J. Math. Soc. Japan
Vol. 58, No. 2, 2006

Weighted harmonic Bergman kernel on half-spaces

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(Received Mar. 16, 2005)

Abstract. On the setting of the upper half-space $\boldsymbol{H}$ of the Euclidean n-space, we study weighted harmonic Bergman functions as follows. First, we define the fractional derivatives of some functions defined on $\boldsymbol{H}$. Next, we find the explicit formula for weighted Bergman kernel through the fractional derivative of the extended Poisson kernel and then we give the size estimates for derivatives of this kernel.

1. Introduction.

For a fixed positive integer $n \geq 2$, let $\boldsymbol{H} = \boldsymbol{R}^{n-1} \times \boldsymbol{R}_+$ be the upper half-space where $\boldsymbol{R}_+$ denotes the set of all positive real numbers. We write point $z \in \boldsymbol{H}$ as $z = (z', z_n)$ where $z' \in \boldsymbol{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$, $1 \leq p < \infty$, and $\Omega \subset \boldsymbol{R}^n$, let $b^p_\alpha(\Omega)$ denote *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on Ω such that

$$\|u\|_{L^p_\alpha(\Omega)} := \left(\int_\Omega |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = \operatorname{dist}(z, \partial\Omega)^\alpha \, dz$, $\operatorname{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary of Ω and dz is the Lebesgue measure on $\boldsymbol{R}^n$. We let $b^p_\alpha = b^p_\alpha(\boldsymbol{H})$ and $b^p = b^p_0$. Then we can check easily that the space b^p_α is a Banach space with the usual weighted L^p-norm.

Harmonic Bergman spaces are not studied as extensively as their holomorphic counterparts and most work on Bergman spaces has been done for bounded domains. [**4**], for example, is a good reference for weighted holomorphic Bergman spaces on the setting of the unit disc. Recently, $b^p_0(\Omega)$ is studied in [**6**] and [**5**] on the setting of upper half-space and bounded smooth domain in $\boldsymbol{R}^n$, respectively. Although $b^p_\alpha(B)$ where B is the open unit ball in $\boldsymbol{R}^n$ is studied in [**3**], this work is done via the series representation of harmonic Bergman kernel for nonnegative integer α.

Because $\boldsymbol{H}$ is a unbounded domain, it causes some problems. For example, the weighted harmonic Bergman kernel is not even integrable unlike the case of bounded domains. However $\boldsymbol{H}$ is a product space, so we can use the integration by parts (especially) with respect to the last component and this gives us reproducing properties of weighted harmonic Bergman functions. Furthermore, $\boldsymbol{H}$ is invariant under dilations, i.e., for every $r > 0$,

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2000 *Mathematics Subject Classification.* Primary 31B05; Secondary 31B10, 30D45, 30D55.

Key Words and Phrases. weighted Bergman kernel, harmonic Bergman functions, fractional derivative, upper half-space.

This research is supported by the Korea Research Foundation Grant (KRF-2004-015-C00022).

$$\{rz \mid z \in \boldsymbol{H}\ \} = \boldsymbol{H}.$$

Therefore we can use change of variable with respect to the last coordinate which helps us to estimate the size of some integrals that appear in this paper.

This paper is organized as follows. In section 2, we review some results about the extended Poisson kernel including its related facts and then we introduce fractional derivatives of some harmonic functions. In section 3, we prove some basic results and then we find the explicit formula for the Bergman kernel of $b^2_\alpha(\boldsymbol{H})$ through the fractional derivative of the extended Poisson kernel (Corollary 3.9). We also give the size estimates of derivatives of this kernel (Theorem 3.7).

Constants. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular functions, points or parameters under consideration. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Preliminary results and fractional derivatives.

In this section, we start with some preliminary results about the extended Poisson kernel on the upper half-space and its related facts, because its explicit form of the Bergman kernel for b^2_α takes the fractional derivative of the extended Poisson kernel.

