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One parameter families of Riemann surfaces and presentations of elements of mapping class group by Dehn twists

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Abstract. We obtain a presentation of a certain orientation preserving periodic homeomorphism of a compact real surface of genus $g \geq 2$ by a product of right handed Dehn twists using a splitting family. It was expected that a presentation of a homeomorphism by right handed Dehn twists obtained from a splitting family is one of the shortest presentation. In this paper, we give a counter example of this conjecture.

Introduction.

Let $f\colon S \to B$ be a proper flat morphism from a two dimensional complex manifold S to a one dimensional complex manifold B. If all but a finite number of fibers of f are nonsingular algebraic curves of genus $g \geq 2$, then we call a triple (f, S, B) *a family of curves* of genus g. We call P *a critical point* of f when $f^{-1}(P)$ is singular. We denote the set of all critical points of f by D_f and its cardinality by $^{\#}D_f$. If each singular fiber has only one node as singularity, we call it *a family of Lefschetz type* or *a Lefschetz fibration*. We call a family of curves *a degeneration of curves*, when $^{\#}D_f = 1$ and B is a small disk on $\boldsymbol{C}$. Two degenerations (f_1, S_1, B_1) and (f_2, S_2, B_2) are said to be *topologically equivalent* if there exist orientation preserving homeomorphisms $h\colon S_1 \to S_2$ and $h'\colon B_1 \to B_2$ satisfying $f_2 \circ h = h' \circ f_1$. For the topological equivalence class of a degeneration (f, S, B), we can determine the topological monodromy as follows (see [**MM**]); Let $l\colon [0, 2\pi] \to B$ be a simple closed curve on B with counter clockwise orientation rounding the critical point and Σ_g a compact real surface of genus g. Fixing a homeomorphism $\psi_0^l\colon \Sigma_g \to f^{-1}(l(0))$, we can naturally define $\psi_\theta^l\colon \Sigma_g \to f^{-1}(l(\theta))$ continuously along l. Then we see that *a monodromy homeomorphism* $\psi_f^l := (\psi_{2\pi}^l)^{-1} \circ \psi_0^l$ *along* l is an orientation preserving homeomorphism. Note that ψ_f^l is a homeomorphism of $f^{-1}(l(0))$ when we set $\psi_0^l := \mathrm{id}_{f^{-1}(l(0))}$. In this case, we call $f^{-1}(l(0))$ *a reference fiber*. It is well-known that the conjugacy class $[\psi_f^l]$ of the isotopy class of ψ_f^l in the mapping class group can be uniquely determined for the topological equivalence class of (f, S, B). We call $[\psi_f^l]$ *the topological monodromy* (the monodromy, for short) of (f, S, B). The most simplest monodromy is the conjugacy class of a right handed Dehn twist D_C in a simple closed curve C on Σ_g. (a left hand Dehn twist cannot occur). The monodromy of a degeneration is the conjugacy class of a right handed Dehn twist if and only if it is of Lefschetz type (See [**DK**] or [**Ka**]).

By Lickorish ([**Li**]), we see that each element of the mapping class group of genus $g \geq 2$ is a product of Dehn twists (it might need both right and left hand Dehn twists).

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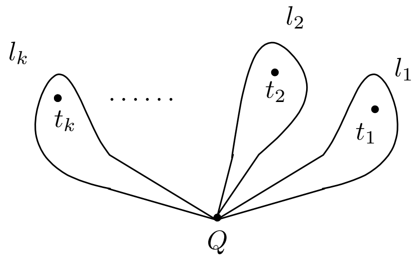

Figure 1.

Thus, it is natural to study presentations of monodromy homeomorphisms (representative of monodromies) by products of Dehn twists. To investigate the presentations by Dehn twists, we use deformations of degenerations called *splitting families* (see Definition 1.2).

If a degeneration (f, S, B) has a splitting family, there exists a family of curves (f', S', B') which is a deformation of (f, S, B) satisfying ${}^{\#}D_{f'} = k \geq 2$. This phenomenon can be considered that the unique singular fiber of (f, S, B) splits into k-singular fibers. Thus we see that the monodromy $[\psi]$ of (f, S, B) is split into k-monodromy homeomorphisms satisfying $\psi = \psi_{f'}^{l_k} \circ \psi_{f'}^{l_{k-1}} \circ \cdots \circ \psi_{f'}^{l_1}$, where $\psi_{f'}^{l_j}$ is a monodromy homeomorphism along l_j rounding only one critical point on B'. In particular, when (f', S', B') is of Lefschetz type, we obtain a presentation of ψ by a product of right handed Dehn twists (we sometimes say *a presentation of ψ by Dehn twists*, for short). In this way, presentations of monodromy homeomorphisms by Dehn twists were studied in certain cases by Matsumoto and Ito (cf. [**M**] and [**I**]).

