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Homotopy classes of self-maps and induced homomorphisms of homotopy groups

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Abstract. For a based space X, we consider the group $\mathscr{E}_{\#n}(X)$ of all self homotopy classes α of X such that $\alpha_{\#} = \mathrm{id} : \pi_i(X) \to \pi_i(X)$, for all $i \leq n$, where $n \leq \infty$, and the group $\mathscr{E}_{\Omega}(X)$ of all α such that $\Omega\alpha = \mathrm{id}$. Analogously, we study the semigroups $\mathscr{Z}_{\#n}(X)$ and $\mathscr{Z}_{\Omega}(X)$ defined by replacing 'id' by '0' above. There is a chain of containments of the $\mathscr{E}$-groups and the $\mathscr{Z}$-semigroups, and we discuss examples for which the containment is proper. We then obtain various conditions on X which ensure that the $\mathscr{E}$-groups and the $\mathscr{Z}$-semigroups are equal. When X is a group-like space, we derive lower bounds for the order of these groups and their localizations. In the last section we make specific calculations for the $\mathscr{E}$-groups and $\mathscr{Z}$-groups of certain low dimensional Lie groups.

1. Introduction.

Let X and Y be topological spaces with base point. A major objective of homotopy theory is to investigate and understand the set $[X, Y]$ of homotopy classes of based maps from X to Y. Typically, one restricts the spaces in order to put more structure on the sets $[X, Y]$. In this paper we consider the case $X = Y$, so that there is a binary operation in $[X, X]$ obtained from composing homotopy classes. We study certain subgroups and subsemigroups of $[X, X]$. More specifically, we consider the monoid $[X, X]$ and its group of units $\mathscr{E}(X)$. Then $\mathscr{E}(X)$ is the group of homotopy classes of homotopy equivalences $X \to X$. Define subgroups $\mathscr{E}_{\Omega}(X)$ and $\mathscr{E}_{\#n}(X)$ of $\mathscr{E}(X)$ by

$$\mathscr{E}_{\Omega}(X) = \{\alpha \in \mathscr{E}(X),\ \Omega\alpha = \mathrm{id}\} \quad \text{and}$$
$$\mathscr{E}_{\#n}(X) = \{\alpha \in \mathscr{E}(X),\ \alpha_{\#} = \mathrm{id} : \pi_i(X) \to \pi_i(X),\ \text{for all } i \leq n\},$$

where Ω is the loop-space functor, $\alpha_{\#}$ is the induced homomorphism of homotopy groups and id is the identity homomorphism. Furthermore, we allow $n = \infty$, that is,

$$\mathscr{E}_{\#\infty}(X) = \{\alpha \in \mathscr{E}(X),\ \alpha_{\#} = \mathrm{id} : \pi_i(X) \to \pi_i(X),\ \text{for all } i\}.$$

In addition, if X is a CW-complex of dimension n, define the subgroup $\mathscr{E}_{\#}(X)$ of $\mathscr{E}(X)$ by $\mathscr{E}_{\#}(X) = \mathscr{E}_{\#n}(X)$. Then there is a chain of subgroups of $\mathscr{E}(X)$:

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$$\mathscr{E}_\Omega(X) \subseteq \mathscr{E}_{\#\infty}(X) \subseteq \mathscr{E}_\#(X). \tag{1.1}$$

We refer to them collectively as $\mathscr{E}$-groups. They have been studied extensively (for some of the references, see [**1**], [**2**], [**3**], [**4**], [**6**], [**8**], [**11**], [**12**], [**13**], [**21**], [**22**], [**23**], [**30**], [**31**], [**33**]).

We also define subsets of $[X, Y]$ by

$$\mathscr{Z}_\Omega(X, Y) = \{\alpha \in [X, Y],\ \Omega\alpha = 0\} \quad \text{and}$$
$$\mathscr{Z}_{\#n}(X, Y) = \{\alpha \in [X, Y],\ \alpha_\# = 0 : \pi_i(X) \to \pi_i(Y) \text{ for all } i \leq n\},$$

where $n \leq \infty$. When $X = Y$ we obtain subsemigroups of $[X, X]$ by setting $\mathscr{Z}_\Omega(X) = \mathscr{Z}_\Omega(X, X)$ and $\mathscr{Z}_{\#n}(X) = \mathscr{Z}_{\#n}(X, X)$. When X is a CW-complex of dimension n, we define $\mathscr{Z}_\#(X) = \mathscr{Z}_{\#n}(X)$. Then we have a chain of subsemigroups of $[X, X]$:

$$\mathscr{Z}_\Omega(X) \subseteq \mathscr{Z}_{\#\infty}(X) \subseteq \mathscr{Z}_\#(X). \tag{1.2}$$

These semigroups have also been widely studied [**5**], [**6**], [**7**], [**23**], [**24**], [**25**]. We refer to them collectively as $\mathscr{Z}$-semigroups.

It is natural to ask if there are spaces X for which containments in (1.1) and (1.2) are proper. Several known results show that three of the four inclusions can be proper, and we complete the answer to this question by giving an example in Proposition 2.1 which shows that the fourth inclusion can be proper. The full result is stated as Proposition 2.1. Most of the spaces which serve as examples are finite complexes. However, the only known space X for which $\mathscr{E}_\Omega(X) \subsetneq \mathscr{E}_{\#\infty}(X)$ is an infinite-dimensional complex. The obvious analogy between the $\mathscr{Z}$-groups and the corresponding $\mathscr{E}$-groups and the fact that there is a finite complex X with $\mathscr{Z}_\Omega(X) \subsetneq \mathscr{Z}_{\#\infty}(X)$ have led us to make the following conjecture, which also appears in [**31**, p. 680].

Conjecture 1.1. *There is a finite-dimensional CW-complex X such that $\mathscr{E}_\Omega(X) \subsetneq \mathscr{E}_{\#\infty}(X)$.*

When the space X is group-like, this analogy is more precise, for there is a bijection between the $\mathscr{Z}$-groups and the corresponding $\mathscr{E}$-groups (Proposition 3.1). On the other hand, finite-dimensional group-like spaces are strongly related to products of odd-dimensional spheres, for which $\mathscr{E}_\Omega = \mathscr{E}_{\#\infty}$ (Corollary 2.8). Thus we make the following additional conjecture, which is supported by the calculations of §4.

Conjecture 1.2. *If X is a finite-dimensional group-like space, then $\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X)$.*

After discussing the examples mentioned above, we consider in §2 the general problem of determining when the $\mathscr{E}$-groups are all equal and when the $\mathscr{Z}$-semigroups are all equal. We first present an alternate characterization of $\mathscr{E}_\Omega(X)$ and $\mathscr{Z}_\Omega(X)$. Our main result in §2 (Theorem 2.13) is that the $\mathscr{E}$-groups of X are equal and the $\mathscr{Z}$-semigroups are equal if X is a product of spheres and projective spaces. We then consider localization and show that the $\mathscr{E}$-groups and $\mathscr{Z}$-semigroups are equal for a rational space and for a group-like space localized at a regular prime. In §3 we obtain lower bounds for the order of the localizations of the $\mathscr{E}$-groups and $\mathscr{Z}$-groups of a group-like space. We further specialize to Lie groups in §4 and give conditions which are equivalent to the triviality of an $\mathscr{E}$-group or a $\mathscr{Z}$-group. We conclude the paper by explicitly calculating these groups for low-dimensional Lie groups such as $U(2)$, $S^1 \times SO(3)$, $SU(3)$, $Sp(2)$, $S^3 \times SO(3)$ and $SO(4)$.

We end this section by describing our notation and assumptions. Each space is to be connected, based and have the homotopy type of a based CW-complex. All maps and homotopies are to preserve the base point, which is denoted $*$. We do not distinguish notationally between a map and its homotopy class. A *nilpotent space* is one such that the fundamental group is nilpotent and which acts nilpotently on the higher homotopy groups [**17**, p. 62]. The identity map of X is denoted id_X or simply id and the constant map is $0 : X \to Y$. A space X is a co-H-space if there is a map $\phi : X \to X \vee X$ (the wedge of X with itself) whose composition with each of the two projections $X \vee X \to X$ is id_X. If $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f = \mathrm{id}_X$ then X is a retract of Y; g is called a *retraction* of f and f is called a *section* of g. A space is *group-like* if it satisfies all the axioms of a group up to homotopy [**35**, p. 118]. A map $f : X \to Y$ induces functions $f_* : [A, X] \to [A, Y]$ and $f^* : [Y, B] \to [X, B]$ by composition, for all A and B. The homomorphism of homotopy groups $\pi_n(X) \to \pi_n(Y)$ induced by f is denoted $f_\#$ or $f_{\#n}$. The standard notation of homotopy theory will be used: '$\equiv$' for same homotopy type, 'Σ' for (reduced) suspension, 'Ω' for loop-space, '$\vee$' for wedge and '$\wedge$' for smashed product. The natural isomorphism between $[\Sigma X, Y]$ and $[X, \Omega Y]$ is called the *adjoint* isomorphism. Finally $\mathrm{nil}\, G$ denotes the nilpotency (class) of the group G, and '$\cong$' denotes isomorphism of groups or Lie groups.

