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Orbifold elliptic genera and rigidity

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Abstract. Rigidity theorems for equivariant elliptic genus and equivariant orbifold elliptic genus of almost complex orbifolds with compact connected group actions are formulated and proved.

1. Introduction.

Let M be a closed almost complex manifold on which a compact connected Lie group G acts non-trivially. If the first Chern class of M is divisible by an integer N greater than 1, then its equivariant elliptic genus $\varphi(M)$ of level N is rigid, i.e., it is constant as a virtual character of G. This result was predicted by Witten [**15**] and proved by Taubes [**13**], Bott-Taubes [**1**] and Hirzebruch [**6**]. Elliptic genus can be defined even for almost complex orbifolds. Moreover another genus called orbifold elliptic genus is defined for orbifolds. A natural question is whether the rigidity property holds for these genera on orbifolds or not. It turns out that the answer is no in general. In [**5**] we were concerned with related topics.

In this note a modified orbifold elliptic genus of level N will be defined for closed almost complex orbifolds such that N is relatively prime to the orders of all isotropy groups. One of main results, Theorem 3.1, states that the modified orbifold elliptic genus $\breve{\varphi}(X)$ of level N of an almost complex orbifold X of dimension $2n$ such that $\Lambda^n TX = L^N$ for some *orbifold* line bundle L is rigid for non-trivial G action. As to the orbifold elliptic genus itself Theorem 3.3 states that the orbifold elliptic genus $\hat{\varphi}(X)$ of level N of X is rigid for non-trivial G action if $\Lambda^n TX = L^N$ for some *genuine* G line bundle L. Furthermore the orbifold elliptic genus $\hat{\varphi}(X)$ is rigid for non-trivial G action if some positive power of $\Lambda^n TX$ is trivial as an orbifold line bundle (Theorem 3.4). The last result is essentially due to Dong, Liu and Ma [**2**].

Liu [**10**] gave a proof of rigidity by using modular property of elliptic genera for manifolds. Our proof of the rigidity for the genera $\breve{\varphi}(X)$ and $\hat{\varphi}(X)$ also uses Liu's method.

The organization of the paper is as follows. In Section 2 we review basic materials concerning orbifolds in general. The notion of sectors is particularly relevant for later use. In Section 3 we give the definitions of orbifold elliptic genus and modified orbifold elliptic genus and the main theorems are stated here. Section 4 is devoted to exhibiting fixed point formulae for the above genera. The proof of the main results will be given in Section 5 and Section 6. In Section 6 some additional results related to vanishing property are

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given. Main results in this section are Propositions 6.6, 6.8 and 6.10. Section 7 concerns the orbifold T_y genus and its modified one. They are always rigid for non-trivial actions of compact connected Lie groups and take special forms when the orbifold elliptic genera vanish. In Section 8 the generalization to the case of stably almost complex orbifolds are discussed and it will be shown that main results in Section 3, Section 6 and Section 7 also hold for stably almost complex orbifolds.

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2. Sectors.

We first recall some basic facts about orbifolds. We refer to [**11**] and [**3**] for relevant definitions and to [**5**] for notations used here. In [**11**] orbifolds were called V-manifolds.

Let X be a closed orbifold of dimension n and let $\mathscr{U}$ be an atlas of X. $\mathscr{U}$ is a collection of (orbifold) charts $\{(V_\mu, U_\mu, H_\mu, p_\mu)\}$ where U_μ is an open set in X, V_μ is a smooth manifold of dimension n, H_μ is a finite group acting on V_μ and p_μ is a map $V_\mu \to U_\mu$ inducing a homeomorphism from V_μ/H_μ onto U_μ. The collection $\{U_\mu\}$ is assumed to contain a basis of neighborhoods for each point $x \in X$. Let $(V_i, U_i, H_i, p_i),\ i = 1, 2$, be two orbifold charts in $\mathscr{U}$ such that $U_1 \subset U_2$. A pair of injective group homomorphism $\rho : H_1 \to H_2$ and ρ-equivariant open embedding $\psi : V_1 \to V_2$ covering the inclusion map $U_1 \to U_2$ with the following property

$$\{h \in H_2 \mid h(\psi(V_1)) \cap \psi(V_1) \neq \varnothing\} = \rho(H_1) \tag{1}$$

is called an injection of charts and will be written as

$$\Psi = (\rho, \psi) : (V_1, U_1, H_1, p_1) \to (V_2, U_2, H_2, p_2).$$

Note that, if $h \in H_2$, then $(c_h \circ \rho, h \circ \psi)$ is an injetion of charts, where $c_h : H_2 \to H_2$ denotes the conjugation by h. These charts must satisfy the following compatibility condition. If U_1 and U_2 have a non-empty intersection, then, for each $x \in U_1 \cap U_2$, there are a chart $(V, U, H, p) \in \mathscr{U}$ with $x \in U \subset U_1 \cap U_2$ and injections of charts $\Psi_i = (\rho_i, \psi_i) : (V, U, H, p) \to (V_i, U_i, H_i, p_i),\ i = 1, 2$.

Note. We do not assume the action of H_μ on V_μ is effective. If the effectiveness is assumed then the condition (1) automatically follows.

When one considers additional structures like Riemannian structure or almost complex structure, the V_μ are assumed to have the structures in question, and the action of H_μ and the maps ψ are assumed to preserve those structures.

Let $\Psi = (\rho, \psi) : (V_1, U_1, H_1, p_1) \to (V_2, U_2, H_2, p_2)$ be an injection of charts. One readily sees that the isotropy subgroup $H_{1,v}$ at a point $v \in V_1$ is isomorphic to that of $H_2 \in V_2$ at $\psi(v)$ as a consequence of (1). Thus the isomorphism class of the group $H_{1,v}$ depends only on $x = p_1(v) \in X$. It is called the *isotropy group* of the orbifold X at x

and will be denoted by H_x.

For each $x \in X$ there is an orbifold chart (V_x, U_x, H_x, p_x) such that $p^{-1}(x)$ is a single point, and hence H_x is the isotropy group at x. Moreover we can take U_x small enough so that V_x is modelled on an open ball in $\boldsymbol{R}^n$ with a linear H_x action. Then the fixed point set V_x^h of each $h \in H_x$ is connected. Such a chart will be called a *reduced chart* centered at x. Hereafter we sometimes write simply (V, U, H) for an orbifold chart (V, U, H, p), when the meaning of p is clear from the context. We also assume that orbifolds considered hereafter are closed (compact, connected and without boundary) orbifolds.

Fix a finite group H and set $X_H = \{x \in X \mid H_x \cong H\}$. Then X_H is a smooth manifold. In fact, if (V_x, U_x, H_x, p_x) is a reduced chart centered at $x \in X_H$, then p_x induces a homeomorphism between $V_x^{H_x}$ and $X_H \cap U_x$. A connected component of X_H is called a *stratum* of X. The totality of strata is called a stratification of X. If X is connected then there is a unique stratum S_H such that H is the minimum with respect to the obvious ordering among isotropy groups induced by inclusions. This stratum is called principal stratum of X and the order $|H|$ is called the multiplicity of the connected orbifold X and is denoted by $m(X)$. The principal stratum is open and dense in X. In case the actions of isotropy groups are effective we have $m(X) = 1$.

A map $f : X \to X'$ from an orbifold X to another orbifold X' is called *smooth* if, for each orbifold chart (V, U, H, p) around x and (V', U', H', p') around $f(x)$ such that $f(U) \subset U'$, there is a pair (ρ, ψ) of group homomorphism $\rho : H \to H'$ and ρ-equivariant smooth map $\psi : V \to V'$ satisfying the relation $p' \circ \psi = f \circ p$. The totality of these pairs (ρ, ψ) is required to satisfy the obvious compatibility relation with respect to injections of orbifold charts for X and X'. Such a pair (ρ, ψ) is called a map of orbifold chart covering f and will be written $(\rho, \psi) : (V, U, H, p) \to (V', U', H', p')$.

Let G be a Lie group. An action of G on an orbifold X is a smooth map $G \times X \to X$ which satisfies the usual rule of group action.

Let X' be an orbifold with atlas $\mathscr{U}'$. Let X be a subspace of X' such that, for each chart $(V', U', H, p) \in \mathscr{U}'$ of X', $p^{-1}(X \cap U')$ is an H-invariant submanifold V of V' of dimension n. Then $\mathscr{U} = \{(V, X \cap U', H, p|V) \mid (V', U', H, p) \in \mathscr{U}'\}$ defines an orbifold structure on X. With this structure X is called a suborbifold of X'.

A triple (W, X, p) of orbifolds W, X and smooth map $\pi : W \to X$ is called an orbifold vector bundle over X if it satisfies the following condition. For any orbifold chart (V, U, H, p) for X with U sufficiently small there is an orbifold chart $(\tilde{V}, \tilde{U}, \tilde{H}, \tilde{p})$ for W with $\tilde{U} = \pi^{-1}(U)$ and $\tilde{H} = H$ together with a map of orbifold chart (ρ, ψ) : $(\tilde{V}, \tilde{U}, \tilde{H}, \tilde{p}) \to (V, U, H, p)$ covering $\pi : W \to X$ such that $\rho =$ identity and $\tilde{p} : \tilde{V} \to V$ is a vector bundle with compatible H action. The tangent bundle TX of an orbifold X is a typical example of vector bundles. Locally its orbifold chart is given by $(TV, TV/H, H)$. A smooth map $s : X \to W$ is a section if it is locally an H_x-invariant section. If s is a section then $\pi \circ s$ equals the identity map. A differential form of degree q is a section of the q-th exterior power $\Lambda^q T^* X$ of the cotangent bundle $T^* X$.

We are now in a position to give the definition of so called sectors of an orbifold X. The notion was first introduced by Kawasaki in [**8**] in order to describe the index theorem for orbifolds. We set

$$\hat{X} = \bigsqcup_{x \in X} \mathscr{C}(H_x),$$

where $\mathscr{C}(H)$ denotes the set of conjugacy classes of a group H. We define $\pi : \hat{X} \to X$ by $\pi(\mathscr{C}(H_x)) = x$. An orbifold structure is endowed on $\hat{X}$ in the following way. Let (V, U, H, p) be an orbifold chart for X. We set

$$\hat{V} = \{(v, h) \in V \times H \mid hv = v\} \text{ and } \hat{U} = \bigsqcup_{x \in U} \mathscr{C}(H_x) \subset \hat{X}.$$

The group H acts on $\hat{V}$ by

$$g(v, h) = (gv, ghg^{-1}).$$

We define $\hat{p} : \hat{V} \to \hat{U}$ by $\hat{p}(v, h) = [h] \in \mathscr{C}(H_{p(v)})$. Then $\hat{p}$ induces a bijection $\hat{V}/H \to \hat{U}$. By this bijection we identify $\hat{V}/H$ and $\hat{U}$. Using this identification one can give a topology on $\hat{X}$ and the collection of quadruples $\{(\hat{V}, \hat{U}, H, \hat{p})\}$ defines an orbifold structure on $\hat{X}$. Moreover, if we define $\hat{\pi} : \hat{V} \to V$ by $\hat{\pi}(v, h) = v$, then $(\mathrm{id}, \hat{\pi})$ defines a map of orbifold chart $(\hat{V}, \hat{U}, H, \hat{p}) \to (V, U, H, p)$. It follows that the map $\pi : \hat{X} \to X$ is a smooth map of orbifolds.

$\hat{X}$ is not connected unless X is a (connected) smooth manifold. Its connected component are sometimes called *sectors*. We shall call $\hat{X}$ the *total sector* of X. To get reduced charts of $\hat{X}$ centered at $\hat{x} \in \hat{X}$ we proceed as follows. Note that, if $C(h)$ denotes the centralizer of $h \in H$, then $C(h)$ acts on the fixed point set V^h of h. If h and h' are conjugate each other in H, then there is a canonical homeomorphism between $V^h/C(h)$ and $V^{h'}/C(h')$. So we can associate the space $V^h/C(h)$ to each conjugacy class $[h]$. We also see that there is a disjoint sum decomposition of $\hat{U} = \hat{V}/H$

$$\hat{U} = \bigsqcup_{[h] \in \mathscr{C}(H)} V^h/C(h).$$

Take a reduced orbifold chart (V_x, U_x, H_x) centered at x. Then V_x^h is connected and hence $(V_x^h, V_x^h/C(h), C(h))$ gives us a reduced chart of $\hat{X}$ centered at $[h] \in \mathscr{C}(H_x)$.

NOTE. The projection $\hat{U} \to U$ is covered by the inclusion map $V_x^h \to V_x$. Hence $\pi : \hat{X} \to X$ is an immersion of orbifold.

When X is connected and its multiplicity $m(X)$ is equal to 1, there is a unique component of $\hat{X}$ which is mapped isomorphically onto X by π, where the point over $x \in X$ is the identity element in H_x regarded as an elemnet in $\mathscr{C}(H_x)$. Components with lower dimensions are called *twisted sectors*.

In order to look more closely at a sector, consider a reduced chart (V_x, U_x, H_x, p_x) centered at x and take a point y in U_x. $p_x^{-1}(y)$ is an orbit of H_x and the isotropy subgroup $(H_x)_v$ of H_x at each point $v \in p_x^{-1}(y)$ is isomorphic to H_y. This determines an injection of H_y into H_x unique up to conjugations by elements of H_x. This injection induces a canonical map $\rho_{x,y} : \mathscr{C}(H_y) \to \mathscr{C}(H_x)$.

Let $\{\hat{X}_{\hat{\lambda}}\}_{\hat{\lambda} \in \hat{\Lambda}}$ denote the totality of sectors. Take $\gamma_x \in \hat{X}_{\hat{\lambda}}$ and take a reduced chart (V_x, U_x, H_x, p_x) centered at $x = \pi(\gamma_x) \in X$. Then the following lemma gives another local expression of a sector.

LEMMA 2.1.

$$\hat{X}_{\hat{\lambda}} \cap \pi^{-1}(U_x) = \{\gamma_y \mid y \in U_x, \gamma_y \in \pi^{-1}(y), \rho_{x,y}(\gamma_y) = \gamma_x\}.$$

PROOF. It is clear from the definition that the left hand side is contained in the right hand side. Conversely take an element $\gamma_y \in \pi^{-1}(U_x)$ with $\pi(\gamma_y) = y$ and $\rho_{x,y}(\gamma_y) = \gamma_x$. Let v be a point in V_x such that $p_x(v) = y$, and identify $(H_x)_v$ with H_y. If $h \in H_x$ is a representative of γ_x, then h belongs to $(H_x)_v = H_y$ since $\rho_{x,y}(\gamma_y) = \gamma_x$. Hence $v \in V_x^h$. This means that $\gamma_y = [h]$ belongs to the same component $\hat{X}_{\hat{\lambda}}$ as γ_x. □

We set

$$\hat{H}_x = \{h \in H_x \mid V_x^h = V_x^{H_x}\}.$$

This set is closed under conjugations. The conjugacy classes in $\hat{H}_x$ is denoted by $\mathscr{C}(\hat{H}_x)$. The following equality is immediate from the definition.

$$\mathscr{C}(\hat{H}_x) = \{\gamma_x \in \mathscr{C}(H_x) \mid \gamma_x \notin \rho_{x,y}(\mathscr{C}(H_y)) \text{ for any } y \in U_x \setminus (U_x \cap p_x(V_x^{H_x}))\}.$$

In this paper we shall make the assumption

(#) *the fixed point set V_x^H of each subgroup H of H_x has even codimension in V_x*

throughout. This is the case of stably almost complex orbifolds.