Assume that a degeneration with the monodromy $[\psi_{f'}^{l_j}]$ has a splitting family. Observing the splitting family as above, we can obtain a presentation of $\psi_{f'}^{l_j}$ by a composite of monodromy homeomorphisms in principle. Substituting this presentation to $\psi = \psi_{f'}^{l_k} \circ \psi_{f'}^{l_{k-1}} \circ \cdots \circ \psi_{f'}^{l_1}$, we obtain a new presentation of ψ. If we obtain $\psi = D_{C_K} D_{C_{K-1}} \cdots D_{C_1}$, a presentation of ψ by Dehn twists by repeating this "geometric" operation, we call it *a geometric presentation of ψ*. Since presentations described in [**M**] and [**I**] are very simple, it is expected that a geometric presentation is one of the shortest presentation by Dehn twists, i.e., any presentation of ψ by Dehn twists needs more than or equal to K-Dehn twists (This conjecture is first proposed by Cadavid ([**Ca**])).

In this paper, first, we review the theory of monodromy and give the definition of splitting families. Next, we obtain a degeneration with a certain periodic monodromy $[\psi]$ by a double covering of $\mathbf{P}^1 \times \Delta$. Deforming the branch locus of the double covering, we obtain a splitting family and a geometric presentation of $[\psi]$ (Theorem 2.1). We also obtain an odd chain relation immediately from this Theorem 2.1 (cf. Corollary 2.2 and Remark 2.3). Using this presentation and chain relations, we give an example which shows that a geometric presentation is not necessarily the shortest.

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1. Review of monodromies and splitting family.

1.1. Review of periodic monodromies.

In this section, we review two invariants of periodic monodromies and the configurations of the singular fibers of normally minimal degenerations with periodic monodromies (see [**MM**] and [**Ni**]).

We call a degeneration *a normally minimal* if the reduced scheme of the singular fiber has only nodes as singularities and any (-1)-curve in the special fiber intersects the other components at at least three points. Note that any degeneration is bimeromorphic to a normally minimal degeneration because we can obtain a normally minimal degeneration by a succession of blowing-ups. Thus, we see that the normally minimal model of a degeneration is unique.

An orientation preserving homeomorphism $\psi\colon \Sigma_g \to \Sigma_g$ is said to be *periodic* if there exists an integer n such that ψ^n is isotopic to the identity map. The smallest positive such integer n is called *the period* of ψ. By Kerckhoff's theorem (cf. [**Ke**]), for each periodic homeomorphism ψ, there exist a Riemann surface C of genus g and a holomorphic automorphism $\bar{\psi}\colon C \to C$ isotopic to ψ. We introduce the notion of the total valency using this holomorphic map $\bar{\psi}$. See the original definition of the valency in [**Ni**]. For each point P on C, we denote by r_P the cardinality of the orbit of P under $\bar{\psi}$, and let $l_P := n/r_P$. Let δ_P be the smallest non-negative integer such that $(\bar{\psi})^{r_P}$ is the rotation of angle $2\pi\delta_P/l_P$ near each point in the orbit. Denote by s_P the smallest positive integer satisfying $\delta_P s_P \equiv 1 \pmod{l_P}$ if $\delta_P \neq 0$, and set $s_P := 0$ when $\delta_P = 0$. The symbol s_P/l_P is called *the valency* of the orbit of P.

Note that the valencies of all but a finite number of orbits are zero. The set of the positive valencies is called *the total valency* of $\bar{\psi}$ and expressed as the formal sum $\sum s_P/l_P$ of symbols. We define the total valency of the conjugacy class $[\psi]$ of ψ as the total valency of $\bar{\psi}$ and denote it by $\mathrm{tv}([\psi])$. By the main theorem of [**MM**], we see that there exists a degeneration of curves whose monodromy is the conjugacy class $[\psi]$ of ψ. From the period n and total valency $\mathrm{tv}([\psi]) = n_1/m_1 + n_2/m_2 + \cdots + n_k/m_k$ of $[\psi]$, we obtain the configuration of the singular fiber of a normally minimal degeneration with monodromy $[\psi]$ as follows; Let g' be the genus of the quotient $\Sigma_g/\langle\psi\rangle$, where $\langle\psi\rangle$ is the cyclic group generated by ψ. Let $(n_i^{(0)}, n_i^{(1)}, \cdots, n_i^{(s_i)})$ be the sequence of positive integers which satisfies the following:

(i) $n_i^{(0)} = n$ and $n_i^{(1)} = nn_i/m_i$.
(ii) $n_i^{(j)}$ is the smallest positive integer such that $n_i^{(j)} + n_i^{(j-2)}$ is a multiple of $n_i^{(j-1)}$.

