mu^*(x) = x \otimes 1 + 1 \otimes x + y \otimes yz$. We see $\lambda^*(x) = \rho^*(x) = -x$.

PROPOSITION 2.12. *If $H^*(X)$ is finite dimensional, then the following two conditions are equivalent.*

(i) $\lambda^*(x_i) = -x_i$.
(ii) $\lambda \simeq \rho$.

PROOF. From Corollary 2.7, we have $\lambda \simeq \rho$ when $\lambda^*(x_i) = -x_i$.

Assume that $\lambda \simeq \rho$. Since $H^*(X)$ is finite dimensional, $|x_i|$ must be odd. Let i be the least integer such that $\lambda^*(x_i) \neq -x_i$. We write

$$\mu^*(x_i) = x_i \otimes 1 + 1 \otimes x_i + Q_1 \otimes Q_2,$$

where Q_1, Q_2 are polynomials in $x_j (j < i)$ having positive degrees. Then we have

$$\lambda^*(x_i) = -x_i - P,$$

where we denote $\lambda^*(Q_1)Q_2$ by P. For dimensional reasons, P has degree greater than 3 and odd word length. Since P is a polynomial in $x_j (j < i)$, it follows that $\lambda^*(P) = -P$. Therefore

$$(\lambda^*)^2(x_i) = \lambda^*(-x_i + P) = x_i - 2P.$$

The statement follows from Corollary 2.7. □

REMARK 2.13. Proposition 2.12 does not always hold if $H^*(X)$ is infinite dimensional. Consider the Sullivan model $(\bigwedge(x, y), 0)$, where $|x| = 4$ and $|y| = 2$. Define its multiplication μ^* such that $\mu^*(x) = x \otimes 1 + 1 \otimes x + y \otimes y$ and $\mu^*(y) = y \otimes 1 + 1 \otimes y$. We see that $\lambda^*(x) = \rho^*(x) = -x + y^2$.

3. Model for the path space fibration.

Let X be a 1-connected space. In order to investigate the multiplication of ΩX by means of a minimal model for X, we first recall a KS-model for the path space fibration

$\Omega X \to PX \to X$ (see [**TO**, Remark 5.5]).

Let $(\bigwedge V, d)$ be a minimal model for X. Then the following is a (not minimal) Sullivan model for the free path space of X:

$$\Big(\bigwedge(V \oplus V' \oplus \delta V'), d\Big), \quad dv' = \delta v', d\delta v' = 0,$$

where $V' = \{v' | v \in V\}(|v'| = |v| - 1)$ and $\delta V' = \{\delta v' | v \in V\}$. We define a derivation I on the Sullivan model by $I(v) = v', I(v') = 0 = I(\delta v')$. Then the automorphism $e^{I \circ d + d \circ I}$ of the model for X^I is defined by

$$e^{I \circ d + d \circ I} = 1 + d \circ I + \sum_{n=1} \frac{(I \circ d)^n}{n!}.$$

We denote $\sum_{n=1} \frac{(I \circ d)^n}{n!} v$ by $\Omega(v)$.

Let $\hat{v} = e^{I \circ d + d \circ I} v$ and $\hat{V} = \{\hat{v} | v \in V\}$ then there exists a DGA $(\bigwedge(V \oplus V' \oplus \hat{V}), d)$ such that

$$\Big(\bigwedge(V \oplus V' \oplus \delta V'), d\Big) \cong \Big(\bigwedge(V \oplus V' \oplus \hat{V}), d\Big).$$

Lemma 3.1. *We define a DGA as follows*:

$$\Big(\bigwedge V \otimes \bigwedge V', D\Big), \quad Dv = dv, \quad Dv' = v - \tau\Omega(v),$$

where $\tau : (\bigwedge(V \oplus V' \oplus \hat{V}), d) \to (\bigwedge V \otimes \bigwedge V', D)$ *is a DGA map defined by* $\tau(v) = 0, \tau(\hat{v}) = v, \tau(v') = v'$. *Then this DGA has the following properties.*

(i) $D^2 = 0$. (*D is actually differential.*)
(ii) $\mathbf{Im}(D) \subset \bigwedge^{\geq 1} V \otimes \bigwedge V'$.
(iii) $\tau\Omega(v) \equiv \tau(\sum_n \frac{1}{n!}(I \circ d_1)^n v)$, *where '$\equiv$' means the components in* $V \otimes \bigwedge V'$ *are equal.*

Proof.