Let $\hat{X}_{\hat{\lambda}}$ be a sector. A point γ_x in $\hat{X}_{\hat{\lambda}}$ such that $\gamma_x \in \mathscr{C}(\hat{H}_x)$ where $x = \pi(\gamma_x)$ will be called *generic*.

LEMMA 2.2. *The set of all generic points in $\hat{X}_{\hat{\lambda}}$ is a connected and dense open set in $\hat{X}_{\hat{\lambda}}$.*

PROOF. Let $(V_x^h, V_x/C(h), C(h))$ be a reduced chart centered at $\gamma_x = [h]$. If γ_x lies in the principal stratum of $\hat{X}_{\hat{\lambda}}$, then $V_x^h = V_x^{C(h)}$. Assume that $V_x^h \neq V_x^{H_x}$ and hence $V_x^h \supsetneqq V_x^{H_x}$. Then there exists a point $x_1 \in V_x$ such that $C(h) \subset H_{x_1} \subsetneqq H_x$.

If $V_{x_1}^h \neq V_{x_1}^{H_{x_1}}$ further, then there is a sequence of points $x_1, x_2, \ldots \in V_x$ such that

$$H_x \supsetneqq H_{x_1} \supsetneqq H_{x_2} \supsetneqq \cdots.$$

Since H_x is a finite group, this sequence terminates at a finite step and there is a point $y \in V_x$ and $h \in H_y$ such that $V_y^h = V_y^{H_y}$. Then $\gamma_y = [h] \in \mathscr{C}(\hat{H}_y)$ is generic. This proves the existence of generic points.

We shall denote by $\hat{X}_{\hat{\lambda}}^{gen}$ the set of generic points in $\hat{X}_{\hat{\lambda}}$. If γ_x lies in $\hat{X}_{\hat{\lambda}} \setminus \hat{X}_{\hat{\lambda}}^{gen}$, then there is a point $\gamma_{x_0} = [h_0] \in \hat{X}_{\hat{\lambda}}^{gen}$ near γ_x such that

$$V_x^{h_0} \supsetneqq V_x^{H_x},$$

where $x = \pi(\gamma_x)$. Since $V_x^{H_x}$ has even codimension in $V_x^{h_0}$ by the assumption (#), it follows that $V_x^{h_0} \setminus V_x^{H_x}$ is connected. This implies that $\hat{X}_{\hat{\lambda}}^{gen}$ is connected. It is also open and dense in $\hat{X}_{\hat{\lambda}}$. □

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be the totality of strata of X. The isomorphism class of isotropy group at any point in S_λ depends only on S_λ and is denoted by H_λ. The index set Λ is a poset by the ordering $\prec$ defined by

$$\lambda \prec \lambda' \Leftrightarrow S_\lambda \subset \bar{S}_{\lambda'}.$$

Note that, if $\lambda \prec \lambda'$, then $H_\lambda \supset H_{\lambda'}$ in the sense that there is an injective homomorphism $H_{\lambda'} \to H_\lambda$. When X is connected, Λ has a unique maximal element λ_0. S_{λ_0} is the principal stratum and H_{λ_0} is the minimum isotropy group.

Since $\pi(\hat{X}_{\hat{\lambda}}^{gen})$ is connected, there is a unique $\lambda \in \Lambda$ such that $\pi(\hat{X}_{\hat{\lambda}}^{gen}) \subset S_\lambda$. This λ will be denoted by $\pi(\hat{\lambda})$. Note that $\pi : \hat{X} \to X$ maps $\hat{X}_{\hat{\lambda}}$ onto $\bar{S}_\lambda$. The restriction of π on $\hat{X}_{\hat{\lambda}}^{gen}$ is a covering map onto S_λ.

Remark 2.3. An orbifold X is orientable by definition if V can be given an orientation and the action of H on V is orientation preserving for each orbifold chart (V, U, H), and ψ preserves the given orientations of V and V' for each injection $(\rho, \psi) : (V, U, H) \to (V', U', H')$. In this case X regarded as a $\boldsymbol{Q}$-homology manifold is orientable. It should be noticed however that strata and sectors of X are not orientable in general even if the assumption (#) is satisfied.

3. Orbifold elliptic genus.

In this section X will be an almost complex closed orbifold. We shall give the definition of elliptic genus $\varphi(X)$, orbifold elliptic genus $\hat{\varphi}(X)$ and modified orbifold elliptic genus $\breve{\varphi}(X)$.

If (V, U, H) is an orbifold chart of X, then V is an almost complex manifold and the action of each element $h \in H$ on V preserves almost complex structure. Hence V^h is also an almost complex manifold. It follows that the sectors of X are all almost complex orbifolds. Let $\hat{X}_{\hat{\lambda}}$ be a sector of X, and let $W_{\hat{\lambda}}$ be the normal bundle of the immersion $\pi : \hat{X}_{\hat{\lambda}} \to X$. Take $\gamma_x \in \hat{X}_{\hat{\lambda}}$ with $\pi(\gamma_x) = x$. Let $h \in H_x$ be a representative of γ_x. Then h acts on the normal bundle $\tilde{W}_x$ of V_x^h in V_x. Let

$$\tilde{W}_x = \bigoplus \tilde{W}_{x,i} \tag{2}$$

be the eigen-bundle decompositions with respect to this action of h, where h acts on $\tilde{W}_{x,i}$ with weight $m_{\hat{\lambda},i}$, i.e., by the multiplication of $e^{2\pi\sqrt{-1}m_{\hat{\lambda},i}} \neq 1$. $m_{\hat{\lambda},i}$ is determined modulo integers and depends only on $[h] = \gamma_x$. If y is near x and γ_y lies in $\hat{X}_{\hat{\lambda}}^{gen}$, then $\rho_{x,y}(\gamma_y) = \gamma_x$ by Lemma 2.1. Hence $m_{\hat{\lambda},i}$ mod $\boldsymbol{Z}$ is a locally constant function of γ_x and consequnetly it is constant on $\hat{X}_{\hat{\lambda}}$. It follows that the decomposition (2) gives the one for the bundle $W_{\hat{\lambda}}$:

$$W_{\hat{\lambda}} = \bigoplus W_{\hat{\lambda},i}. \tag{3}$$

The $m_{\hat{\lambda},i}$ with $0 < m_{\hat{\lambda},i} < 1$ is written by $f_{\hat{\lambda},i}$. We set $f_{\hat{\lambda}} = \sum_i f_{\hat{\lambda},i} \dim W_{\hat{\lambda},i}$.

Let $\tau, \sigma \in \boldsymbol{C}$ with $\Im(\tau) > 0$. We set $q = e^{2\pi\sqrt{-1}\tau}, \zeta = e^{2\pi\sqrt{-1}\sigma}$. Let TX be the complex tangent bundle of the almost complex orbifold X. We define formal vector bundles $\mathscr{T} = \mathscr{T}(\sigma), \mathscr{T}_{\hat{\lambda}} = \mathscr{T}_{\hat{\lambda}}(\sigma), \mathscr{W}_{\hat{\lambda},i} = \mathscr{W}_{\hat{\lambda},i}(\sigma)$ and $\mathscr{W}_{\hat{\lambda}} = \mathscr{W}_{\hat{\lambda}}(\sigma)$ by

$$\mathscr{T}(\sigma) = \Lambda_{-\zeta} T^*X \otimes \bigotimes_{k=1}^{\infty} \big(\Lambda_{-\zeta q^k} T^*X \otimes \Lambda_{-\zeta^{-1}q^k} TX \otimes S_{q^k} T^*X \otimes S_{q^k} TX\big),$$

$$\mathscr{T}_{\hat{\lambda}}(\sigma) = \Lambda_{-\zeta} T^*\hat{X}_{\hat{\lambda}} \otimes \bigotimes_{k=1}^{\infty} \big(\Lambda_{-\zeta q^k} T^*\hat{X}_{\hat{\lambda}} \otimes \Lambda_{-\zeta^{-1}q^k} T\hat{X}_{\hat{\lambda}} \otimes S_{q^k} T^*\hat{X}_{\hat{\lambda}} \otimes S_{q^k} T\hat{X}_{\hat{\lambda}}\big),$$

$$\begin{aligned}\mathscr{W}_{\hat{\lambda},i}(\sigma) = \Lambda_{-\zeta q^{f_{\hat{\lambda},i}}} W^*_{\hat{\lambda},i} &\otimes \bigotimes_{k=1}^{\infty} \big(\Lambda_{-\zeta q^{f_{\hat{\lambda},i}+k}} W^*_{\hat{\lambda},i} \otimes \Lambda_{-\zeta^{-1} q^{-f_{\hat{\lambda},i}+k}} W_{\hat{\lambda},i}\big)\\ &\otimes S_{q^{f_{\hat{\lambda},i}}} W^*_{\hat{\lambda},i} \otimes \bigotimes_{k=1}^{\infty} \big(S_{q^{f_{\hat{\lambda},i}+k}} W^*_{\hat{\lambda},i} \otimes S_{q^{-f_{\hat{\lambda},i}+k}} W_{\hat{\lambda},i}\big),\\ \mathscr{W}_{\hat{\lambda}}(\sigma) = \bigotimes_i \mathscr{W}_{\hat{\lambda},i}(\sigma).&\end{aligned}$$

Here $\Lambda_t W = \bigoplus_i \Lambda^i W$ and $S_t W = \bigoplus_i S^i W$ denote the total exterior power and total symmetric power of a vector bundle W. We can write $\mathscr{T}$ and $\mathscr{T}_{\hat{\lambda}} \otimes \mathscr{W}_{\hat{\lambda}}$ as formal power series in q and $q^{\frac{1}{r}}$

$$\begin{aligned}\mathscr{T}(\sigma) &= \sum_{k=0}^{\infty} R_k(\sigma) q^k,\\ \mathscr{T}_{\hat{\lambda}}(\sigma) \otimes \mathscr{W}_{\hat{\lambda}}(\sigma) &= \sum_{k=0}^{\infty} \hat{R}_{\hat{\lambda},k}(\sigma) q^{\frac{k}{r}}\end{aligned} \tag{4}$$

with coefficients $R_k(\sigma), \hat{R}_{\hat{\lambda},k}(\sigma) \in K_{orb}(X) \otimes \boldsymbol{Z}[\zeta, \zeta^{-1}]$ where $K_{orb}(X)$ denotes the Grothendieck group of orbifold vector bundles and r is the least common multiple of the orders $|H_\lambda|$ of the isotropy groups.

Let W be a complex orbifold vector bundle over X and $D \otimes W$ a spin-c Dirac operator twisted by W in the sense of [**3**]. It is an elliptic differential operator

$$D \otimes W : \Gamma(X, E^+ \otimes W) \to \Gamma(X, E^- \otimes W), \tag{5}$$

where

$$E^+ = \bigoplus_{i:even} \Lambda^i TX \text{ and } E^- = \bigoplus_{i:odd} \Lambda^i TX.$$

$D \otimes W$ is constructed from hermitian metrics and connections on various vector bundles associated to TX and W. Its principal symbol is such that, when X is a complex orbifold and W is a holomorphic vector bundle, then $D \otimes W$ has the same principal symbol (up to multiplicative constant) as $\bar{\partial} + \bar{\partial}^*$ twisted by W acting on the sections of $E^+ \otimes W$.

We now define

$$\varphi(X) = \zeta^{-\frac{n}{2}} \operatorname{ind}(D \otimes \mathscr{T}(\sigma)),$$
$$\hat{\varphi}(X) = \zeta^{-\frac{n}{2}} \sum_{\hat{\lambda} \in \hat{\Lambda}} \zeta^{f_{\hat{\lambda}}} \operatorname{ind}\big(D_{\hat{X}_{\hat{\lambda}}} \otimes \mathscr{T}_{\hat{\lambda}}(\sigma) \otimes \mathscr{W}_{\hat{\lambda}}(\sigma)\big),$$

where $\dim X = 2n$ and $D_{\hat{X}_{\hat{\lambda}}}$ is the spin-c Dirac operator for $\hat{X}_{\hat{\lambda}}$. More precisely

$$\begin{aligned} \varphi(X) &= \zeta^{-\frac{n}{2}} \sum_{k=0}^{\infty} \operatorname{ind}(D \otimes R_k(\sigma)) q^k, \\ \hat{\varphi}(X) &= \zeta^{-\frac{n}{2}} \sum_{\hat{\lambda} \in \hat{\Lambda}} \zeta^{f_{\hat{\lambda}}} \sum_{k=0}^{\infty} \operatorname{ind}\big(D_{\hat{X}_{\hat{\lambda}}} \otimes \hat{R}_{\hat{\lambda},k}(\sigma)\big) q^{\frac{k}{r}}, \end{aligned} \tag{6}$$

where R_k and $\hat{R}_k$ are given in (4). $\varphi(X)$ and $\hat{\varphi}(X)$ are called *elliptic genus* and *orbifold elliptic genus* of the orbifold X respectively.

Let $N > 1$ be an integer. If $f = \frac{s}{r}$ is a rational number with r relatively prime to N, let d be an integer such that $dr \equiv 1 \bmod N$. We define $\breve{f} \in \boldsymbol{Z}$ by

$$\breve{f} = ds.$$

$\breve{f}$ is determined modulo N. It satisfies $\breve{f}_1 + \breve{f}_2 \equiv f_1 \breve{+} f_2 \pmod N$.

Assume that N is relatively prime to $|H_\lambda|$ for all $\lambda \in \Lambda$. Then $\breve{f}_{\hat{\lambda}}$ is defined since $f_{\hat{\lambda}}$ can be written in the form $f_{\hat{\lambda}} = \frac{s}{|H_{\pi(\hat{\lambda})}|}$. Under the above assumption we put $\sigma = \frac{k}{N}$ with $0 < k < N$ and define the *modified orbifold elliptic genus* $\breve{\varphi}(X)$ of *level* N by

$$\breve{\varphi}(X) = \zeta^{-\frac{n}{2}} \sum_{\hat{\lambda} \in \hat{\Lambda}} \zeta^{\breve{f}_{\hat{\lambda}}} \operatorname{ind}\big(D_{\hat{X}_{\hat{\lambda}}} \otimes \mathscr{T}_{\hat{\lambda}} \otimes \mathscr{W}_{\hat{\lambda}}\big). \tag{7}$$

The genus $\zeta^{\frac{n}{2}}\varphi(X)$ belongs to $(\boldsymbol{Z}[\zeta, \zeta^{-1}])[[q]]$ and the genus $\zeta^{\frac{n}{2}}\hat{\varphi}(X)$ to $(\boldsymbol{Z}[\zeta^{\frac{1}{r}}, \zeta^{-\frac{1}{r}}])[[q^{\frac{1}{r}}]]$. Similarly the genus $\zeta^{\frac{n}{2}}\breve{\varphi}(X)$ belongs to $(\boldsymbol{Z}[\zeta, \zeta^{-1}]/(\zeta^N))[[q^{\frac{1}{r}}]]$. When it is necessary to make explicit the parameters τ and σ we write $\varphi(X;\tau,\sigma), \hat{\varphi}(X;\tau,\sigma)$ and $\breve{\varphi}(X;\tau,\sigma)$ for $\varphi(X), \hat{\varphi}(X)$ and $\breve{\varphi}(X)$ respectively.