Then, the singular fiber $X_{[\psi]}$ of a degeneration with monodromy $[\psi]$ can be written down as $X_{[\psi]} = n\Theta + \Sigma_{i=1}^{k}\Sigma_{j=1}^{s_i} n_i^{(j)} E_i^j$ which satisfies the following conditions:

(1) Θ is a smooth curve of genus g' and E_i^j are nonsingular rational curves.
(2) $\Theta \cdot E_i^1 = E_i^1 E_i^2 = \cdots = E_i^{s_i-1} E_i^{s_i} = 1$, $E_i^j E_i^{j'} = 0$ $(|j - j'| \geq 2)$, $E_{i_1}^j E_{i_2}^{j'} = 0$ $(i_1 \neq i_2)$.

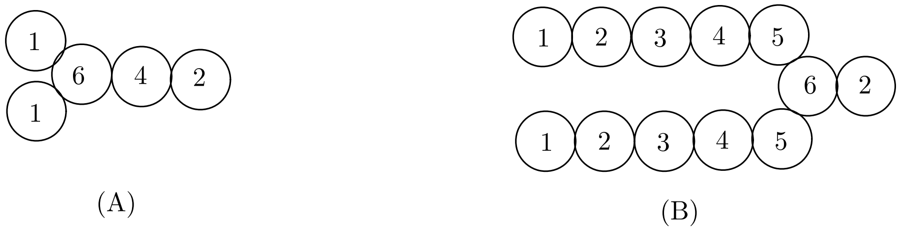


Figure 2.

Example 1.1. Let $F\colon C \to C$ be a holomorphic automorphism of a genus two curve with the period 6 and $\mathrm{tv}([F]) = 1/6 + 1/6 + 2/3$. There actually exist such a genus two curve and a holomorphic automorphism (cf. [**AI**]). Then, a normally minimal degeneration with monodromy $[F]$ has the singular fiber as shown in Figure 2(A). In Figure 2, circles and natural numbers in circles mean nonsingular rational curves and their multiplicities, respectively. From the definition of the total valency, we see that there exist three orbits $\{P_1\}$, $\{P_2\}$ and $\{P_3, P_4\}$ whose valencies are not zero. Moreover, taking a suitable coordinates at each of the above points, F is the rotation of angle $2\pi(1/6)$ near P_1 and P_2, and F^2 is the rotation of angle $2\pi(2/3)$ near P_3 and P_4. Then we can easily see that F^5 and $(F^5)^2$ are the rotations of angle $2\pi(5/6)$ near P_1 and P_2, $2\pi(1/3)$ near P_3 and P_4, respectively. Thus, we see that F^5 is periodic with the period 6 and $\mathrm{tv}([F^5]) = 5/6 + 5/6 + 1/3$. The singular fiber of normally minimal degeneration with monodromy $[F^5]$ is as shown in Figure 2(B).

1.2. Splitting families and monodromies.

Let ψ and $[\psi]$ be an element of the mapping class group of genus g and its conjugacy class, respectively. A product of right handed Dehn twists $D_k D_{k-1} \cdots D_1$ is said to be *a positive presentation of* $[\psi]$ *by Dehn twists* if there exists a representative $\widetilde{\psi}$ of $[\psi]$ satisfying $\widetilde{\psi} = D_k D_{k-1} \cdots D_1$. If $\widetilde{\psi} = \psi$, we call it a positive presentation of ψ by Dehn twists. A positive presentation $D_k D_{k-1} \cdots D_1$ of $[\psi]$ by Dehn twists is called *minimal* if any positive presentation $D_{k'} D_{k'-1} \cdots D_1$ of $[\psi]$ by Dehn twists satisfies $k \le k'$. We call such natural number k *the length of* $[\psi]$ and denote it $L([\psi])$. We set $L([\mathrm{id}_{\Sigma_g}]) = 0$.

Let B and $\Delta_\varepsilon := \{s \in \boldsymbol{C} \,|\, |s| < \varepsilon\}$ be an open set and a small disk on $\boldsymbol{C}$, respectively. Let $\Psi\colon \mathscr{S} \to B \times \Delta_\varepsilon$ be a proper flat morphism from three dimensional complex manifold $\mathscr{S}$ to $B \times \Delta_\varepsilon$ whose general fibers are algebraic curves of genus g. We set $B_s := B \times \{s\}$, $\mathscr{S}_s := \Psi^{-1}(B_s)$ and $\Psi_s := \Psi|_{\mathscr{S}_s}$. Assume that $(\Psi_s, \mathscr{S}_s, B_s)$ is a family of curves of genus g for all s. Let D_{Ψ_s} and $^{\#}D_{\Psi_s}$ be the set of critical points of Ψ_s and its cardinality, respectively.