(i) We see $D^2(v) = D^2(v') = 0$ for $v \in V, v' \in V'$.

$$\begin{aligned} D^2(v) &= d^2(v) = 0 \\ D^2(v') &= dv - D\tau\Omega(v) \\ &= \tau d(\hat{v} - \Omega(v)) \\ &= \tau d(v + \delta v') = 0. \end{aligned}$$

(ii) First we observe $\Omega(v) \in \bigwedge(V^{<|v|} \oplus V'^{<|v'|} \oplus \delta V'^{<|\delta v'|})$. By induction on $|v|$, we show $\tau\Omega(v) \in \bigwedge^{\geq 1} V \otimes \bigwedge V'$, which is equivalent to $\Omega(v) \in \bigwedge^{\geq 1}(V \oplus \hat{V}) \otimes \bigwedge V'$.

Since $\delta v' = \hat{v} - v - \Omega(v)$, by induction, it is enough to show $\Omega(v) \in \bigwedge^{\geq 1}(V \oplus \delta V') \otimes \bigwedge V'$. Since $d : \bigwedge^n(V \oplus \delta V') \otimes \bigwedge V' \to \bigwedge^{\geq n+1}(V \oplus \delta V') \otimes \bigwedge V'$ and $I : \bigwedge^n(V \oplus \delta V') \otimes \bigwedge V' \to \bigwedge^{\geq n-1}(V \oplus \delta V') \otimes \bigwedge V'$, it follows that

$$(I \circ d)^n : \bigwedge^{n}(V \oplus \delta V') \otimes \bigwedge V' \to \bigwedge^{\geq n}(V \oplus \delta V') \otimes \bigwedge V'.$$

Therefore we get $\Omega(v) \in \bigwedge^{\geq 1}(V \oplus \delta V') \otimes \bigwedge V'$.

(iii) We observe $\tau^{-1}(V \otimes \bigwedge V') \subset \delta V' \otimes \bigwedge V' \subset (\bigwedge^1(V \oplus \delta V') \otimes \bigwedge V', d)$. We extend the derivation d_1 of $(\bigwedge V, d)$ to a derivation of $\bigwedge(V \oplus V' \oplus \delta V')$ by the canonical way. Since $I \circ d - I \circ d_1 : \bigwedge^n(V \oplus \bigwedge \delta V') \otimes \bigwedge V' \to \bigwedge^{\geq n+1}(V \otimes \bigwedge \delta V') \otimes \bigwedge V'$, it follows

$$(I \circ d)^n - (I \circ d_1)^n : \bigwedge^{1}\left(V \oplus \bigwedge \delta V'\right) \otimes \bigwedge V' \to \bigwedge^{\geq 2}\left(V \otimes \bigwedge \delta V'\right) \otimes \bigwedge V'.$$

Therefore, $\Omega(v) \equiv \sum_n \frac{1}{n!}(I \circ d_1)^n v$, where '$\equiv$' means the components in $\delta V' \otimes \bigwedge V'$ are equal. This completes the proof. $\square$

Proposition 3.2. *The following is a minimal model for the path space fibration $X \leftarrow PX \leftarrow \Omega X$:*

$$(\bigwedge V, d) \xrightarrow{\quad i \quad} (\bigwedge V \otimes \bigwedge V', D) \xrightarrow{\epsilon \otimes 1} (\bigwedge V', 0),$$

where i is the inclusion and ϵ is the augmentation.