When a compact connected Lie group G acts on X preserving almost complex structure, G acts naturally on vector bundles $\mathscr{T}, \mathscr{T}_{\hat{\lambda}}$ and $\mathscr{W}_{\hat{\lambda}}$. This is clear for $\mathscr{T}$. As to the other two it should be noticed that the action of a connected group preserves each stratum and each conjugacy class of the isotropy group of the stratum. It follows that the action of G lifts to the action on each sector, and hence on $\mathscr{T}_{\hat{\lambda}}$. Moreover some finite covering group of G acts on each $W_{\hat{\lambda}}$ compatibly with the decomposition (3) as we shall see in Section 4, Lemma 4.1. Then it also acts on $\mathscr{W}_{\hat{\lambda}}$. We shall assume here that G itself acts on each $\mathscr{W}_{\hat{\lambda}}$. By using a G invariant spin-c Dirac operator the formulae (6) and

(7) define equivariant genera which will be denoted by the same symbols. In this case $\zeta^{\frac{n}{2}}\varphi(X)$ belongs to $(R(G)\otimes \boldsymbol{Z}[\zeta,\zeta^{-1}])[[q]]$, $\zeta^{\frac{n}{2}}\hat{\varphi}(X)$ to $(R(G)\otimes \boldsymbol{Z}[\zeta^{\frac{1}{r}},\zeta^{-\frac{1}{r}}])[[q^{\frac{1}{r}}]]$ and $\zeta^{\frac{n}{2}}\breve{\varphi}(X)$ to $(R(G)\otimes \boldsymbol{Z}[\zeta,\zeta^{-1}]/(\zeta^N))[[q^{\frac{1}{r}}]]$, where $R(G)$ is the character ring of the group G. The value at $g\in G$ of $\varphi(X)$ is denoted by $\varphi_g(X)$, and similarly by $\hat{\varphi}_g(X), \breve{\varphi}_g(X)$ for $\hat{\varphi}(X), \breve{\varphi}(X)$.

Let $N>1$ be an integer. When $\sigma=\frac{k}{N}$, $0<k<N$, the genera $\varphi(X)$ and $\hat{\varphi}(X)$ are also called of *level* N. We can now state the main theorems of the present paper.

THEOREM 3.1. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Let $N>1$ be an integer such that N is relatively prime to the orders of all isotropy groups H_λ. Assume that there is an orbifold line bundle L with a lifted action of G over X such that $\Lambda^nTX=L^N$. Then the equivariant modified elliptic genus $\breve{\varphi}_g(X)$ of level N is rigid, that is, $\breve{\varphi}_g(X)$ is constant as a function on G for $\sigma=\frac{k}{N}$, $0<k<N$.*

REMARK 3.2. The Picard group of orbifold line bundles over X is isomorphic to the second cohomology group $H^2(X,\boldsymbol{Z}_X)$ where $\boldsymbol{Z}_X$ is a certain sheaf over X, cf. [**12**]. This correspondence can be considered as assigning the first Chern class to orbifold line bundles. In this sense there is an orbifold line bundle L satisfying the condition $\Lambda^nTX=L^N$ if and only if the first Chern class $c_1(\Lambda^nTX)$ is divisible by N in $H^2(X,\boldsymbol{Z}_X)$. The class $c_1(\Lambda^nTX)$ can be called the first Chern class of the almost complex orbifold X and be written $c_1(X)$ as the manifold case.

Let $L\to X$ be an orbifold line bundle and let $\tilde{\pi}:(\tilde{V},\tilde{U},H)\to(V,U,H)$ be an orbifold chart of L. L will be called a *genuine* line bundle if h acts trivially on the fiber $\tilde{\pi}^{-1}(v)$ over v for any h such that $hv=v$. In this case $\pi: L\to X$ becomes a line bundle over the space X in the usual sense. The orbifold elliptic genus is not rigid in general. However we have the following theorem.

THEOREM 3.3. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Let $N>1$ be an integer. Assume that there is a genuine line bundle L with a lifted action of G over X such that $\Lambda^nTX=L^N$. Then the equivariant elliptic genus $\hat{\varphi}_g(X)$ of level N is rigid, that is, $\hat{\varphi}_g(X)$ is constant as a function on G for $\sigma=\frac{k}{N}$, $0<k<N$.*

NOTE. Under the assumption of Theorem 3.3 each $f_{\hat{\lambda}}$ is an integer. See Note after Lemma 6.7.

The next Theorem concerns the case where Λ^nTX is a torsion element in the Picard group of orbifold line bundles.

THEOREM 3.4. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Assume that there is a positive integer d such that $(\Lambda^nTX)^d$ is trivial as an orbifold line bundle. Then the equivariant elliptic genus $\hat{\varphi}_g(X)$ is rigid, that is, $\hat{\varphi}_g(X)$ is constant as a function on G.*

Theorem 3.4 is essentially due to Dong, Liu and Ma [**2**] in a more general setting. It seems the hypothesis concerning the vanishing of first Chern class in [**2**] was erroneously stated.

The proof of Theorem 3.1, Theorem 3.3 and Theorem 3.4 will be given in Section 5 and Section 6.

4. Vergne's fixed point formula.

In this section we review the fixed point formula due to Vergne [**14**].

Let X be an almost complex closed orbifold of dimension $2n$ and $\pi : W \to X$ a complex orbifold vector bundle. Introduce a hermitian metric and a hermitian connection on W, and let ω be the curvature form of the connection. It is a differential form with values in $\mathrm{End}(W)$ where $\mathrm{End}(W)$ is the vector bundle of skew hermitian endomorphisms of W. Locally it is an H_x-invariant differential form ω_V on V with values in $\mathrm{End}(\tilde{V})$, where (V, U, H) is a chart for X and $(\tilde{V}, \tilde{U}, H)$ is a chart for the bundle $W \to X$. These ω_V behave compatibly with injections of charts. With respect to an orthonormal basis ω_V is expressed as a skew-hermitian matrix valued form.

The form $\Gamma(W) = -\frac{1}{2\pi\sqrt{-1}}\omega$ is called the *Chern matrix*, cf. [**3**]. The form

$$c(W) = \sum_{i=1}^{d} c_i(W) = \det(1 + \Gamma(W)) \quad (d = \operatorname{rank} W)$$

is the total Chern form of W. It is convenient to write formally

$$c(W) = \prod_{i=1}^{d}(1 + x_i). \tag{8}$$

The x_i are called the Chern roots and (8) is called the formal splitting of the total Chern form. The Chern character of W is defined by

$$\mathrm{ch}(W) = \operatorname{tr} e^{\Gamma(W)} = \sum_{i=1}^{d} e^{x_i}.$$

The Todd form of the almost complex orbifold X is given by

$$Td(X) = \det\left(\frac{\Gamma(TX)}{1 - e^{-\Gamma(TX)}}\right) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}},$$

with $c(TX) = \prod_i (1 + x_i)$.

Now suppose that a compact connected Lie group G acts on X preserving almost complex structure. Let W be a complex orbifold vector bundle over X with a compatible action of G. Let $F = X^G$ be the fixed point set of the action. F is an almost complex orbifold and the normal bundle of each component of F is a complex orbifold vector bundle. We can say more about this.

Lemma 4.1. *Let (V_x, U_x, H_x) be a reduced chart centered at $x \in F$ such that U_x*

is invariant under the action of G. Then the action lifts to an action of some finite covering group $\tilde{G} \to G$ on V_x. That action commutes with the action of H_x.

PROOF. From the fact that G acts on X it follows that there are an automorphism ρ of the group H_x, a small neighborhood O of the identity element e of G, and a local transformation group $\psi : O \times V_x \to V_x$ such that

$$\psi_g \circ h = \rho(h) \circ \psi_g,$$

where $\psi_g : V_x \to V_x$ is defined by $\psi_g(v) = \psi(g, v)$. We may assume that ψ_e is the identity map by conjugating by an element of H_x if necessary. It follows that ρ is the identity automorphism, and hence ψ_g commutes with the action of H_x. The local transformation group ψ extends to an action of the universal covering group G^{univ} of G. Let H_0 be the normal subgroup of H_x consisting of elements which act trivially on V_x. The quotient H_x/H_0 acts effectively on V_x and if v is a generic point in V_x, then H_x/H_0 acts simply transitively on the orbit of v. We identify $\pi_1(G)$ with the kernel of the projection $G^{univ} \to G$. Then $\pi_1(G)$ acts through ψ on each orbit of H_x and the action ψ commutes with that of H_x. Hence we get a homomorphism $\alpha : \pi_1(G) \to H_x/H_0$ defined by

$$\alpha(g)v = \psi_g(v).$$

Let Γ be the kernel of α. Then $\tilde{G} = G^{univ}/\Gamma$ is a finite covering of $G = G^{univ}/\Gamma$ and the action ψ_g induces the action of $\tilde{G}$ on V_x. □

Let F_ν be a component of F and $x \in F_\nu$. If a finite covering group $\tilde{G}_\nu$ of G acts on V_x, then $\tilde{G}_\nu$ also acts on V_y for any $y \in U_x$. It follows that $\tilde{G}_\nu$ acts on the normal bundle $N(F_\nu, X)$ of F_ν in X by (fiberwise) automorphisms. Since there are only a finite number of components in F and there is a common finite covering group $\tilde{G}$ of all the $\tilde{G}_\nu$, it follows that the group $\tilde{G}$ acts on each $N(F_\nu, X)$. Similar consideration yields that we can take a finite covering group $\tilde{G}$ which acts on the normal bundle $W_{\hat{\lambda}}$ of each sector $X_{\hat{\lambda}}$. Hereafter we assume that G *itself acts on* V_x for any $x \in F$ replacing G by $\tilde{G}$ if necessary. Then G acts also on the normal bundle $N(F, X)$ of F in X by (fiberwise) automorphisms.

For the fixed point formula it is enough to assume that the group G is a torus T. We then take a topological generator $g \in T$. Let $\{\hat{F}_{\hat{\lambda}}\}_{\hat{\lambda} \in \hat{\Lambda}_F}$ be the totality of sectors of $F = X^T$. Charts of $\hat{F}_{\hat{\lambda}}$ are of the form

$$\big(V_x^{g,h}, V_x^{g,h}/C(h), C(h)\big),$$

where $V_x^{g,h} = (V_x^g)^h = (V_x^h)^g$. The normal bundle of the immersion $\pi : \hat{F}_{\hat{\lambda}} \to F$ is denoted by $N_{\hat{\lambda}}$. We also set $\hat{N}_{\hat{\lambda}} = \pi^* N(F, X)|\hat{F}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}} = \pi^* W|\hat{F}_{\hat{\lambda}}$. If $\gamma_x = [h] \in \mathscr{C}(H_x)$, then h acts on each fiber of $N_{\hat{\lambda}}, \hat{N}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}}$ in the sense as explained in Section 3. Also T acts on $\hat{N}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}}$.

We define three differential forms on $\hat{F}_{\hat{\lambda}}$ by

$$D_h\big(N_{\hat\lambda}\big) = \det\big(1 - h^{-1}e^{-\Gamma(N_{\hat\lambda})}\big),$$
$$D_{g,h}\big(\hat N_{\hat\lambda}\big) = \det\big(1 - g^{-1}h^{-1}e^{-\Gamma(\hat N_{\hat\lambda})}\big),$$
$$\mathrm{ch}_{g,h}\big(\hat W_{\hat\lambda}\big) = \mathrm{tr}\,\big(ghe^{\Gamma(\hat W_{\hat\lambda})}\big).$$

Vergne's fixed point formula is stated in the following form. Let D be a spin-c Dirac operator on an almost complex orbifold X. Then we have

$$\mathrm{ind}_t(D\otimes W) = \sum_{\hat\lambda\in\hat\Lambda_F} \frac{1}{m\big(\hat F_{\hat\lambda}\big)} \int_{\hat F_{\hat\lambda}} \frac{Td\big(\hat F_{\hat\lambda}\big)\,\mathrm{ch}_{g,h}\,\big(\hat W_{\hat\lambda}\big)}{D_h\big(N_{\hat\lambda}\big)D_{g,h}\big(\hat N_{\hat\lambda}\big)}. \tag{9}$$

Note that $m(\hat F_{\hat\lambda}) = |C(h)|$ where $[h]=\gamma_x$ is an arbitrary point in $\hat F^{gen}_{\hat\lambda}$. Let h' be another representative of $[h]$. Then the identification $V_x^h/C(h) = V_x^{h'}/C(h')$ induces the equality

$$\frac{\mathrm{ch}_{g,h}\,\big(\hat W_{\hat\lambda}\big)}{D_h\big(N_{\hat\lambda}\big)D_{g,h}\big(\hat N_{\hat\lambda}\big)} = \frac{\mathrm{ch}_{g,h'}\,\big(\hat W_{\hat\lambda}\big)}{D_{h'}\big(N_{\hat\lambda}\big)D_{g,h'}\big(\hat N_{\hat\lambda}\big)}.$$

Since $|\gamma_x||C(h)| = |H_x| = |H_{\pi(\hat\lambda)}|$ we obtain from (9)

$$\mathrm{ind}_t(D\otimes W) = \sum_{\hat\lambda\in\hat\Lambda_F} \frac{1}{\big|H_{\pi(\hat\lambda)}\big|} \int_{\hat F_{\hat\lambda}} \sum_{h\in\gamma_x} \frac{Td\big(\hat F_{\hat\lambda}\big)\,\mathrm{ch}_{g,h}\,\big(\hat W_{\hat\lambda}\big)}{D_h\big(N_{\hat\lambda}\big)D_{g,h}\big(\hat N_{\hat\lambda}\big)}, \tag{10}$$

where $\gamma_x \in \hat F^{gen}_{\hat\lambda}$. Since $\hat F_{\hat\lambda}\setminus \hat F^{gen}_{\hat\lambda}$ has at least codimension two, the above integral is well-defined.

The following observation is useful later. Suppose that $T=S^1$. We put $t = e^{2\pi\sqrt{-1}z}$ with $z\in\boldsymbol{R}$. $\hat N_{\hat\lambda}$ decomposes into the direct sum

$$\hat N_{\hat\lambda} = \bigoplus_{i,j} E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}} \quad \text{(finite sum)}$$

of eigen-bundle where the sum is extended over the pairs of characters $(\chi_i^{S^1},\chi_j^{\langle h\rangle})$ of S^1 and $\langle h\rangle$(= cyclic subgroup of H_x generated by h), and (t,h) acts on $E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}}$ by multiplication by $\chi_i^{S^1}(t)\chi_j^{\langle h\rangle}(h)$. Write

$$\chi_i^{S^1}(t) = e^{2\pi\sqrt{-1}m_iz} \text{ with } t = e^{2\pi\sqrt{-1}z} \text{ and } \chi_j^{\langle h\rangle}(h) = e^{2\pi\sqrt{-1}m_j(h)},$$

and $c(E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}}) = \prod(1+x_{i,j,k})$ formally as before. We put $x_{i,j,k} = 2\pi\sqrt{-1}y_{i,j,k}$. Then

$$D_{t,h}\big(\hat N_{\hat\lambda}\big) = \prod_{i,j} D_{t,h}\Big(E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}}\Big)$$

with

$$D_{t,h}\left(E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}}\right) = \det\left(1 - t^{-1}h^{-1}e^{-\Gamma(E_{\chi_i^{S^1},\chi_j^{\langle h\rangle}})}\right)$$
$$= \prod_k \left(1 - e^{2\pi\sqrt{-1}(-m_i z - m_j(h) - y_{i,j,k})}\right). \tag{11}$$

Similar formulae for $D_h(N_{\hat{\lambda}})$ and $\mathrm{ch}_{t,h}(\hat{W}_{\hat{\lambda}})$ are available.