Definition 1.2. We call $(\Psi, \mathscr{S}, B \times \Delta_\varepsilon)$ *a simple splitting family* of a degeneration (f, S, B) if it satisfies the following conditions;

(i) (Ψ_0, S_0, B_0) and (f, S, B) are topologically equivalent,
(ii) $^{\#}D_{\Psi_s}$ is greater than one for all $s \in \Delta_\varepsilon \setminus \{0\}$ and the function $\Delta_\varepsilon \setminus \{0\} \to \boldsymbol{N}$ $(s \mapsto {}^{\#}D_{\Psi_s})$ is a constant function.

The conjugacy class $[\psi]$ is said to be *splittable* if there exist a degeneration (f, S, B) with monodromy $[\psi]$ and a splitting family $(\Psi, \mathscr{S}, B \times \Delta_\varepsilon)$ of (f, S, B).

We fix s $(\in \Delta_\varepsilon \setminus \{0\})$ and set $D_{\Psi_s} := \{t_j\}_{j=1,2,\cdots,k}$. Let l_j $(j = 1, 2, \cdots, k)$ be a simple closed curve on B_s rounding only one critical point t_j with a common initial points Q (see Figure 1) and $\psi^{l_j}_{\Psi_s}$ monodromy homeomorphism along l_j with the reference fiber $\Psi_s^{-1}(Q)$. Take a point Q' on B_0 and a path l on $B \times \Delta_\varepsilon \setminus \{\bigcup_{s\in\Delta_\varepsilon} D_{\Psi_s}\}$ connecting Q' with Q. Since we see that $l^{-1} \circ l_k \circ \cdots \circ l_1 \circ l$ is homotopic to a simple closed curve on B_0 rounding the critical point of Ψ_0, the composite $\psi^{l_k}_{\Psi_s} \circ \cdots \circ \psi^{l_1}_{\Psi_s}$ is isotopic to a monodromy homeomorphism ψ of (f, S, B).

If $[\psi^{l_j}_{\Psi_s}]$ is splittable, we can express $\psi^{l_j}_{\Psi_s}$ as a composite of homeomorphisms using splitting family in principle. Substituting this expression of $\psi^{l_j}_{\Psi_s}$ to $\psi = \psi^{l_k}_{\Psi_s} \circ \cdots \circ \psi^{l_1}_{\Psi_s}$, we express again ψ as a composite of homeomorphisms. If we obtain $\psi = \psi_K \circ \cdots \circ \psi_1$ such that the isotopy class of ψ_n $(n = 1, 2, \cdots, K)$ is a right handed Dehn twist by repeating the above "geometric" process, we obtain a positive presentation of the monodromy of (f, S, B) by Dehn twists. We call this presentation *a geometric presentation.*

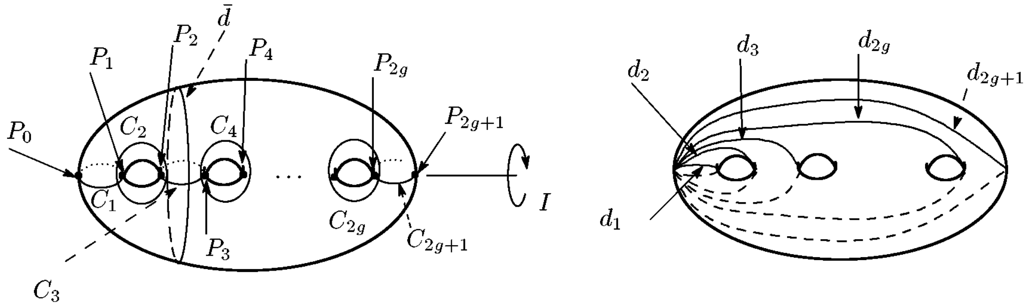
2. Presentations of monodromies by Dehn twists.

2.1. Construction of a splitting family.

In this section, we give a geometric presentation of a certain periodic monodromy. We use the same notation as in the previous section.