Proof. Minimality follows from previous Lemma. We have to show $H^{\geq 1}(\bigwedge V \otimes \bigwedge V', D) = 0$. We consider the spectral sequence associated to the word length filtration. The E_1-term has the form $H^*(\bigwedge(V \oplus V'), D_0)$, and the cochain complex $(\bigwedge(V \oplus V'), D_0)$ is obviously acyclic. $\square$

4. Nilpotency of loop spaces.

Definition 4.1. The *commutator* φ of an associative H-space (X, μ) is the composition of the following maps:

$$X \times X \xrightarrow{\Delta \times \Delta} X \times X \times X \times X \xrightarrow{1 \times t \times 1} X \times X \times X \times X \xrightarrow{\lambda \times \lambda \times 1 \times 1} X \times X \times X \times X \xrightarrow{\mu \times \mu} X \times X \xrightarrow{\mu} X,$$

where $t : X \times X \to X \times X$ is the map defined by $t(x, y) = (y, x)$. Thus the Sullivan representative φ^* is expressed as the composition

$$(\bigwedge V, 0) \xrightarrow{\mu^*} (\bigwedge V, 0) \otimes (\bigwedge V, 0) \xrightarrow{\mu^* \otimes \mu^*} (\bigwedge V, 0)^{\otimes 4} \xrightarrow{\lambda^* \otimes \lambda^* \otimes 1 \otimes 1} (\bigwedge V, 0)^{\otimes 4} \xrightarrow{1 \otimes t^* \otimes 1} (\bigwedge V, 0)^{\otimes 4} \xrightarrow{\Delta^* \otimes \Delta^*} (\bigwedge V, 0) \otimes (\bigwedge V, 0),$$

where $t^* : v_1 \otimes v_2 \mapsto (-1)^{|v_1||v_2|} v_2 \otimes v_1$.

As for the definition of the *n-fold commutator*, $\varphi_0 = 1, \varphi_1 = \varphi$ and $\varphi_n = \varphi \circ (1 \times \varphi_{n-1})$ $(n \geq 2)$.

Definition 4.2. The *nilpotency* of an associative H-space (X, μ) is the least n such that φ_{n+1} is null homotopic. We denote it by $\mathbf{nil}X$.

For an Hopf algebra $\bigwedge V$ with an associative multiplication μ^*, $\mathbf{nil}(\bigwedge V, \mu^*)$ is defined by the least n such that φ^*_{n+1} is 0.

We investigate the nilpotency of the loop space ΩX for a 1-connected space X. To this end, we consider the path space fibration $\Omega X \to PX \xrightarrow{p} X$. The following is also a fibration:

$$\Omega X \times \Omega X \longrightarrow PX \times \Omega X \xrightarrow{p \circ p_L} X,$$

where p_L is the projection onto the left factor.

We constructed in the previous section a minimal model for the path space fibration.

$$\left(\bigwedge V, d\right) \to \left(\left(\bigwedge V \otimes \bigwedge V'\right), D\right) \to \left(\bigwedge V', 0\right),$$

where $Dv = dv, Dv' = v - \tau\Omega(v)$. We regard $(\bigwedge V', 0)$ as an associative Hopf algebra with the multiplication μ^* induced from the multiplication of ΩX.

The action $\phi : PX \times \Omega X \to PX$ gives the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p} & PX & \longleftarrow & \Omega X \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \phi}\uparrow & & {\scriptstyle \mu}\uparrow \\ X & \xleftarrow{p \cdot p_L} & PX \times \Omega X & \longleftarrow & \Omega X \times \Omega X. \end{array}$$

Then we can choose a Sullivan representative for ϕ which makes the following diagram commutative:

$$\begin{array}{ccccc} (\bigwedge V, d) & \xrightarrow{incl} & (\bigwedge V \otimes \bigwedge V', D) & \xrightarrow{\varepsilon \otimes 1} & (\bigwedge V', 0) \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \phi^*}\downarrow & & {\scriptstyle \mu^*}\downarrow \\ (\bigwedge V, d) & \xrightarrow{incl} & (\bigwedge V \otimes \bigwedge V', D) \otimes (\bigwedge V', 0) & \xrightarrow{\varepsilon \otimes 1 \otimes 1} & (\bigwedge V', 0) \otimes (\bigwedge V', 0). \end{array}$$