2. Equality of the $\mathscr{E}$-groups and $\mathscr{Z}$-semigroups.

We begin this section by discussing CW-complexes X for which the inequalities in (1.1) and (1.2) are strict containments. We then turn our attention to proving results which guarantee that the containments of (1.1) and (1.2) are actually equalities. The bulk of this work is done in subsection 2.2, where we are concerned with results that are valid for general spaces. These results can be localized, and in the third subsection we study in more detail spaces that have been localized; of particular interest here are some special properties of group-like spaces.

2.1. Proper containment.

Our first result gives examples showing that the containments in (1.1) and (1.2) can be proper. All of these except (4) are already known.

Proposition 2.1.

(1) *There is an infinite-dimensional CW-complex X such that $\mathscr{E}_\Omega(X) \subsetneq \mathscr{E}_{\#\infty}(X)$.*
(2) *There is a finite complex X such that $\mathscr{E}_{\#\infty}(X) \subsetneq \mathscr{E}_\#(X)$.*
(3) *There is a finite complex X such that $\mathscr{Z}_\Omega(X) \subsetneq \mathscr{Z}_{\#\infty}(X)$.*
(4) *There is a finite complex X such that $\mathscr{Z}_{\#\infty}(X) \subsetneq \mathscr{Z}_\#(X)$.*

All of these spaces can be chosen to be simply-connected.

Proof. For (1), the example can be found in [**13**]; an example for (2) is given in [**4**, Corollary 4.13 or Proposition 6.3]; and examples for (3) can be found in [**6**] or [**7**, p. 395].

We now turn our attention to part (4). Let $\mathbf{2} : S^{n-1} \to S^{n-1}$ be the map of degree 2, $M = S^{n-1} \cup_{\mathbf{2}} e^n$ a Moore space of type $(\boldsymbol{Z}_2, n-1)$ and $X = S^n \vee M$, where $n > 3$. Let $q : M \to S^n$ collapse S^{n-1} to a point. Let $p_2 : X \to M$ be the projection and $i_1 : S^n \to X$ the injection, and set $\alpha = i_1 \circ q \circ p_2 : X \to X$. We will show that $\alpha_{\#r} = 0$ for all $r \leq n$, and that $\alpha_{\#n+1} \neq 0$. For these assertions, it suffices to show that $q_{\#r} = 0 : \pi_r(M) \to \pi_r(S^n)$ for all $r \leq n$, and that $q_{\#n+1} \neq 0 : \pi_{n+1}(M) \to \pi_{n+1}(S^n)$. For $r < n$, this is obvious because $\pi_r(S^n) = 0$. Let r be n or $n+1$. Let $\Sigma^\infty : \pi_k(Y) \to \pi_k^s(Y) = \lim_{l\to\infty} \pi_{k+l}(\Sigma^l Y)$ be the stabilization homomorphism. We have the following commutative diagram in which the horizontal sequence is an exact sequence of abelian groups:

$$\begin{tikzcd}
 & & \pi_r(M) \arrow[r, "q_{\#r}"] \arrow[d, "\Sigma_1^\infty"] & \pi_r(S^n) \arrow[d, "\Sigma_2^\infty", "\cong"'] & \\
\pi_r^s(S^{n-1}) \arrow[r, "\mathbf{2}^s_{\#r}"] & \pi_r^s(S^{n-1}) \arrow[r, "i^s_{\#r}"] & \pi_r^s(M) \arrow[r, "q^s_{\#r}"] & \pi_r^s(S^n) \arrow[r, "\mathbf{2}^s_{\#r}"] & \pi_r^s(S^n).
\end{tikzcd}$$

By the Freudenthal suspension theorem [**35**, p. 369], Σ_2^∞ is an isomorphism, and Σ_1^∞ is an isomorphism for $r = n$ and an epimorphism for $r = n+1$. Since $\pi_n^s(S^n) \cong \boldsymbol{Z}$, $\pi_n^s(S^{n-1}) \cong \pi_{n+1}^s(S^n) \cong \boldsymbol{Z}_2$, and $\mathbf{2}^s_{\#r}$ is multiplication by 2, we have $\pi_n^s(M) \cong \mathbb{Z}_2$, so $q_{\#n} = 0$ by the commutativity of the above diagram. Also $q^s_{\#n+1}$ is a surjection and hence $q_{\#n+1}$ is non-zero. Therefore $\alpha \notin \mathscr{Z}_{\#\infty}(X)$. $\square$

2.2. General results.

In this subsection, we find conditions that are sufficient to guarantee that the inclusions in (1.1) and (1.2) are equalities. These results will be applied to products of spheres and projective spaces and will play a role in the calculations of §4.

We begin with a simple, but useful, observation.

Remark 2.2. Let $\phi : W \to X$, where W is a wedge of spheres with $\dim(W) \leq N$. Then

(1) if $f \in \mathscr{E}_{\#N}(X)$ then $f \circ \phi = \phi$ and
(2) if $f \in \mathscr{Z}_{\#N}(X)$ then $f \circ \phi = 0$.

Now we give an alternate characterization of $\mathscr{E}_\Omega$ and $\mathscr{Z}_\Omega$.

Proposition 2.3. *For any two spaces X and Y,*

$$\mathscr{E}_\Omega(X) = \{\alpha \in \mathscr{E}(X),\ \alpha_* = \mathrm{id} : [\Sigma A, X] \to [\Sigma A, X] \text{ for every space } A\} \quad \text{and}$$

$$\mathscr{Z}_\Omega(X, Y) = \{\alpha \in [X, Y],\ \alpha_* = 0 : [\Sigma A, X] \to [\Sigma A, Y] \text{ for every space } A\}.$$

Proof. We only prove the statement for $\mathscr{E}_\Omega$, since the proof for $\mathscr{Z}_\Omega$ is analogous. Let $\alpha \in \mathscr{E}_\Omega(X)$ and $\beta \in [\Sigma A, X]$. Let $\hat{\beta} : A \to \Omega X$ be the adjoint of β. Then $(\Omega\alpha) \circ \hat{\beta} = \hat{\beta}$, since $\Omega\alpha = \mathrm{id}$. Taking the adjoint gives $\alpha \circ \beta = \beta$.

Next let $\alpha \in \mathscr{E}(X)$ be such that $\alpha_* = \mathrm{id} : [\Sigma A, X] \to [\Sigma A, X]$ for every space A. Let $A = \Omega X$ and let $p : \Sigma\Omega X \to X$ be the canonical map. Then $\alpha \circ p = p$. By taking adjoints we obtain $\Omega\alpha = \mathrm{id}_{\Omega X}$. □

Using Proposition 2.3, we can give short proofs (and some easy generalizations) of some results of Pavešić on $\mathscr{E}_\Omega$ which were originally proved by spectral sequence arguments [**31**]. The first of these concerns co-H-spaces. The second one, Corollary 2.8, concerns products of spheres. The third one deals with rational spaces, and appears below in Proposition 2.17.

Corollary 2.4 (Corollary 3.1 of [**31**]). *If X is a co-H-space, then $\mathscr{E}_\Omega(X) = \{\mathrm{id}\}$ and $\mathscr{Z}_\Omega(X) = \{0\}$.*

Proof. We will just prove the result for $\mathscr{E}_\Omega$; the proof for $\mathscr{Z}_\Omega$ is analogous. Since X is a co-H-space, X is a retract of some suspension ΣA [**16**, p. 209]. Therefore there are maps $i : X \to \Sigma A$ and $r : \Sigma A \to X$ such that $r \circ i = \mathrm{id}_X$. If $f \in \mathscr{E}_\Omega(X)$, then $f \circ r = f_*(r) = r$ by Proposition 2.3. Applying i to both sides, we get $f = \mathrm{id}_X$. □

Remark 2.5. One possible approach to Conjecture 1.1 would be to find a finite co-H-space X such that $\mathscr{E}_{\#\infty}(X) \neq \{\mathrm{id}\}$ (cf. [**31**, p. 680]).