5. Proof of main theorems; I fixed point formula.

In order to prove Theorem 3.1, Theorem 3.3 and Theorem 3.4 it is enough to assume that G is a circle group S^1 since the character of a compact connected Lie group G is determined by those of all circle subgroups of G. Let X be a connected stably almost complex closed orbifold with a non-trivial S^1 action. Let $\hat{X}_{\hat{\lambda}}$ be a sector of X where $\hat{\lambda} \in \hat{\Lambda}$ as in Section 2. As remarked in Section 3, S^1 may be assumed to act on $\hat{X}_{\hat{\lambda}}$. We first investigate sectors of the orbifold $\hat{X}_{\hat{\lambda}}^{S^1}$. For that purpose we need some notations. Let H be a finite group. We define

$$CM(H) = \{(h_1, h_2) \in H \times H \mid h_1h_2 = h_2h_1\}.$$

The group acts on $CM(H)$ by conjugations on each factor. Let $\hat{\mathscr{C}}(H)$ be the set of conjugacy classes by that action. If $\delta = [h_1, h_2] \in \hat{\mathscr{C}}(H)$ is the conjugacy class of (h_1, h_2), we define $\delta^{(i)} \in \mathscr{C}(H)$, $i = 1, 2$, by $\delta^{(1)} = [h_1]$, $\delta^{(2)} = [h_2]$. A group homomorphism $H \to H'$ induces an obvious map $\hat{\mathscr{C}}(H) \to \hat{\mathscr{C}}(H')$.

Let $\hat{\pi} : \hat{\hat{F}} \to \hat{F} = \hat{X}^{S^1}$ be the total sector.

Lemma 5.1. *The inverse image $(\pi \circ \hat{\pi})^{-1}(x)$ for $x \in F = X^{S^1}$ can be identified with $\hat{\mathscr{C}}(H_x)$ in a canonical way. With this identification made, an element $\delta \in \hat{\mathscr{C}}(H_x)$ belongs to a sector of $\hat{X}_{\hat{\lambda}}$ if and only if $\delta^{(1)} \in \hat{X}_{\hat{\lambda}}$.*

Proof. Take an element $\hat{\gamma}_x \in (\pi \circ \hat{\pi})^{-1}(x)$ and put $\gamma_x = \hat{\pi}(\hat{\gamma}_x)$. If $h_1 \in H_x$ is a representative of γ_x then $\hat{\gamma}_x$ is identified with a conjugacy class $[h_2]$ of the group $C(h_1)$. We assign the equivalence class $[h_1, h_2] \in \hat{\mathscr{C}}(H_x)$ of (h_1, h_2) to $\hat{\gamma}_x$. This assignment is well-defined and bijective as is easily seen.

If δ corresponds to $\hat{\gamma}_x$ by this identification, then $\delta^{(1)} = \gamma_x$. Hence $\hat{\pi}(\delta) \in \hat{X}_{\hat{\lambda}}$ if and only if $\delta^{(1)} \in \hat{X}_{\hat{\lambda}}$. □

Let $\mathscr{H}$ be the upper half plane. We consider the function $\Phi(z, \tau)$ defined on $\boldsymbol{C} \times \mathscr{H}$ given by the following formula.

$$\Phi(z, \tau) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \prod_{k=1}^{\infty} \frac{(1 - tq^k)(1 - t^{-1}q^k)}{(1 - q^k)^2},$$

where $t = e^{2\pi\sqrt{-1}z}$ and $q = e^{2\pi\sqrt{-1}\tau}$. Note that $|q| < 1$.

The group $SL_2(\boldsymbol{Z})$ acts on $\boldsymbol{C} \times \mathscr{H}$ by

$$A(z,\tau)=\left(\frac{z}{c\tau+d},A\tau\right)=\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right),\quad A=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

Φ is a Jacobi form and satisfies the following transformation formulae, cf. [**7**].

$$\begin{aligned}\Phi(A(z,\tau))&=(c\tau+d)^{-1}e^{\frac{\pi\sqrt{-1}cz^2}{c\tau+d}}\Phi(z,\tau),\\ \Phi(z+m\tau+n,\tau)&=(-1)^{m+n}e^{-\pi\sqrt{-1}(2mz+m^2\tau)}\Phi(z,\tau)\end{aligned}\tag{12}$$

where $m,n\in\boldsymbol{Z}$.

For $\sigma\in\boldsymbol{C}$ we set

$$\phi(z,\tau,\sigma)=\frac{\Phi(z+\sigma,\tau)}{\Phi(z,\tau)}=\zeta^{-\frac{1}{2}}\frac{1-\zeta t}{1-t}\prod_{k=1}^{\infty}\frac{(1-\zeta tq^k)(1-\zeta^{-1}t^{-1}q^k)}{(1-tq^k)(1-t^{-1}q^k)},$$

where $\zeta=e^{2\pi\sqrt{-1}\sigma}$. From (12) the following transformation formulae for ϕ follow:

$$\begin{aligned}\phi(A(z,\tau),\sigma)&=e^{\pi\sqrt{-1}c(2z\sigma+(c\tau+d)\sigma^2)}\phi(z,\tau,(c\tau+d)\sigma),\\ \phi(z+m\tau+n,\tau,\sigma)&=e^{-2\pi\sqrt{-1}m\sigma}\phi(z,\tau,\sigma)=\zeta^{-m}\phi(z,\tau,\sigma).\end{aligned}\tag{13}$$

For the later use we extend the domain of the function ϕ. Let w be a polynomial in an indeterminate y. We put

$$\phi(w,\tau,\sigma)=\zeta^{-\frac{1}{2}}\frac{1-\zeta e^{2\pi\sqrt{-1}w}}{1-e^{2\pi\sqrt{-1}w}}\prod_{k=1}^{\infty}\frac{(1-\zeta e^{2\pi\sqrt{-1}w}q^k)(1-\zeta^{-1}e^{-2\pi\sqrt{-1}w}q^k)}{(1-e^{2\pi\sqrt{-1}w}q^k)(1-e^{-2\pi\sqrt{-1}w}q^k)}$$

and consider it formally as an element of $\boldsymbol{C}[[y]]$. In the above extended meaning the function ϕ still satisfies the same transformation laws:

$$\begin{aligned}\phi(A(w,\tau),\sigma)&=e^{\pi\sqrt{-1}c(2w\sigma+(c\tau+d)\sigma^2)}\phi(w,\tau,(c\tau+d)\sigma),\\ \phi(w+m\tau+n,\tau,\sigma)&=e^{-2\pi\sqrt{-1}m\sigma}\phi(w,\tau,\sigma)=\zeta^{-m}\phi(w,\tau,\sigma).\end{aligned}\tag{14}$$

In fact, if we substitute an arbitrary complex number for y then the above equalities hold by (13). Hence the equalities (14) hold in $\boldsymbol{C}[[y]]$. If $A^*=\sum_{l=0}^{\infty}A^{2l}$ is a commutative graded algebra over $\boldsymbol{C}$ with even grading and with $A^0=\boldsymbol{C}$, then $\phi(w,\tau,\sigma)$ can be defined for $w\in A^*$ and it satisfies (14) since one may think of w as a specilization of an element in $\boldsymbol{C}[y]$.

Hereafter we assume that X is an almost complex closed orbifold of dimension $2n$ with a non-trivial action of S^1. Let $\hat{\hat{F}}_{\hat{\lambda}}$ be a sector of $\hat{F}=\hat{X}^{S^1}$, that is a component of $\hat{\hat{F}}$. Its orbifold charts are of the form

$$\big(V_x^{t,h_1,h_2}, V_x^{t,h_1,h_2}/(C(h_1)\cap C(h_2)), C(h_1)\cap C(h_2)\big), \quad (h_1,h_2)\in CM(H_x).$$

Note that $C(h_1)\cap C(h_2)$ coincides with $C_{C(h_1)}(h_2)$, the centralizer of h_2 in $C(h_1)$. The image $\hat{\pi}(\hat{\hat{F}}_{\hat{\lambda}})$ is contained in a unique sector $\hat{X}_{\hat{\lambda}}$ of the orbifold X. We formally write

$$\begin{aligned} c\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big) &= \prod_{i=1}^{r_1}(1+x_i), \\ c\Big(N_{\hat{\lambda}}\Big) &= \prod_{i=r_1+1}^{r_1+r_2}(1+x_i), \\ c\Big(\hat{\hat{N}}_{\hat{\lambda}}\Big) &= \prod_{i=r_1+r_2+1}^{r_1+r_2+r_3}(1+x_i), \\ c\Big(\hat{W}_{\hat{\lambda}}\Big) &= \prod_{i=r_1+r_2+r_3+1}^{r_1+r_2+r_3+r_4}(1+x_i), \end{aligned} \tag{15}$$

where $N_{\hat{\lambda}}$ is the normal bundle of the immersion $\hat{\hat{F}}_{\hat{\lambda}} \to \hat{X}_{\hat{\lambda}}^{S^1}$, $\hat{\hat{N}}_{\hat{\lambda}} = \hat{\pi}^{-1}N(\hat{X}_{\hat{\lambda}}^{S^1}, \hat{X}_{\hat{\lambda}})|\hat{\hat{F}}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}} = \hat{\pi}^{-1}W_\lambda|\hat{\hat{F}}_{\hat{\lambda}}$. Recall that W_λ is the normal bundle of the immersion $\hat{X}_{\hat{\lambda}} \to X$, cf. Section 3. Note that $r_1+r_2+r_3+r_4 = n$. We denote the Euler class of the orbifold $\hat{\hat{F}}_{\hat{\lambda}}$ by $e(\hat{\hat{F}}_{\hat{\lambda}})$. It is equal to the top Chern class $c_{r_1}(\hat{\hat{F}}_{\hat{\lambda}})$ and is written

$$e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big) = \prod_{i=1}^{r_1} x_i.$$

Finally we put

$$x_i = 2\pi\sqrt{-1}y_i.$$

Take a point $\hat{\gamma}_x \in \hat{\hat{F}}_{\hat{\lambda}}$ and identify it with $\delta = [h_1,h_2] \in \hat{\mathscr{C}}(H_x)$. Then h_1 acts on W_λ and hence on $\hat{W}_{\hat{\lambda}}$ as was explained in Section 3. Also as was explained in Section 4, h_2 acts on $N_{\hat{\lambda}}, \hat{\hat{N}}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}}$. Furthermore $t = e^{2\pi\sqrt{-1}z} \in S^1$ acts on $\hat{\hat{N}}_{\hat{\lambda}}$ and $\hat{W}_{\hat{\lambda}}$. The weights of these actions can be taken compatibly with the formal splitting (15). We write them in the following form

$$\begin{cases} m_i^{S^1} & \text{for } t = e^{2\pi\sqrt{-1}z}, \\ m_i^{(h_1,h_2)}(h_1) & \text{for } h_1, \\ m_i^{(h_1,h_2)}(h_2) & \text{for } h_2 \end{cases} \tag{16}$$

for $i = 1, \dots, r_1 + r_2 + r_3 + r_4$. Note that $m_i^{S^1} \in \boldsymbol{Z}$ and $m_i^{(h_1,h_2)}(h_j)$, $j = 1, 2$, is a rational number determined modulo $\boldsymbol{Z}$. In the sequel we shall fix one representative $m_i^{(h_1,h_2)}(h_j)$, $j = 1, 2$, for each pair $(h_1, h_2) \in CM(H_x)$. We make the convention that

$$\begin{aligned} m_i^{S^1} &= 0 \quad \text{for } 1 \le i \le r_1 + r_2, \\ m_i^{(h_1,h_2)}(h_1) &= 0 \quad \text{for } 1 \le i \le r_1 + r_2 + r_3, \\ m_i^{(h_1,h_2)}(h_2) &= 0 \quad \text{for } 1 \le i \le r_1. \end{aligned} \tag{17}$$

Hereafter we shall write $\hat{\varphi}(X; z, \tau, \sigma)$ and $\breve{\varphi}(X; z, \tau, \sigma)$ instead of $\hat{\varphi}_t(X; \tau, \sigma)$ and $\breve{\varphi}_t(X; \tau, \sigma)$. Let $\{\hat{\hat{F}}_{\hat{\lambda}}\}_{\hat{\lambda} \in \hat{\Lambda}}$ be the totality of sectors of $\hat{F}$.

PROPOSITION 5.2. *Let X be an almost complex orbifold with a non-trivial action of S^1. Then the equivariant elliptic orbifold elliptic genus $\hat{\varphi}(X; z, \tau, \sigma)$ is given by*

$$\begin{aligned} \hat{\varphi}(X; z, \tau, \sigma) = \sum_{\hat{\lambda} \in \hat{\Lambda}} \frac{1}{|H_{\pi \circ \hat{\pi}(\delta)}|} \int_{\hat{\hat{F}}_{\hat{\lambda}}} \sum_{(h_1,h_2) \in \delta} e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big) \prod_{i=1}^{n} e^{2\pi\sqrt{-1} m_i^{(h_1,h_2)}(h_1)\sigma} \\ \cdot \phi\Big(- y_i - m_i^{S^1} z + m_i^{(h_1,h_2)}(h_1)\tau - m_i^{(h_1,h_2)}(h_2), \tau, \sigma \Big). \end{aligned} \tag{18}$$

Here δ is a point in $\hat{\hat{F}}_{\hat{\lambda}}$ as in Lemma 5.1.

NOTE. The above expressions give well-defined functions independent of the choice of representatives $m_i^{(h_1,h_2)}(h_1), m_i^{(h_1,h_2)}(h_2)$ as is easily seen from (13). They are meromorphic functions in the variables z, τ, σ.

PROPOSITION 5.3. *Let $N > 1$ be an integer. We assume that $|H_x|$ is relatively prime to N for all $x \in X$. Then the modified orbifold elliptic genus $\breve{\varphi}(X; z, \tau, \sigma)$ of level N is given by*

$$\begin{aligned} \breve{\varphi}(X; z, \tau, \sigma) = \sum_{\hat{\lambda} \in \hat{\Lambda}} \frac{1}{|H_{\pi \circ \hat{\pi}(\delta)}|} \int_{\hat{\hat{F}}_{\hat{\lambda}}} \sum_{(h_1,h_2) \in \delta} e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big) \prod_{i=1}^{n} e^{2\pi\sqrt{-1} \breve{m}_i^{(h_1,h_2)}(h_1)\sigma} \\ \cdot \phi\Big(- y_i - m_i^{S^1} z + m_i^{(h_1,h_2)}(h_1)\tau - m_i^{(h_1,h_2)}(h_2), \tau, \sigma \Big). \end{aligned} \tag{19}$$

PROOF. Let $\hat{X}_{\hat{\lambda}}$ be a sector of X. Recall that the contribution to $\hat{\varphi}(X)$ from $\hat{X}_{\hat{\lambda}}$ is

$$\zeta^{-\frac{1}{2}} \zeta^{f_{\hat{\lambda}}} \operatorname{ind}\big(D_{\hat{X}_{\hat{\lambda}}} \otimes \mathscr{T}_{\hat{\lambda}} \otimes \mathscr{W}_{\hat{\lambda}}\big).$$