For $x' \in V'$, we write

$$\mu^*(x') = x' \otimes 1 + 1 \otimes x' + \sum_i \Phi_i \otimes \Psi_i$$

and

$$\phi^*(x') = 1 \otimes \mu^*(x') + \sum_i A_i \otimes B_i \otimes C_i,$$

where $\Phi_i, \Psi_i \in \bigwedge^{\geq 1} V', A_i \in \bigwedge^{\geq 1} V, \quad B_i, C_i \in \bigwedge V'$. Then we obtain

$$\phi^* Dx' = x \otimes 1 \otimes 1 - \phi^* \tau \Omega(x)$$

and

$$(D \otimes 1)\phi^* x' = Dx' \otimes 1 + \sum_i D\Phi_i \otimes \Psi_i + \sum_i (DA_i \otimes B_i \otimes C_i + (-1)^{|A_i|} A_i \wedge DB_i \otimes C_i).$$

From above commutative diagram, $\phi^* Dx' = (D \otimes 1)\phi^* x'$. This equation is the key to the rest of this section.

Suppose that the graded vector space V has a filtration $\{V_i\}$ such that

$$V = \bigcup V_i, \quad V_0 \subset V_1 \subset \cdots, \quad d_1 : V_i \to \bigwedge V_{i-1}.$$

This gives a filtration of the graded vector space V' by $(V')_n = (V_n)'$. Then we have the following Lemma.

LEMMA 4.3.

$$\mu^*(x') - x' \otimes 1 - 1 \otimes x' \in \bigwedge V'_n \otimes \bigwedge V'_n, \quad x' \in V'_{n+1}.$$

PROOF. It follows from Lemma 3.1 that the components of $\phi^* Dx'$ in $V_{n+1} \otimes \bigwedge V' \otimes \bigwedge V'$ is $x \otimes 1 \otimes 1$. On the other hand, components of $(D \otimes 1)\phi^* x'$ in $V_{n+1} \otimes \bigwedge V' \otimes \bigwedge V'$ lies in $x \otimes 1 \otimes 1 + \sum_i D_0 \Phi_i \otimes \Psi_i$. Hence $\sum_i D_0 \Phi_i \otimes \Psi_i$ doesn't contain terms in $V_{n+1} \otimes \bigwedge V' \otimes \bigwedge V'$, that is, Φ_i does not contain an element of V'_{n+1} as its factor.

Considering the other path space fibration with converse start point and end point, we get Ψ_i does not contain an element of V'_{n+1} as its factor. □

COROLLARY 4.4. *If $(\bigwedge V, 0)$ is a minimal model for an H-space (X, μ), then all elements of V' are primitive.*

PROOF. We can choose a filtration of V so that $V_0 = V$. □

REMARK 4.5. The converse of Corollary 4.4 is not true. Consider a minimal model for CP^2:

$$\Big(\bigwedge(x, y), \quad dx = 0, dy = x^3\Big), \quad |x| = 2, |y| = 5.$$

For dimensional reasons, we see that the elements x' and y' are primitive in $H^*(\Omega CP^2)$.

We give an upper bound of $\mathbf{nil}\Omega X$.

LEMMA 4.6. *For a minimal model $(\bigwedge V, 0)$ for an associative H-space (X, μ),*

$$\mathbf{Im}\varphi^* \in \bigwedge^{\geq 1} V \otimes \bigwedge^{\geq 1} V.$$

Proof. For $v \in V$ we write $\mu^*(v) = v \otimes 1 + 1 \otimes v + \sum_i P_i \otimes Q_i$. Then the components of $\varphi^*(v)$ in $\bigwedge V \otimes 1$ is $\lambda^*(v) \otimes 1 + v \otimes 1 + \sum_i \lambda^*(P_i)Q_i \otimes 1 = 0$. Similarly we have that the components of $\varphi^*(v)$ in $1 \otimes \bigwedge V$ is zero. □

Proposition 4.7. *If X has a minimal model $(\bigwedge V, d)$ with a filtration $\{V_i\}_{i\leq n}$ of V such that $V = \bigcup_{i\leq n} V_i$, $0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots$ and $d_1 : V_i \to \bigwedge V_{i-1}$, then $\mathbf{nil}\Omega X \leq n$.*

Proof. We show that $\varphi^*_{i+1}x' = 0$ in $\bigwedge V'_{\leq i}$ by induction on i. We only have to show this for the generators.