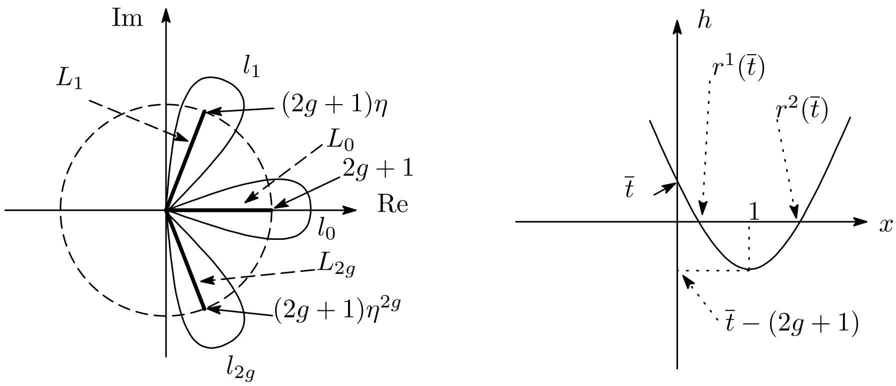
Let $(X_0 : X_1)$ be a homogeneous coordinates of $\mathbf{P}^1$ and (t, s) a coordinates of $\Delta \times \Delta' := \{(t, s) \in \mathbf{C} \times \mathbf{C} | |t| < 4(2g + 1), |s| < 2\}$. Let D be the divisor on $\mathbf{P}^1 \times \Delta \times \Delta'$ defined by the equation $h(X_0 : X_1, t, s) := X_0^{2g+2} - (2g+2)sX_0X_1^{2g+1} + tX_1^{2g+2} = 0$. We denote the associated line bundle of D by $[D]$. Since $[D]$ is an even bundle on $\mathbf{P}^1 \times \Delta \times \Delta'$, i.e., there exists a line bundle L satisfying $L^{\otimes 2} \simeq [D]$, we can construct a double covering $\Theta\colon \mathscr{S} \to \mathbf{P}^1 \times \Delta \times \Delta'$ branched along D. Note that $\mathscr{S}$ is constructed in the total space of L as a hypersurface. Since D is nonsingular, $\mathscr{S}$ is nonsingular variety of dimension three. Let $p_{23}\colon \mathbf{P}^1 \times \Delta \times \Delta' \to \Delta \times \Delta'$ be the natural projection. First, we determine the place of the singular fibers of $\Psi := p_{23} \circ \Theta\colon \mathscr{S} \to \Delta \times \Delta'$. Set $U := \{(X_0 : X_1, t, s) | X_1 \neq 0\}$, $x := X_0/X_1$, $h(x, t, s) := h(x, 1, t, s) = x^{2g+2} - (2g + 2)sx + t$, $\eta := e^{2\pi i/(2g+1)}$ and $\Theta_{(t,s)} := \Theta|_{\Theta^{-1}(\mathbf{P}^1 \times \{t\} \times \{s\})}$. It is sufficient to observe $\mathscr{S}$ over U because D does not intersect the divisor defined by $X_1 = 0$.

Let y be a fiber coordinates of L over U. Then, the defining equation of $\Theta^{-1}(U)$ is $y^2 = h(x, t, s)$. Since the natural morphism $\Theta_{(t,s)}\colon \Psi^{-1}(t, s) \to \mathbf{P}^1$ is of degree two and branched at the roots of $h(x, t, s) = 0$ for each $(t, s) \in \Delta \times \Delta'$, we see that $\Psi^{-1}(t, s)$ is nonsingular if and only if the equation $h(x, t, s)$ has distinct $2g + 2$-roots. Since the discriminant of the equation is $t^{2g+1} - (2g + 1)^{2g+1}s^{2g+2}$, $\Psi_s\colon \mathscr{S}_s \to \Delta^s (:= \Delta \times \{s\})$ has $2g + 1$ distinct singular fibers for each $s \in \Delta' \setminus \{0\}$. The critical points of Ψ_s are $t = (2g+1)s \cdot s^{1/(2g+1)}\eta^j$ $(j = 0, 1, \cdots, 2g)$ for each $s \neq 0$, where $s^{1/(2g+1)}$ is the $(2g+1)$-st root of s. Moreover, by easy calculations, we see that each singular fiber of Ψ_s has only one node as singularity. When $s = 0$, $(\Psi_0, \mathscr{S}_0, \Delta_0)$ has a unique singular fiber over the origin of Δ_0. Then, we see that $\Psi\colon \mathscr{S} \to \Delta \times \Delta'$ is a simple splitting family of $\Psi_0\colon \mathscr{S}_0 \to \Delta_0$. Thus, describing the monodromy homeomorphisms of $\Psi_s\colon \mathscr{S}_s \to \Delta_s$ $(s \neq 0)$, we obtain a geometric presentation of the monodromy of $(\Psi_0, \mathscr{S}_0, \Delta_0)$. Calculating the normally minimal model of $\Psi_0\colon \mathscr{S}_0 \to \Delta_0$, we see that the monodromy of this degeneration is



(A) (B)

Figure 3.