If $\hat{\hat{F}}_{\hat{\lambda}}$ is a sector of $\hat{X}_{\hat{\lambda}}$ and $\delta \in \hat{\hat{F}}_{\hat{\lambda}}$, then $\delta^{(1)}$ lies in $\hat{X}_{\hat{\lambda}}$. For a moment we fix a representative h_1 of $\delta^{(1)}$. Then $\delta^{(2)}$ is a conjugacy class of $C(h_1)$. We apply the fixed point formula (10) to this and get

$$\zeta^{-\frac{n}{2}}\zeta^{f_{\hat\lambda}}\,\mathrm{ind}\left(D_{\hat X_{\hat\lambda}}\otimes\mathscr{T}_{\hat\lambda}\otimes\mathscr{W}_{\hat\lambda}\right)$$
$$=\zeta^{-\frac{n}{2}}\sum_{\hat{\hat\lambda}\in\hat{\Lambda},\hat\pi(\hat{\hat\lambda})=\hat\lambda}\frac{1}{|C(h_1)|}\zeta^{f_{\hat\lambda}}\int_{\hat{\hat F}_{\hat\lambda}}\sum_{h_2\in\delta^{(2)}}\frac{Td\left(\hat{\hat F}_{\hat{\hat\lambda}}\right)ch_{t,h_2}\left(\mathscr{T}_{\hat{\hat\lambda}}\right)ch_{t,h_1,h_2}\left(\mathscr{W}_{\hat{\hat\lambda}}\right)}{D_{h_2}\left(N_{\hat{\hat\lambda}}\right)D_{t,h_2}\left(\hat{\hat N}_{\hat{\hat\lambda}}\right)}$$

where $\mathscr{T}_{\hat{\hat\lambda}}=\hat\pi^*(\mathscr{T}_{\hat\lambda})|\hat{\hat F}_{\hat{\hat\lambda}}$ and $\mathscr{W}_{\hat{\hat\lambda}}=\hat\pi^*(\mathscr{W}_{\hat\lambda})|\hat{\hat F}_{\hat{\hat\lambda}}$. Note that $\hat\pi^*(T\hat X_{\hat\lambda})|\hat{\hat F}_{\hat{\hat\lambda}}=T\hat{\hat F}_{\hat{\hat\lambda}}\oplus N_{\hat{\hat\lambda}}\oplus\hat{\hat N}_{\hat{\hat\lambda}}$. Using (11), (15) and (16) we have

$$Td\left(\hat{\hat F}_{\hat{\hat\lambda}}\right)=\prod_{i=1}^{r_1}\frac{x_i}{1-e^{-2\pi\sqrt{-1}y_i}},$$
$$D_{h_2}\left(N_{\hat{\hat\lambda}}\right)=\prod_{i=r_1+1}^{r_1+r_2}\left(1-e^{-2\pi\sqrt{-1}(y_i+m_i^{(h_1,h_2)}(h_2))}\right),$$
$$D_{t,h_2}\left(\hat{\hat N}_{\hat{\hat\lambda}}\right)=\prod_{i=r_1+r_2+1}^{r_1+r_2+r_3}\left(1-e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right).$$

We also have

$$ch_{t,h_2}\left(\mathscr{T}_{\hat{\hat\lambda}}\right)=\prod_{i=1}^{r_1+r_2+r_3}\Bigg(\left(1-\zeta e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)$$
$$\cdot\prod_{k=1}^{\infty}\frac{\left(1-\zeta q^ke^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)\left(1-\zeta^{-1}q^ke^{2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}{\left(1-q^ke^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)\left(1-q^ke^{2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}\Bigg),$$

and

$$ch_{t,h_1,h_2}\left(\mathscr{W}_{\hat{\hat\lambda}}\right)=\prod_{i=r_1+r_2+r_3+1}^{r_1+r_2+r_3+r_4}\Bigg(\frac{\left(1-\zeta q^{f_{\hat\lambda,i}}e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}{\left(1-q^{f_{\hat\lambda,i}}e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}$$
$$\cdot\prod_{k=1}^{\infty}\frac{\left(1-\zeta q^{f_{\hat\lambda,i}+k}e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)\left(1-\zeta^{-1}q^{-f_{\hat\lambda,i}+k}e^{2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}{\left(1-q^{f_{\hat\lambda,i}+k}e^{-2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)\left(1-q^{-f_{\hat\lambda,i}+k}e^{2\pi\sqrt{-1}(y_i+m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}\right)}\Bigg).$$

Gathering these together we obtain

$$\zeta^{-\frac{n}{2}}\zeta^{f_{\hat\lambda}}\,\mathrm{ind}\left(D_{\hat X_{\hat\lambda}}\otimes\mathscr{T}_{\hat\lambda}\otimes\mathscr{W}_{\hat\lambda}\right)=\sum_{\hat{\hat\lambda}\in\hat{\Lambda},\hat\pi(\hat{\hat\lambda})=\hat\lambda}\frac{1}{|C(h_1)|}\int_{\hat{\hat F}_{\hat\lambda}}\sum_{h_2\in\delta^{(2)}}e\left(\hat{\hat F}_{\hat{\hat\lambda}}\right)\prod_{i=1}^{n}e^{2\pi\sqrt{-1}f_{\hat\lambda,i}\sigma}$$
$$\cdot\phi\Big(-y_i-m_i^{S^1}z+f_{\hat\lambda,i}\tau-m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big).\quad(20)$$

Note that $f_{\hat\lambda,i}\equiv m_i^{(h_1,h_2)}(h_1)\mod\boldsymbol{Z}$ for $r_1+r_2+r_3+1\le i\le r_1+r_2+r_3+r_4$ and $f_{\hat\lambda,i}=0$ for $i\le r_1+r_2+r_3$. Then, by (14),

$$e^{2\pi\sqrt{-1}f_{\hat\lambda,i}\sigma}\phi\Big(-y_i - m_i^{S^1}z + f_{\hat\lambda,i}\tau - m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big)$$
$$= e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)\sigma}\phi\Big(-y_i - m_i^{S^1}z + m_i^{(h_1,h_2)}(h_1)\tau - m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big).$$

Putting this into (20) we get

$$\zeta^{-\frac{1}{2}}\zeta^{f_{\hat\lambda}}\operatorname{ind}\big(D_{\hat X_{\hat\lambda}}\otimes\mathscr{T}_{\hat\lambda}\otimes\mathscr{W}_{\hat\lambda}\big)$$
$$= \sum_{\hat{\hat\lambda}\in\hat\Lambda,\hat\pi(\hat{\hat\lambda})=\hat\lambda}\frac{1}{|C(h_1)|}\int_{\hat{\hat F}_{\hat\lambda}}\sum_{h_2\in\delta^{(2)}} e\Big(\hat{\hat F}_{\hat\lambda}\Big)\prod_{i=1}^{n}e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)\sigma}$$
$$\cdot\,\phi\Big(-y_i - m_i^{S^1}z + m_i^{(h_1,h_2)}(h_1)\tau - m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big).$$

So far we fixed a representative h_1 in $\delta^{(1)}$ and $\hat X_{\hat\lambda}$ in which $\delta^{(1)}$ lies. We now move h_1 and $\hat\lambda$, and sum up. Then we obtain the equality in Proposition 5.2.

The equality in Proposition 5.3 is proved in a parallel way. One has only to observe that

$$\breve m_i^{(h_1,h_2)}(h_1) - m_i^{(h_1,h_2)}(h_1) \equiv \breve f_{\hat\lambda,i} - f_{\hat\lambda,i} \bmod N. \qquad \square$$

6. Proof of main theorems; II modular property.

For $A \in SL_2(\boldsymbol{Z})$ we define $\hat\varphi^A(X;z,\tau,\sigma)$ by

$$\hat\varphi^A(X;z,\tau,\sigma) = \hat\varphi(X;A(z,\tau),\sigma).$$

Similarly $\breve\varphi^A(X;z,\tau,\sigma)$ is defined by

$$\breve\varphi^A(X;z,\tau,\sigma) = \breve\varphi(X;A(z,\tau),\sigma).$$

Lemma 6.1. *Let $N > 1$ be an integer. Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action. Assume that there exists an orbifold line bundle L such that $\Lambda^n TX = L^N$. Then the action of S^1 can be lifted to an action of some finite covering group $\tilde S^1$ on L.*

Proof. S^1 acts on $L^N = \Lambda^n TX$. Locally L is an N-fold covering of L^N off zero-section. Hence, the action of N-fold covering $\tilde S^1 \to S^1$ on L^N lifts to an action on L. $\square$

Hereafter we assume *the action of S^1 itself lifts to L* under the situation of Lemma 6.1. This causes no loss of generality in view of Lemma 6.1; we may replace S^1 by a suitable finite covering $\tilde S^1$ if necessary.

Lemma 6.2. *Under the situation of Lemma* 6.1*, let m^{S^1} be the weight of the action* *of S^1 on L at $x \in F = X^{S^1}$. Then the tangential weights $m_i^{S^1}$ satisfy the relation*

$$\sum_{i=1}^{n} m_i^{S^1} = N m^{S^1} + l \tag{21}$$

where l belongs to $\boldsymbol{Z}$ and is independent of $x \in F$.

Proof. The weight of S^1-action on $\Lambda^n TX$ at x is $\sum_i m_i^{S^1}$. The weight of S^1-action on L^N at x is Nm^{S^1}. Since $\Lambda^n TX = L^N$ and the both actions cover the same action on X, they differ only by the fiberwise action on $\Lambda^n TX$ which is of the form $g \cdot v = g^l v,\ g \in S^1$, with $l \in \boldsymbol{Z}$. This implies (21). $\square$

Locally L is given by a line bundle over V_x with a lifted action of H_x. The group H_x acts on the fiber over $\tilde{x} = p_x^{-1}(x)$. Let $m(h)$ be a weight of that action for $h \in H_x$. It is determined modulo integers. Since $\Lambda^n TX = L^N$ and the the weights $m_i^{(h_1,h_2)}(h)$ are determined modulo $\boldsymbol{Z}$ we may assume that $m_i^{(h_1,h_2)}(h)$ satisfy the equality

$$\sum_{i=1}^{n} m_i^{(h_1,h_2)}(h) = N m(h). \tag{22}$$

If y denotes the first Chern form of L, then we may also assume that

$$\sum_i y_i = Ny, \tag{23}$$

since only the integral concerns in the sequel.

Lemma 6.3. *Let $N > 1$ be an integer. Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action such that $|H_x|$ is relatively prime to N for all $x \in X$. Assume that there exists an orbifold line bundle L such that $\Lambda^n TX = L^N$. Then the modified orbifold elliptic genus $\breve{\varphi}(X; z, \tau, \sigma)$ of level N with $\sigma = \frac{k}{N},\ 0 < k < N$, is transformed by $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\boldsymbol{Z})$ in the following way.*

$$\begin{aligned}
\breve{\varphi}^A(X; z, \tau, \sigma) = e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2 - 2clz\sigma)} \sum_{\hat{\hat{\lambda}} \in \hat{\hat{\Lambda}}} \frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|} \int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}} \sum_{(h_1,h_2)\in\delta} e\Big(\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big) \\
\cdot e^{-2\pi\sqrt{-1}m(h_1)dk} e^{-2\pi\sqrt{-1}(y + m^{S^1}z + m(h_2))ck} \\
\cdot \prod_{i=1}^{n} e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)(c\tau+d)\sigma} \\
\cdot \phi\Big(-y_i - m_i^{S^1}z + m_i^{(h_1,h_2)}(h_1)\tau - m_i^{(h_1,h_2)}(h_2), \tau, (c\tau+d)\sigma\Big),
\end{aligned} \tag{24}$$

where $c_1(L) = 2\pi\sqrt{-1}y$, $m^{\langle h_1,h_2\rangle}(h_1)$ is the weight of H_x on L and $\delta \in \hat{\hat{F}}_{\hat{\hat{\lambda}}}$.

Proof. By Proposition 5.3 we have

$$\begin{aligned}\breve{\varphi}^A(X;z,\tau,\sigma) = &\sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\lambda}}}\sum_{(h_1,h_2)\in\delta} e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big)\prod_{i=1}^{n} e^{2\pi\sqrt{-1}\breve{m}_i^{\langle h_1,h_2\rangle}(h_1)\sigma}\\
&\cdot\phi\Big(-y_i - m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big)\\
= &\sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\lambda}}}\sum_{(h_1,h_2)\in\delta}\frac{1}{(c\tau+d)^{r_1}}\Big(\prod_{i=1}^{r_1}(c\tau+d)x_i\Big)\\
&\cdot\prod_{i=1}^{n} e^{2\pi\sqrt{-1}\breve{m}_i^{\langle h_1,h_2\rangle}(h_1)\sigma}\phi\Big(-\frac{(c\tau+d)y_i}{c\tau+d} - m_i^{S^1}\frac{z}{c\tau+d}\\
&\qquad + m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big).\end{aligned}$$

The term of degree $2j$ of

$$\prod_{i=1}^{r_1}(c\tau+d)x_i\prod_{i=1}^{n}\phi\Big(-\frac{(c\tau+d)y_i}{c\tau+d} - m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big)$$

is equal to $(c\tau+d)^j$ times that of

$$\prod_{i=1}^{r_1}x_i\prod_{i=1}^{n}\phi\Big(-\frac{y_i}{c\tau+d} - m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big).$$

Since $\dim\hat{\hat{F}}_{\hat{\lambda}} = 2r_1$ we obtain

$$\begin{aligned}\breve{\varphi}^A(X;z,\tau,\sigma) = &\sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\lambda}}}\sum_{(h_1,h_2)\in\delta} e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big)\prod_{i=1}^{n} e^{2\pi\sqrt{-1}\sum_i\breve{m}_i^{\langle h_1,h_2\rangle}(h_1)\sigma}\\
&\cdot\phi\Big(-\frac{y_i}{c\tau+d} - m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big).\end{aligned}\tag{25}$$

Using (14), (21) and (23), we see that

$$\begin{aligned}&\prod_{i=1}^{n}\phi\Big(-\frac{y_i}{c\tau+d} - m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau - m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big)\\
&\quad=\prod_{i=1}^{n}\phi\Big(\frac{-y_i - m_i^{S^1}z+(a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2)}{c\tau+d},A\tau,\sigma\Big)\\
&\quad= e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)}e^{-2\pi\sqrt{-1}(y+m^{S^1}z)ck}e^{2\pi\sqrt{-1}\sum_i((a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2))c\sigma}\\
&\qquad\cdot\prod_i\phi\Big(-y_i - m_i^{S^1}z+(a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2),\tau,(c\tau+d)\sigma\Big).\end{aligned}\tag{26}$$

In (26) we have

$$\Big((a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2)\Big)c$$
$$=-m_i^{(h_1,h_2)}(h_1)+\Big(am_i^{(h_1,h_2)}(h_1)-cm_i^{(h_1,h_2)}(h_2)\Big)(c\tau+d),$$

and

$$(a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2)$$
$$=\Big(am_i^{(h_1,h_2)}(h_1)-cm_i^{(h_1,h_2)}(h_2)\Big)\tau+bm_i^{(h_1,h_2)}(h_1)-dm_i^{(h_1,h_2)}(h_2).$$

We consider the map $\rho : CM(H_x)\to CM(H_x)$ defined by

$$\rho(h_1,h_2)=(\bar h_1,\bar h_2)=\big(h_1^a h_2^{-c},h_1^{-b}h_2^d\big).$$

It is a bijection and its inverse is given by

$$\rho^{-1}(\bar h_1,\bar h_2)=\big(\bar h_1^d\bar h_2^c,\bar h_1^b\bar h_2^a\big).$$