Let $P(z,w)$ be the extended Poisson kernel on $\boldsymbol{H}$, i.e.,

$$P_z(w) := P(z,w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \overline{w}|^n} \tag{2.1}$$

where $z \in \boldsymbol{H}$, $w \in \overline{\boldsymbol{H}}$, $\overline{w} = (w', -w_n)$ and $V(B)$ is the volume of the unit ball in $\boldsymbol{R}^n$. Then it is well known that for each $z \in \boldsymbol{H}$ and for every $w \in \overline{\boldsymbol{H}}$,

$$\int_{\partial \boldsymbol{H}} P(z,w)\, dw' = 1. \tag{2.2}$$

Here $\partial \boldsymbol{H} = \boldsymbol{R}^{n-1}$ denotes the boundary of $\boldsymbol{H}$. Note that for each $j = 1, \dots, n-1$, $D_{z_j} P(z,w) = -D_{w_j} P(z,w)$ and $D_{z_n} P(z,w) = D_{w_n} P(z,w)$. Therefore we can show from (2.1) that for multi-indices $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$,

$$\begin{aligned} D_z^\beta D_w^\gamma P(z,w) &= D_{z_1}^{\beta_1} \cdots D_{z_n}^{\beta_n} D_{w_1}^{\gamma_1} \cdots D_{w_n}^{\gamma_n} P(z,w) \\ &= (-1)^{\gamma_1 + \cdots + \gamma_{n-1}} D_{z_1}^{\beta_1 + \gamma_1} \cdots D_{z_n}^{\beta_n + \gamma_n} P(z,w) \\ &= (-1)^{\gamma_1 + \cdots + \gamma_{n-1}} \frac{f_{\beta,\gamma}(z - \overline{w})}{|z - \overline{w}|^{n+2|\beta|+2|\gamma|}}, \end{aligned} \tag{2.3}$$

where $f_{\beta,\gamma}$ is a homogeneous polynomial of degree $1 + |\beta| + |\gamma|$.

The Poisson integral of $f \in L^p(\partial \boldsymbol{H})$, for $1 \le p \le \infty$, is the function $P[f]$ on $\boldsymbol{H}$ defined by

$$P[f](z) = \int_{\partial \boldsymbol{H}} P\big(z, (x, 0)\big) f(x)\, dx.$$

Let k be a nonnegative integer and let D denote the differentiation with respect to the last component. If $u \in b^p_\alpha(\Omega)$, then we know from the mean value property, Jensen's inequality and then Cauchy's estimate that

$$|D^k u(z)| \lesssim \operatorname{dist}(z, \partial\Omega)^{-(n+\alpha)/p-k} \tag{2.4}$$

for each $z \in \Omega$. This shows that if $u \in b^p_\alpha$, then u is a bounded harmonic function on every proper half-space contained in $\boldsymbol{H}$. Thus we have

$$P[u(\cdot, z_n)](z', t) = u(z', z_n + t) \tag{2.5}$$

for $t > 0$. (See [**1**] for details.)

Before we define fractional derivatives, we first define fractional integration on some function space defined on $\boldsymbol{H}$. Let $\mathscr{F}_\beta$ $(\beta > 0)$ be the collection of all measurable functions v on $\boldsymbol{H}$ satisfying $|v(z)| \lesssim z_n^{-\beta}$ and let $\mathscr{F} = \cup_{\beta>0} \mathscr{F}_\beta$. If $v \in \mathscr{F}$, then $v \in \mathscr{F}_\beta$ for some $\beta > 0$. In this case, we define the fractional integration of v of order s by

$$\mathscr{D}^{-s} v(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} v(z', z_n + t)\, dt$$

for the range $0 < s < \beta$. Here, Γ is a Gamma function.

Now we define the fractional derivative of $u \in b^p_\alpha$ of order s for $s \ge 0$ by

$$\mathscr{D}^s u = \mathscr{D}^{-(\lceil s \rceil - s)} D^{\lceil s \rceil} u,$$

where $\lceil s \rceil$ is the smallest integer greater than or equal to s and $\mathscr{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < \lceil s \rceil - s - 1 < 0$ and $\lceil s \rceil \ge 1$. Thus we know from (2.4) that for each $z \in \boldsymbol{H}$ and for every $u \in b^p_\alpha$,

$$\mathscr{D}^s u(z) = \frac{1}{\Gamma(\lceil s \rceil - s)} \int_0^\infty t^{\lceil s \rceil - s - 1} D^{\lceil s \rceil} u(z', z_n + t)\, dt$$

always makes sense. Our definition of fractional integration is similar to that of [**2**]. However in [**2**], fractional integration is defined with weight: For a suitable function v on $\boldsymbol{H}$ and for $z \in \boldsymbol{H}$, $\mathscr{D}^{-s} v(z) = 1/\Gamma(s) \int_0^\infty t^{s-1} e^{-t} v(z', z_n + t)\, dt$.