(A) (B)

Figure 4.

periodic with the period $2g+2$ and total valency $1/(2g+2)+1/(2g+2)+g/(g+1)$.

We observe monodromy homeomorphisms of $\Psi_1 \colon \mathscr{S}_1 \to \Delta_1$, for simplicity. Let α be the real root of $x^{2g+1}-(2g+2)=0$. We take $\Psi_1^{-1}(0)$ as a reference fiber. Note that the defining equation of $\Psi_1^{-1}(0)$ is $y^2 = x^{2g+2}-(2g+2)x$ and the morphism $\Theta_{(0,1)} \colon \Psi_1^{-1}(0) \to \mathbf{P}^1$ of degree two is branched at $(2g+2)$-points $Q_0 := \{x=0\}$, $Q_j := \{x = \alpha\eta^j\}$ $(j = 0, 2, \cdots, 2g)$. Let $\{P_j\}$ $(j = 0, 1, \cdots, 2g+1)$ be points such that $\Theta_{(0,1)}(P_j) = Q_j$. We draw a picture of the reference fiber indicating the place of $\{P_j\}$ (See Figure 3(A)). In Figure 3, I means the hyperelliptic involution of the reference fiber.

Let l_j and L_j $(j = 0, 1, \cdots, 2g)$ be simple closed curves and the segments on Δ_1 as shown in Figure 4(A). Each simple closed curve l_j is beginning at the origin and rounds the critical point $t = (2g+1)\eta^j$. Each L_j is the path connecting the origin with $t = (2g+1)\eta^j$.

THEOREM 2.1. *The notation is as above. The isotopy class of the monodromy homeomorphism along l_j with reference fiber $\Psi_1^{-1}(0)$ is the right handed Dehn twist in*

d_{j+1} described in Figure 3(B). Thus, $D_{d_{2g+1}}D_{d_{2g}}\cdots D_{d_2}D_{d_1}$ is a positive presentation of the periodic monodromy with the period $2g+2$ and the total valency $1/(2g+2)+1/(2g+2)+g/(g+1)$ by Dehn twist.

PROOF. Let B_j ($j=0,1,\cdots,2g$) be a simply connected open subset on Δ^1 that contains l_j and does not contain the critical points of Ψ_1 except $t=(2g+1)\eta^j$. Thus, $\Psi_1|_{\Psi_1^{-1}(B_j)}\colon \Psi_1^{-1}(B_j)\to B_j$ is a degeneration of Lefschetz type. We first observe a monodromy homeomorphism along l_0. Let $\bar t$ ($0\le \bar t\le 2g+1$) be a parameter of L_0. We consider $h(x,1,\bar t,1)$ as a real valued function $h(x,1,\bar t,1)\colon \boldsymbol{R}\to\boldsymbol{R}$ ($x\mapsto x^{2g+2}-(2g+2)x+\bar t$). The graph of this function is as shown in Figure 4(B). Let $r^1(\bar t)$ and $r^2(\bar t)$ be two real roots of $h(x,1,\bar t,1)=0$ satisfying $r^1(\bar t)\le r^2(\bar t)$. From the graph, we see that $|r^1(\bar t)-r^2(\bar t)|$ is a monotone decreasing function of $\bar t$ and $|r^1(2g+1)-r^2(2g+1)|=0$. Since the points $x=r^1(\bar t)$ and $x=r^2(\bar t)$ on $\mathbf{P}^1$ are branch points of $\Theta_{(\bar t,1)}$, the lift $\langle r^1(\bar t),r^2(\bar t)\rangle$ of the segment connecting $x=r^1(\bar t)$ with $x=r^2(\bar t)$ by $\Theta_{(\bar t,1)}$ is a simple closed curve on $\Psi_1^{-1}(\bar t)$ that is not homotopic to zero for each $\bar t(\ne 2g+1)$. When $\bar t=2g+2$, the limit of $\langle r^1(\bar t),r^2(\bar t)\rangle$ is the node on the singular fiber $\Psi_1^{-1}(2g+1)$, i.e., as $\bar t$ increase, $\langle r^1(\bar t),r^2(\bar t)\rangle$ becomes smaller, and finally, it becomes the node on the singular fiber. Thus, we see that $\{\langle r^1(\bar t),r^2(\bar t)\rangle | t\in[0,2g+1)\}$ are the vanishing cycles of the Lefschetz fibration $\Psi_1|_{\Psi^{-1}(B_0)}\colon \Psi_1^{-1}(B_0)\to B_0$. By Kas's theorem ([**Ka**]), the isotopy class of a monodromy homeomorphism with reference fiber $\Psi_1^{-1}(0)$ is the right handed Dehn twist in $\langle r^1(0),r^2(0)\rangle$. Since $r^1(0)=0$ and $r^2(0)=\alpha$, we see that $\langle r^1(0),r^2(0)\rangle$ is homotopic to d_1 as shown in Figure 3(B).