ρ induces a bijection of $\hat{\mathscr{C}}(H_x)$ onto itself which we shall also denote by ρ. It in turn induces a permutation ρ of $\{\hat{\hat F}_{\hat\lambda}\}_{\hat\lambda\in\hat\Lambda}$ in the following way. If δ lies in $\hat{\hat F}_{\hat\lambda}$, then $\rho(\delta)$ lies in $\hat{\hat F}_{\rho(\hat\lambda)}$. Note that $\delta\mapsto\rho(\delta)$ defines an isomorphism of orbifolds from $\hat{\hat F}_{\hat\lambda}$ onto $\hat{\hat F}_{\rho(\hat\lambda)}$. In fact, if $(V_x^{t,h_1,h_2},V_x^{t,h_1,h_2}/(C(h_1)\cap C(h_2)),C(h_1)\cap C(h_2))$ is a chart for $\hat{\hat F}_{\hat\lambda}$ and $(V_x^{t,\bar h_1,\bar h_2},V_x^{t,\bar h_1,\bar h_2}/(C(\bar h_1)\cap C(\bar h_2)),C(\bar h_1)\cap C(\bar h_2))$ is a chart for $\rho(\hat{\hat F}_{\hat\lambda})$, then $V_x^{t,h_1,h_2}=V_x^{t,\bar h_1,\bar h_2}$ and $C(h_1)\cap C(h_2)=C(\bar h_1)\cap C(\bar h_2)$. It follows that the identity map $V_x^{t,h_1,h_2}\to V_x^{t,\bar h_1,\bar h_2}$ induces $\rho:\hat{\hat F}_{\hat\lambda}\to\hat{\hat F}_{\rho(\hat\lambda)}$. $\rho:\hat{\hat F}_{\hat\lambda}\to\hat{\hat F}_{\rho(\hat\lambda)}$ respects $\pi\circ\hat\pi$ but not necessarily $\hat\pi$. Moreover $am_i^{(h_1,h_2)}(h_1)-cm_i^{(h_1,h_2)}(h_2)$ is a weight of $\bar h_1$ and $-bm_i^{(h_1,h_2)}(h_1)+dm_i^{(h_1,h_2)}(h_2)$ is a weight of $\bar h_2$. We shall use these for weights on the transformed sector $\rho(\hat{\hat F}_{\hat\lambda})$ assigned for the pair $(\bar h_1,\bar h_2)$ and write them

$$m_i^{(\bar h_1,\bar h_2)}(\bar h_1),\ m_i^{(\bar h_1,\bar h_2)}(\bar h_2).$$

Thus

$$\Big((a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2)\Big)c$$
$$=-m_i^{(h_1,h_2)}(h_1)+m_i^{(\bar h_1,\bar h_2)}(\bar h_1)(c\tau+d),\tag{27}$$

and

$$(a\tau+b)m_i^{(h_1,h_2)}(h_1)-(c\tau+d)m_i^{(h_1,h_2)}(h_2)=m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)\tau-m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_2). \tag{28}$$

Also we have

$$m_i^{(h_1,h_2)}(h_1)=dm_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)+cm_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_2).$$

Hence, by (22)

$$\sum_i m_i^{(h_1,h_2)}(h_1)\sigma=dkm(\bar{h}_1)+ckm(\bar{h}_2). \tag{29}$$

Finally, since $\sum_i \breve{m}_i^{\langle h_1,h_2\rangle}(h_1)=N\breve{m}(h_1)$ by (22), we get

$$e^{2\pi\sqrt{-1}\sum_i \breve{m}_i^{\langle h_1,h_2\rangle}(h_1)\sigma}=1 \tag{30}$$

for $\sigma=\frac{k}{N}$ with $0<k<N$.

Then, by using (27), (28) and (29), we can rewrite (26) as follows.

$$\begin{aligned}
&\prod_{i=1}^n \phi\Big(-\frac{y_i}{c\tau+d}-m_i^{S^1}\frac{z}{c\tau+d}+m_i^{(h_1,h_2)}(h_1)A\tau-m_i^{(h_1,h_2)}(h_2),A\tau,\sigma\Big)\\
&\quad=e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)}e^{-2\pi\sqrt{-1}m(\bar{h}_1)dk}\rho^*\\
&\qquad\cdot\Big(e^{-2\pi\sqrt{-1}(y+m^{S^1}z+m(\bar{h}_2))ck}\prod_i e^{2\pi\sqrt{-1}m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)(c\tau+d)\sigma}\\
&\qquad\quad\cdot\phi\Big(-y_i-m_i^{S^1}+m_i^{\langle\bar{h}_1,\bar{h}_2\rangle}(\bar{h}_1)-m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_2),\tau,(c\tau+d)\sigma\Big)\Big).
\end{aligned} \tag{31}$$

We also have

$$e\Big(\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)=\rho^*\Big(e\Big(\hat{\hat{F}}_{\rho(\hat{\hat{\lambda}})}\Big)\Big). \tag{32}$$

Putting (30), (31) and (32) into (25), we obtain

$$\begin{aligned}
\breve{\varphi}^A(X;z,\tau,\sigma)&=e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)}\\
&\quad\cdot\sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(\bar{h}_1,\bar{h}_2)\in\rho(\delta)}e^{-2\pi\sqrt{-1}m(\bar{h}_1)dk}\\
&\quad\cdot\rho^*\Big(e\Big(\hat{\hat{F}}_{\rho(\hat{\hat{\lambda}})}\Big)e^{-2\pi\sqrt{-1}(y+m^{S^1}z+m(\bar{h}_2))ck}\prod_{i=1}^n e^{2\pi\sqrt{-1}m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)(c\tau+d)\sigma}\\
&\qquad\cdot\phi\Big(-y_i-m_i^{S^1}z+m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)\tau-m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_2),\tau,(c\tau+d)\sigma\Big)\Big)
\end{aligned}$$

$$= e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)} \sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}} \frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}$$

$$\cdot \int_{\hat{\hat{F}}_{\rho(\hat{\hat{\lambda}})}} \sum_{(\bar{h}_1,\bar{h}_2)\in\rho(\delta)} e\Big(\hat{\hat{F}}_{\rho(\hat{\hat{\lambda}})}\Big) e^{-2\pi\sqrt{-1}m(\bar{h}_1)dk} e^{-2\pi\sqrt{-1}(y+m^{S^1}z+m(\bar{h}_2))ck}$$

$$\cdot \prod_{i=1}^{n} e^{2\pi\sqrt{-1}m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)(c\tau+d)\sigma} \phi\Big(-y_i - m_i^{S^1}z + m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_1)\tau - m_i^{(\bar{h}_1,\bar{h}_2)}(\bar{h}_2), \tau, (c\tau+d)\sigma\Big).$$

Replacing $\rho(\hat{\hat{\lambda}})$ by $\hat{\hat{\lambda}}$ and $\bar{h}_1$ and $\bar{h}_2$ by h_1 and h_2 respectively we obtain (24). $\square$

The definition of vector bundles $\mathscr{T}_{\hat{\lambda}}$ and $\mathscr{W}_{\hat{\lambda}}$ depended on a parameter σ with $\zeta = e^{2\pi\sqrt{-1}\sigma}$. In case it is necessary to specify the parameter σ, we shall write $\mathscr{T}_{\hat{\lambda}}(\sigma)$ and $\mathscr{W}_{\hat{\lambda}}(\sigma)$.

Corollary 6.4. *Under the situation of Lemma* 6.3 *we have*

$$\breve{\varphi}^A(X; z, \tau, \sigma) = e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)} \sum_{\hat{\lambda}\in\hat{\Lambda}} e^{-2\pi\sqrt{-1}m(h_1)dk} e^{2\pi\sqrt{-1}f_{\hat{\lambda}}(c\tau+d)\sigma}$$

$$\cdot \operatorname{ind}\Big(D_{\hat{X}_{\hat{\lambda}}} \otimes (L^*)^{ck} \otimes \mathscr{T}_{\hat{\lambda}}((c\tau+d)\sigma) \otimes \mathscr{W}_{\hat{\lambda}}((c\tau+d)\sigma)\Big), \tag{33}$$

where $[h_1] \in \hat{X}_{\hat{\lambda}}$.

Proof. Apply the index formula (10) to Lemma 6.3. The details of proof are similar to that of Proposition 5.2. $\square$

The following Lemma is crucial for the subsequent discussion.

Lemma 6.5. *The meromorphic function* $\breve{\varphi}^A(X; z, \tau, \sigma)$ *in* z *has no poles at* $z \in \boldsymbol{R}$.

Proof. In view of (33) it suffices to show that each

$$\operatorname{ind}\Big(D_{\hat{X}_{\hat{\lambda}}} \otimes (L^*)^{ck} \otimes \mathscr{T}_{\hat{\lambda}}((c\tau+d)\sigma) \otimes \mathscr{W}_{\hat{\lambda}}((c\tau+d)\sigma)\Big)$$

has no poles at $z \in \boldsymbol{R}$. Furthermore we may replace $(L^*)^{ck}$ by an orbifold line bundle L' and $(c\tau+d)\sigma$ by σ without loss of generality. Considering $\operatorname{ind}(D_{\hat{X}_{\hat{\lambda}}} \otimes L' \otimes \mathscr{T}_{\hat{\lambda}}(\sigma) \otimes \mathscr{W}_{\hat{\lambda}}(\sigma))$ as a function of z, we write it $\varphi(z)$ and expand it as a power series:

$$\varphi(z) = \sum_k b_k(z) q^{\frac{k}{r}},$$

where $b_k(z)$ is of the form $\operatorname{ind}(\hat{R}'_{\hat{\lambda},k}(\sigma)) \in R(S^1) \otimes \boldsymbol{C}$ as in (6). It follows that $b_k(z)$ has no poles at $z \in \boldsymbol{R}$.

On the other hand (24) compared with (33) shows that $\varphi(z)$ has the following expression

$$\varphi(z)=\zeta^{\frac{n}{2}}\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}},\hat{\pi}(\hat{\hat{F}}_{\hat{\hat{\lambda}}})\subset\hat{X}_{\hat{\lambda}}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(h_1,h_2)\in\delta}e\Big(\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)e^{2\pi\sqrt{-1}(y+m^{S^1}z+m(h_2))}$$
$$\cdot\prod_{i=1}^{n}\phi\Big(-y_i-m_i^{S^1}z+f_{\hat{\lambda},i}\tau-m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big),\tag{34}$$

where y is the first Chern form of L', and m^{S^1} and $m(h)$ are the weights of the actions of S^1 and $h\in H_x$ on L'_x respectively. If $\sum_k b_{k,\hat{\hat{\lambda}}}(z)q^{\frac{k}{r}}$ is the contribution from $\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}$ to (34), then

$$b_k(z)=\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}b_{k,\hat{\hat{\lambda}}}(z).$$

The term of degree zero in the denominator of $\phi(-y_i-m_i^{S^1}z+f_{\hat{\lambda},i}\tau-m_i^{(h_1,h_2)}(h_2),\tau,\sigma)$ is

$$\Big(1-e^{2\pi\sqrt{-1}(-m_i^{S^1}z-m_i^{(h_1,h_2)}(h_2))}q^{f_{\hat{\lambda},i}\tau}\Big)$$
$$\cdot\prod_{k=1}^{\infty}\Big(1-e^{2\pi\sqrt{-1}(-m_i^{S^1}z-m_i^{(h_1,h_2)}(h_2))}q^{f_{\hat{\lambda},i}+k}\Big)\Big(1-e^{2\pi\sqrt{-1}(m_i^{S^1}z+m_i^{(h_1,h_2)}(h_2))}q^{-f_{\hat{\lambda},i}+k}\Big).$$

Therefore poles of $b_{k,\hat{\hat{\lambda}}}(z)$ lie in $\boldsymbol{R}$ but they cancel out in the sum $b_k(z)=\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}b_{k,\hat{\hat{\lambda}}}(z)$ as noted above. Assume $z_0\in\boldsymbol{R}$ is a pole of $\varphi(z)$. Then there is an open set U containing z_0 such that the power series $\sum_k b_{k,\hat{\hat{\lambda}}}(z)q^{\frac{k}{r}}$ converges uniformly on any compact set in $U\setminus\{z_0\}$ and $b_k(z)=\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}b_{k,\hat{\hat{\lambda}}}(z)$ is holomorphic in U. In such a situation one can conclude that $\varphi(z)=\sum_k b_k(z)q^{\frac{k}{r}}$ has no poles in $\boldsymbol{R}$. We refer to Lemma in Section 7 of [**6**]. See also Section 5 of [**5**]. $\square$

We now proceed to the proof of Theorem 3.1. We regard $\breve{\varphi}(X;z,\tau,\sigma)$ as a meromorphic function of z. By the transformation law (13) $\phi(z,\tau,\sigma)$ is an elliptic function in z with respect to the lattice $\boldsymbol{Z}\cdot N\tau\oplus\boldsymbol{Z}$ for $\sigma=\frac{k}{N}$ with $0<k<N$. Hence the equivariant modified orbifold elliptic genus $\breve{\varphi}(X;z,\tau,\sigma)$ of level N is also an elliptic function in z. Thus, in order to show that $\breve{\varphi}(X;z,\tau,\sigma)$ is a constant it suffices to show that it does not have poles.

Assume that z is a pole. Then $1-t^mq^r\alpha=0$ for some integer $m\neq 0$, some rational number r and a root of unity α. Consequently there are intergers $m_1\neq 0$ and k_1 such that $m_1z+k_1\tau\in\boldsymbol{Z}$. Then there is an element $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\boldsymbol{Z})$ such that

$$\frac{z}{c\tau+d}\in\boldsymbol{R}.$$

Since

$$\breve{\varphi}(X;z,\tau,\sigma)=\breve{\varphi}\Big(X;A^{-1}\Big(\frac{z}{c\tau+d},A\tau\Big),\sigma\Big)=\breve{\varphi}^{A^{-1}}\Big(X;\frac{z}{c\tau+d},A\tau,\sigma\Big),$$

the function $\breve{\varphi}^{A^{-1}}(X;w,A\tau,\sigma)$ must have a pole $w=\frac{z}{c\tau+d}\in\boldsymbol{R}$. But this contradicts Lemma 6.5. This contradiction proves that $\breve{\varphi}(X;z,\tau,\sigma)$ can not have a pole and Theorem 3.1 follows.

Proposition 6.6. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action. Let $N>1$ be an integer relatively prime to the orders of all isotropy groups $|H_\lambda|$. Assume that there exists an orbifold line bundle L such that $\Lambda^nTX=L^N$. If the integer l in* (21) *is relatively prime to N, then the modified orbifold elliptic genus $\breve{\varphi}(X)$ of level N vanishes.*

Proof. By Theorem 3.1 the equivariant modified elliptic genus $\breve{\varphi}_t(X;\tau,\sigma)=\breve{\varphi}(X;z,\tau,\sigma)$ is constant and equal to $\breve{\varphi}(X;\tau,\sigma)$. By (13), Proposition 5.3 and (21) we have

$$\breve{\varphi}(X;\tau,\sigma)=\breve{\varphi}(X;z+\tau,\tau,\sigma)=\zeta^l\breve{\varphi}(X;z,\tau,\sigma)=\zeta^l\breve{\varphi}(X;\tau,\sigma).$$

Since l is relatively prime to N, ζ^l is not equal to 1. Hence $\breve{\varphi}(X;\tau,\sigma)$ must vanish. □

By similar calculations to the ones used in the proof of Lemma 6.3 we obtain the following

Lemma 6.7. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action and let $N>1$ be an integer. Assume that there exists a genuine line bundle L such that $\Lambda^nTX=L^N$. Then the orbifold elliptic genus $\hat{\varphi}(X;z,\tau,\sigma)$ of level N with $\sigma=\frac{k}{N}$, $0<k<N$, is transformed by $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\boldsymbol{Z})$ in the following way.*

$$\begin{aligned}\hat{\varphi}^A(X;z,\tau,\sigma)=e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)}&\sum_{\hat{\hat{\lambda}}\in\hat{\Lambda}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{F}_{\hat{\lambda}}}\sum_{(h_1,h_2)\in\delta}e\Big(\hat{\hat{F}}_{\hat{\lambda}}\Big)\\&\cdot e^{-2\pi\sqrt{-1}(y+m^{S^1}z)ck}\prod_{i=1}^n e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)(c\tau+d)\sigma}\\&\cdot\phi\Big(-y_i-m_i^{S^1}z+m_i^{(h_1,h_2)}(h_1)\tau-m_i^{(h_1,h_2)}(h_2),\tau,(c\tau+d)\sigma\Big).\end{aligned}\tag{35}$$

Note. Under the assumption of Lemma 6.7 (including the case $N=1$) each $f_{\hat{\lambda}}$ is an integer. In fact $\sum_i m_i^{(h_1,h_2)}(h)=Nm(h)$ is an integer as the weight of h on a genuine line bundle L^N by (22). $f_{\hat{\lambda}}$ is congruent to $\sum_i m_i^{(h_1,h_2)}(h)$ for $[h]\in\hat{X}_{\hat{\lambda}}$.