When $i = 0$, by Corollary 4.4 we have $\varphi^* = 0$. Suppose that $\varphi^*_{i+1}x' = 0$ if $x' \in V'_{<i}$. For $x' \in V'_i$, by Lemma 4.6, we can write $\varphi^*x' = A \otimes B$, where $A, B \in \bigwedge^{\geq 1} V'$. By Lemma 4.3, if φ^*x' would contain generators in $V_{\geq i}$, it must be x'. However this is impossible for the dimensional reasons. □

Next we investigate a lower bound of $\mathbf{nil}\Omega X$.

Proposition 4.8. *If $d_1x = \sum_i u_i \wedge v_i$, $x, u_i, v_i \in V$, we have*

$$\sum_i \pi(\Phi_i) \otimes \pi(\Psi_i) = -\sum_i \big((-1)^{|u_i|}u'_i \otimes v'_i + (-1)^{(|u_i|+1)|v_i|}v'_i \otimes u'_i\big),$$

where $\pi : \bigwedge^{\geq 1} V' \to V'$ is the quotient.

Proof. We compare the components in $V \otimes 1 \otimes V'$ of the equation $\phi^*(Dx') = (D \otimes 1)\phi^*(x')$. From the proof of Lemma 3.1,

$$\begin{aligned}\phi^*(Dx') &\equiv \phi^*\bigg(-\frac{1}{2}\sum_i((-1)^{|u_i|}u_i \wedge v'_i + u'_i \wedge v_i)\bigg)\\ &\equiv -\frac{1}{2}\sum_i \big((-1)^{|u_i|}u_i \wedge \phi^*(v'_i) + (-1)^{(|u_i|+1)|v_i|}v_i \wedge \phi^*(u'_i)\big)\\ &\equiv -\frac{1}{2}\sum_i \big((-1)^{|u_i|}u_i \otimes 1 \otimes v'_i + (-1)^{(|u_i|+1)|v_i|}v_i \otimes 1 \otimes u'_i\big),\end{aligned}$$

where '$\equiv$' means the components in $V \otimes 1 \otimes V'$ are equal. On the other hand, since $DA_i \otimes B_i \otimes C_i, A_i \wedge DB_i \otimes C_i \in \bigwedge^{\geq 2} V \otimes \bigwedge V' \otimes \bigwedge V'$, the component of $(D \otimes 1)\phi^*(x')$ in $V \otimes 1 \otimes V'$ is $\sum D_0\pi(\Phi_i) \otimes 1 \otimes \pi(\Psi_i)$.

Comparing these completes the proof. □

We calculate the first terms of the commutator of ΩX from the quadratic part of the differential of a minimal model for X.

Proposition 4.9. *If $d_1x = \sum_i u_i \wedge v_i$ then we have*

$$\varphi^* x' \equiv -\sum_i \big((-1)^{|u_i|}(u_i' \otimes v_i' + (-1)^{(|u_i|+1)|v_i|} v_i' \otimes u_i')\big),$$

where '≡' means the components in $V' \otimes V'$ are equal.

Proof. Word length argument gives the component of $\varphi^* x'$ in $V' \otimes V'$ is determined by the component of μ^* in $V' \otimes V'$. Direct calculation using the result of previous Proposition completes the proof. □

Definition 4.10. The Whitehead length of X, written $WL(X)$, is the least integer n such that all $(n+1)$-fold Whitehead products vanish.

Now we consider a lower bound of the nilpotency.