Next we consider when $\mathscr{E}_\# = \mathscr{E}_\Omega$ and $\mathscr{Z}_\# = \mathscr{Z}_\Omega$ for certain products of spaces. For this we will make use of the following lemma, which may be of independent interest.

Lemma 2.6. *Let $f, g : A_1 \times \cdots \times A_r \to X$ be two maps, let $j : A_1 \vee \cdots \vee A_r \hookrightarrow A_1 \times \cdots \times A_r$ be the canonical inclusion and consider the conditions*

(1) *$f \circ j = g \circ j$ and*
(2) *$\Omega f = \Omega g$.*

Then (1) *implies* (2). *Furthermore, if each A_i is a co-H-space, then* (2) *implies* (1).

Proof. Assume (1) and let $V = A_1 \vee \cdots \vee A_n$ and $P = A_1 \times \cdots \times A_n$. Consider the homomorphism $j_* : [\Sigma\Omega P, V] \to [\Sigma\Omega P, P]$ and the canonical map $p \in [\Sigma\Omega P, P]$. Since j_* is easily seen to be onto, there is $\theta \in [\Sigma\Omega P, V]$ such that $j \circ \theta = p$. By taking adjoints we get $\Omega j \circ \hat\theta = \hat p = \mathrm{id}_{\Omega P}$. From hypothesis (1) we have $(\Omega f) \circ (\Omega j) = (\Omega g) \circ (\Omega j)$, so

$$\Omega f = \Omega f \circ \mathrm{id}_{\Omega P} = (\Omega f \circ \Omega j) \circ \hat\theta = (\Omega g \circ \Omega j) \circ \hat\theta = \Omega g.$$

Thus (2) holds.

Now assume that each A_i is a co-H-space and that $\Omega f = \Omega g$. We claim that $f|_{A_i} = g|_{A_i}$ for each i. For this we use the commutative diagram

$$\begin{tikzcd}
\Sigma\Omega A_i \arrow[r, "\Sigma\Omega j_i"] \arrow[d] & \Sigma\Omega P \arrow[r, "\Sigma\Omega f = \Sigma\Omega g"] \arrow[d] & \Sigma\Omega X \arrow[d, "p"] \\
A_i \arrow[u, bend left, "s"] \arrow[r, "j_i"] & P \arrow[r, "f,\ g"] & X
\end{tikzcd}$$

where j_i is the inclusion of the i^{th} factor. The section s exists because A_i is a co-H-space [**16**, p. 209]. Now

$$f \circ j_i = p \circ \Sigma\Omega f \circ \Sigma\Omega j_i \circ s = p \circ \Sigma\Omega g \circ \Sigma\Omega j_i \circ s = g \circ j_i,$$

which proves (1). □

Lemma 2.6 has several corollaries. The first is well-known.

Corollary 2.7. *If $q : A_1 \times \cdots \times A_n \to A_1 \wedge \cdots \wedge A_n$ is the projection, then $\Omega q = 0$.*

Proof. Since $q \circ j = 0 = 0 \circ j$, Lemma 2.6 shows that $\Omega q \simeq \Omega(0) = 0$. □

Pavešić [**31**, Theorem 3.6] has proved $\mathscr{E}_\Omega(S^k \times S^l) = \mathscr{E}_\#(S^k \times S^l)$. Using Lemma 2.6 we can easily generalize this to arbitrary finite products of spheres.

Corollary 2.8. *Let $X = S^{k_1} \times \cdots \times S^{k_r}$ and write $N = \max\{k_1, \ldots, k_r\}$. Then*

$$\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X) = \mathscr{E}_\#(X) = \mathscr{E}_{\#N}(X) \quad \textit{and}$$
$$\mathscr{Z}_\Omega(X) = \mathscr{Z}_{\#\infty}(X) = \mathscr{Z}_\#(X) = \mathscr{Z}_{\#N}(X).$$

Proof. We will only prove $\mathscr{E}_\Omega(X) = \mathscr{E}_{\#N}(X)$. Let $j : S^{k_1} \vee \cdots \vee S^{k_r} \hookrightarrow X$. If $f \in \mathscr{E}_{\#N}(X)$, then $f \circ j = j = \mathrm{id} \circ j$ by Remark 2.2. Therefore Lemma 2.6 shows that $\Omega f \simeq \Omega(\mathrm{id}) = \mathrm{id}$. □

In order to further generalize Corollary 2.8, we give a definition.

Definition 2.9. For a space X, we say $\pi_*(X)$ is *spherically generated in dimensions* $\leq N$ if there is a wedge of spheres W with $\dim(W) \leq N$ and a map $\phi : W \to X$ such that $\phi_\# : \pi_k(W) \to \pi_k(X)$ is surjective for all k.

Proposition 2.10. *Let X and Y be spaces. Assume that $\pi_*(X)$ is spherically generated in dimensions $\leq N$ and that $\pi_*(Y)$ is spherically generated in dimensions $\leq M$. Then*

1. *$\mathscr{E}_{\#\infty}(X) = \mathscr{E}_{\#N}(X)$ and $\mathscr{Z}_{\#\infty}(X) = \mathscr{Z}_{\#N}(X)$,*
2. *$\pi_*(X \times Y)$ is spherically generated in dimensions $\leq \max(M, N)$ and*
3. *$\pi_*(S^n)$ is spherically generated in dimensions $\leq n$, and $\pi_*(\boldsymbol{F}\mathrm{P}^{n-1})$ is spherically generated in dimensions $\leq nd - 1$, where $d = 1, 2$ or 4 according as $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$.*

Proof. We let $f \in \mathscr{E}_{\#N}(X)$ and $\alpha \in \pi_n(X)$ for some n and show that $f \circ \alpha = \alpha$. Since $\pi_*(X)$ is spherically generated in dimensions $\leq N$, there is an wedge of spheres W of dimension $\leq N$ and a map $\phi : W \to X$ such that $\phi_{\#i}$ is surjective for all i. Thus $\alpha = \phi_\#(\beta)$ for some $\beta \in \pi_n(W)$. Using Remark 2.2, we have $f \circ \phi = \phi$ because $\dim(W) \leq N$. Therefore

$$f_\#(\alpha) = f_\#(\phi_\#(\beta)) = (f \circ \phi)_\#(\beta) = \phi_\#(\beta) = \alpha,$$

which completes the proof of the first assertion of (1). The proof of the second assertion is similar.

For (2), observe that if W and V are wedges of spheres and $W \to X$ and $V \to Y$ are maps which induce surjections on homotopy groups, then the composite $W \vee V \to W \times V \to X \times Y$ is also surjective on homotopy groups.

In (3), only the statements about the projective spaces require proof. We begin with the fibration sequence

$$\begin{tikzcd} \cdots \arrow[r, "0"] & \Omega S^{nd-1} \arrow[r, "\Omega p"] & \Omega \boldsymbol{F}\mathrm{P}^{n-1} \arrow[r] & S^{d-1} \arrow[l, bend right, "s"'] \arrow[r, "0"] & S^{nd-1} \arrow[r, "p"] & \boldsymbol{F}\mathrm{P}^{n-1}, \end{tikzcd}$$

where s is a section of $\Omega \boldsymbol{F}\mathrm{P}^{n-1} \to S^{d-1}$. Then we have the well-known homotopy equivalence $\Omega S^{nd-1} \times S^{d-1} \equiv \Omega \boldsymbol{F}\mathrm{P}^{n-1}$ determined by Ωp and s [**14**], and so

$$\pi_k(\boldsymbol{F}\mathrm{P}^{n-1}) \cong \pi_{k-1}(\Omega \boldsymbol{F}\mathrm{P}^{n-1}) \cong \mathrm{Im}(\Omega p_{\#}) \oplus \mathrm{Im}(s_{\#}).$$

Now let $j : S^d \hookrightarrow \boldsymbol{F}\mathrm{P}^{n-1}$ be the inclusion of the lowest dimensional cell. Since $j_{\# d}$, and hence $(\Omega j)_{\# d-1}$, is surjective, there is a lift λ in the diagram

$$\begin{tikzcd} & \Omega S^d \arrow[d, "\Omega j"] \\ S^{d-1} \arrow[ur, dashed, "\lambda"] \arrow[r, "s"'] & \Omega \boldsymbol{F}\mathrm{P}^{n-1}. \end{tikzcd}$$

From this it follows that $\mathrm{Im}(s_{\#}) \subseteq \mathrm{Im}(\Omega j_{\#})$, and so $\pi_{k-1}(\Omega \boldsymbol{F}\mathrm{P}^{n-1}) \cong \mathrm{Im}(\Omega p_{\#}) + \mathrm{Im}(\Omega j_{\#})$ (the sum may not be direct). Therefore $\pi_k(\boldsymbol{F}\mathrm{P}^{n-1}) \cong \mathrm{Im}(p_{\#}) + \mathrm{Im}(j_{\#})$, and hence the map $(j, p) : S^d \vee S^{nd-1} \to \boldsymbol{F}\mathrm{P}^{n-1}$ determined by j and p is surjective on all homotopy groups. $\square$

Remark 2.11. There are other interesting spaces X for which $\pi_*(X)$ is spherically generated in dimensions $\leq N$ for some N. One important case occurs when X is the orbit space S^n/G of a free action of a finite group G on S^n, $(n \geq 1)$. Examples of such actions can be found in [**34**, Chapter 6]. Another important case consists of spaces of the form $S^{4n+3}/N(S^1)$, where $N(S^1)$ is the normalizer of S^1 in S^3 and the action is the restriction of the standard one of S^3 on S^{4n+3} [**10**, Chapter III].