The proof of Theorem 3.3 can be given in a similar way to that of Theorem 3.1 by using Lemma 6.7. In fact, under the assumption that the orbifold line bundle L is genuine, the term $e^{-2\pi\sqrt{-1}(y+m^{S^1}z)ck}$ in the integrand of (35) is equal to $e^{-2\pi\sqrt{-1}(y+m^{S^1}z+m(h_2))ck}$ so that we have

$$\hat{\varphi}^A(X;z,\tau,\sigma) = e^{\pi\sqrt{-1}(nc(c\tau+d)\sigma^2-2clz\sigma)} \sum_{\hat{\lambda}\in\hat{\Lambda}} e^{2\pi\sqrt{-1}f_{\hat{\lambda}}(c\tau+d)\sigma}$$
$$\cdot \operatorname{ind}\Big(D_{\hat{X}_{\hat{\lambda}}} \otimes (L^*)^{ck} \otimes \mathscr{T}_{\hat{\lambda}}((c\tau+d)\sigma) \otimes \mathscr{W}_{\hat{\lambda}}((c\tau+d)\sigma)\Big).$$

The rest of the proof is entirely similar to that of Theorem 3.1. One sees that $\hat{\varphi}(X;z,\tau,\sigma)$ can not have a pole and consequently it is constant.

Proposition 6.8. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action and let $N > 1$ be an integer. Assume that there exists a genuine line bundle L such that $\Lambda^n TX = L^N$. If the integer l in* (21) *is relatively prime to N, then the orbifold elliptic genus $\hat{\varphi}(X)$ of level N vanishes.*

Proof. In a similar way to the proof of Proposition 6.6, we have

$$\hat{\varphi}(X;\tau,\sigma) = \hat{\varphi}(X;z+\tau,\tau,\sigma) = \zeta^l\hat{\varphi}(X;z,\tau,\sigma) = \zeta^l\hat{\varphi}(X;\tau,\sigma).$$

Since $\zeta^l \neq 1$, $\hat{\varphi}(X;\tau,\sigma)$ must vanish. $\square$

At this point it is to be noted that the fact that the number l in (21) is an integer is not used in the proof of Theorem 3.3. For example suppose that there is a genuine line bundle L with a lifted action of G such that $(\Lambda^n TX)^d = L^{dN}$ for some positive integer N and d. Then the equality (21) hold with a rational number l such that dl is an integer. This observation leads to the following generalization of Theorem 3.3.

Proposition 6.9. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action and let $N > 1$ be an integer. Assume that there exist a genuine orbifold line bundle L and a positive integer d such that $(\Lambda^n TX)^d = L^{dN}$. Then the orbifold elliptic genus $\hat{\varphi}(X)$ of level N is rigid.*

Note. The assumption in Proposition 6.9 is almost the same as saying that $c_1(TX) = Nc_1(L)$ in $H^2(X, \boldsymbol{Z}_X) \otimes \boldsymbol{Q}$. It should be noticed that the numbers $f_{\hat{\lambda}}$ are not necessarily integers under the assumption of Proposition 6.9.

The proof of Theorem 3.4 goes as follows. The assumption that $(\Lambda^n TX)^d$ is trivial as an orbifold line bundle is equivalent to taking L to be a trivial line bundle in Proposition 6.9. Then the assumption of Proposition 6.9 is satisfied by any integer $N > 1$. It follows that $\hat{\varphi}(X;z,\tau,\sigma)$ is constant for $\sigma = \frac{k}{N}, 0 < k < N$. Since this is true for any integer $N > 1$ and $\sigma = \frac{k}{N}$, $\hat{\varphi}(X;z,\tau,\sigma)$ must be constant.

Proposition 6.10. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial S^1 action. Assume that $(\Lambda^n TX)^d$ is trival for some positive integer d. If the number $l \in \boldsymbol{Z}/d$ in* (21) *is not zero, then $\hat{\varphi}(X)$ vanishes.*

Proof. If l is not equal to 0, take an integer N relatively prime to dl. Then $\zeta^{dl} \neq 1$ for $\sigma = \frac{k}{N}, 0 < k < N$. A similar argument to the proof of Proposition 6.6 proves that $\hat{\varphi}(X;\tau,\sigma) = \zeta^{dl}\hat{\varphi}(X;\tau,\sigma)$ and consequently $\hat{\varphi}(X;\tau,\sigma) = 0$ for such σ. Since there are infinitely many integers N relatively prime to dl, this implies that $\hat{\varphi}(X;\tau,\sigma)$ must vanish. $\square$

7. Orbifold T_y-genus.

Let X be an almost complex closed orbifold. The T_y-genus $T_y(X)$ is defined to be the index

$$T_y(X) = \operatorname{ind}(D \otimes \Lambda_y T^* X) \in \boldsymbol{Z}[y, y^{-1}].$$

We further consider the orbifold T_y-genus

$$\hat{T}_y(X) = \sum_{\hat{\lambda} \in \hat{\Lambda}} (-y)^{f_{\hat{\lambda}}} T_y\big(\hat{X}_{\hat{\lambda}}\big) = \sum_{\hat{\lambda} \in \hat{\Lambda}} (-y)^{f_{\hat{\lambda}}} \operatorname{ind}\Big(D_{\hat{X}_{\hat{\lambda}}} \otimes \Lambda_y T^* \hat{X}_{\hat{\lambda}}\Big). \tag{36}$$

In case the orders of all isotropy groups H_x are relatively prime to an integer $N > 1$ the modified orbifold T_y-genus $\breve{T}_y(X)$ of level N is defined by

$$\breve{T}_y(X) = \sum_{\hat{\lambda} \in \hat{\Lambda}} (-y)^{\breve{f}_{\hat{\lambda}}} T_y\big(\hat{X}_{\hat{\lambda}}\big) = \sum_{\hat{\lambda} \in \hat{\Lambda}} (-y)^{\breve{f}_{\hat{\lambda}}} \operatorname{ind}\Big(D_{\hat{X}_{\hat{\lambda}}} \otimes \Lambda_y T^* \hat{X}_{\hat{\lambda}}\Big), \tag{37}$$

where $-y = e^{2\pi\sqrt{-1}\frac{k}{N}},\ 0 < k < N$.

Note. $T_y(X)$ and $\hat{T}_y(X)$ are the constant terms in the power series expansions (6) of the elliptic genus $\varphi(X)$ and orbifold elliptic genus $\hat{\varphi}(X)$ with ζ replaced by $-y$. Similarly $\breve{T}_y(X)$ is the constant term of the q-expansion of $\breve{\varphi}(X)$ with ζ replaced by $-y$ when $\zeta = e^{2\pi\sqrt{-1}\frac{k}{N}},\ 0 < k < N$.

When a compact group G acts on X one can consider the corresponding equivariant genera. It is known [**9**] that the T_y-genus is rigid for closed manifolds with compact connected group action.

Proposition 7.1. *The T_y-genus, the $\hat{T}_y$-genus and the $\breve{T}_y$-genus are rigid for closed orbifolds with compact connected Lie group action.*

Proof. It is enough to prove the statement for T_y-genus in view of (36) and (37). We may further assume that G is the circle group S^1. Let $T_{y,t}(X) \in \boldsymbol{Z}[y, y^{-1}] \otimes R(S^1)$ be the equivariant T_y-genus of X. We shall use the notations in Section 5 and put $\zeta = -y$. By the fixed point formula (10) we have

$$\begin{aligned} T_{y,t}(X) = &\sum_{\hat{\lambda} \in \hat{\Lambda}_F} \frac{1}{|H_{\pi(\hat{\lambda})}|} \int_{\hat{F}_{\hat{\lambda}}} \sum_{h \in \gamma_x} Td\Big(\hat{F}_{\hat{\lambda}}\Big) \prod_{i=1}^{r_1} \big(1 - \zeta e^{-2\pi\sqrt{-1}y_i}\big) \\ &\cdot \prod_{i=r_1+1}^{n} \frac{\Big(1 - \zeta t^{-m_i^{S^1}} e^{-(2\pi\sqrt{-1}y_i + m_i^{(1,h)}(h))}\Big)}{\Big(1 - t^{-m_i^{S^1}} e^{-(2\pi\sqrt{-1}y_i + m_i^{(1,h)}(h))}\Big)}. \end{aligned}$$

Note that r_1 and $m_i^{S^1}$ depend on $\hat{\lambda}$, and $m_i^{S^1} \neq 0$ for $r_1 + 1 \leq i \leq n$. Put

$$\mu(\hat{\lambda}) = \#\{i \mid m_i^{S^1} > 0\}.$$

Then $\#\{i \mid m_i^{S^1} < 0\} = n - r_1 - \mu(\hat{\lambda})$. We regard

$$\alpha_{\hat{\lambda}}(t) = \frac{1}{\left|H_{\pi(\hat{\lambda})}\right|} \int_{\hat{F}_{\hat{\lambda}}} \sum_{h \in \gamma_x} Td\left(\hat{F}_{\hat{\lambda}}\right) \prod_{i=1}^{r_1} \left(1 - \zeta e^{-2\pi\sqrt{-1}y_i}\right)$$
$$\cdot \prod_{i=r_1+1}^{n} \frac{\left(1 - \zeta t^{-m_i^{S^1}} e^{-(2\pi\sqrt{-1}y_i + m_i^{(1,h)}(h))}\right)}{\left(1 - t^{-m_i^{S^1}} e^{-(2\pi\sqrt{-1}y_i + m_i^{(1,h)}(h))}\right)}$$

as a rational function of t. Then we see easily that

$$\alpha_{\hat{\lambda}}(0) = \frac{1}{\left|H_{\pi(\hat{\lambda})}\right|} \int_{\hat{F}_{\hat{\lambda}}} \sum_{h \in \gamma_x} Td\left(\hat{F}_{\hat{\lambda}}\right) \prod_{i=1}^{r_1} \left(1 - \zeta e^{-2\pi\sqrt{-1}y_i}\right) \zeta^{\mu(\hat{\lambda})},$$
$$\alpha_{\hat{\lambda}}(\infty) = \frac{1}{\left|H_{\pi(\hat{\lambda})}\right|} \int_{\hat{F}_{\hat{\lambda}}} \sum_{h \in \gamma_x} Td\left(\hat{F}_{\hat{\lambda}}\right) \prod_{i=1}^{r_1} \left(1 - \zeta e^{-2\pi\sqrt{-1}y_i}\right) \zeta^{n - r_1 - \mu(\hat{\lambda})}.$$

Thus $T_{y,t}(X) = \sum_{\hat{\lambda}} \alpha_{\hat{\lambda}}(t)$ takes finite values at $t = 0$ and $t = \infty$. Since $T_{y,t}(X)$ belongs to $\boldsymbol{Z}[y, y^{-1}] \otimes R(S^1) = \boldsymbol{Z}[y, y^{-1}] \otimes \boldsymbol{Z}[t, t^{-1}]$, it must be a constant which is equal to $T_y(X)$. □

In the above proof we have in fact proved the following

Corollary 7.2.

$$T_y(X) = \sum_{k=0}^{n-r_1} (-y)^k \sum_{\hat{\lambda}:\mu(\hat{\lambda})=k} T_y\left(\hat{F}_{\hat{\lambda}}\right) = \sum_{k=0}^{n-r_1} (-y)^k \sum_{\hat{\lambda}:\mu(\hat{\lambda})=n-r_1-k} T_y\left(\hat{F}_{\hat{\lambda}}\right).$$

Note that $T_y(X)$ is a polynomial in $-y$ of degree at most $n = \frac{\dim X}{2}$ with integer coefficients.

Lemma 7.3. *If $\Lambda^n TX$ is a genuine line bundle, then $\hat{T}_y(X)$ is a polynomial in $-y$ of degree at most n with integer coefficients.*

Proof. Each $f_{\hat{\lambda}}$ is an integer by Note after Lemma 6.7. Therefore $\hat{T}_y(X)$ is a polynomial in $-y$ with integer coefficients. If $\dim \hat{X}_{\hat{\lambda}} = 2k < 2n$, then $f_{\hat{\lambda}} = \sum_{i=1}^{n-k} f_{\hat{\lambda},i}$ with $0 < f_{\hat{\lambda},i} < 1$. Therefore $0 < f_{\hat{\lambda}} < n - k$ and the degree of $(-y)^{f_{\hat{\lambda}}} T_y(\hat{X}_{\hat{\lambda}})$ is less than $n - k + k = n$. □

Proposition 7.4. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Let $N > 1$ be an integer. Assume that there exists a genuine line bundle L with a lifted action of G such that $\Lambda^n TX = L^N$. If the integer l in (21) is relatively prime to N, then the orbifold $\hat{T}_y$-genus*

$\hat{T}_y(X)$ is a polynomial in $-y$ divisible by

$$\sum_{k=0}^{N-1}(-y)^k.$$

Moreover if $\hat{T}_y(X)\neq 0$, then $N\leq n+1$.

PROOF. By Proposition 6.8 the orbifold elliptic genus $\hat{\varphi}(X)$ of level N vanishes. In particular its degree 0 term $\hat{T}_y(X)=\sum_{\hat{\lambda}}(-y)^{f_{\hat{\lambda}}}T_y(\hat{X}_{\hat{\lambda}})$ vanishes for $-y=e^{2\pi\sqrt{-1}\frac{k}{N}}$, $0<k<N$. Since $\hat{T}_y(X)$ is a polynomial with integer coefficients in $-y$ it must be divisible by $\sum_{k=0}^{N-1}(-y)^k$.

Assume that $\hat{T}_y(X)\neq 0$. Since its degree is at most n by Lemma 7.3 and it is divisible by $\sum_{k=0}^{N-1}(-y)^k$, we must have $N-1\leq n$. □

REMARK 7.5. Suppose that the multiplicity of X is equal to 1 and Λ^nTX is a genuine line bundle. The constant term of $\hat{T}_y(X)$ considered as a polynomial of $-y$ is equal to the constant term $T_0(X)$ of $T_y(X)$ which is nothing but the Todd genus of X, since $f_{\hat{\lambda}}>0$ for twisted sectors. Thus, if $T_0(X)\neq 0$, then $\hat{T}_y(X)$ does not vanish.

Situations like Proposition 7.4 occur when an n-dimensional torus acts on X, cf. [**4**].

