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# On Alexander polynomials of torus curves

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**Abstract.** Let $p$ and $q$ be integers such that $p > q \geq 2$ and $q$ divides $p$. Let $\varphi(q)$ be the Euler number of $q$. We exhibit a Zariski $\varphi(q)$-ple, distinguished by the Alexander polynomial, whose curves are tame torus curves of type $(p, q)$, with $q$ smooth irreducible components of degree $p$, and one single singular point topologically equivalent to the Brieskorn-Pham singularity $v^q + u^{qp^2} = 0$.

## Introduction.

Let $C \subset \boldsymbol{P}^2$ be a projective plane curve defined over the complex numbers. By the *Hyperplane section theorem* of Zariski [**29**] (completed by Hamm and Lê [**7**]), the study of the fundamental group $\pi_1(\boldsymbol{P}^2 \backslash C)$ lies at the heart of an understanding of the fundamental group of the complement of a hypersurface of any dimension $V \subset \boldsymbol{P}^N$. Namely, by induction, $\pi_1(\boldsymbol{P}^N \setminus V)$ is isomorphic to the fundamental group of the complement of the plane curve $C$ obtained by intersecting $V$ with a generic linear space of dimension two.

Let us assume that the curve $C$ is given in affine coordinates by $f(x, y) = 0$. For any integer $n$, one may consider the cyclic branched coverings of $\boldsymbol{P}^2$ defined by the surface $W_n : t^n - f(x, y) = 0$ in $\boldsymbol{P}^3$. Zariski proved that simple homological invariants of $W_n$ provide non-trivial invariants of the complement to the branch locus, linking the algebraic-geometric and knot-theoretic situations together, as follows.

Let $\xi \in C$ be a singular point. Let us assume for simplicity that the germ $(C, \xi)$ has only one branch. Let $S^3_\varepsilon$ be the 3-sphere centered at $\xi$ and with radius $\varepsilon$. For suitably small $\varepsilon$, the intersection $K = C \cap S^3_\varepsilon$ is an iterated torus knot (called an *algebraic knot*), completely determined by the Puiseux pairs of the singularity $(C, \xi)$. It is classical in knot theory to consider, for all integer $n$, the compact 3-manifold $M^3_n$, which is a cyclic $n$-fold covering of $S^3_\varepsilon$ ramified along $K$. The manifold $M^3_n$ is intimately related to the *Alexander polynomial* of the knot $K$ (see [**1**]). Zariski studied algebraic knots of the singularities and 3-manifolds $M^3_n$ in [**30**] and observed that there is a formula connecting the Alexander polynomials of knots and links of the singularities of the curve $C$ with the first Betti number $b_1(M^3_n)$ of $M^3_n$.

In the spirit of Zariski's works, A. Libgober introduced the *global* Alexander polynomial $\Delta_C(t)$ of the curve $C$, with respect to a general line, whose main property is the following (see [**10**]): The (global) Alexander polynomial $\Delta_C(t)$ of $C$ divides the product $\Delta_1(t) \cdots \Delta_k(t)$ of the (local) Alexander polynomials of the singularities of $C$.

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It is in general very difficult to calculate the fundamental group $\pi_1(\boldsymbol{P}^2 \setminus C)$, but the Alexander polynomial $\Delta_C(t)$ is an invariant of $\pi_1(\boldsymbol{P}^2 \setminus C)$, which is easier to be computed. A. Libgober announced in [**11**] how to compute $\Delta_C(t)$ by the data of the degree of $C$, the topological type of the singularities of $C$ and their relative positions, without passing by the calculation of $\pi_1(\boldsymbol{P}^2 \setminus C)$. The first published complete proof of the results of [**11**] is due to Loeser and Vaquié [**12**], using R. Randell's interpretation [**23**] of the Alexander polynomial $\Delta_C(t)$ as the characteristic polynomial of the monodromy acting on the Milnor fiber of the affine cone of $C$, and H. Esnault's description [**5**] of the mixed Hodge structure of this Milnor fiber. In the present work, we will use the computational method which can be extracted from the results of H. Esnault [**5**] and E. Artal [**2**], in terms of data coming from the resolution of the singularities of the curve $C$.

The first non-trivial example studied extensively by Zariski is the case of a sextic with six cusps. If $n$ is a multiple of 6, the irregularity $q(W_n) = b_1(W_n)/2$ of the associated cyclic branched covering $W_n$ is 1 or 0, depending on whether or not the six cusps lie on a conic. If $q(W_n) = 1$, the sextic is a *torus curve* of type $(3,2)$, with fundamental group the free product of cyclic groups $\boldsymbol{Z}/2\boldsymbol{Z} * \boldsymbol{Z}/3\boldsymbol{Z}$ and Alexander polynomial $t^2 - t + 1$. If $q(W_n) = 0$, the fundamental group of the sextic is abelian (assuming the irreducibility of the moduli of such sextics), isomorphic to the cyclic group $\boldsymbol{Z}/6\boldsymbol{Z}$, and with Alexander polynomial 1.

This striking topological phenomenon has been studied further by many authors (see e.g. [**2**], [**20**], [**24**], [**25**]), and leads E. Artal to the following formulation [**2**]. A pair of reduced curves $\{C', C''\}$ is called a *Zariski pair* if the curves have the same degree, the configurations of local singularities are the same up to topological equivalence, and there exist regular neighbourhoods $N(C')$ and $N(C'')$ of $C'$ and $C''$ respectively, such that the pairs $(N(C'), C')$ and $(N(C''), C'')$ are homeomorphic but the pairs $(\boldsymbol{P}^2, C')$ and $(\boldsymbol{P}^2, C'')$ are not homeomorphic. Similarly, $k$-ple of plane curves $\{C_1, \dots, C_k\}$ is called a *Zariski $k$-ple* if each pair $\{C_i, C_j\}$ is a Zariski pair for $1 \leq i < j \leq k$.

Let $p$ and $q$ be integers such that $p \geq q \geq 2$. A *torus curve* of type $(p,q)$ and degree $pq$ is by definition a curve $C$ which admits an equation of the form

$$(f_q(x,y))^p + (f_p(x,y))^q = 0,$$

where $f_p(x,y)$ and $f_q(x,y)$ are polynomials of degree $p$ and $q$ respectively. The geometry of such a curve $C$ is strongly related to the geometry of the intersection of the curves $C_p : f_p(x,y) = 0$ and $C_q : f_q(x,y) = 0$. A torus curve is said to be *tame* if its singular locus coincides with the intersection of the associated curves $C_p$ and $C_q$. The case of tame generic torus curves, i.e. when the associated curves $C_p$ and $C_q$ intersect transversally at $pq$ distinct points, is classical (see [**27**], [**15**]). The study of the other extreme case, i.e. when $C_p$ and $C_q$ intersect at one single point, is the purpose of the present paper. One of our main results may be stated as follows (the definition of Zariski multiple is straightforward ).

Theorem. *Let $p$ and $q$ be integers such that $p > q \geq 2$ and $q$ divides $p$. Let $\varphi(q)$ be the Euler number of $q$ and let $\nu \geq 1$ be a divisor of $q$. Let $f_\nu(x,y)$ and $f_p(x,y)$ be polynomials of degree $\nu$ and $p$ respectively. Assume that the torus curve $\Gamma_\nu$ : $((f_\nu(x,y))^{q/\nu})^p + (f_p(x,y))^q = 0$ satisfies the following conditions*:

(1) $\mathrm{Sing}(\Gamma_\nu) = \{f_p(x,y)=0\} \cap \{f_\nu(x,y)=0\} = \{(0,0)\}$;
(2) $\{f_p = 0\}$ *and* $\{f_\nu = 0\}$ *are smooth at* $(0,0)$, *and the intersection multiplicity of* $\{f_\nu = 0\}$ *with its tangent line at* $(0,0)$ *is equal to* $\nu$.

*Then the family of torus curves* $\{\Gamma_\nu\}_{\nu|q}$ *is a Zariski* $\varphi(q)$*-ple, distinguished by the Alexander polynomial, whose curves have* $q$ *smooth irreducible components of degree* $p$*, and one single singular point topologically equivalent to the Brieskorn-Pham singularity* $B_{q,qp^2} : v^q + u^{qp^2} = 0$.

The content of our paper is the following. We state our results in Section 1, where we compare two extreme cases of tame torus curves: the generic case and the maximal contact case, and eventually exhibit from the latter Zariski multiples. In Section 2, we prove our main theorem by an explicit calculation of Alexander polynomial. The calculation reduces to a Diophantine equation, which can be solved in an elementary way. In Section 3, we give the proof of the description of the moduli space of torus curves of maximal contact and maximal flex order. Finally, Section 4 is devoted to a new proof of the classical formula of the Alexander polynomial of generic tame torus curves (see [**15**], [**19**]).

Notations. For any rational number $a/b$, let us note [a/b] its integral part defined by $a/b \ -1 < [a/b] \leq a/b$. For any integer $m \geq 1$, let $\varphi(m)$ be the number of strictly positive divisors of $m$ (the Euler number of $m$).

## 1. Statements of the results.

### 1.1. Alexander polynomial.

Let $X$, $Y$, $Z$ be homogeneous coordinates of the complex projective plane $\boldsymbol{P}^2$ and let $\{Z = 0\}$ be the line at infinity. Let $x = X/Z$, $y = Y/Z$ be affine coordinates of the affine plane $\boldsymbol{C}^2 = \boldsymbol{P}^2 \setminus \{Z = 0\}$. Let $C \subset \boldsymbol{P}^2$ be a reduced projective plane curve of degree $d$. Let us assume that $\{Z = 0\}$ is general position with respect to $C$, i.e. that $\{Z = 0\}$ intersects $C$ at $d$ distinct points. One associates with the fundamental group $\pi_1(\boldsymbol{P}^2 \setminus (C \cup \{Z = 0\})) = \pi_1(\boldsymbol{C}^2 \setminus C)$ an infinite cyclic covering whose Alexander polynomial is by definition the (generic) Alexander polynomial of the curve $C$ (see [**10**]).

The definition of the Alexander polynomial of a curve $C$ does not depend on the choice of a general line for $C$, but for simplicity we will consider projective curves for which the line at infinity is in general position, and compute Alexander polynomial relatively to $\{Z = 0\}$.