Now we turn our attention to analogous properties of the following collection of spaces:

$$\mathscr{A} = \{X \mid \Sigma\Omega X \equiv \text{ a wedge of spheres}\}.$$

Proposition 2.12. *Let X and Y be spaces such that $X, Y \in \mathscr{A}$. Then*

(1) $\mathscr{Z}_\Omega(X) = \mathscr{Z}_{\#\infty}(X)$ *and* $\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X)$ (*cf.* [**6**, Proposition 5.1]),
(2) $X \times Y \in \mathscr{A}$ *and*
(3) $S^n, \boldsymbol{F}\mathrm{P}^n \in \mathscr{A}$ *for all* $n \geq 1$ *and* $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ *or* $\boldsymbol{H}$.

Proof. (1) Suppose $\lambda : \bigvee S^{n_i} \to \Sigma\Omega X$ is a homotopy equivalence, and let $f \in \mathscr{Z}_{\#\infty}(X)$. Write $p : \Sigma\Omega X \to X$ for the canonical map. Then $f \circ p \circ \lambda = 0$, and so $f \circ p = 0$. By taking adjoints we find that $\Omega f = 0$. The second assertion of (1) is similarly proved.

(2) Clearly $\Omega(X \times Y) = \Omega(X) \times \Omega(Y)$. For any spaces A and B, we have $\Sigma(A \times B) \equiv \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$ by [**16**, 11.10] and so

$$\Sigma(\Omega(X) \times \Omega(Y)) \equiv \Sigma\Omega(X) \vee \Sigma\Omega(Y) \vee (\Sigma\Omega(X) \wedge \Omega(Y)).$$

Then (2) follows.

(3) Since $\Sigma\Omega S^n$ has the homotopy type of a wedge of spheres according to [**18**], $S^n \in \mathscr{A}$. Also $\Omega \boldsymbol{F}\mathrm{P}^n \equiv S^{d-1} \times \Omega S^{(n+1)d-1}$, where $d = 1, 2$ or 4, according as $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$, as noted earlier. The result follows from the decomposition for $\Sigma(A \times B)$ used in (2). □

We now put all of these results together.

Theorem 2.13. *Let $X = X_1 \times X_2 \times \cdots \times X_r$ where each X_i is either a sphere or a projective space $\boldsymbol{F}\mathrm{P}^n$ with $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$ and $r, n \geq 1$. Let $N = 3 + \max\{\dim(X_i)\}$. Then*

$$\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X) = \mathscr{E}_{\#N}(X) \quad \textit{and}$$
$$\mathscr{Z}_\Omega(X) = \mathscr{Z}_{\#\infty}(X) = \mathscr{Z}_{\#N}(X).$$

Proof. Since each X_i is in $\mathscr{A}$ by Proposition 2.12(3), so is the product X by Proposition 2.12(2). Now Proposition 2.12(1) shows that $\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X)$ and $\mathscr{Z}_\Omega(X) = \mathscr{Z}_{\#\infty}(X)$.

Analogously, each $\pi_*(X_i)$ is spherically generated in dimensions $\leq \dim(X_i) + 3$ by Proposition 2.10(3), so $\pi_*(X)$ is spherically generated in dimensions $\leq N$ by Proposition 2.10(2). Hence Proposition 2.10(1) applies to show that $\mathscr{E}_{\#\infty}(X) = \mathscr{E}_{\#N}(X)$ and $\mathscr{Z}_{\#\infty}(X) = \mathscr{Z}_{\#N}(X)$. □

Remark 2.14. Generally, $N \leq \dim(X)$, where N the number in the previous theorem. This is the case, for example, if either there are at least 4 spaces in the product, or if there are at least 2 factors with dimension at least 3. Furthermore, the term '$3 + \max\{\dim(X_i)\}$' can be replaced with '$1 + \max\{\dim(X_i)\}$' if no X_i is equal to $\boldsymbol{H}\mathrm{P}^n$; it can be replaced with '$0 + \max\{\dim(X_i)\}$' if none of the factors is $\boldsymbol{H}\mathrm{P}^n$ or $\boldsymbol{C}\mathrm{P}^n$.

The latter special case of this remark will be used in the following sections, so we state it as a separate corollary.

Corollary 2.15. *If each X_i is either a sphere or a real projective space, then the product $X = X_1 \times \cdots \times X_r$ $(r \geq 1)$ satisfies*

$$\mathscr{E}_\Omega(X) = \mathscr{E}_{\#\infty}(X) = \mathscr{E}_{\#}(X) \quad \textit{and}$$
$$\mathscr{Z}_\Omega(X) = \mathscr{Z}_{\#\infty}(X) = \mathscr{Z}_{\#}(X).$$

2.3. Localized spaces.

We now turn to results that are specific to p-local spaces or to rational spaces (for details on localization, see [**17**]). We write $A \to A_{(0)}$ for rationalization of groups or spaces and $A \to A_{(p)}$ for localization of groups or spaces at a prime p.

To begin, we observe that many of the above results are true p-locally.

Remark 2.16. The proofs of all of the results of §2.2 are valid for p-local spaces, where p is either a prime number or zero.

Proposition 2.17 (Corollary 3.2 of [**31**]). *If X is the rationalization of a nilpotent CW-complex, then $\mathscr{E}_{\Omega}(X) = \mathscr{E}_{\#\infty}(X)$ and $\mathscr{Z}_{\Omega}(X) = \mathscr{Z}_{\#\infty}(X)$.*

Proof. In fact, since $\Sigma\Omega X$ is the suspension of a rational space, it is homotopy equivalent to a wedge of rational spheres [**15**, p. 167]. Therefore, we apply the rational version of Proposition 2.12, and the proposition is proved. □

We next consider group-like spaces, which enjoy special localization properties. In particular, since group-like spaces are nilpotent, they *can* be localized. Furthermore, if a finite complex X is group-like, then for any sufficiently large prime number p, $X_{(p)} \equiv S^{k_1}_{(p)} \times \cdots \times S^{k_r}_{(p)}$ [**19**, p. 73]. Such primes are called *regular* for X.

Corollary 2.18. *Let X be a finite group-like complex of dimension N and p a regular prime for X. Then $\mathscr{E}_{\Omega}(X_{(p)}) = \mathscr{E}_{\#N}(X_{(p)})$ and $\mathscr{Z}_{\Omega}(X_{(p)}) = \mathscr{Z}_{\#N}(X_{(p)})$.*

Proof. If p is regular, then $X_{(p)} \equiv S^{k_1}_{(p)} \times \cdots \times S^{k_r}_{(p)}$. Now apply (the p-local version of) Corollary 2.8. □

We conclude this section with a general discussion of localization and the $\mathscr{E}$-groups and $\mathscr{Z}$-semigroups. Let p denote either 0 or a prime number and let X be a finite nilpotent CW-complex of dimension N. Then the groups $\mathscr{E}_{\#}(X)$ and $\mathscr{E}_{\#\infty}(X)$ are nilpotent [**12**], so they may be localized. On the other hand, localization of spaces defines a homomorphism $\mathscr{E}_{\#}(X) \to \mathscr{E}_{\#N}(X_{(p)})$. It is known (see [**21**] and [**23**, Theorem 2.7]) that these homomorphisms are in fact p-localization homomorphisms. Furthermore, if X is an H_0-space (i.e., $X_{(0)}$ is an H-space), then $\mathscr{E}_{\#\infty}(X) \to \mathscr{E}_{\#\infty}(X_{(p)})$ is also p-localization [**23**, Corollary 2.10].