Finally as corollaries of Proposition 6.6 and Proposition 6.10 we have

PROPOSITION 7.6. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Let $N>1$ be an integer relatively prime to the orders of all isotropy groups $|H_\lambda|$. Assume that there exists an orbifold line bundle L with a lifted action of G such that $\Lambda^nTX=L^N$. If the integer l in* (21) *is relatively prime to N, then the modified $\check{T}_y$-genus $\check{T}_y(X)$ of level N vanishes for $-y=e^{2\pi\sqrt{-1}\frac{k}{N}}$, $0<k<N$.*

PROPOSITION 7.7. *Let X be an almost complex closed orbifold of dimension $2n$ with a non-trivial action of a compact connected Lie group G. Assume that there is a positive integer d such that $(\Lambda^nTX)^d$ is trivial as an orbifold line bundle. If the number $l\in\boldsymbol{Z}/d$ in* (21) *is not equal to* 0, *then the orbifold $\hat{T}_y$-genus $\hat{T}_y(X)$ vanishes.*

8. Appendix: stably almost complex orbifolds.

A stably almost complex orbifold is an oriented orbifold with a stably almost complex structure on the tangent bundle. More precisely, let $\mathscr{U}$ be an orbifold atlas of X. Let k be a positive integer and let $\boldsymbol{k}$ denote the trivial real vector bundle of dimension k endowed with the standard orientation. Then a complex structure on $TV\oplus\boldsymbol{k}$ is given for each chart $(V,U,H)\in\mathscr{U}$ in such a way that it is compatible with the orientation of TV followed by that of $\boldsymbol{k}$, and it is preserved by each element $h\in H$. Here h acts trivially on $\boldsymbol{l}$. It is also required that these complex structures are compatible with injections of charts. Two complex structures $TV\oplus\boldsymbol{k}$ and $TV\oplus\boldsymbol{k'}$ are considered as equivalent if the Whitney sums of $\boldsymbol{l_C}$ and $\boldsymbol{l'_C}$ to them give the same complex structures on $TV\oplus(\boldsymbol{k}+\boldsymbol{2l})=TV\oplus(\boldsymbol{k'}+\boldsymbol{2l'})$, where $\boldsymbol{l_C}$ and $\boldsymbol{l'_C}$ are trivial complex vector bundles of dimension l and l' such that $k+2l=k'+2l'$. An equivalence class is the stably almost complex structure. Note that each sector $\hat{X}_{\hat{\lambda}}$ is also a stably almost complex orbifold. In fact, for $\gamma_x=[h]\in\hat{X}_{\hat{\lambda}}$ with $h\in H_x\subset H$, the complex vector bundle $(TV\oplus\boldsymbol{k})^h=TV^h\oplus\boldsymbol{k}$ gives the stably almost complex structure on $\hat{X}_{\hat{\lambda}}$. The normal bundle $W_{\hat{\lambda}}$ of the immersion $\pi:\hat{X}_{\hat{\lambda}}\to X$ has a canonical complex structure and the eigen-bundle decomposition by the action of h just like (3).

In general let X be an orbifold. For an orbifold complex vector bundle W of rank l we put $\tilde{W}=W-\boldsymbol{l}_{\boldsymbol{C}}\in K_{orb}(X)$. If X is a stably almost complex orbifold, $\widetilde{TX\oplus\boldsymbol{k}}$ is a well-defined element in $K_{orb}(X)$ which we simply denote by $\tilde{T}X$. Similarly we put $\tilde{T}^*X=\widetilde{T^*X\oplus\boldsymbol{k}}$.

We then define formal vector bundles $\tilde{\mathscr{T}}=\tilde{\mathscr{T}}(\sigma)$ and $\tilde{\mathscr{T}}_{\hat{\lambda}}=\tilde{\mathscr{T}}_{\hat{\lambda}}(\sigma)$ by

$$\tilde{\mathscr{T}}(\sigma)=\Lambda_{-\zeta}\tilde{T}^*X\otimes\bigotimes_{k=1}^{\infty}\Big(\Lambda_{-\zeta q^k}\tilde{T}^*X\otimes\Lambda_{-\zeta^{-1}q^k}\tilde{T}X\otimes S_{q^k}\tilde{T}^*X\otimes S_{q^k}\tilde{T}X\Big),$$

$$\tilde{\mathscr{T}}_{\hat{\lambda}}(\sigma)=\Lambda_{-\zeta}\tilde{T}^*\hat{X}_{\hat{\lambda}}\otimes\bigotimes_{k=1}^{\infty}\Big(\Lambda_{-\zeta q^k}\tilde{T}^*\hat{X}_{\hat{\lambda}}\otimes\Lambda_{-\zeta^{-1}q^k}\tilde{T}\hat{X}_{\hat{\lambda}}\otimes S_{q^k}\tilde{T}^*\hat{X}_{\hat{\lambda}}\otimes S_{q^k}\tilde{T}\hat{X}_{\hat{\lambda}}\Big).$$

It is to be noted that

$$\Lambda_\lambda\tilde{W}=\Lambda_\lambda W/(1+\lambda)^{\operatorname{rank}W},\quad S_\lambda\tilde{W}=S_\lambda W/(1+\lambda)^{\operatorname{rank}W},$$

because of multiplicative properties of total exterior power operation and symmetric power operation. It follows that

$$\tilde{\mathscr{T}}(\sigma)=\mathscr{T}(\sigma)/\big(-\zeta^{\frac{1}{2}}\Phi(\sigma,\tau)\big)^{\frac{\dim X}{2}},\quad \tilde{\mathscr{T}}_{\hat{\lambda}}(\sigma)=\mathscr{T}(\sigma)_{\hat{\lambda}}/\big(-\zeta^{\frac{1}{2}}\Phi(\sigma,\tau)\big)^{\frac{\dim X_{\hat{\lambda}}}{2}},$$

when X is an almost complex orbifold. Since we have eigen-bundle decomposition (3) as in the case of almost complex orbifold, we can define $\mathscr{W}_{\hat{\lambda},i}=\mathscr{W}_{\hat{\lambda},i}(\sigma)$ and $\mathscr{W}_{\hat{\lambda}}=\mathscr{W}_{\hat{\lambda}}(\sigma)$ just as in section 3. We then define

$$\tilde{\mathscr{W}}_{\hat{\lambda}}=\mathscr{W}_{\hat{\lambda}}/\big(-\zeta^{\frac{1}{2}}\Phi(\sigma,\tau)\big)^{\operatorname{rank}W_{\hat{\lambda}}}.$$

On a stably almost complex orbifold a spin-c Dirac operator can be introduced as in the case of almost complex manifolds. It is an operator of the same form as (5) with

$$E^+=\bigoplus_{i:even}\Lambda^i(TX\oplus\boldsymbol{k})\text{ and }E^-=\bigoplus_{i:odd}\Lambda^i(TX\oplus\boldsymbol{k}).$$

Hereafter we assume that X is a stably almost complex orbifold of dimension $2n$, and a complex vector bundle structure of rank $n+s$ is given on $T'X=TX\oplus\boldsymbol{2s}$ for some s.

We define the stabilized elliptic genus $\varphi_{st}(X)$ and stabilized orbifold elliptic genus

$\hat{\varphi}_{st}(X)$ of X by

$$\varphi_{st}(X) = \mathrm{ind}(D \otimes \tilde{\mathscr{T}}(\sigma)),$$
$$\hat{\varphi}_{st}(X) = \sum_{\hat{\lambda} \in \hat{\Lambda}} \zeta^{f_{\hat{\lambda}}} \, \mathrm{ind}\left(D_{\hat{X}_{\hat{\lambda}}} \otimes \tilde{\mathscr{T}}_{\hat{\lambda}}(\sigma) \otimes \tilde{\mathscr{W}}_{\hat{\lambda}}(\sigma) \right).$$

When X is an almost complex orbifold of dimension $2n$ we have

$$\varphi(X) = \varphi_{st}(X)\big(- \Phi(\sigma, \tau)\big)^n, \quad \hat{\varphi}(X) = \hat{\varphi}_{st}(X)\big(- \Phi(\sigma, \tau)\big)^n. \tag{38}$$

We may define $\varphi(X)$ and $\hat{\varphi}(X)$ for stably almost complex orbifold X by (38).

Suppose that $N > 1$ is an integer relatively prime to every $|H_x|$. We define modified stabilized orbifold elliptic genus $\check{\varphi}_{st}(X)$ by

$$\check{\varphi}_{st}(X) = \sum_{\hat{\lambda} \in \hat{\Lambda}} \zeta^{\check{f}_{\hat{\lambda}}} \, \mathrm{ind}\left(D_{\hat{X}_{\hat{\lambda}}} \otimes \tilde{\mathscr{T}}_{\hat{\lambda}}(\sigma) \otimes \tilde{\mathscr{W}}_{\hat{\lambda}}(\sigma) \right),$$

and set $\check{\varphi}(X) = \check{\varphi}_{st}(X)(-\Phi(\sigma, \tau))^n$.

The cohomology classes of $c(T'X) \in H^*(X, \boldsymbol{Z}_X)$ and $Td(T'X) \in H^*(X, \boldsymbol{Q}_X)$ are well-defined classes depending only on $\tilde{T}X$, where

$$Td(T'X) = \det\left(\frac{\Gamma(T'X)}{1 - e^{-\Gamma(T'X)}} \right) = \prod_{i=1}^{n+s} \frac{x_i}{1 - e^{-x_i}},$$

with $c(T'X) = \prod_i (1 + x_i)$ as before.

When a compact connected Lie group G acts on X it is always assumed that the action preserves the stably almost complex structure. Then we can consider equivariant genera corresponding $\hat{\varphi}(X)$ etc. The fixed point set X^G is a stably almost complex orbifold. Vergne's fixed point formulas (9) and (10) still hold for stably almost complex orbifolds by replacing $Td(\hat{F}_{\hat{\lambda}})$ by $Td(T'\hat{F}_{\hat{\lambda}})$.

Suppose that $N > 1$ is an integer and there is an orbifold line bundle L such that $\Lambda^{n+s}T'X = L^N$. This condition is equivalent to saying that the first orbifold Chern class $c_1(\tilde{T}X) \in H^2(X, \boldsymbol{Z}_X)$ is divisible by N. Similarly the condition that $\Lambda^{n+s}T'X$ is trivial means that $c_1(\tilde{T}X) \in H^2(X, \boldsymbol{Z}_X)$ vanishes. Theorems 3.1, 3.3 and 3.4 have meanings for stably almost complex manifolds by replacing $\Lambda^n TX$ by $\Lambda^{n+s}T'X$, and they in fact hold in this extended sense. Similarly Propositions 6.6, 6.8, 6.9 and 6.10 hold for stably almost complex orbifolds.

Proofs are almost verbatim. We work with stabilized genera $\hat{\varphi}_{st}(X)$ and so on. We put

$$\phi_{st}(z, \tau, \sigma) = -\phi(z, \tau, \sigma)/\Phi(\sigma, \tau) = -\Phi(z + \sigma, \tau)/\Phi(z, \tau)\Phi(\sigma, \tau).$$

$\phi_{st}(z, \tau, \sigma)$ satisfies the following transformation law.

$$\phi_{st}(A(z,\tau),\sigma)=(c\tau+d)e^{2\pi\sqrt{-1}cz\sigma}\phi_{st}(z,\tau,(c\tau+d)\sigma)\quad\text{for } A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{Z}),$$

$$\phi_{st}(z+m\tau+n,\tau,\sigma)=e^{-2\pi\sqrt{-1}m\sigma}\phi_{st}(z,\tau,\sigma)\quad\text{for } m,n\in\boldsymbol{Z}.$$

We put $e\big(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\big)=\prod_{i=1}^{r_1}x_i$ where $T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}=T\hat{\hat{F}}_{\hat{\hat{\lambda}}}\oplus\boldsymbol{2s}$ and $r_1=\mathrm{rank}_{\boldsymbol{C}}\left(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\right)=\frac{\dim\hat{\hat{F}}_{\hat{\hat{\lambda}}}}{2}+s$. We use the same notions and conventions as in (15), (16) and (17). Then (18) and (19) are replaced by

$$\hat{\varphi}_{st}(X;z,\tau,\sigma)=\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(h_1,h_2)\in\delta}e\Big(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)\prod_{i=1}^{n}e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)\sigma}$$
$$\cdot\phi_{st}\Big(-y_i-m_i^{S^1}z+m_i^{(h_1,h_2)}(h_1)\tau-m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big)\tag{39}$$

and

$$\check{\varphi}_{st}(X;z,\tau,\sigma)=\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(h_1,h_2)\in\delta}e\Big(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)\prod_{i=1}^{n}e^{2\pi\sqrt{-1}\check{m}_i^{\langle h_1,h_2\rangle}(h_1)\sigma}$$
$$\cdot\phi_{st}\Big(-y_i-m_i^{S^1}z+m_i^{(h_1,h_2)}(h_1)\tau-m_i^{(h_1,h_2)}(h_2),\tau,\sigma\Big)\tag{40}$$

respectively.

As for (24) in Lemma 6.3 it is replaced by

$$\check{\varphi}^A_{st}(X;z,\tau,\sigma)=(c\tau+d)^ne^{-2\pi\sqrt{-1}clz\sigma}\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(h_1,h_2)\in\delta}e\Big(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)$$
$$\cdot e^{-2\pi\sqrt{-1}m(h_1)dk}e^{-2n\pi\sqrt{-1}(y+m^{S^1}z+m(h_2))ck}\prod_{i=1}^{n}e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)(c\tau+d)\sigma}$$
$$\cdot\phi_{st}\Big(-y_i-m_i^{S^1}z+m_i^{(h_1,h_2)}(h_1)\tau-m_i^{(h_1,h_2)}(h_2),\tau,(c\tau+d)\sigma\Big).\tag{41}$$

Similarly (35) in Lemma 6.7 is replaced by

$$\hat{\varphi}^A_{st}(X;z,\tau,\sigma)=(c\tau+d)^ne^{-2\pi\sqrt{-1}clz\sigma}\sum_{\hat{\hat{\lambda}}\in\hat{\hat{\Lambda}}}\frac{1}{|H_{\pi\circ\hat{\pi}(\delta)}|}\int_{\hat{\hat{F}}_{\hat{\hat{\lambda}}}}\sum_{(h_1,h_2)\in\delta}e\Big(T'\hat{\hat{F}}_{\hat{\hat{\lambda}}}\Big)$$
$$\cdot e^{-2\pi\sqrt{-1}(y+m^{S^1}z)ck}\prod_{i=1}^{n}e^{2\pi\sqrt{-1}m_i^{(h_1,h_2)}(h_1)(c\tau+d)\sigma}$$
$$\cdot\phi_{st}\Big(-y_i-m_i^{S^1}z+m_i^{(h_1,h_2)}(h_1)\tau-m_i^{(h_1,h_2)}(h_2),\tau,(c\tau+d)\sigma\Big).\tag{42}$$

By using (39), (40), (41) and (42) proofs of stably almost complex versions of Theorems 3.1, 3.3 and 3.4 can be easily completed.

Todd genus $T_0(X)$ is an invariant of stably almost complex orbifolds. It is defined as the index of a spin-c Dirac operator D and written as the integral $T_0(X) = \int_X Td(T'X)$. We define the stabilized T_y-genus and stabilized $\hat{T}_y$-genus of X by

$$\begin{aligned} T_{y,st}(X) &= \operatorname{ind}\left(D \otimes \Lambda_y \tilde{T}^* X\right) \\ \hat{T}_{y,st}(X) &= \sum_{\hat{\lambda}\in\hat{\Lambda}} (-y)^{f_{\hat{\lambda}}} T_{y,st}\left(\hat{X}_{\hat{\lambda}}\right). \end{aligned}$$

They are the degree zero terms in the q-expansions of $\zeta^{\frac{n}{2}}\varphi_{st}(X)$ and $\zeta^{\frac{n}{2}}\hat{\varphi}_{st}(X)$. If $N > 1$ is an integer such that it is relatively prime to all $|H_x|$, then we define

$$\check{T}_{y,st}(X) = \sum_{\hat{\lambda}\in\hat{\Lambda}} (-y)^{\check{f}_{\hat{\lambda}}} T_{y,st}\left(\hat{X}_{\hat{\lambda}}\right).$$

Note that, if X is an almost complex orbifold of dimension $2n$, then

$$T_{y,st}(X) = T_y(X)/(1+y)^n,\ \hat{T}_{y,st}(X) = \hat{T}_y(X)/(1+y)^n,\ \check{T}_{y,st}(X) = \check{T}_y(X)/(1+y)^n.$$

We may define $T_y(X)$, $\hat{T}_y(X)$ and $\check{T}_y(X)$ for stably almost complex orbifolds using the above equalities. With these understandings the results in Section 7 still hold for stably almost complex orbifolds.

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