Let us recall the following classical example. Let $p$ and $q$ be integers such that $p \geq q \geq 2$. The Brieskorn-Pham singularity $C : x^p + y^q = 0$ has $r = \gcd(p,q)$ branches. The corresponding link $K = C \cap S^3_\varepsilon$ consists of $r$ torus knots of type $(p/r, q/r)$. The Alexander polynomial of the curve $C$ is equal to the product of cyclotomic polynomials $\Delta_{p,q}(t)$ given by the formula (see [**4**], [**22**] and also [**21**])

$$\Delta_{p,q}(t) = \frac{(t^{pq/r} - 1)^r \, (t-1)}{(t^p - 1)(t^q - 1)}.$$

### 1.2. Torus curves.

Let $p$ and $q$ be integers such that $p \geq q \geq 2$ and let $d \geq 2$ be a common multiple of $p$ and $q$. Let $m = d/p$ and $n = d/q$. A torus curve $C$ of type $(p, q; d)$ is classically a reduced projective curve of degree $d$, given in affine coordinates by an equation of the form

$$(f_m(x, y))^p + (f_n(x, y))^q = 0,$$

where $f_m(x, y)$ and $f_n(x, y)$ are polynomials of degree $m$ and $n$ respectively. Different torus expressions $f_m^p + f_n^q$ and $g_m^p + g_n^q$ may define the same curve. To clear this ambiguity, let us attach a fixed torus decomposition, each time we consider a torus curve, as stated in the following definition.

Definition 1.2.1. Let $p$ and $q$ be integers such that $p \geq q \geq 2$ and let $d \geq 2$ be a common multiple of $p$ and $q$. Let $m = d/p$ and $n = d/q$. A torus curve $C$ of type $(p, q; d)$ consists of the data of two fixed polynomials $f_m(x, y)$ and $f_n(x, y)$ of degree $m$ and $n$ respectively, such that $(f_m(x, y))^p + (f_n(x, y))^q$ is a reduced polynomial of degree $d$. We associate to a torus curve of type $(p, q; d)$ the curves $C_m : f_m(x, y) = 0$ and $C_n : f_n(x, y) = 0$.

### 1.3. Singular locus.

Let $C$ be a torus curve of type $(p, q; d)$. Any intersection point of the associated curves $C_m$ and $C_n$ is a singular point of $C$. Conversely, suppose that we are given polynomials $f_m(x, y)$ and $f_n(x, y)$ of degree $m$ and $n$ respectively, such that $mp = nq = d$. They define a pencil of torus curves of type $(p, q; d)$, with $(\lambda, \mu) \in \boldsymbol{P}^1$, as follows

$$\lambda\ (f_m(x, y))^p + \mu\ (f_n(x, y))^q = 0.$$

By Bertini theorem for a pencil (see e.g. [**6**, p. 137]), the curves of the pencil of torus curves of type $(p, q; d)$ defined by the polynomials $f_m(x, y)$ and $f_n(x, y)$ are, except a finite number of curves of the pencil, smooth away from the base locus, which is nothing but the intersection of the curves $C_m : f_m(x, y) = 0$ and $C_n : f_n(x, y) = 0$.

Definition 1.3.1. A torus curve of type $(p, q; d)$ is said to be *tame* if its singular locus coincides with the intersection of the associated curves $C_n$ and $C_m$.

Let $C$ be a torus curve of type $(p, q; d)$. Let $\xi \in C_n \cap C_m$ be a singular point of $C$. If the curve $C_n$ is smooth at $\xi$ then the topological type of the singularity $(C, \xi)$ is determined by the intersection multiplicity of the curves $C_n$ and $C_m$ at $\xi$. More precisely one has the following lemma which extends [**17**, Lemma 23, p. 264].

Lemma 1. *Let $C$ be a torus curve of type $(p, q; d)$. Let $\xi \in C_n \cap C_m$ such that $C_n$ is non-singular at $\xi$ and let $\iota$ be the intersection multiplicity $I(C_n, C_m; \xi)$. Then the singularity $(C, \xi)$ is topologically equivalent to the Brieskorn-Pham singularity $B_{q,p\iota}$ : $v^q + u^{p\iota} = 0$.*

Proof. The assumption implies that there exist local complex analytic coordinates

$(u, v)$ such that $f_n = v$ and $f_m \equiv cu^\iota$ modulo $(v, u^{\iota+1})$, for some constant $c \in \mathbf{C}^*$. Then

$$f_m^p + f_n^q = c^p u^{p\iota} + v^q + \sum_{qi+p\iota j > \iota pq} a_{i,j} u^i v^j.$$

As the Newton principal part is Newton non-degenerate, the topological type of the germ at the origin is determined by the Newton principal part and does not depend on the terms with higher degree with respect to the Newton filtration (see [**8**], [**26**] combined with [**9**]). □

### 1.4. Generic torus curves.

Let us recall some known results about torus curves.

Definition 1.4.1. A torus curve $C$ of type $(p, q; d)$ is said to be *generic* if the associated curves $C_m$ and $C_n$ intersect transversally at $mn$ distinct points. The singular points of $C$ given by the intersection of the curves $C_m$ and $C_n$ are topologically equivalent to the Brieskorn-Pham singularity (see Lemma 1)

$$B_{q,p} : v^q + u^p = 0.$$

The fundamental group of the complement of a generic tame torus curve was computed by O. Zariski [**27**] for the case of sextics, i.e. $(p, q; d) = (3, 2; 6)$ and by M. Oka [**15**], [**19**] for general $(p, q; d)$. For example, when the integers $p$ and $q$ are coprime and $d = pq$, the fundamental group $\pi_1(\boldsymbol{P^2} \setminus C)$ is isomorphic to the free product $\boldsymbol{Z}/p\boldsymbol{Z} * \boldsymbol{Z}/q\boldsymbol{Z}$ (see [**15**]). When $\{Z = 0\}$ is in general position with respect to $C$, the fundamental groups $\pi_1(\boldsymbol{C}^2 \setminus C)$ and $\pi_1(\boldsymbol{P^2} \setminus C)$ are related by the central extension (see [**28**]), and the Alexander polynomial is obtained using a result of A. Nemethi (see [**19**]).

Theorem 1 ([**15**], [**19**]). *Let $C$ be a generic tame torus curve of type $(p, q; d)$. The Alexander polynomial $\Delta_C(t)$ of the curve $C$ is equal to $\Delta_{p,q}(t)$.*

Remark 1.4.2. The Alexander polynomial of a generic tame torus curve of type $(p, q; d)$ depends only on the integers $p$ and $q$, and not on the degree $d$.

### 1.5. Torus curves of maximal contact.

Our interest in the present paper is the other extreme case.

Definition 1.5.1. We say that a torus curve $C$ of type $(p, q; d)$ is a torus curve of *maximal contact* if the associated curves $C_m$ and $C_n$ satisfy the following conditions.

(1) The intersection of the curves $C_m$ and $C_n$ is a single point, which is noted by $\xi_0$;
(2) The curve $C_n$ is smooth at the point $\xi_0$.

Let us remark that the intersection multiplicity $I(C_n, C_m; \xi_0)$ is equal to $nm$, by Bézout theorem, and the singularity $(C, \xi_0)$ is topologically equivalent to the Brieskorn-Pham singularity (see Lemma 1)

$$B_{q,qn^2} \ : \ v^q + u^{qn^2} = 0.$$

Example 1.5.2. Let $f_m(x,y) = y - x^m$ and $f_n(x,y) = y^n + y - x^m$ such that $m \leq n$ and $mp = nq = d$. The curves $C_m : f_m(x,y) = 0$ and $C_n : f_n(x,y) = 0$ are irreducible, smooth at the origin, and intersect only at the origin, with intersection multiplicity $mn$. The curves of the pencil of torus curves of type $(p,q;d)$ defined by the polynomials $f_m(x,y)$ and $f_n(x,y)$ are, except a finite number of curves of the pencil, tame torus curves of maximal contact.

1.5.3. *Tangent cone.* The reduced tangent cone at the singular point $\xi_0$ of a torus curve of maximal contact is given by the following proposition.

Proposition 1. *Let $C$ be a torus curve of type $(p,q;d)$ and of maximal contact, and let $\xi_0$ be the intersection of the associated curves $C_n : f_n(x,y) = 0$ and $C_m : f_m(x,y) = 0$. The tangent line of $C_n$ at $\xi_0$, the reduced tangent cone of $C_m$ at $\xi_0$ and the reduced tangent cone of the curve $C$ at $\xi_0$ all coincide.*

Proof. The proof is left to the reader. □

### 1.6. Moduli spaces.

A projective plane curve of degree $d$, which does not contain the line at infinity, is given by a polynomial $f(x,y)$ of degree $d$, in affine coordinates $x = X/Z$, $y = Y/Z$. The number of monomials $x^i y^j$ such that $0 \leq i+j \leq d$ is equal to $\binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$.

Definition 1.6.1. Let us define the *moduli space* $\mathscr{M}(d)$ of reduced projective plane curves of degree $d$, which do not contain the line at infinity $\{Z = 0\}$, as the set of reduced polynomials $f(x,y)$ of degree $d$, considered as a Zariski-open subset of $\boldsymbol{C}^{\binom{d+2}{2}}$. The topology of $\mathscr{M}(d) \subset \boldsymbol{C}^{\binom{d+2}{2}}$ is the induced transcendental topology.

One may consider spaces of plane curves of degree $d$, with a prescribed (up to topological equivalence) configuration of singularities.

Definition 1.6.2. Let $\mathscr{M}(B_{q,qn^2}; d)$ be the submoduli space of $\mathscr{M}(d)$ which consists of curves with a single singular point topologically equivalent to the Brieskorn-Pham singularity

$$B_{q,qn^2} \ : \ v^q + u^{qn^2} = 0.$$

The germ $v^q + u^{qn^2} = 0$ at the origin is Newton non-degenerate. The Milnor number is $(q-1)(qn^2-1)$.

The moduli space $\mathscr{M}(B_{q,qn^2}; d)$ has several connected components, but we do not know how many in general. However, for the case of sextics, i.e. when $(p,q;d) = (3,2;6)$, the moduli space $\mathscr{M}(B_{2,18}; 6)$ has four irreducible components (see [**2**], [**14**], [**13**]).

### 1.7. Deformation with constant Milnor number.

Let us recall the following classical fact. It follows from Lê D. T. [**9**] that an analytic family of germs of plane curves with constant Milnor number is equisingular, and, by the theory of equisingularity of Zariski, has a simultaneous resolution. This observation leads, via a partition of unity argument, to a geometric proof of the invariance of the embedded topological type of projective plane curves, which stay in a fixed connected component of a space of plane curves with a prescribed configuration of singularities.

### 1.8. Torus curves of maximal contact and flex points.

Let us first fix a notation for the space of torus curves of maximal contact.

DEFINITION 1.8.1. Let $\mathscr{M}_{torus}^{\max}(p,q;d)$ be the submoduli space of $\mathscr{M}(d)$ which consists of torus curves of type $(p,q;d)$ and of maximal contact. For $C \in \mathscr{M}_{torus}^{\max}(p,q;d)$, let us note the single intersection point of the associated curves $C_m$ and $C_n$ by $\xi_0$.