If, in addition, X is a group-like space, then the sets $\mathscr{Z}_{\Omega}(X)$, $\mathscr{Z}_{\#\infty}(X)$ and $\mathscr{Z}_{\#}(X)$ have a nilpotent group structure obtained from the additive nilpotent group $[X, X]$ [**35**, p. 464], and so they may be localized. It has been proved that the natural maps

$$\mathscr{Z}_{\#}(X) \to \mathscr{Z}_{\#N}(X_{(p)}) \qquad \text{and} \qquad \mathscr{Z}_{\#\infty}(X) \to \mathscr{Z}_{\#\infty}(X_{(p)})$$

are also p-localization homomorphisms [**23**, Lemma 1.6 and Corollary 1.7].

This discussion suggests the following questions.

Question 2.19. Let X be a finite nilpotent complex, and let p be a prime number or zero.

(1) Is the natural map $\mathscr{E}_{\Omega}(X) \to \mathscr{E}_{\Omega}(X_{(p)})$ p-localization?
(2) If X is also group-like, is the natural map $\mathscr{Z}_{\Omega}(X) \to \mathscr{Z}_{\Omega}(X_{(p)})$ p-localization?

3. Grouplike spaces.

In this section we consider group-like spaces in more detail. In the first subsection, we establish a close link between the $\mathscr{Z}$-groups and the $\mathscr{E}$-groups for group-like spaces. We then use commutator subgroups to give lower bounds for the order of some of these groups and their localizations.

Throughout this section, X denotes a finite group-like complex and so X is a nilpotent space and the additive group $[X, X]$ is nilpotent.

3.1. $\mathscr{Z}$ and $\mathscr{E}$ for grouplike spaces.

As was mentioned in the introduction, there is a strong analogy between the $\mathscr{E}$-groups and the $\mathscr{Z}$-groups when X is group-like. In this case, $\Omega : [X, X] \to [\Omega X, \Omega X]$ is a homomorphism of groups. Thus $\mathscr{Z}_\Omega(X) = \Omega^{-1}(0) = \operatorname{Ker}\Omega$ is a subgroup of $[X, X]$ and $\mathscr{E}_\Omega(X) = \Omega^{-1}(\mathrm{id})$ is a coset of $\operatorname{Ker}\Omega$ of the group $[X, X]$. Similarly, we obtain homomorphisms

$$r_n : [X, X] \to \bigoplus_{k \le n} \operatorname{Hom}(\pi_k(X), \pi_k(X))$$

and we deduce that $\mathscr{Z}_{\#n}(X) = \ker(r_n)$ is a subgroup and $\mathscr{E}_{\#n}(X)$ is a coset of $\mathscr{Z}_{\#n}(X)$. Furthermore, since r_m factors through r_n for $m < n$ and r_n factors through Ω for all $n \le \infty$, we have the following proposition.

Proposition 3.1. *If X is group-like, then the function $\alpha \mapsto \mathrm{id} + \alpha$ defines bijections Θ_Ω and Θ_n making the diagram*

$$\begin{tikzcd}
\mathscr{Z}_\Omega(X) \arrow[r, hook] \arrow[d, "\Theta_\Omega"'] & \mathscr{Z}_{\#n}(X) \arrow[r, hook] \arrow[d, "\Theta_n"'] & \mathscr{Z}_{\#m}(X) \arrow[d, "\Theta_m"] \\
\mathscr{E}_\Omega(X) \arrow[r, hook] & \mathscr{E}_{\#n}(X) \arrow[r, hook] & \mathscr{E}_{\#m}(X)
\end{tikzcd}$$

commutative for each $m \le n \le \infty$.

Remark 3.2. Proposition 3.1 has been observed by several people (e.g., [**23**, p. 51] and [**8**, p. 693]). It follows from Theorem 4.3(2) that $\Theta_\Omega : \mathscr{Z}_\Omega(X) \to \mathscr{E}_\Omega(X)$ is not necessarily a homomorphism.

3.2. Grouplike spaces and commutators.

Commutators feature prominently in this section, so we adopt the notation $\mathscr{H}(X) = [X, X]$ in order to avoid possible confusion of $[X, X]$ with the commutator subgroup of X, when X is a topological group.

We begin by establishing our notation for commutators. If Γ is a group and if $A, B \subseteq \Gamma$, then we write $[A, B]$ for the subgroup of Γ generated by all commutators $[a, b] = aba^{-1}b^{-1}$ with $a \in A$ and $b \in B$. The lower central series of Γ is the sequence of subgroups

$$\Gamma = \Gamma^{(1)} \supseteq \Gamma^{(2)} \supseteq \cdots \supseteq \Gamma^{(k)} \supseteq \Gamma^{(k+1)} \supseteq \cdots$$

defined by setting $\Gamma^{(1)} = \Gamma$ and $\Gamma^{(i)} = [\Gamma, \Gamma^{(i-1)}]$ for $i \geq 2$. If k is the smallest integer such that $\Gamma^{(k+1)} = \{\mathrm{id}\}$ for some k, then Γ is nilpotent with nilpotency k, written $\mathrm{nil}\,\Gamma = k$.

In what follows, we write $|S|$ to denote the number of elements of a set S.

Lemma 3.3. *Let Γ be a nilpotent group with $\mathrm{nil}\,\Gamma = k$, and let p be a prime number or 0. Then*

(1) $|\Gamma^{(2)}| \geq 2^{k-1}$;
(2) *If Γ is a p-local group, then* $|\Gamma^{(2)}| \geq p^{k-1}$;
(3) *If Γ is a finitely-generated infinite nilpotent group, then $\Gamma_{(p)}$ is infinite.*

Proof. (1) Since the case $k = 1$ is trivial, we assume $k \geq 2$. Then we have

$$\Gamma^{(1)} \supsetneqq \Gamma^{(2)} \supsetneqq \cdots \supsetneqq \Gamma^{(k)} \supsetneqq \Gamma^{(k+1)} = 1.$$

By [**17**, Corollary 2.6 and Theorem 2.7], the groups $\Gamma^{(i)}/\Gamma^{(i+1)}$ are nontrivial for $2 \leq i \leq k$. It follows that $|\Gamma^{(i)}/\Gamma^{(i+1)}| \geq 2$ for $2 \leq i \leq k$, and so $|\Gamma^{(2)}| \geq 2^{k-1}$.

Assertion (2) follows from the same argument because the nontrivial groups $\Gamma^{(i)}/\Gamma^{(i+1)}$ are p-local and so they must have at least p elements each.

We prove (3) by induction on the nilpotency of Γ. If $\mathrm{nil}\,\Gamma = 1$, then Γ is abelian. Hence $\Gamma_{(p)} = \Gamma \otimes \boldsymbol{Z}_{(p)}$, and so $\Gamma_{(p)}$ is infinite. Suppose that the result is true for all groups with nilpotency $< k$. Let $\mathrm{nil}\,\Gamma = k$ and let $\Gamma^{(k)} = [\Gamma, \Gamma^{(k-1)}]$ and consider the exact sequence $1 \to \Gamma^{(k)} \to \Gamma \to \Gamma/\Gamma^{(k)} \to 1$, with $\Gamma^{(k)}$ and $\Gamma/\Gamma^{(k)}$ both finitely-generated and of nilpotency at most $k-1$. This gives rise to an exact sequence $1 \to (\Gamma^{(k)})_{(p)} \to \Gamma_{(p)} \to (\Gamma/\Gamma^{(k)})_{(p)} \to 1$ of p-localized groups [**17**, p. 12]. If Γ is infinite, either $\Gamma^{(k)}$ or $\Gamma/\Gamma^{(k)}$ is infinite. By the inductive hypothesis, at least one of $(\Gamma^{(k)})_{(p)}$ or $(\Gamma/\Gamma^{(k)})_{(p)}$ must be infinite and thus $\Gamma_{(p)}$ is infinite. □

Our next lemma establishes the link between commutators and $\mathscr{Z}_\Omega(G)$ (and therefore, by Proposition 3.1, with $\mathscr{E}_\Omega(G)$).

Lemma 3.4. *If X is a group-like space, then $\mathscr{H}(X)^{(2)} \subseteq \mathscr{Z}_\Omega(X)$.*

Proof. If $f, g \in \mathscr{H}(X)$, we have $\Omega[f, g] = [\Omega f, \Omega g] = 0$, because ΩX is homotopy-commutative. Therefore, $[f, g] \in \mathscr{Z}_\Omega(X)$, and since $\mathscr{H}(X)^{(2)}$ is the subgroup generated by these commutators, we have $\mathscr{H}(X)^{(2)} \subseteq \mathrm{Ker}(\Omega) = \mathscr{Z}_\Omega(X)$. □

Proposition 3.5. *Let X be a finite group-like complex, and p be a prime number. Then*

(1) (a) $|\mathscr{Z}_\Omega(X)_{(p)}| \geq p^{\mathrm{nil}\,\mathscr{H}(X)_{(p)}-1}$;
  (b) $|\mathscr{E}_\Omega(X)_{(p)}| \geq p^{\mathrm{nil}\,\mathscr{H}(X)_{(p)}-1}$;
  (c) $|\mathscr{Z}_\Omega(X)| \geq 2^{\mathrm{nil}\,\mathscr{H}(X)-1}$.
(2) $|\mathscr{Z}_\Omega(X_{(p)})| \geq p^{\mathrm{nil}\,\mathscr{H}(X)_{(p)}-1}$.