Lemma 4.11. *Let $(\bigwedge V, d)$ be a minimal model for X. The least number n such that the component of $\varphi^*_{n+1}(x')$ in ${V'}^{\otimes n+2}$ vanishes, equals $\mathbf{WL}(X)$.*

Proof. Let $(\bigwedge W_i, d)(1 \leq i \leq n+2)$ be a minimal model for $S^{m_i}(m_i \geq 1)$. We observe that the natural quasi-isomorphisms $(\bigwedge W_i, d) \to H^*(S^{m_i})$ define the bijection

$$\begin{aligned}
[S^{m_1} \times \cdots \times S^{m_{n+2}}, \Omega X]_0 &\cong \Big[\Big(\bigwedge V', 0\Big), \Big(\bigwedge W_1, d\Big) \otimes \cdots \otimes \Big(\bigwedge W_{n+2}, d\Big)\Big] \\
&\cong \Big[\bigwedge V', H^*(S^{m_1}) \otimes \cdots \otimes H^*(S^{m_{n+2}})\Big] \\
f &\mapsto H^*(f).
\end{aligned}$$

Since $\mathrm{Im}\varphi^*_{n+1} \subset \bigwedge^{>n+1} V'$, we have $\varphi^*_{n+1} \equiv 0$ in ${V'}^{\otimes n+2}$ if and only if $H^*(f_1) \otimes \cdots \otimes H^*(f_{n+2})\varphi^*_{n+1} \equiv 0$ in $[\bigwedge V', H^*(S^{m_1}) \otimes \cdots \otimes H^*(S^{m_{n+2}})]$ for any maps $f_i : S^{m_i} \to \Omega X$. By the bijection above, this is equivalent to the Lemma. □

Definition 4.12. d_1-depth of a minimal model $(\bigwedge V, d)$ is the least number n such that $V_n = V_{n+1}$, where

$$V_{-1} = 0, \quad V_n = \Big\{v \in V | d_1 v \in \bigwedge V_{n-1}\Big\}, \quad V = \bigcup_i V_i.$$

If such an integer doesn't exist, we define d_1-depth$(\bigwedge V, d) = \infty$.

Remark 4.13. d_1-depth is a rational homotopy invariant. Indeed, any DGA map between minimal models $f^* : (\bigwedge V, d) \to (\bigwedge W, d)$ preserves the filtration mentioned above, that is, $f^* : \bigwedge V_n \to \bigwedge W_n$. Hence, if f^* is an isomorphism, then $f^* : \bigwedge(V_n \setminus V_{n-1}) \to \bigwedge(W_n \setminus W_{n-1})$. Therefore we define d_1-depth of a space X by d_1-depth of its minimal model.

Remark 4.14. There is a coformal space X_{cf} such that $\pi_*(\Omega X)$ is isomorphic to $\pi_*(\Omega X_{cf})$ as a Lie algebra. Such a space is called the associated coformal space of X. Topologically, d_1-depth(X) can be considered as the height of the generalized Postnikov tower of X_{cf}.

Theorem 4.15. *For a 1-connected space X we have $\mathbf{WL}(X) = \mathbf{nil}(\Omega X) = d_1$-depth$(X)$.*

Proof. By Lemma 4.11, we have $\mathbf{WL}(X) \leq \mathbf{nil}(\Omega X)$. By Proposition 4.7, we have $\mathbf{nil}(\Omega X) \leq d_1$-depth$(X)$. We show $\mathbf{WL}(X) \geq d_1$-depth(X).