REMARK 1.8.2. Let us remark that by Lemma 1, the germ of a curve $C$ at the singular point $\xi_0$, if $C \in \mathscr{M}_{torus}^{\max}(p,q;d)$, is topologically equivalent to the Brieskorn-Pham singularity $B_{q,qn^2} : v^q + u^{qn^2} = 0$. Also, by Bertini theorem for a pencil of curves, the subspace of the tame curves of the moduli space $\mathscr{M}_{torus}^{\max}(p,q;d)$ is a Zariski-open subspace, which is contained in $\mathscr{M}(B_{q,qn^2};d)$.

For $C \in \mathscr{M}_{torus}^{\max}(p,q;d)$, let $\ell$ be the reduced tangent cone of the associated curve $C_m$ at $\xi_0$. The moduli space $\mathscr{M}_{torus}^{\max}(p,q;d)$ splits into disjoint subspaces, according to the contact between the curve $C_m$ and the line $\ell$ at $\xi_0$, as follows.

DEFINITION 1.8.3. Let $C$ be a curve in $\mathscr{M}_{torus}^{\max}(p,q;d)$.

(1) We say that the associated curve $C_m$ has *flex-order* $s$ at $\xi_0$ if the intersection multiplicity $I(C_m,\ell;\xi_0)$ is equal to $s$. Note that $2 \le s \le m$. We use the notation

$$\mathrm{flex}(C_m,\xi_0) = I(C_m,\ell;\xi_0).$$

(2) For $2 \le s \le m$, let $\mathscr{N}_s(p,q;d)$ be the moduli space of torus curves $C$ of type $(p,q;d)$ and of maximal contact, such that $\mathrm{flex}(C_m,\xi_0) = s$. The moduli space of torus curves of maximal contact decomposes into

$$\mathscr{M}_{torus}^{\max}(p,q;d) = \bigcup_{2\le s\le m} \mathscr{N}_s(p,q;d).$$

The description of the submoduli space $\mathscr{N}_s(p,q;d)$ seems to be extremely difficult in general. However the submoduli space $\mathscr{N}_m(p,q;d)$ has a simpler structure. For each divisor $\nu \ge 1$ of the integer $m$, the moduli space $\mathscr{N}_m(p,q;d)$ contains a subspace $\mathscr{N}_m^\nu(p,q;d)$ that we are able to describe easily. Let us define $\mathscr{N}_m^\nu(p,q;d)$ the subset of $\mathscr{N}_m(p,q;d)$ such that $f_m(x,y)$ takes the form $f_m(x,y) = f_\nu(x,y)^{m/\nu}$, for a polynomial $f_\nu(x,y)$ of degree $\nu$, and the curve $C_\nu : f_\nu(x,y) = 0$ is smooth at $\xi_0$, and $\mathrm{flex}(C_\nu,\xi_0) = \nu$. Note that $f_m$ is not reduced if $\nu \ne m$. It is easy to see that $\mathscr{N}_m(p,q;d) = \cup_{\nu|m}\mathscr{N}_m^\nu(p,q;d)$.

To describe each subspace, we take the following normal slice. Let $\mathscr{N}_m^\nu(p,q;d)(\xi_0)$ be the subset of $\mathscr{N}_m^\nu(p,q;d)$ such that $\xi_0 = (0,0)$ and $\{y - \delta_{\nu,1}\ x = 0\}$ is the tangent line of $C_\nu$ at $(0,0)$, where $\delta_{\nu,1} = 1$ if $\nu = 1$, or 0 otherwise. One can easily observe that $PGL(3,\boldsymbol{C}) \cdot \mathscr{N}_m^\nu(p,q;d)(\xi_0) = \mathscr{N}_m^\nu(p,q;d)$. The slices $\mathscr{N}_m^\nu(p,q;d)(\xi_0)$ are described by the following lemma.

Lemma 2. *Let $\nu \geq 1$ be a divisor of $m$. The slice $\mathcal{N}_m^{\nu}(p,q;d)(\xi_0)$ is described as follows.*

(1) *For each curve $C : f_m^p + f_n^q = 0$ in $\mathcal{N}_m^{\nu}(p,q;d)(\xi_0)$ with $f_m = f_\nu^{m/\nu}$, there exist a unique expression*

$$f_n(x,y) = \sum_{i=1}^{[n/\nu]} r_i(x,y) f_\nu(x,y)^i + c_0\, y^n,$$

*where $c_0$ is a non-zero complex number and $r_i(x,y), i = 1,\dots,[n/\nu]$ are polynomials with*

$$\deg r_i(x,y) \leq n - i\nu,\ \deg_x r_i(x,y) < \nu, \quad \text{and } r_1(0,0) \neq 0.$$

*In particular, $f_n(x,y) \equiv c_0 y^n$ modulo $f_\nu(x,y)$. Conversely for a given curve $f_\nu = 0$ of degree $\nu$ which is smooth at $\xi_0$ and $I(f_\nu, y - \delta_{\nu,1}x; \xi_0) = \nu$, and for any $f_n$ which is given by the above equality, the curve $C :\ f_m^p + f_n^q = 0$, $(f_m = f_\nu^{m/\nu})$ belongs to the slice $\mathcal{N}_m^{\nu}(p,q;d)(\xi_0)$.*

(2) *The dimension of the space of $f_m$'s is equal to the number $\alpha$ of monomials $x^a y^b$ such that $0 \leq a+b \leq \nu, (a,b) \neq (0,0),\dots,(\nu-1,0)$. The dimension of the space of $f_n$'s (when $f_\nu$ is fixed) is equal to the sum $\beta$ of $\beta_i$, for $i = 1,\dots,[n/\nu]$, where $\beta_i$ is the number of monomials $x^a y^b$ such that $0 \leq a+b \leq n - i\nu$, $a < \nu$. The space $\mathcal{N}_m^{\nu}(p,q;d)(\xi_0)$ is identified with a Zariski-open subset of $\boldsymbol{C}^{(\alpha-2)+(\beta-1)} \times (\boldsymbol{C}^*)^4$.*

(3) *The tame curves of $\mathcal{N}_m^{\nu}(p,q;d)$ have $\gcd(d/\nu, q)$ irreducible components. In particular, tame curves in $\mathcal{N}_m^{m}(p,q;d)$ have $\gcd(p,q)$ irreducible components.*

The proof is given in §3. For $\nu = 1$, we have chosen $y - x = 0$ as the reduced tangent cone of $f_m = 0$ for the simplicity of the description.

Example 1.8.4. Let $m \leq n$ such that $mp = nq = d$ and let $\nu \geq 1$ be divisor of $m$. Let $f_\nu(x,y) = y - x^\nu$, and $f_m(x,y) = (f_\nu(x,y))^{m/\nu}$, and $f_n(x,y) = y^n + y - x^\nu$. The curves $C_\nu : f_\nu(x,y) = 0$ and $C_n : f_n(x,y) = 0$ are irreducible, smooth at the origin, and intersect only at the origin, with intersection multiplicity $\nu n$. Furthermore $\mathrm{flex}(C_m, O) = \nu m/\nu = m$. The curves of the pencil of torus curves of type $(p,q;d)$ defined by the polynomials $f_m(x,y) = (y - x^\nu)^{m/\nu}$ and $f_n(x,y) = y^n - y - x^\nu$ are, except a finite number of curves of the pencil, tame torus curves which belong to the moduli space $\mathcal{N}_m^{\nu}(p,q;d)$.

Remark 1.8.5. By Lemma 2, (2), and §1.7, the embedded topological type of the tame torus curves in the subspace $\mathcal{N}_m^{\nu}(p,q;d)$ is completely determined by the embedded topological type of the tame torus curves of Example 1.8.4.

### 1.9. Main Results.

Theorem 2. *Let $\Delta(t)$ be the Alexander polynomial of a tame curve in $\mathcal{N}_m^{m}(p,q;d)$. Then $\Delta(t)$ is equal to*

$$\Delta_{p,q}(t) = \frac{(t^{pq/r}-1)^r(t-1)}{(t^p-1)(t^q-1)},$$

*which is equal to the Alexander polynomial of a generic tame torus curve of type* $(p, q; d)$ *if* $p > q$ *(see* §1 *and Theorem* 1*).*
*The case* $p = q$ *is exceptional and* $\Delta(t)$ *is given by* $\frac{(t^{p^2}-1)^{p-1}(t-1)}{(t^p-1)}$.

For the proof, we may consider the tame torus curves of the pencil of torus curves defined by the polynomials $f_m(x, y) = y - x^m$ and $f_n(x, y) = y^n + y - x^m$, by the moduli space description in §1.8.

Let $m < n$ such that $mp = nq = d$ and let $\nu \geq 1$ be a divisor of $m$. Let $f_\nu(x, y) = y - x^\nu$, and $f_m(x, y) = (f_\nu(x, y))^{m/\nu}$, and $f_n(x, y) = y^n + y - x^\nu$ and we consider the pencil of the curves $C^\nu(s) : f_m^p + s\, f_n^q = f_\nu^{pm/\nu} + s\, f_n^q = 0$. Let us remark that $C^\nu(s) \in \mathscr{N}_m^\nu(p, q; d) = \mathscr{N}_\nu^\nu(pm/\nu, q; d)$ for generic $s \neq 0$. Thus by Theorem 2, we obtain

Corollary 1. *The Alexander polynomial of* $C^\nu(s)$ *is, except for a finite number of curves of the pencil, equal to* $\Delta_{pm/\nu,q}(t)$*, and* $\{\Delta_{pm/\nu,q}(t),\ \nu|m\}$ *are mutually distinct.*

As the Alexander polynomial of the tame curves in one fixed connected component of the submoduli space $\mathscr{N}_m(p, q; d)$ of $\mathscr{M}_{torus}^{\max}(p, q; d)$ remains the same (see §1.7), one gets the following result (see §1.8, Lemma 2).

Corollary 2. *Let us assume that* $p > q$*. The submoduli space* $\mathscr{N}_m(p, q; d)$ *of* $\mathscr{M}_{torus}^{\max}(p, q; d)$ *has at least* $\varphi(m)$ *connected components, given by the submoduli space* $\mathscr{N}_m^\nu(p, q; d)$*, where* $\nu$ *is a divisor of* $m$*.*

1.9.1. *Zariski multiple.* Let $p$ and $q$ be integers such that $p > q \geq 2$, and let $d \geq 2$ be a common multiple of $p$ and $q$. Put $m = d/p$ and $n = d/q$. Let $\nu \geq 1$ be a divisor of $m$. Let $f_\nu(x, y)$, and $f_m(x, y) = (f_\nu(x, y))^{m/\nu}$, and $f_n(x, y)$ be polynomials of degree $\nu$, $m$ and $n$ respectively, such that the torus curve $C : (f_m(x, y))^p + (f_n(x, y))^q = 0$ of type $(p, q; d)$ is tame, and belongs to the moduli space $\mathscr{N}_m^\nu(p, q; d)$ (see §1.8, Lemma 2).