Proof. (1)(a) We have $\mathscr{H}(X)^{(2)} \subseteq \mathscr{Z}_\Omega(X)$ by Lemma 3.4, and so by [**17**, Theorem 2.7 on p. 20], $(\mathscr{H}(X)_{(p)})^{(2)} = (\mathscr{H}(X)^{(2)})_{(p)} \subseteq (\mathscr{Z}_\Omega(X))_{(p)}$. Then by Lemma 3.3(2), we have $p^{\mathrm{nil}\,\mathscr{H}(X)_{(p)}-1} \leq |(\mathscr{H}(X)_{(p)})^{(2)}| \leq |\mathscr{Z}_\Omega(X)_{(p)}|$.

(1)(b) This does not follow immediately from (1)(a) since there is only a bijection, not necessarily an isomorphism, between $\mathscr{E}_\Omega(X)$ and $\mathscr{Z}_\Omega(X)$. If $\mathscr{Z}_\Omega(X)$ is infinite, so is $\mathscr{E}_\Omega(X)$ by Proposition 3.1. Therefore by Lemma 3.3(3), $\mathscr{E}_\Omega(X)_{(p)}$ is infinite, and the inequality holds. If $\mathscr{Z}_\Omega(X)$ is finite, so is $\mathscr{E}_\Omega(X)$, and both nilpotent groups have the same order. But the p-localization of a finite, nilpotent group is its unique p-Sylow subgroup. Therefore $|\mathscr{E}_\Omega(X)_{(p)}| = |\mathscr{Z}_\Omega(X)_{(p)}|$, and the result now follows from (1)(a).

(1)(c) We assume without loss of generality that $\mathscr{Z}_\Omega(X)$ is finite. The result then follows form Lemma 3.3(1).

(2) Since $\mathscr{H}(X)_{(p)} \cong \mathscr{H}(X_{(p)})$ by [**9**, Proposition 5.3], we have $|\mathscr{Z}_\Omega(X_{(p)})| \geq |(\mathscr{H}(X)_{(p)})^{(2)}| \geq p^{\operatorname{nil}\mathscr{H}(X)_{(p)}-1}$ by Lemmas 3.3 and 3.4. □

Remark 3.6. (1) By (1.2) and subsection 2.3, the results of Proposition 3.5 immediately provide lower bounds for the order of: $\mathscr{Z}_{\#\infty}(X)_{(p)} \cong \mathscr{Z}_{\#\infty}(X_{(p)})$, $\mathscr{Z}_{\#}(X)_{(p)}$ and $\mathscr{Z}_{\#\infty}(X)$. By (1.1) and subsection 2.3, this is also true with $\mathscr{E}$ replacing $\mathscr{Z}$.

(2) The inequalities in Proposition 3.5(1)(a) and (1)(b) can be strict: $\mathscr{H}(Sp(2))_{(3)}$ is commutative by [**27**, Theorem 2] and $\mathscr{E}_\Omega(Sp(2))_{(3)} \cong \mathscr{Z}_\Omega(Sp(2))_{(3)} \cong \boldsymbol{Z}_3$ by Theorem 4.3.

4. Lie groups.

In this final section we specialize further and study Lie groups. Our first theorem gives the equivalence of several statements including the triviality of the $\mathscr{Z}$-semigroups and the $\mathscr{E}$-groups. We then give explicit computations for some low-dimensional Lie groups.

Throughout this section, G denotes a compact Lie group. This implies, in particular, that G is a finite nilpotent complex.

We shall need the following lemma in Theorem 4.2.

Lemma 4.1. *If X_1 and $X_2 \in \{S^1, S^3, SO(3)\}$ and $q : X_1 \times X_2 \to X_1 \wedge X_2$ is the projection inducing $q^* : [X_1 \wedge X_2, X_1 \times X_2] \to [X_1 \times X_2, X_1 \times X_2]$, then*

$$\mathscr{Z}_\Omega(X_1 \times X_2) = \mathscr{Z}_{\#\infty}(X_1 \times X_2) = \mathscr{Z}_{\#}(X_1 \times X_2) = \operatorname{Im}(q^*) \cong [X_1 \wedge X_2, X_1 \times X_2].$$

Proof. First of all, the exact sequence

$$0 \longrightarrow [X_1 \wedge X_2, X_1 \times X_2] \xrightarrow{q^*} \mathscr{H}(X_1 \times X_2) \xrightarrow{i^*} [X_1 \vee X_2, X_1 \times X_2] \longrightarrow 0$$

shows that $\operatorname{Im}(q^*) \cong [X_1 \wedge X_2, X_1 \times X_2]$. Since $SO(3) \cong \boldsymbol{R}\mathrm{P}^3$, we know

$$\mathscr{Z}_\Omega(X_1 \times X_2) = \mathscr{Z}_{\#\infty}(X_1 \times X_2) = \mathscr{Z}_{\#}(X_1 \times X_2)$$

by Corollary 2.15. Since $\Omega q = 0$ by Corollary 2.7, we clearly have $\operatorname{Im}(q^*) \subseteq \mathscr{Z}_\Omega(X_1 \times X_2)$.

For the reverse containment, let $f \in \mathscr{Z}_{\#}(X_1 \times X_2)$. We claim that f is in $\operatorname{Im}(q^*)$. In every case, $X_1 \times X_2$ has one of $S^1 \times S^1, S^1 \times S^3$ or $S^3 \times S^3$ as a covering space (depending on $\dim(X_1 \times X_2)$); fix a covering map $p : S^a \times S^b \to X_1 \times X_2$. Since $f_{\#1} = 0$, there is a lift $\hat{f}$ in the diagram

$$\begin{tikzcd} & S^a \times S^b \arrow[d, "p"] \\ X_1 \times X_2 \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"'] & X_1 \times X_2. \end{tikzcd}$$

Since p is a covering, $p_{\#} : \pi_*(S^a \times S^b) \to \pi_*(X_1 \times X_2)$ is injective, and so $\hat{f}_{\#n} = 0$ for all n.

We next show that $\hat{f}|_{X_1 \vee X_2} = 0$. It is well-known that $[\boldsymbol{R}\mathrm{P}^3, S^1] \cong H^1(\boldsymbol{R}\mathrm{P}^3; \boldsymbol{Z}) = 0$ and $[\boldsymbol{R}\mathrm{P}^3, S^3] \cong H^3(\boldsymbol{R}\mathrm{P}^3; \boldsymbol{Z}) \cong \boldsymbol{Z}$, generated by the quotient map $\boldsymbol{R}\mathrm{P}^3 \to \boldsymbol{R}\mathrm{P}^3/\boldsymbol{R}\mathrm{P}^2 = S^3$. This map, and all of its nonzero multiples, is nontrivial on π_3. Therefore, $\hat{f}|_{X_i} = 0$ if $X \equiv \boldsymbol{R}\mathrm{P}^3$. Clearly, if X_i is a sphere ($i = 1$ or 2 or both), then $\hat{f}|_{X_i} = 0$. Therefore $f \in \mathrm{Im}(q^*)$ as claimed. □

The next theorem shows the equivalence of several statements about Lie groups G, most of which have been previously proved elsewhere. We denote the torus of dimension n by T^n, $n \geq 1$, and let T^0 be the trivial group.

Theorem 4.2. *The following statements are equivalent*:

(1) $\mathscr{Z}_\Omega(G) = \{0\}$ (*or, equivalently,* $\mathscr{E}_\Omega(G) = \{\mathrm{id}\}$).
(2) $\mathscr{Z}_{\#\infty}(G) = \{0\}$ (*or, equivalently,* $\mathscr{E}_{\#\infty}(G) = \{\mathrm{id}\}$).
(3) $\mathscr{Z}_{\#}(G) = \{0\}$ (*or, equivalently,* $\mathscr{E}_{\#}(G) = \{\mathrm{id}\}$).
(4) *The left distibutive law holds in* $\mathscr{H}(G)$: $a \circ (b + c) = a \circ b + a \circ c$ *for* $a, b, c \in \mathscr{H}(G)$.
(5) G *is isomorphic to one of* T^n $(n \geq 0)$, S^3 *or* $SO(3)$.
(6) $\mathscr{H}(G)$ *is commutative and* G *is not isomorphic to* $\mathrm{T}^n \times S^3$ $(n = 1, 2)$.