By similar way, we observe the behavior of the roots of $h(x,1,t,1)=0$ when t moves along L_j and determine the vanishing cycles. For each $\bar t$ ($0\le\bar t\le 2g+1$), we see that $\eta^j r^1(\bar t)$ and $\eta^j r^2(\bar t)$ are roots of $h(x,1,\eta^j\bar t,1)=0$. Thus, the lift $\langle \eta^j r^1(\bar t),\eta^j r^2(\bar t)\rangle$ of the segment connecting $x=\eta^j r^1(\bar t)$ with $x=\eta^j r^2(\bar t)$ on $\Psi_1^{-1}(\eta^j\bar t)$ is a simple closed curve for each $\bar t\ne 2g+1$. We see that these curves are vanishing cycles of Lefschetz fibration $\Psi_1|_{\Psi^{-1}(B_j)}\colon \Psi_1^{-1}(B_j)\to B_j$ and the monodromy homeomorphisms are isotopic to the Dehn twists in $\langle \eta^j r^1(0),\eta^j r^2(0)\rangle$. Since $\eta^j r^1(0)=0$ and $\eta^j r^2(0)=\alpha\eta^j$, we see that $\langle \eta^j r^1(0),\eta^j r^2(0)\rangle$ is homotopic to d_{j+1} and we obtain the assertion. □

Let $C_1,C_2,\cdots,C_{2g+1}$ be simple closed curves as shown in Figure 3(A). By easy calculations, we see that d_j is homotopic to $D_{C_j}^{-1}D_{C_{j-1}}^{-1}\cdots D_{C_2}^{-1}(C_1)$. Thus, by Lemma 1 in [**Li**], we obtain that

$$D_{d_j}=D_{C_j}^{-1}D_{C_{j-1}}^{-1}\cdots D_{C_2}^{-1}D_{C_1}D_{C_2}\cdots D_{C_{j-1}}D_{C_j}. \tag{1}$$

COROLLARY 2.2. *The product of the Dehn twists $D_{C_1}D_{C_2}\cdots D_{C_{2g+1}}$ is also a positive presentation of the periodic monodromy of Theorem 2.1.*

PROOF. We denote the right handed Dehn twist in C_i by i. For example, $3^{-1}12$ means $D_{C_3}^{-1}D_{C_1}D_{C_2}$. We know the braid relations that $ij=ji$ ($|i-j|>1$) and $j(j+1)j=(j+1)j(j+1)$. Thus, we obtain the following (we use the braid relations at the terms with underlines);

$$\begin{aligned}(n^{-1}\cdots 3^{-1}2^{-1}123\cdots n)(123\cdots(n-1)) &= (n^{-1}\cdots 3^{-1}2^{-1}\underline{121}\cdots n)(23\cdots(n-1))\\ &= (n^{-1}\cdots 3^{-1}2^{-1}212\cdots n)(23\cdots(n-1))\\ &= 1(n^{-1}\cdots 3^{-1}23\cdots n)(23\cdots(n-1))\\ &= 1(n^{-1}\cdots 3^{-1}\underline{232}4\cdots n)(34\cdots(n-1))\\ &= 12(n^{-1}\cdots 4^{-1}34\cdots n)(34\cdots(n-1))\\ &\ \ \vdots\\ &= 123\cdots n.\end{aligned}$$

Using this relation and (1), we obtain the assertion. □

Remark 2.3. We immediately obtain the relation $\{123\cdots(2g)(2g+1)\}^{2g+2}=\mathrm{id}_{\Sigma_g}$ from Corollary 2.2. This relation is called *the odd chain relation* (c.f. [**J**]).

2.2. Example of non-minimal geometric presentation.

In this section, we introduce an example such that a geometric presentation of a monodromy is not minimal.

Definition 2.4. Let $f\colon S\to B$ be a family of curves of genus g. We denote the topological Euler number of a fiber $f^{-1}(P)$ $(P\in B)$ by $E(f^{-1}(P))$. Setting $E_f:=\Sigma_{P\in\Delta}\{E(f^{-1}(P))-(2-2g)\}$ and we call it *the Euler contribution* of the family.