**Hypothesis:** Let us assume that $q$ divides $p$.

Let us remark that if $q|p$ and $\nu|m$, then $\nu|n$ and the curve $C$ has $q$ irreducible components (see (3) of Lemma 2), as it can be seen from the equality $f_m^p + f_n^q = (f_\nu^{n/\nu})^q + f_n^q$ (recall that $pm = qn$). Furthermore the irreducible components are smooth and of degree $n$, and intersect at the single singular point of the curve $C$, topologically equivalent to the Brieskorn-Pham singularity $B_{q,qn^2} : v^q + u^{qn^2} = 0$, which has $q$ branches. The irreducible component of the curve $C$ which contains a given branch of the singular point is well-defined, up to the choice of a $q$-th root of unity. Following E. Artal's terminology introduced in [**2**], one says that the tame curves in $\mathscr{N}_m^\nu(p, q; d)$, for $\nu|m$, have the same combinatorics (data coming from the resolution of the singularity, which determine the topology of a regular neighbourhood of the curve), independently from the integer $\nu$. Thus, one gets the following result (see also [**3**] for other Zariski multiples).

Theorem 3. *Let* $p$ *and* $q$ *be integers such that* $p > q \geq 2$ *and let* $d \geq 2$ *be a common multiple of* $p$ *and* $q$*. Let* $m = d/p$ *and* $n = d/q$*. Let us assume that* $q$ *divides* $p$*.*

*Then any $\varphi(m)$-ple $\{B_\nu,\ \nu|m\}$, where $B_\nu$ is a tame torus curve in $\mathscr{N}_m^\nu(p,q;d)$ for each $\nu|m$, is a Zariski $\varphi(m)$-ple, distinguished by the Alexander polynomial, whose curves have $q$ smooth irreducible components of degree $n$, and one single singular point topologically equivalent to the Brieskorn-Pham singularity $B_{q,qn^2} : v^q + u^{qn^2} = 0$.*

Example 1.9.2. The curves of $\mathscr{M}_{torus}^{\max}(4,2;8)$ are the union of two smooth irreducible quartics, which intersect at a single point. There are only two possibilities for the embedded topological type of the tame curves of $\mathscr{M}_{torus}^{\max}(4,2;8)$, described according to the decomposition

$$\mathscr{M}_{torus}^{\max}(4,2;8) = \mathscr{N}_2^1(4,2;8) \bigcup \mathscr{N}_2^2(4,2;8)$$

as follows.

(1) The tame curves in $\mathscr{N}_2^1(4,2;8)$ represented by the curves of the pencil of torus curves of type $(4,2;8)$, except a finite number of curves of the pencil, defined by the polynomials $f_2(x,y) = (y-x)^2$ and $f_4(x,y) = y^4 + y - x$, whose Alexander polynomial is given by

$$\Delta_1(t) = (t-1)(t^2+1)(t^4+1);$$

(2) The tame curves in $\mathscr{N}_2^2(4,2;8)$ represented by the curves of the pencil of torus curves of type $(4,2;8)$, except a finite number of curves of the pencil, defined by the polynomials $f_2(x,y) = y - x^2$ and $f_4(x,y) = y^4 + y - x^2$, whose Alexander polynomial is given by

$$\Delta_2(t) = (t-1)(t^2+1).$$

## 2. Proof of Theorem 2.

### 2.1. Calculation of Alexander Polynomials.

What follows is directly extracted from [**5**]. Let $f \in \boldsymbol{C}[X,Y,Z]$ be a homogeneous polynomial of degree $d$, and $\boldsymbol{X} = \{(x,y,z) \in \boldsymbol{C}^3 \mid f(x,y,z) = 1\}$ the associated Milnor fiber. Let $\overline{\boldsymbol{X}} = \{(X,Y,Z,T) \in \boldsymbol{P}^3 \mid T^d - f(X,Y,Z) = 0\}$ so that the projection $\overline{\tau} : \overline{\boldsymbol{X}} \longrightarrow \boldsymbol{P}^2$ is a ramified covering of $\boldsymbol{P}^2$, with branch locus the projective plane curve $C : f(X,Y,Z) = 0$ (we assume that $C$ is reduced). The quasi-projective subvariety $\overline{\boldsymbol{X}} \setminus \overline{\tau}^{-1}(C)$ of $\boldsymbol{P}^3$ is isomorphic to the Milnor fiber $\boldsymbol{X}$. Let $\omega = \exp(2\pi\sqrt{-1}/d)$, a primitive root of unity. The monodromy of the covering $\overline{\tau}|\boldsymbol{X} : \boldsymbol{X} \longrightarrow \boldsymbol{P}^2 \setminus C$ is given by $(X,Y,Z) \longmapsto (\omega X, \omega Y, \omega Z)$, and coincides with the monodromy of the Milnor fiber $\boldsymbol{X}$.

Let $\sigma : Y \longrightarrow \boldsymbol{P}^2$ be an embedded resolution of the curve $C$. One has $D = \sigma^* C = C' + \sum \nu_k E_k$, where $C'$ is the normalization (strict transform) of $C$ and $D$ is a divisor with normal crossings. Let $\tau' : Y' \longrightarrow Y$ be the pull-back of $\overline{\tau}$ by $\sigma$, $\overline{Y}$ the normalization of $Y'$, and $Z$ a desingularization of $\overline{Y}$ whose boundary $\Delta = Z \setminus \boldsymbol{X}$ is a divisor with normal crossings. One has $H^*(\boldsymbol{X}) = H^*(Z \setminus \Delta)$, and by Deligne's mixed Hodge theory for smooth quasi-projective varieties, one has the decomposition:

$$H^1(Z \setminus \Delta) = H^0(Z, \Omega^1_Z(\log \Delta)) \oplus H^1(Z, \mathscr{O}_Z),$$

where $\Omega^*_Z(\log \Delta)$ is the complex of holomorphic differential forms with logarithmic poles along the divisor $\Delta$. Moreover, this decomposition is compatible with the monodromy action. For each index $j$ such that $0 \le j \le d-1$,

$$H^1(\boldsymbol{X})_{\omega^j} = H^1(Z \setminus \Delta)_{\omega^j} = H^0(Z, \Omega^1_Z(\log \Delta))_{\omega^j} \oplus H^1(Z, \mathscr{O}_Z)_{\omega^j},$$

where we note by subscript $\omega^j$ the eigenspace of the monodromy action corresponding to the eigenvalue $\omega^j$.

Definition 2.1.1 ([**5**, p. 479]). Let $\mathscr{L} = \sigma^* \mathscr{O}_{\boldsymbol{P}^2}(1)$, and for $j = 0, \ldots, d-1$, let $\mathscr{L}^{(j)} = \mathscr{L}^{\otimes j} \otimes \mathscr{O}_Y(-\sum_k [\nu_k j/d] E_k)$. Let us define $l_j = \dim H^1(Y, \mathscr{L}^{(j)-1})$.

By [**5**, Lemma 2, p. 479], $H^1(Z, \mathscr{O}_Z)_{\omega_j}$ is isomorphic to $H^1(Y, \mathscr{L}^{(j)-1})$. By [**5**, Corollary 4, p. 481 and Lemma 7, p. 485], $\dim H^0(Z, \Omega^1_Z(\log \Delta))_{\omega_j}$ is equal to $\dim H^1(Y, \mathscr{L}^{(d-j)-1}) = l_{d-j}$ (see [**12**, proposition 4.6] for more details).

For $j = 0, \ldots, d-1$, let us define $j_c = d - j$. From the above discussion, one has

$$\dim H^1(\boldsymbol{X})_{\omega^j} = l_j + l_{j_c}.$$

Let $r$ be the number of irreducible components of the curve $C$. For $j = 0$, classical homological arguments (see e.g. [**19**]), or again results of [**5**], show that $\dim H^1(\boldsymbol{X})_{\omega^0} = r - 1$. The Alexander polynomial $\Delta_C(t)$ of the curve $C$ is given by the following formula (via [**23**])

$$\text{(ALEX)} \qquad \Delta_C(t) = (t-1)^{r-1} \prod_{j=1}^{d-1} \left(t - \exp(2\pi j \sqrt{-1}/d)\right)^{l_j + l_{j_c}}.$$

We will use E. Artal's Theorem [**2**, Theorem 2.7] to compute the $l_j$'s, in terms of the singularities of the curve $C$ and their relative positions (see [**18**, Lemma 3, §2.1] for a description in the Newton non-degenerate setting).

### 2.2. Proof of Theorem 2.

Let $p$ and $q$ be integers such that $p \ge q \ge 2$ and let $d \ge 2$ be a common multiple of $p$ and $q$. Put $m = d/p$ and $n = d/q$. Let $f_m(x, y) = y - x^m$ and $f_n(x, y) = y^n + y - x^m$ (let us remark that $1 \le m \le n$). By Bertini theorem, the curves of the pencil of torus curves of type $(p, q; d)$ defined by the polynomials $f_m(x, y)$ and $f_n(x, y)$ are, except a finite number of curves of the pencil, tame torus curves that belong to the moduli space $\mathscr{N}^m_m(p, q; d)$ (see §1.8, Lemma 2). Let $C$ be such a curve. In particular, the curve $C$ has $r = \gcd(p, q)$ irreducible components. We calculate the Alexander polynomial of $C$ with respect to a generic line at infinity $\{Z = 0\}$.