Let $(\bigwedge V, d)$ be a Sullivan model for X and $V = \{V_i\}_{i \leq n}$ be the filtration which gives d_1-depth. We denote the component of φ_i^* in $V^{\otimes i+1}$ by $\bar{\varphi}_i^*$. We show that $\bar{\varphi}_i^*(x') \neq 0$ for $x' \in V_{i+1} \setminus V_i$ by induction on i. Let $\{v_j\}$ be a basis of V. We can write $\bar{\varphi}^*(x') = \sum_j v_j \otimes U_j$, where $U_j \in V$. It follows from Proposition 4.9 that there exists an integer j such that $U_j \in V_i \setminus V_{i-1}$. By induction hypothesis, $\bar{\varphi}_i^*(x') = \sum_j v_j \otimes \bar{\varphi}_{i-1}^*(U_j) \neq 0$. This completes the proof. □

Example 4.16. We give a space X with $\mathbf{nil}(\Omega X) = n$.
Define a Sullivan model $(\bigwedge\{V_i\}_{i \leq n}, d)$ as follows.

$$\begin{aligned} V_i = \{x_{\alpha_i}\}, &\quad V_0 = \{x_{\alpha_0}, x_0\} \\ d : V_i &\to \bigwedge V_{i-1} \\ x_{\alpha_i} &\mapsto x_{\alpha_{i-1}} \wedge x_0, (1 \leq i \leq n) \\ x_{\alpha_0} &\mapsto 0 \\ x_0 &\mapsto 0, \end{aligned}$$

where $|x_0|$ is odd. By Theorem 4.15, $\mathbf{nil}(\bigwedge V') = n$.

5. Nilpotency of homotopy associative *H*-spaces.

In this section, we investigate the nilpotency of a connected homotopy associative H-space G.

Let L be a connected graded Lie algebra. We regard L as a differential graded Lie algebra(DGL) with zero differential. First, we recall the functor $\mathscr{C}^*$ [**FHT**, §23], which sends L to a minimal model for a coformal space Z such that $\pi_*(\Omega Z) \cong L$ as a graded Lie algebra. We denote the functor DGA $\to$ DGL taking the primitive space by $\mathscr{P}$. By Theorem 4.5 of [**Qui**, Appendix B], $\mathscr{C}^*\mathscr{P}H_*(G)$ is a minimal model for a coformal space Z such that $\pi_*(\Omega Z) \cong \pi_*(G)$. Taking the universal enveloping algebra and the dual, we have an isomorphism of Hopf algebras $H^*(G) \cong H^*(\Omega Z)$. Therefore by Theorem 4.15, we have

Theorem 5.1.

$$\mathbf{nil}(G) = d_1\text{-}depth(\mathscr{C}^*\mathscr{P}H_*(G)).$$

In other words,

$$\mathbf{nil}G = \mathbf{nil}\pi_*(G),$$

where $\pi_*(G)$ is considered as a Lie algebra equipped with the Samelson product.

REMARK 5.2. If G is homotopy commutative, then $\mathscr{P}H_*(G)$ is abelian. Therefore, $\mathscr{C}^*\mathscr{P}H_*(G)$ has zero differential. This implies that there is an H-equivalence $G \simeq \Omega^2 Y$ for some space Y.

References

[Ark] M. Arkowitz, The generalized Whitehead product, Pacific J. Math., **12** (1962), 7–23.

[AOS] M. Arkowitz, H. Oshima and J. Strom, The Inverse of an H-space, Manuscripta Math., **108** (2002), 399–408.

[BG] I. Berstein and T. Ganea, Homotopical nilpotency, Illinois J. Math., **5** (1961), 99–130.

[FHT] Y. Félix, S. Halperin and J. Thomas, Rational Homotopy Theory, Graduate Texts in Math., **205**, Springer, New York, 2001.

[Jam] I. M. James, On H-spaces and their homotopy groups, Quart. J. Math., **11** (1960), 161–179.

[KY] K. Kuribayashi and T. Yamaguchi, A rational splitting of a based mapping space, preprint.

[Qui] D. Quillen, Rational homotopy theory, Ann. of Math., **90** (1969), 205–295.

[Sal] P. Salvatore, Rational homotopy nilpotency of self-equivalences, Topology Appl., **77** (1997), 37–50.

[TO] A. Tralle and J. Oprea, Symplectic manifolds with no Kähler structure, Lecture Notes in Math., **1661**, Springer, Berlin, 1997.

Shizuo KAJI

Department of Mathematics, Faculty of Science
Kyoto University
Kitashirakawa, Kyoto
Japan