Proof. In [**20**, Theorem 1.1], it is proved that (2), (4), (5) and (6) are equivalent. Obviously (3) implies (2) and (2) implies (1). Therefore it suffices to show that (5) implies (3) and (1) implies (5).

To prove that (5) implies (3), let G be T^n, S^3 or $SO(3)$, and let k be 1 or 3 according to whether or not G is T^n. Then the map $f \mapsto f_{\#k}$ induces isomorphisms $\mathscr{H}(G) \cong \mathrm{Hom}(\pi_k(G), \pi_k(G))$ (see [**27**, Proposition 4.1] for $SO(3)$). Hence $\mathscr{Z}_{\#k}(G) = \{0\}$. Since $\mathscr{Z}_{\#}(\mathrm{T}^n) = \mathscr{Z}_{\#1}(\mathrm{T}^n)$, we have $\mathscr{Z}_{\#}(G) = \{0\}$.

We conclude by showing that (1) implies (5), that is, if $G \ncong \mathrm{T}^n$, S^3 or $SO(3)$, then $\mathscr{E}_\Omega(G) \neq \{\mathrm{id}\}$. First suppose $G \ncong \mathrm{T}^n$, $\mathrm{T}^m \times S^3$ $(m = 1, 2)$, S^3 or $SO(3)$. Then $\mathscr{H}(G)$ is not commutative by [**20**, Theorem 1.1], so that $\mathscr{E}_\Omega(G) \neq \{\mathrm{id}\}$ by Proposition 3.5(1)(b). Next suppose that $G \cong \mathrm{T}^m \times S^3$ with $m = 1, 2$. The following square is commutative

$$\begin{tikzcd} \mathscr{E}(S^1 \times S^3) \arrow[r, "i'_*"] \arrow[d, "\Omega"] & \mathscr{E}(S^1 \times S^1 \times S^3) \arrow[d, "\Omega"] \\ \mathscr{E}(\Omega(S^1 \times S^3)) \arrow[r, "i''_*"] & \mathscr{E}(\Omega(S^1 \times S^1 \times S^3)), \end{tikzcd}$$

where i'_* and i''_* are the injective homomorphisms defined by $i'_*(f) = \mathrm{id}_{S^1} \times f$ and $i''_*(f) = \mathrm{id}_{\Omega S^1} \times f$. Hence to show $\mathscr{E}_\Omega(\mathrm{T}^m \times S^3)$ nontrivial for $m = 1, 2$, it suffices to prove that $\mathscr{E}_\Omega(S^1 \times S^3) \neq \{\mathrm{id}\}$. But $|\mathscr{E}_\Omega(S^1 \times S^3)| = |\mathscr{Z}_\Omega(S^1 \times S^3)| = |\pi_4(S^1 \times S^3)| = 2$ by Lemma 4.1, and so the proof is complete. □

We conclude with some concrete calculations for low-dimensional Lie Groups.

Theorem 4.3.

(1) *If G is isomorphic to one of $S^1 \times S^3$, $U(2)$, $S^1 \times SO(3)$, $SU(3)$, $Sp(2)$ or $S^3 \times S^3$, then $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G) \cong \mathscr{Z}_\Omega(G) = \mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G)$, and this common group is explicitly given as follows*:

$$\begin{array}{llll} \boldsymbol{Z}_2 & \text{if } G \cong S^1 \times S^3 \text{ or } U(2), & \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 & \text{if } G \cong S^1 \times SO(3), \\ \boldsymbol{Z}_{12} & \text{if } G \cong \mathrm{SU}(3), & \boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_{12} & \text{if } G \cong S^3 \times S^3, \\ \boldsymbol{Z}_{120} & \text{if } G \cong Sp(2). & & \end{array}$$

(2) *If G is isomorphic to $S^3 \times SO(3)$ or $SO(4)$, then*

$$\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G) \cong M_4 \oplus (\boldsymbol{Z}_3)^2 \qquad \text{and}$$
$$\mathscr{Z}_\Omega(G) = \mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G) \cong (\boldsymbol{Z}_4)^4 \oplus (\boldsymbol{Z}_3)^2,$$

where M_4 is the noncommutative group of order 2^8 defined in [**30**].

Proof. First we make the elementary observation that the groups in question depend only on the homotopy type of G, and not on its structure as a Lie group.

(1) We begin by considering the groups $S^1 \times S^3$ and $U(2)$. Since they are homeomorphic, it suffices to prove the result for $S^1 \times S^3$. Lemma 4.1 shows that the $\mathscr{Z}$-groups are all isomorphic to $\pi_4(S^3 \times S^1) \cong \pi_4(S^3) \cong \boldsymbol{Z}_2$. The result for the $\mathscr{E}$-groups now follows from Proposition 3.1.

Now consider $G = S^1 \times SO(3)$. Since $SO(3)$ is homeomorphic to $\boldsymbol{R}\mathrm{P}^3$, we have $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G)$ and $\mathscr{Z}_\Omega(G) = \mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G)$ by Corollary 2.15. By Lemma 4.1, these latter groups are isomorphic to $q^*[S^1 \wedge \boldsymbol{R}\mathrm{P}^3, S^1 \times \boldsymbol{R}\mathrm{P}^3]$ where $q : S^1 \times \boldsymbol{R}\mathrm{P}^3 \to S^1 \wedge \boldsymbol{R}\mathrm{P}^3$ is the quotient map. It is known that $q^*[S^1 \wedge \boldsymbol{R}\mathrm{P}^3, S^1 \times \boldsymbol{R}\mathrm{P}^3] \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ by [**20**, Lemma 7.3]. To complete the proof for $G = S^1 \times SO(3)$ it suffices to show that $\Theta_\Omega : \mathscr{Z}_\Omega(G) \to \mathscr{E}_\Omega(G)$ is a homomorphism. This follows from the methods in the proof of [**30**, Proposition 3.1] and is similar to the proof below that $\Theta_\Omega : \mathscr{Z}_\Omega(S^3 \times S^3) \to \mathscr{E}_\Omega(S^3 \times S^3)$ is a homomorphism.

For $G = SU(3)$, let $q : G \to S^8$ be the quotient map obtained by collapsing the 7-skeleton. Recall from [**28**, Theorem 4.1] that $\pi_8(G) \cong \boldsymbol{Z}_{12}$ and from [**27**, p. 85] that $q^* : \pi_8(G) \to \mathscr{H}(G)$ is a monomorphism whose image is generated by a commutator, and so $\mathrm{Im}(q^*) \subseteq \mathscr{Z}_\Omega(G)$. On the other hand, it follows from [**27**, Theorem 5.1] that $\mathscr{Z}_{\#\infty}(G) \subseteq \mathrm{Im}(q^*)$, and so $\mathscr{Z}_\Omega(G) = \mathscr{Z}_{\#\infty}(G) = \mathrm{Im}(q^*) \cong \boldsymbol{Z}_{12}$. Also, by [**24**, Theorem 3.3], $\mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G)$. Hence $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G)$. By [**27**, Proposition 7.2(2)], the composite of

$$\pi_8(G) \xrightarrow[\cong]{q^*} \operatorname{Im}(q^*) = \mathscr{Z}_{\#\infty}(G) \xrightarrow{\Theta_\infty} \mathscr{E}_{\#\infty}(G)$$

is an isomorphism of groups. Hence $\mathscr{E}_{\#\infty}(G) \cong \pi_8(G)$ and the assertion for $G = SU(3)$ follows.