By implicit function theorem at the origin, there exists a unique convergent power series without constant term $\varphi_m(x)$, which is the solution in $y$ of the equation $y^n + y - x^m = 0$. This power series has the expansion

$$\varphi_m(x) = x^m - x^{nm} + (\text{higher terms}).$$

We take the following change of local coordinates $\Phi : (\boldsymbol{C}^2, O) \longrightarrow (\boldsymbol{C}^2, O)$ near the origin

$$(x, y) = (u, v + \varphi_m(u)),$$

and define $(\Phi^* f)(u, v) = f(u, v + \varphi_m(u))$, the pull-back by $\Phi$ of a polynomial $f(x, y)$. The germ of the curve $(C, O)$ is topologically equivalent to the Brieskorn-Pham singularity $v^q + u^{qn^2} = 0$ (see Lemma 1, §1.2), as we have $(\Phi^* f_n)(u, v) = v$ modulo $(uv, v^2)$, or equivalently, the Newton principal part of $(\Phi^* f_n)(u, v)$ is reduced to the linear coordinate $v$, and

$$(\Phi^* f_m)(u, v) = v - u^{nm} + \sum_{i > nm} a_i u^i.$$

Let $Q = (1, n^2)$ be the primitive weight covector of the one-dimensional face of the Newton boundary of $v^q + u^{qn^2} = 0$. For each index $j$ such that $1 \leq j \leq d - 1$, let us define

$$\alpha(j) = \left[\frac{j}{d}\,\nu(v^q + u^{qn^2}, Q)\right] - |Q| + 1$$

where $\nu(-, Q) : \boldsymbol{C}\{u, v\} \longrightarrow \boldsymbol{Z}_{\geq 0}$ is the multiplicity of a function germ in the ring of power series $\boldsymbol{C}\{u, v\}$, with respect to the weight covector $Q$ (see [**16**, p. 106]), and $|Q| = 1 + n^2$. One has $\nu(v^q + u^{qn^2}, Q) = qn^2$, thus the integer $\alpha(j)$ is given by

$$\alpha(j) = n(j - n).$$

The *ideals of quasi-adjunction* $\mathscr{I}_j$ of the function germ $v^q + u^{qn^2}$ at the origin are the ideals of the ring $\boldsymbol{C}\{u, v\}$ given by the following property

$$g(u, v) \in \mathscr{I}_j \Leftrightarrow \nu(g(u, v), Q) \geq \alpha(j).$$

For $j = 1, \dots, d-1$, let $\boldsymbol{C}[x, y]_{\leq j-3}$ be the vector space of polynomials $P(x, y)$ such that $\deg P(x, y) \leq j - 3$ (if $\deg P(x, y) < 0$, then by convention $P(x, y) = 0$). Let $\Phi_j^*$ be the canonical morphism of vector spaces

$$\Phi_j^* : \boldsymbol{C}[x, y]_{\leq j-3} \longrightarrow \boldsymbol{C}\{u, v\}/\mathscr{I}_j$$

induced by the change of local coordinates $\Phi$.

*The multiplicity* $l_j$ (*see Definition* 2.1.1) *is equal to the dimension of the cokernel of the linear map* $\Phi_j^*$ (*see* [**2**, Theorem 2.7] *and* [**18**, Lemma 3, §2.1]).

The main idea is now to consider an appropriate basis of the vector space $\boldsymbol{C}[x, y]_{\leq j-3}$.

LEMMA 3. *For any non-negative integers $\gamma$, $\delta$, $\alpha$ and $\beta$ such that $0 \le \alpha \le m-1$ and $0 \le \beta \le n-1$, let us define the polynomials*

$$M_{\alpha,\beta,\gamma,\delta}(x,y) = x^\alpha y^\beta (y^n + y - x^m)^\gamma (y - x^m)^\delta.$$

*The degree of $M_{\alpha,\beta,\gamma,\delta}$ is equal to $\alpha + \beta + n\gamma + m\delta$. Then the set of polynomials $M_{\alpha,\beta,\gamma,\delta}(x,y)$ such that $\alpha + \beta + n\gamma + m\delta \le j-3$ is a basis of $\boldsymbol{C}[x,y]_{\le j-3}$.*

PROOF. Let $x^a y^b$ be a monomial such that $a + b \le j - 3$. Let us assume that $a \ge m$. Then we have $x^a = x^{a-m}(-y + x^m) + yx^{a-m}$. If $b \ge n$, we use the equality $y^b = y^{b-n}(y^n + y - x^m) - y^{b-n}(y - x^m)$. Using these equalities inductively, we can write $x^a y^b$ as a linear combination of $M_{\alpha,\beta,\gamma,\delta}$ whose degree is not greater than $a+b$. This shows that $\{M_{\alpha,\beta,\gamma,\delta}; \deg M_{\alpha,\beta,\gamma,\delta} \le j-3\}$ generates $\boldsymbol{C}[x,y]_{\le j-3}$. The linear independence of the $M_{\alpha,\beta,\gamma,\delta}$'s is shown as follows.

By the definition of $\varphi_m(u)$, we observe that $\Phi^*(y^n + y - x^m)(u,v) = v$ modulo $(uv, v^2)$, and $\Phi^*(y - x^m)(u,v) = v - u^{nm} + \sum_{i>nm} a_i u^i$. Thus, the Newton principal part of $(\Phi^* M_{\alpha,\beta,\gamma,\delta})(u,v)$ is reduced to the single monomial $u^{\alpha+\beta m+\delta nm} v^\gamma$, which implies that the multiplicity with respect to the weight vector $Q = (1, n^2)$ is given by

$$\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v), Q) = \alpha + \beta m + \delta nm + \gamma n^2.$$

In particular, if $M_{\alpha,\beta,\gamma,\delta} \ne M_{\alpha',\beta',\gamma',\delta'}$, the Newton principal parts of $\Phi_j^* M_{\alpha,\beta,\gamma,\delta}$ and $\Phi_j^* M_{\alpha',\beta',\gamma',\delta'}$ are different. Let us assume that $\psi = \sum c_{\alpha,\beta,\gamma,\delta} M_{\alpha,\beta,\gamma,\delta} = 0$. Then by the above observation, the Newton principal part of $\Phi_j^*(\psi)$ must be 0, which implies that the corresponding coefficient is 0. Thus, repeating the argument, we get $\psi = 0$. $\square$

At the target space of $\Phi_j^*$, a basis of the vector space $\boldsymbol{C}\{u,v\}/\mathscr{I}_j$ is given by the monomials $u^l v^\gamma$ such that $\nu(u^l v^c, Q) \le \alpha(j) - 1$, with $\nu(u^l v^\gamma, Q) = l + \gamma n^2$. There is a bijective correspondence induced by $\Phi_j^*$ between the polynomials $M_{\alpha,\beta,\gamma,\delta}(x,y)$ such that $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}, Q) \le \alpha(j) - 1$, and the monomials $u^l v^\gamma$ such that $\nu(u^l v^\gamma, Q) \le \alpha(j) - 1$ (see Lemma 8, §2.3). Let us consider the following subspaces of $\boldsymbol{C}[x,y]_{\le j-3}$

$$V' = \langle M_{\alpha,\beta,\gamma,\delta} | \deg M_{\alpha,\beta,\gamma,\delta} \le j-3,\ \nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}, Q) \le \alpha(j) - 1 \rangle$$

$$V'' = \langle M_{\alpha,\beta,\gamma,\delta} | \deg M_{\alpha,\beta,\gamma,\delta} \le j-3,\ \nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}, Q) > \alpha(j) - 1 \rangle.$$

Then $\boldsymbol{C}[x,y]_{\le j-3}$ is the direct sum of $V'$ and $V''$, and the above considerations show that the kernel of $\Phi_j^*$ is $V''$.

LEMMA 4 (KEY LEMMA). *If $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v), Q) \le \alpha(j) - 1$, then $\deg M_{\alpha,\beta,\gamma,\delta}(x,y) \le j-2$. Furthermore the degree of the polynomial $M_{\alpha,\beta,\gamma,\delta}(x,y)$ is equal to $j-2$ if and only if*

- *either* (a): $n > m$, $\alpha = m-1$, $\beta = n-1$ *and* $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v), Q) = \alpha(j) - 1$,
- *or* (b): $n = m$, $\alpha = m-1$ *and* $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v), Q) = \alpha(j) - 1$.

PROOF OF LEMMA 4. Let us assume that $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v), Q) \le \alpha(j) - 1$. We will show that $n((j-2)-\deg(M_{\alpha,\beta,\gamma,\delta}))\geq 0$. As $\alpha(j)=n\ (j-n)$ the hypothesis is equivalent to $nj\geq \nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta},Q)+n^2+1$. Then,

$$\begin{aligned} n\ [(j-2)-\deg(M_{\alpha,\beta,\gamma,\delta})] &= nj-2n-n\ \deg(M_{\alpha,\beta,\gamma,\delta}) \\ &\geq n^2+1-2n+\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta},Q)-n\ \deg(M_{\alpha,\beta,\gamma,\delta}). \end{aligned}$$

As $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta},Q)-n\deg(M_{\alpha,\beta,\gamma,\delta})=-\alpha(n-1)+\beta(m-n)$ and $0\leq\beta\leq n-1$, one gets (let us recall that $n\geq m$)

$$\begin{aligned} n\ [(j-2)-\deg(M_{\alpha,\beta,\gamma,\delta})] &\geq (n-1)(n-1-\alpha)+(n-1)(m-n) \\ &= (n-1)(m-1-\alpha). \end{aligned}$$

The last term is positive, because $\alpha\leq m-1$. It is straightforward to see that the equality

$$n\ [(j-2)-\deg(M_{\alpha,\beta,\gamma,\delta})]=0$$

holds if and only if either (a) $n>m$, and $\alpha=m-1$, and $\beta=n-1$, and $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v),Q)=\alpha(j)-1$, or (b) $n=m$, and $\alpha=m-1$ and $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v),Q)=\alpha(j)-1$. □

According to Lemma 4, let us distinguish two cases.

2.2.1. *The case $n>m$ (or equivalently $p>q$).* Let us assume that $n>m$, and $\alpha=m-1$, and $\beta=n-1$, and that $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}(u,v),Q)=\alpha(j)-1$. Then $\gamma$ and $\delta$ satisfy the equation

$$j=n(\gamma+1)+m(\delta+1). \tag{1}$$

By Lemma 4, the multiplicity $l_j$ (see Definition 2.1.1) is equal to the number of solutions $(\gamma,\delta)$ of the equation (1). Because of the restriction on the integer $j$, namely $1\leq j\leq d-1$, the integers $\gamma$ and $\delta$ must satisfy $0<\gamma+1<d/n=q$ and $0<\delta+1<d/m=p$. The determination of the integer $l_j$ thus becomes an arithmetic problem. Let us define $j_c=d-j$. By Lemma 6 and Lemma 7 of §2.3, the sum of the multiplicities $l_j+l_{j_c}$ is given as follows.

Let $r'=\gcd(n,m)$ and $r=\gcd(p,q)$. Let us recall that $r$ is the number of irreducible components of the curve. The integers $r$ and $r'$ satisfy the equality $rd=pqr'$.

- If $r'$ does not divide $j$, then $l_j+l_{j_c}=0$;
- If $r'$ divides $j$:
  - If $n$ does not divide $j$ and $m$ does not divide $j$, then $l_j+l_{j_c}=r$;
  - If $n$ divides $j$ and $m$ does not divide $j$, or if $n$ does not divide $j$ and $m$ divides $j$, then $l_j+l_{j_c}=r-1$;
  - If $n$ divides $j$ and $m$ divides $j$, then $l_j+l_{j_c}=r-2$.