If $f^{-1}(P)$ is nonsingular, $E(f^{-1}(P))-(2-2g)=0$. Since E_f is a topological invariant, it is invariant under deformations. In particular, for any simple splitting $\Psi\colon\mathscr{S}\to\Delta\times\Delta'$, we see that $E_{\Psi_0}=E_{\Psi_s}$ $(s\in\Delta')$. Note that $\{E(f^{-1}(P))-(2-2g)\}=1$ when $f^{-1}(P)$ has a node as singularity. Thus, if the monodromy of $\Psi_0\colon\mathscr{S}_0\to\Delta_0$ has a geometric presentation, it is expressed as a product of E_{Ψ_0}-Dehn twists.

Let $f\colon S\to\Delta$ be a degeneration of curves of genus two whose monodromy is periodic with the period 6 and the total valency $5/6+5/6+1/3$ (cf. Example 1.1). Set $\psi:=D_{C_1}D_{C_2}D_{C_3}D_{C_4}D_{C_5}$. We write it $\psi=12345$, for short. Since $[\psi]$ is periodic with the period 6 and $\mathrm{tv}[\psi]=1/6+1/6+2/3$ by Theorem 2.1 and Corollary 2.2, we see that the monodromy of $f\colon S\to\Delta$ is $[\psi^5]$, i.e., the conjugacy class of $(12345)^5$ from Example 1.1. By [**H**] and [**AA**], we see that any monodromy has a geometric presentation in the case when $g=2$. Thus, $[\psi^5]$ has a geometric presentation. Since $E_f=15$, we see that any splitting of $f\colon S\to\Delta$ gives a geometric presentation of $[\psi^5]$ by 15 Dehn twists. If a geometric presentation always gave a minimal presentation, any positive presentation of $[\psi^5]$ would need at least 15 Dehn twists. However, using the chain relations and braid relations, we obtain a positive presentation of $[\psi^5]$ by 14 Dehn twists as follows (mainly, we use the relation $321=123213^{-1}2^{-1}$ and $432=234324^{-1}3^{-1}$ obtained from the braid relations);

$$\begin{aligned}\psi^5 &= 12345\cdot\underline{12}345\cdot\underline{1}2345\cdot(12345)^2\\ &= 121\underline{32}1\cdot\underline{4}5\underline{3}452\underline{2}345\cdot(12345)^2\\ &= (12)(12)\cdot\underline{321}3^{-1}\cdot 432\underline{545}345\cdot(12345)^2\end{aligned}$$

$$
\begin{aligned}
&= (12)^6 321 \cdot \underline{3^{-1}2^{-1}3^{-1}} \cdot \underline{2^{-1}3^{-1}2^{-1}} \cdot \underline{3^{-1}2^{-1}3^{-1}} \cdot \underline{432} \cdot 45\underline{434}5 \cdot (12345)^2 \\
&= (12)^6 \cdot (3212^{-1}3^{-1}) \cdot 2^{-1}3^{-1}2^{-1}3^{-1}2^{-1}3^{-1}2^{-1} \cdot (23)^2 432 (4^{-1}3^{-1})^2 \cdot 4\underline{5}3\underline{4}3\underline{5} \cdot (12345)^2 \\
&= (12)^6 \cdot (3212^{-1}3^{-1}) \cdot 2^{-1}3^{-1}2^{-1}\underline{432}(4^{-1}3^{-1})^2 \cdot 43\underline{545}3 \cdot (12345)^2 \\
&= (12)^6 \cdot (3212^{-1}3^{-1}) 2^{-1}3^{-1}2^{-1}24323^{-1}(4^{-1}3^{-1})^2 \underline{434}543 \cdot (12345)^2 \\
&= (12)^6 \cdot (3212^{-1}3^{-1}) \cdot (2^{-1}3^{-1}432) 3^{-1}(4^{-1}3^{-1})^2 343543 \cdot (12345)^2 \\
&= (12)^6 \cdot (3212^{-1}3^{-1}) \cdot (2^{-1}3^{-1}432) \cdot (3^{-1}4^{-1}543) \cdot (12345)^2.
\end{aligned}
$$

Note that, by even chain relation, $(12)^6$ is isotopic to the Dehn twist in $\overline{d}$ as shown in Figure 3(A). Since $3212^{-1}3^{-1}$, $2^{-1}3^{-1}432$ and $3^{-1}4^{-1}543$ are $(32)1(32)^{-1}$, $(32)^{-1}4(32)$ and $(43)^{-1}5(43)$, respectively, each of them is isotopic to a Dehn twist.

Remark 2.5. By easy calculation, $[\psi^3]$ is periodic with the period 2 and $\mathrm{tv}([\psi^3]) = 1/2 + 1/2$. In [**M**], he obtained a geometric presentation of $[\psi^3]$ by Dehn twists whose length is four. Using this, we also see that there exists a presentation of $[\psi^5]$ by 14 Dehn twists.

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