Let $G = Sp(2)$, and consider the quotient map $q : G \to S^{10}$. It is known that $q^* : \pi_{10}(G) \to \mathscr{H}(G)$ is injective, that $\mathscr{Z}_\#(G) = \mathscr{Z}_{\#\infty}(G)$ [**24**, Theorem 3.3] and that $\mathscr{Z}_{\#\infty}(G) = \operatorname{Im}(q^*)$. Thus we have

$$\mathscr{Z}_\Omega(G) \subseteq \mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G) = \operatorname{Im}(q^*) \cong \pi_{10}(G),$$

and so the statement about the $\mathscr{Z}$-groups will be verified once we show that $\operatorname{Im}(q^*) \subseteq \mathscr{Z}_\Omega(G)$. The reduced diagonal $d : G \to G \wedge G$ is the composition of the diagonal $\Delta : G \to G \times G$ and the quotient map $p : G \times G \to G \wedge G$. Since $\Omega p = 0$ by Corollary 2.7, it follows that $\Omega d = 0$. Let $\gamma = \gamma' \circ q$, where γ' is a generator of $\pi_{10}(G) \cong \boldsymbol{Z}_{120}$ [**28**, Theorem 5.1]. By the proof of [**24**, Theorem 3.3], γ factors as

$$G \xrightarrow{d} G \wedge G \xrightarrow{\psi} G$$

for some map ψ. Since $\Omega d = 0$, we have $\gamma \in \mathscr{Z}_\Omega(G)$, and so $\operatorname{Im}(q^*) \subseteq \mathscr{Z}_\Omega(G)$. For the $\mathscr{E}$-groups, we now have that $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G)$. By [**27**, Proposition 7.2(2)], the bijection $\Theta_\infty : \mathscr{Z}_{\#\infty}(G) \to \mathscr{E}_{\#\infty}(G)$ is a homomorphism, and so it is an isomorphism. Hence the assertion for $G = Sp(2)$ follows.

Let $G = S^3 \times S^3$. By Lemma 4.1, $\mathscr{Z}_\Omega(G) = \mathscr{Z}_{\#\infty}(G) = \mathscr{Z}_\#(G) = \operatorname{Im}(q^*) \cong \pi_6(G)$, where $q : S^3 \times S^3 \to S^3 \wedge S^3$ is the quotient map, and $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G)$. Furthermore, $\pi_6(G) \cong \pi_6(S^3) \oplus \pi_6(S^3) \cong \boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_{12}$ [**32**]. To complete the proof we show that the bijection $\Theta_\infty : \mathscr{Z}_{\#\infty}(G) \to \mathscr{E}_{\#\infty}(G)$ defined by $\Theta_\infty(\alpha) = \mathrm{id} + \alpha$ is a homomorphism. For $k = 1, 2$, let $i_k : S^3 \to S^3 \times S^3$ and $p_k : S^3 \times S^3 \to S^3$ be the standard inclusions and projections, respectively. Write $c : S^3 \times S^3 \to S^3$ for the commutator map; then $c = [p_1, p_2]$. It is known that $\mathscr{Z}_{\#\infty}(G) \cong \boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_{12}$ and that $\alpha_k = i_k \circ c$ are generators [**27**, Proposition 3.1]. We first prove $\alpha_k \circ (\mathrm{id} + \alpha_l) = \alpha_k$ for $k, l = 1, 2$. Now

$$\begin{aligned} c \circ (\mathrm{id} + \alpha_1) &= [p_1, p_2] \circ (\mathrm{id} + i_1 \circ [p_1, p_2]) \\ &= [p_1 \circ (\mathrm{id} + i_1 \circ [p_1, p_2]), p_2 \circ (\mathrm{id} + i_1 \circ [p_1, p_2])] \\ &= [p_1 + [p_1, p_2], p_2]. \end{aligned}$$

By [**27**, Proposition 3.1(3)], $[p_1, p_2]$ is a central element, so $[p_1 + [p_1, p_2], p_2] = [p_1, p_2] = c$ and hence $c \circ (\mathrm{id} + \alpha_1) = c$. Similarly, $c \circ (\mathrm{id} + \alpha_2) = c$. It follows that $\alpha_k \circ (\mathrm{id} + \alpha_l) = i_k \circ c \circ (\mathrm{id} + \alpha_l) = i_k \circ c = \alpha_k$ and, more generally, $\alpha_k \circ (\mathrm{id} + \alpha_l)^n = \alpha_k$ for any $n \geq 0$. We now prove by induction on $n \geq 0$ that $(\Theta(\alpha_k))^n = \Theta(n\alpha_k)$. The result is trivial for $n = 0$ or $n = 1$. For the inductive step, we have

$$\begin{aligned}
(\Theta(\alpha_k))^n &= \Theta(\alpha_k) \circ \Theta(\alpha_k)^{n-1} \\
&= (\mathrm{id} + \alpha_k) \circ \Theta(\alpha_k)^{n-1} \\
&= (\mathrm{id} + (n-1)\alpha_k) + \alpha_k(\mathrm{id} + \alpha_k)^{n-1} \\
&= (\mathrm{id} + (n-1)\alpha_k) + \alpha_k \\
&= \mathrm{id} + n\alpha_k \\
&= \Theta(n\alpha_k).
\end{aligned}$$

Hence, for any nonnegative integers n_1, n_2, we have

$$\begin{aligned}
\Theta(\alpha_1)^{n_1} \circ \Theta(\alpha_2)^{n_2} &= (1 + n_1\alpha_1) \circ (1 + n_2\alpha_2) \\
&= 1 + n_2\alpha_2 + n_1\alpha_1 \circ (1 + n_2\alpha_2) \\
&= 1 + n_2\alpha_2 + n_1(\alpha_1 \circ (1 + \alpha_2)^{n_2}) \\
&= 1 + n_2\alpha_2 + n_1\alpha_1 \\
&= \Theta(n_2\alpha_2 + n_1\alpha_1).
\end{aligned}$$

Similarly $\Theta(\alpha_2)^{n_2} \circ \Theta(\alpha_1)^{n_1} = \Theta(n_1\alpha_1 + n_2\alpha_2)$, and so

$$\begin{aligned}
\Theta(\alpha_1)^{n_1} \circ \Theta(\alpha_2)^{n_2} &= \Theta(n_2\alpha_2 + n_1\alpha_1) \\
&= \Theta(n_1\alpha_1 + n_2\alpha_2) \\
&= \Theta(\alpha_2)^{n_2} \circ \Theta(\alpha_1)^{n_1}
\end{aligned}$$

since $\mathscr{Z}_{\#\infty}(G) \cong \boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_{12}$ is an abelian group. Now we have

$$\begin{aligned}
\Theta((m_1\alpha_1 + m_2\alpha_2) + (n_1\alpha_1 + n_2\alpha_2)) &= \Theta(\alpha_1)^{m_1+n_1} \circ \Theta(\alpha_2)^{m_2+n_2} \\
&= \Theta(\alpha_1)^{m_1} \circ \Theta(\alpha_1)^{n_1} \circ \Theta(\alpha_2)^{m_2} \circ \Theta(\alpha_2)^{n_2} \\
&= \Theta(\alpha_1)^{m_1} \circ \Theta(\alpha_2)^{m_2} \circ \Theta(\alpha_1)^{n_1} \circ \Theta(\alpha_2)^{n_2} \\
&= \Theta(m_1\alpha_1 + m_2\alpha_2) \circ \Theta(n_1\alpha_1 + n_2\alpha_2),
\end{aligned}$$

which proves that Θ is a homomorphism.

(2) The groups $S^3 \times SO(3)$ and $SO(4)$ are homeomorphic to $S^3 \times \boldsymbol{R}\mathrm{P}^3$, so it suffices to verify the statement (2) for the space $S^3 \times \boldsymbol{R}\mathrm{P}^3$. In fact, we know that

$$\mathscr{Z}_{\Omega}(S^3 \times \boldsymbol{R}\mathrm{P}^3) = \mathscr{Z}_{\#\infty}(S^3 \times \boldsymbol{R}\mathrm{P}^3) = \mathscr{Z}_{\#}(S^3 \times \boldsymbol{R}\mathrm{P}^3) \cong [S^3 \wedge \boldsymbol{R}\mathrm{P}^3, S^3 \times \boldsymbol{R}\mathrm{P}^3]$$

by Lemma 4.1. To identify this latter group we calculate

$$\begin{aligned}[S^3 \wedge \boldsymbol{R}\mathrm{P}^3, S^3 \times \boldsymbol{R}\mathrm{P}^3] &\cong [S^3 \wedge \boldsymbol{R}\mathrm{P}^3, S^3] \times [S^3 \wedge \boldsymbol{R}\mathrm{P}^3, \boldsymbol{R}\mathrm{P}^3] \\ &\cong (\boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_4) \times (\boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_4)\end{aligned}$$

by [**29**, Lemma 2.1(6)]. This completes the proof for $\mathscr{X}$.

Therefore, by Proposition 3.1, we know that $\mathscr{E}_\Omega(G) = \mathscr{E}_{\#\infty}(G) = \mathscr{E}_\#(G)$. Since it is known [**30**, Theorem 1.1] that $\mathscr{E}_{\#\infty}(G) \cong M_4 \oplus (\boldsymbol{Z}_3)^2$, the proof of (2), and hence of Theorem 4.3, is complete. □

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