Let us substitute in the formula (ALEX) §2.1, and conclude by the following observations:

- $\exp(2\pi j\sqrt{-1}/d)$ is a root of the polynomial $t^{d/r'} - 1 = t^{pq/r} - 1$ if and only if $r'|j$.
- $\exp(2\pi j\sqrt{-1}/d)$ is a root of the polynomial $t^q - 1$ (resp. $t^p - 1$) if and only if $n|j$ (resp. $m|j$).

2.2.2. *The case $m = n$ (or equivalently $p = q$).* According to Lemma 4, let us assume that $n = m$, that $\alpha = m - 1$ and that $\nu(\Phi_j^* M_{\alpha,\beta,\gamma,\delta}, Q) = \alpha(j) - 1$. Then the integers $\beta$, $\gamma$ and $\delta$ satisfy the following equation

$$j = n(\gamma + \delta + 1) + \beta + 1, \tag{2}$$

with $0 \leq \beta \leq n - 1$, and $\gamma \geq 0$, and $\delta \geq 0$. The number of solutions of the equation (2) is precisely, by our construction, equal to the multiplicity $l_j$ (see Definition 2.1.1). The number of solutions $(\beta, \gamma, \delta)$ of the equation (2) is equal to the number of solutions $(\beta', \gamma)$ of the equation $j = n(\gamma+1)+(\beta'+1)$, such that $\beta' \geq 0$ and $\gamma \geq 0$. To see this, let us write in a unique way, by Euclidean division, any positive integer $\beta'$ as $\beta' = \delta n + \beta$, with $0 \leq \beta \leq n - 1$. Then by Lemma 6 and Lemma 7, §2.3, one gets (let us recall that $j_c = d - j$):

- If $n$ does not divide $j$, then $l_j + l_{j_c} = p - 1$;
- If $n$ divides $j$, then $l_j + l_{j_c} = p - 2$.

The number of irreducible components of the curve $C$ is equal to $p$ and we conclude as in the previous case. □

### 2.3. Arithmetic Lemmas.

We collect in this section some technical but elementary arithmetic lemmas we need to complete the proof of Theorem 2. Let $p$ and $q$ be integers such that $p \geq q \geq 2$ and let $d \geq 2$ be a common multiple of $p$ and $q$. Define $m = d/p$ and $n = d/q$. For each index $j$ such that $1 \leq j \leq d - 1$, let us consider the following arithmetic problem: Find the integral solutions in $x$ and $y$ of the equation

$$j = nx + my, \ x > 0, \ y \neq 0, \tag{3}$$

and discuss their number. Because of the condition on the integer $j$, namely $1 \leq j \leq d-1$, the integer $x$ must satisfy $1 \leq x \leq q - 1$. Let us remark that if the greatest common divisor of $n$ and $m$ does not divide the integer $j$, then the equation (3) has no integral solution.

Lemma 5. *Let $r' = \gcd(n, m)$. Let us assume that $r'$ divides $j$ and let us write $j = r'j_1$, $n = r'n_1$ and $m = r'm_1$, where the integers $n_1$ and $m_1$ are coprime. The equation $j_1 = n_1 x + m_1 y$ (or equivalently $j = nx+my$) admits a unique integral solution $(x_1, y_1)$ such that $0 \leq x_1 \leq m_1 - 1$.*

Proof. As the ideal generated by $n_1$ and $m_1$ in the ring of integers is the whole ring, there exist integers $x_0$ and $y_0$ such that $j_1 = n_1 x_0 + m_1 y_0$. By Gauss lemma, the integral solutions of the equation $j_1 = n_1 x + m_1 y$ are of the form $x = x_0 + i m_1$ and $y = y_0 - i n_1$, with $i \in \boldsymbol{Z}$. □

Lemma 6. *Let $r' = \gcd(n, m)$ and $r = \gcd(p, q)$. Let us assume that $r'$ divides $j$ and let us write $j = r'j_1$, $n = r'n_1$ and $m = r'm_1$, where the integers $n_1$ and $m_1$ are coprime. Let $(x_1, y_1)$ be the unique integral solution of the equation $j_1 = n_1x + m_1y$ such that $0 \leq x_1 \leq m_1 - 1$ (see Lemma 5). Then the solutions $(x, y)$ of the equation (3) have to be chosen amongst the following candidates*

$$(x_1 + im_1, y_1 - in_1), \ for\ i = 0, \dots, r - 1.$$

*Moreover the number $N_j$ of solutions of the equation (3) is given according to the following rules.*

- *If $n$ does not divide $j$ and $m$ does not divide $j$, then $N_j = r$;*
- *If $n$ divides $j$ and $m$ does not divide $j$, or if $n$ does not divide $j$ and $m$ divides $j$, then $N_j = r - 1$;*
- *If $n$ divides $j$ and $m$ divides $j$, then $N_j = r - 2$.*

Proof. By Gauss lemma, the integral solutions $(x, y)$ of the equation $j = nx + my$ (or equivalently $j_1 = n_1x + m_1y$) are of the form $x = x_1 + im_1$ and $y = y_1 - in_1$, with $i \in \boldsymbol{Z}$. Because of the restriction on the integer $x$, namely $x < q$, the index $i$ must satisfy the condition $0 \leq i \leq q/m_1 - 1 = r'q/m - 1$. As the integers $r' = \gcd(n, m)$ and $r = \gcd(p, q)$ satisfy the relation $qr' = mr$, we get $r'q/m = r$.

The number of solutions is deduced from the following discussion.

- *Assume that neither $n$ nor $m$ divides $j$.* In this case none of the $x = x_1 + im_1$'s for $0 \leq i \leq r - 1$ is equal to zero, otherwise $m_1$ divides $x_1$ and then $m_1$ divides $j_1 = n_1x_1 + m_1y_1$ i.e. $m$ divides $j$. Similarly, as $n$ does not divide $j$, none of the $y = y_1 - in_1$'s can be equal to zero. Therefore the $r$ candidates have to be considered as true solutions.
- *Assume that $n$ divides $j$ and that $m$ does not divide $j$.* If $m$ does not divide $j$, then $x = x_1 + im_1$ is non-zero. If $n$ divides $j$ (or equivalently $n_1$ divides $j_1$) then by Gauss lemma $n_1$ divides $y_1$. Write $y_1 = i_1n_1$. The only candidate $x = x_1 + i_1m_1$, $y = y_1 - i_1n_1$ is not a true solution, thus $r - 1$ solutions.
- *Assume that $n$ does not divide $j$ and that $m$ divides $j$.* By Gauss lemma, $m_1$ divides $y_1$, but $0 \leq x_1 < m_1$, so $y_1 = 0$. If $y = y_1 - in_1$ is equal to zero then $n_1$ divides $j_1$. So the only candidate $x = 0$, $y = y_1$ is not a true solution and we get $r - 1$ solutions.
- *Assume that both $n$ and $m$ divides $j$.* From the previous cases one gets $x_1 = 0$ and $y_1 = i_1n_1$. The only candidates which are not true solutions are $x = 0$, $y = y_1$ and $x = i_1m_1$, $y = 0$, thus $r - 2$ solutions. □

One may describe the solutions $(x, y)$ of the equation (3) according to the sign of $y$.

Lemma 7. *Let $S_j$ (respectively $S_j^+$) be the set of integral solutions $(x, y)$ of the equation $j = nx + my$ such that $0 < x < q$ and $y \neq 0$ (respectively $0 < x < q$ and $y > 0$). Let us define $j_c = d - j$. There is a bijection between the set $S_j$ and the disjoint union $S_j^+ \sqcup S_{j_c}^+$. In particular if $N_j$ (respectively $l_j$) is the number of elements of $S_j$ (respectively $S_j^+$) then $N_j = l_j + l_{j_c}$.*

Proof. The lemma is a consequence of the following observation. If $(x,y)$ is a solution of $j = nx + my$, then $j_c = n(q-x) + m(-y)$. $\square$

Finally we need the following fact.

Lemma 8. *Let $l$ be a fixed positive integer. There exist unique positive integers $\alpha$, $\beta$ and $\delta$ such that $\alpha \leq m-1$, $\beta \leq n-1$ and $l = \alpha + \beta m + \delta nm$.*

Proof. The integer $\delta$ is equal to the integral part of the rational number $l/nm$. $\square$

## 3. Description of the Moduli Spaces.

### 3.1. Normal form of $f_n(x,y)$.

In this subsection, we give a demonstration of the assertions 1 and 2 of Lemma 2, §1.8. Thus we consider a torus curve of type $(p,q;d)$ and of maximal contact, such that the associated curve $C_m$ has flex-order $m$ at $\xi_0$.

We consider the case $\nu = m$ (the general case is straightforward). Up to a linear change of coordinates, we may assume that $\xi_0 = (0,0)$ and that the line $\{y - \delta_{m,1}\, x = 0\}$ is the common tangent line of $C_n$ and $C_m$ at the origin. By the assumption, $\deg_x f_m(x,0) = m$ and therefore $f_m(x,y)$ is a monic polynomial in $\boldsymbol{C}[y][x]$. By Euclidean division of $f_n(x,y)$ by $f_m(x,y)$ in the polynomial ring $\boldsymbol{C}[y][x]$, there exist polynomials $h_1(x,y)$ and $r_0(x,y)$ such that $f_n(x,y) = h_1(x,y) f_m(x,y) + r_0(x,y)$, with $\deg_x r_0 < m$. Moreover, $\deg h_1 \leq n-m$ and $\deg r_0 \leq m$. If $\deg_x h_1 \geq m$, then $h_1(x,y) = h_2(x,y) f_m(x,y) + r_1(x,y)$, with $\deg_x r_1 < m$, $\deg h_2 \leq n - 2m$ and $\deg r_1 \leq n-m$. By induction, one obtains the following expansion of the polynomial $f_n(x,y)$:

$$f_n(x,y) = \sum_{i=0}^{[n/m]} r_i(x,y) f_m(x,y)^i,$$

with $\deg_x r_i < m$ and $\deg r_i \leq n - im$.

By assumption, the curve $C_m$ is smooth at the origin and its intersection multiplicity with the line $y=0$ at the origin is equal to $m$. Therefore the curve $C_m$ admits a parametrization of the form $(t, \gamma t^m + \sum_{i>m} a_i t^i)$, where $\gamma$ is a non-zero complex number.

The intersection multiplicity at the origin of the curves $C_n$ and $C_m$, which by assumption is equal to $nm$, is given by the valuation of the power series

$$f_n\Big(t, \gamma t^m + \sum_{i>m} a_i t^i\Big) = r_0\Big(t, \gamma t^m + \sum_{i>m} a_i t^i\Big).$$

One has $\deg_x r_0 < m$ and $\deg r_0 \leq n$, so one may write

$$r_0(x,y) = c_0(x) y^n + c_1(x) y^{n-1} + \cdots + c_{n-1}(x) y + c_n(x),$$

where the polynomials $c_i(x)$ satisfy $\deg c_i(x) \leq \min\{i, m-1\}$ (in particular $c_0(x)$ is a constant). The term in $t$ of the lowest degree given by $c_i(x)y^{n-i}$ is of the form $t^{n(i)}\gamma^{n-i}t^{(n-i)m}$, where $n(i)$ is an integer satisfying $0 \le n(i) \le m-1$. For $i = 1,\dots,n$, one has $n(i)+(n-i)m < nm$. Moreover, if $n(i)+(n-i)m = n(j)+(n-j)m$, then $i=j$. Therefore, $c_i(x) = 0$ for $i = 1,\dots,n$, which proves the first assertion of Lemma 2, §1.8.

The dimension of the space of $f_m$'s is given by the number $\alpha$ of monomials $x^a y^b$ such that $0 \le a+b \le m$, with $(a,b) \ne (a,0)$, $a = 0,\dots,m-1$. The dimension of the space of $f_n$'s is given (when $f_m$ is fixed) by the number $\beta = \sum_{i=1}^{[n/m]} \beta_i$, where $\beta_i$ is the number of monomials $x^a y^b$ such that $0 \le a+b \le n-mi$ and $a < m$, for $i = 1,\dots,[n/m]$. By assumption on the curve $C_m$, the polynomial $f_m(x,y)$ contains the monomials $y$ and $x^m$. The curve $C_n$ is smooth at $(0,0)$ with tangent $y=0$, which implies that $r_1(0,0) \ne 0$. We have also observed that $r_0(x,y) = c_0$, with $c_0 \ne 0$. Thus the moduli space $\mathscr{N}_m^m(p,q;d)(\xi_0)$ is identified with a Zariski-open subset of $\boldsymbol{C}^{(\alpha-2)+(\beta-1)} \times (\boldsymbol{C}^*)^4$, which proves the second assertion of Lemma 2, §1.8.

### 3.2. Irreducibility of a certain polynomial.

We prove in this subsection the assertion 3 of Lemma 2, §1.8. The curves of the moduli $\mathscr{N}_m^\nu(p,q;d)$ are in fact torus curves in the moduli $\mathscr{N}_\nu^\nu(d/\nu,q;d)$, thus the problem is reduced to compute the number of irreducible components of the tame curves of the moduli $\mathscr{N}_m^m(p,q;d)$. Then, let us recall that the number of irreducible components of a plane curve is an invariant of the embedded topological type of the curve (it is equal to the multiplicity plus one of the root 1 of the Alexander polynomial of the curve). So, by the assertions 1 and 2 of Lemma 2 which we have just proved in the previous section, it is sufficient to find the number of irreducible components of the tame torus curves of type $(p,q;d)$ of the pencil, with $t \in \boldsymbol{C}$, $(y-x^m)^p + t\,(y^n+y-x^m)^q = 0$. Putting $r = \gcd(p,q)$, one observes that these curves are the union of $r$ torus curves in $\mathscr{N}_m^m(p/r,q/r;d/r)$. Finally tame torus curves in $\mathscr{N}_m^m(p/r,q/r;d/r)$ are irreducible by the following lemma.

Lemma 9. *Let $q$ be an integer $\ge 1$ and $g(x,y)$ be a polynomial. Let us assume that the polynomial equation $g(0,y) = 0$ has at least one simple root. Let $D$ be the affine curve defined by $g(x,y) = 0$ and $D'$ be the curve defined by $g(x^q,y) = 0$. Then, $D$ is irreducible if and only if $D'$ is irreducible.*

In fact, assuming this lemma, the argument goes as follows. The curve $(y-x^m)^{p/r} + t\,(y^n+y-x^m)^{q/r} = 0$, $t \in \boldsymbol{C}$, is irreducible if and only if $(y-x)^{p/r}+t\,(y^n+y-x)^{q/r} = 0$ is irreducible. The latter is equivalent to the irreducibility of the curve $x^{p/r}+t\,(y^n+x)^{q/r} = 0$ after a change of coordinates. This is irreducible as $x^{p/r}+t\,(y+x)^{q/r} = 0$ is irreducible, which is obvious.

Proof of Lemma 9. If $D'$ is irreducible then it follows at once that $D$ is irreducible. Conversely, under the assumption that $D$ is irreducible, we give a topological proof of the irreducibility of $D'$, taking advantage of the fact that a reduced complex analytic set is irreducible if and only if its smooth locus is connected.

Let us consider the projections $\pi : D \longrightarrow \boldsymbol{C}$ and $\pi' : D' \longrightarrow \boldsymbol{C}$ induced by the canonical projection $\boldsymbol{C}^2 \to \boldsymbol{C} : (x,y) \mapsto y$. Let $\Sigma$ be a finite subset of points of $\boldsymbol{C}$ such that the restrictions $\pi : D \setminus \pi^{-1}(\Sigma) \longrightarrow \boldsymbol{C} \setminus \Sigma$ and $\pi' : D' \setminus {\pi'}^{-1}(\Sigma) \longrightarrow \boldsymbol{C} \setminus \Sigma$ induce topological coverings. The total space of a topological covering on $\boldsymbol{C} \setminus \Sigma$ is connected if and only if the fundamental group $\pi_1(\boldsymbol{C} \setminus \Sigma)$ acts transitively on a fixed fiber of the covering.

Let $\gamma_0$ be a simple root of $g(0,y)$ and let $\xi_0$ be a point in $\boldsymbol{C} \setminus \Sigma$, distinct from $\gamma_0$ but arbitrarily close to $\gamma_0$. Let us define $F = \pi^{-1}(\xi_0)$ and $F' = {\pi'}^{-1}(\xi_0)$. Let $n = \deg_x g(x,y)$ be the degree of the covering induced by the projection $\pi$ and note $F = \{(\eta_1, \xi_0), \dots, (\eta_n, \xi_0)\}$, with $\eta_i \neq 0$ for $i = 1, \dots, n$, when $\xi_0$ is sufficiently close to $\gamma_0$. For $i = 1, \dots, n$, let $\eta_{i,1}, \dots, \eta_{i,q}$ be the solutions of $x^q = \eta_i$. The fiber $F'$ is given by the points $(\eta_{i,j}, \xi_0)$, for $i = 1, \dots, n$ and $j = 1, \dots, q$. If the curve $D$ is irreducible, there exists a continuous path $\tau_i : I \to D \setminus \pi^{-1}(\Sigma)$ connecting $(\eta_1, \xi_0)$ to $(\eta_i, \xi_0)$ for $i = 1, \dots, n$, so that it induces the loop $\pi \circ \tau_i : (I, \partial I) \to (\boldsymbol{C} \setminus \Sigma, \xi_0)$. Recall that the action of the loop $\tau_i$ on $\eta_1 \in F$ is nothing but $\eta_i$. Taking the pull-back by the map $\boldsymbol{C}^* \to \boldsymbol{C}^* : x \mapsto x^q$, for $k = 1, \dots, q$, there exists a continuous path in $D' \setminus {\pi'}^{-1}(\Sigma)$ connecting $(\eta_{1,k}, \xi_0)$ to a unique $(\eta_{i,j(k)}, \xi_0)$.

Up to reordering, one may assume that $(\eta_1, \xi_0)$ lies in an arbitrarily small neighbourhood of $(0, \gamma_0)$, and that $\eta_1$ converges to $0$ along the fibers $\pi^{-1}(t\gamma_0 + (1-t)\xi_0)$, $0 \le t \le 1$, when $\xi_0$ converges to $\gamma_0$. To end the proof, we show that $\pi_1(\boldsymbol{C} \setminus \Sigma)$ acts transitively on the set $\{(\eta_{1,1}, \xi_0), \dots, (\eta_{1,q}, \xi_0)\}$. As $\gamma_0$ is a simple root of $g(0,y)$, one has $\partial g/\partial y(0, \gamma_0) \neq 0$. By implicit function theorem, there exist an integer $s \ge 1$ and a power series $\sum_{i \ge s} a_i x^i$, with $a_s \neq 0$, such that the curve $D$ is given, in a small neighbourhood of $(0, \gamma_0)$, by

$$y - \gamma_0 = \sum_{i \ge s} a_i x^i.$$

A local topological model for $D$ (respectively for $D'$) at $(0, \gamma_0)$ is given by $y - \gamma_0 = a_s x^s$ (respectively by $y - \gamma_0 = a_s x^{qs}$). Let $x_1, \dots, x_s$ be the points of the fiber $\pi^{-1}(\xi_0)$, which is described locally around $(0, \gamma_0)$ by the model $\xi_0 - \gamma_0 = a_s x^s$, and let $x_{i,1}, \dots, x_{i,q}$ be the points of the fiber ${\pi'}^{-1}(\xi_0)$, which are over $x_i$. Under a suitable ordering and with the abuse of notation $x_{s+1} = x_1$ and $x_{s+1,j} = x_{1,j}$, we may assume that $x_{i,j}$ is transformed into $x_{i+1,j}$, for $1 \le i \le s-1$, and $x_{s,j}$ is transformed into $x_{1,j+1}$, by the action of the small loop $\tau(t) = \gamma_0 + \epsilon_0 \exp(2\pi\sqrt{-1}t)$, with $\epsilon_0 = |\xi_0 - \gamma_0|$. In particular, the points $x_{1,1}, \dots, x_{1,q}$ are in the same orbit. □

## 4. Appendix. New Proof of Theorem 1.

Let $p$ and $q$ be integers such that $p \ge q \ge 2$, and let $d \ge 2$ be a common multiple of $p$ and $q$. Let $n = d/q$ and $m = d/p$. Let $C$ be the generic tame torus curve of type $(p, q; d)$ defined by the polynomial equation

$$(x^m + 1)^p + (y^n + 1)^q = 0.$$

The singular locus of $C$ coincides with the intersection of the lines $C_m : x^m + 1 = 0$ and $C_n : y^n + 1 = 0$, so that $C$ is a tame torus curve. In this appendix, we calculate the Alexander polynomial of the curve $C$ with respect to the line at infinity $\{Z = 0\}$, which is in general position with respect to the curve $C$. We refer to the proof of Theorem 2 for the notations.

Let $r = \gcd(p, q)$ be the number of irreducible components of the curve $C$. We can write $p = rp_1$ and $q = rq_1$, with $\gcd(p_1, q_1) = 1$, and thus $m = q_1 r'$, $n = p_1 r'$ and $d = rr'p_1q_1$, with $r' = \gcd(m, n)$.

Let $\{a_\lambda\}_{0\le\lambda\le m-1}$ be the set of solutions of $x^m + 1 = 0$, and let $\{b_\mu\}_{0\le\mu\le n-1}$ be the set of solutions of $y^n + 1 = 0$. The singular points of the curve $C$ are the points $P_{\lambda,\mu} = (a_\lambda, b_\mu)$. For any singular point $P_{\lambda,\mu}$, let $\Phi_{\lambda,\mu}$ be the change of local coordinates $\Phi_{\lambda,\mu} : (\boldsymbol{C}^2, P_{\lambda,\mu}) \longrightarrow (\boldsymbol{C}^2, O)$ defined by

$$(u_{\lambda,\mu}, v_{\lambda,\mu}) = (x^m + 1, y^n + 1),$$

and let $\Phi^*_{\lambda,\mu} : \boldsymbol{C}\{x, y\} \longrightarrow \boldsymbol{C}\{u_{\lambda,\mu}, v_{\lambda,\mu}\}$ be the pull-back by $\Phi_{\lambda,\mu}$. The singularity $(C, P_{\lambda,\mu})$ is isomorphic to the Brieskorn-Pham singularity

$$B_{q,p} : v^q_{\lambda,\mu} + u^p_{\lambda,\mu} = 0,$$

which is Newton non-degenerate. Let $Q = (q_1, p_1)$ be the primitive weight covector of the one-dimensional face of the Newton boundary. One has $\nu(u^p_{\lambda,\mu} + v^q_{\lambda,\mu}; Q) = pq_1$. For $j = 1, \dots, d-1$, let $\alpha(j)$ be the integer defined by

$$\alpha(j) = \left[\frac{jpq_1}{d}\right] - (p_1 + q_1) + 1 = \left[\frac{j}{r'}\right] - (p_1 + q_1) + 1.$$

For any singular point $P_{\lambda,\mu}$, the ideal of quasi-adjunction $\mathscr{I}_{j,P_{\lambda,\mu}}$ is generated (as a complex vector space) by the monomials $u^\gamma_{\lambda,\mu} v^\delta_{\lambda,\mu}$, such that $\nu(u^\gamma_{\lambda,\mu} v^\delta_{\lambda,\mu}; Q) \ge \alpha(j)$. Let us recall that $\nu(u^\gamma_{\lambda,\mu} v^\delta_{\lambda,\mu}; Q) = \gamma q_1 + \delta p_1$.

For $j = 1, \dots, d-1$, let $\boldsymbol{C}[x, y]_{\le j-3}$ be the vector space of polynomials $P(x, y)$ such that $\deg P(x, y) \le j - 3$, and for $0 \le \lambda \le m - 1$ and $0 \le \mu \le n - 1$ let $V_{\lambda,\mu} = \boldsymbol{C}\{u_{\lambda,\mu}, v_{\lambda,\mu}\}/\mathscr{I}_{j,P_{\lambda,\mu}}$. We consider the linear mapping

$$\Phi^*_j : \boldsymbol{C}[x, y]_{\le j-3} \longrightarrow \bigoplus_{\lambda,\mu} V_{\lambda,\mu}$$

where the component mapping of $\Phi^*_j$ to $V_{\lambda,\mu}$ is the morphism of vector spaces induced by $\Phi^*_{\lambda,\mu} : \boldsymbol{C}\{x, y\} \longrightarrow \boldsymbol{C}\{u_{\lambda,\mu}, v_{\lambda,\mu}\}$. The multiplicity $l_j$ (see Definition 2.1.1) is equal to the dimension of $\operatorname{coker}(\Phi^*_j)$ (see the proof of Theorem 2, §2.2).

Lemma 10. *For any singular point $P_{\lambda,\mu}$, let $(u_{\lambda\mu}, v_{\lambda\mu})$ be the local coordinates around $P_{\lambda,\mu}$ defined by $u_{\lambda,\mu} = x^m + 1$ and $v_{\lambda,\mu} = y^n + 1$. For any positive integers $\gamma$ and $\delta$ with $\gamma q_1 + \delta p_1 \le \alpha(j) - 1$, let $N_{\gamma,\delta}$ be the vector subspace of $\bigoplus_{\lambda,\mu} V_{\lambda,\mu}$ defined by the direct sum*

$$N_{\gamma,\delta} = \bigoplus_{\lambda,\mu} \langle u^\gamma_{\lambda,\mu} v^\delta_{\lambda,\mu} \rangle \subset \bigoplus_{\lambda,\mu} V_{\lambda,\mu}.$$

*For $j=1,\dots,d-1$, one has an isomorphism of vector spaces*

$$\bigoplus_{\lambda,\mu} V_{\lambda,\mu} \cong \bigoplus_{\gamma q_1+\delta p_1\leq \alpha(j)-1} N_{\gamma,\delta}.$$

PROOF. The proof is straightforward. □

For any positive integers $\gamma$ and $\delta$ and for any integers $\alpha$ and $\beta$ such that $0\leq\alpha\leq m-1$ and $0\leq\beta\leq n-1$, let us consider the polynomials basis

$$M_{\alpha,\beta,\gamma,\delta}=x^\alpha y^\beta (x^m+1)^\gamma (y^n+1)^\delta.$$

The Newton principal part of $(\Phi^*_{\lambda,\mu}M_{\alpha,\beta,\gamma,\delta})(u_{\lambda,\mu},v_{\lambda,\mu})$ is reduced to the single monomial $u^\gamma_{\lambda,\mu}v^\delta_{\lambda,\mu}$, which implies that the multiplicity with respect to the weight covector $Q=(q_1,p_1)$ is given by

$$\nu(\Phi^*_{\lambda,\mu}M_{\alpha,\beta,\gamma,\delta};Q)=\gamma q_1+\delta p_1.$$

LEMMA 11. *For any positive integers $\gamma$ and $\delta$ let $L_{\gamma,\delta}$ be the $mn$-dimensional vector space of polynomials generated by the polynomials*

$$\{M_{\alpha,\beta,\gamma,\delta}\ |\ 0\leq\alpha\leq m-1,\ 0\leq\beta\leq n-1\}.$$

*For $j=1,\dots,d-1$, let $L_j$ be the space of polynomials generated by $M_{\alpha,\beta,\gamma,\delta}$, such that $0\leq\alpha\leq m-1$, and $0\leq\beta\leq n-1$, and $\nu(\Phi^*_{\lambda,\mu}M_{\alpha,\beta,\gamma,\delta};Q)\leq\alpha(j)-1$. By construction*

$$L_j=\bigoplus_{\gamma q_1+\delta p_1\leq\alpha(j)-1} L_{\gamma,\delta},$$

*and the restriction of $\Phi^*_j$ to $L_j$ is an isomorphism.*

PROOF. The vector spaces have the same dimension, therefore it is sufficient to prove injectivity. For this purpose, let $(\Phi^*_j)_0$ be defined by

$$(\Phi^*_j)_0(M_{\alpha,\beta,\gamma,\delta})=\{u^\gamma_{\lambda,\mu}v^\delta_{\lambda,\mu}a^\alpha_\lambda b^\beta_\mu\}_{\lambda,\mu}.$$

In particular $(\Phi^*_j)_0(L_{\gamma,\delta})\subset N_{\gamma,\delta}$. Thus it is sufficient to prove that the restriction $(\Phi^*_j)_0|L_{\gamma,\delta}:L_{\gamma,\delta}\longrightarrow N_{\gamma,\delta}$ is injective. Let $h(x,y)(x^m+1)^\gamma(y^n+1)^\delta$, with $\deg_x(h)\leq m-1$ and $\deg_y(h)\leq n-1$, be an element of $L_{\gamma,\delta}$ such that

$$(\Phi^*_j)_0(h(x,y)(x^m+1)^\gamma(y^n+1)^\delta)=\{u^\gamma_{\lambda,\mu}v^\delta_{\lambda,\mu}h(a_\lambda,b_\mu)\}_{\lambda,\mu}$$

is equal to zero. Then $h(a_\lambda,b_\mu)=0$ for $0\leq\lambda\leq m-1$ and $0\leq\mu\leq n-1$. As $\deg_x h(x,y)\leq m-1$, this implies that $h(x,b_\mu)=0$ for any $b_\mu$, which implies that $h(x,y)$

is divisible by $\prod_\mu (y - b_\mu)$. As $\deg_y\, h(x,y) \le n-1$, this implies $h(x,y) = 0$. □

Lemma 12 (Key Lemma). *If* $\nu(\Phi^*_{\lambda,\mu} M_{\alpha,\beta,\gamma,\delta}; Q) \le \alpha(j) - 1$*, then* $\deg(M_{\alpha,\beta,\gamma,\delta}) \le j-2$. *Furthermore* $\deg(M_{\alpha,\beta,\gamma,\delta}) = j-2$ *if and only if*

$$r'|j,\ \alpha = m-1,\ \beta = n-1, \quad \textit{and} \quad \nu(\Phi^*_{\lambda,\mu} M_{\alpha,\beta,\gamma,\delta}; Q) = \alpha(j) - 1.$$

Proof. Let us assume $\nu(\Phi^*_{\lambda,\mu} M_{\alpha,\beta,\gamma,\delta}; Q) \le \alpha(j) - 1$. Multiplying by $r'$, one gets $\gamma m + \delta n + n + m \le r'[j/r']$. Note that $j \ge r'[j/r']$, and equality holds if and only if $r'|j$. Then

$$\begin{aligned}(j-2) - \deg(M_{\alpha,\beta,\gamma,\delta}) &\ge \gamma m + \delta n + n + m - 2 - \deg(M_{\alpha,\beta,\gamma,\delta}) \\ &= (m-\alpha) + (n-\beta) - 2.\end{aligned}$$

Now, let us assume that $r'|j$, and $\alpha = m-1$, and $\beta = n-1$, and $\nu(\Phi^*_{\lambda,\mu} M_{\alpha,\beta,\gamma,\delta}; Q) = \alpha(j) - 1$. Then by construction, the multiplicity $l_j$ (see Definition 2.1.1) is equal to the number of solutions $(\gamma, \delta)$ of the equation

$$j = (\gamma+1)n + (\delta+1)m. \tag{4}$$

Because of the restriction on the integer $j$, namely $1 \le j \le d-1$, the integers $\gamma$ and $\delta$ must satisfy the inequalities $0 < \gamma + 1 < d/n = q$ and $0 < \delta + 1 < d/m = p$. We conclude by the arithmetic lemmas of §2.3. □

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