3. Bergman kernel and its size estimate.

In this section, we derive the explicit formula for weighted harmonic Bergman kernel of b^2_α and we give the size estimate of derivatives of this kernel. For this purpose, we first prove some basic results.

The following proposition is used in this paper to estimate the size of some integral that appears in (3.14).

Proposition 3.1. *Let $b < 0$ and let $a + b > -1$. Then,*

$$\int_{\boldsymbol{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}\,dw \approx z_n^b$$

as z ranges over all points in $\boldsymbol{H}$.

Proof. Using polar coordinates centered at z' on $\partial\boldsymbol{H}$ and then change of variable $r \mapsto (z_n + w_n)r$, we have

$$\begin{aligned}\int_{\boldsymbol{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}\,dw &= \int_0^\infty \int_{\partial\boldsymbol{H}} \frac{w_n^{a+b}}{\big(|z'-w'|^2+(z_n+w_n)^2\big)^{(n+a)/2}}\,dw'\,dw_n \\ &\approx \int_0^\infty \int_0^\infty \frac{r^{n-2}w_n^{a+b}}{\big(r+(z_n+w_n)\big)^{n+a}}\,dr\,dw_n \\ &= \int_0^\infty \frac{w_n^{a+b}}{(z_n+w_n)^{a+1}} \int_0^\infty \frac{r^{n-2}}{(1+r)^{n+a}}\,dr\,dw_n \\ &\approx \int_0^\infty \frac{w_n^{a+b}}{(z_n+w_n)^{a+1}}\,dw_n, \qquad (3.1)\end{aligned}$$

because $n-2 \geq 0$ and $a+2 > 1$. After applying change of variable $w_n \mapsto z_n t$ once again, we see that (3.1) becomes

$$z_n^b \int_0^\infty \frac{t^{a+b}}{(1+t)^{a+1}}\,dt \approx z_n^b,$$

because $a+b > -1$ and $1-b > 1$. Therefore the proof is complete. □

The following lemma is used in Proposition 3.4 to guarantee switching the order of integration in (3.11). Before we state the lemma, we introduce one notation. Let $\boldsymbol{H}_\delta = \{z \in \boldsymbol{R}^n \mid z_n > -\delta\}$ for $\delta > 0$. Thus for each $\delta > 0$, $\boldsymbol{H}_\delta$ is a half-space that contains $\boldsymbol{H}$ properly.

Lemma 3.2. *Let $\delta > 0$, $\alpha > -1$ and let $1 \leq p < \infty$. Suppose that $s > -1$ is not an integer. Then we have*

$$\int_0^\infty \int_{\boldsymbol{H}} \big|u(w)D^{\lceil s\rceil+1}P_z(w',(1+t)w_n)\big|w_n^{\lceil s\rceil}\,dw\,t^{\lceil s\rceil-s-1}\,dt < \infty$$

for each $u \in b^p_\alpha(\boldsymbol{H}_\delta)$ and for every $z \in \boldsymbol{H}$.

PROOF. Let $u \in b^p_\alpha(\boldsymbol{H}_\delta)$, $z \in \boldsymbol{H}$ and let $k = \lceil s \rceil$. Then k is a nonnegative integer and $k - s > 0$. From (2.3) and (2.4), we get

$$\left|D^{k+1} P_z(w)\right| \lesssim |z - \overline{w}|^{-(n+k)}, \qquad |u(w)| \lesssim (w_n + \delta)^{-(n+\alpha)/p}.$$

Therefore we have

$$\begin{aligned}
&\int_0^\infty \int_{\boldsymbol{H}} \left|u(w) D^{\lceil s \rceil + 1} P_z(w', (1+t)w_n)\right| w_n^{\lceil s \rceil}\, dw\, t^{\lceil s \rceil - s - 1}\, dt \\
&\quad \lesssim \int_0^\infty \int_{\boldsymbol{H}} \frac{w_n^k}{|z - (w', -(1+t)w_n)|^{n+k} (w_n + \delta)^{(n+\alpha)/p}}\, dw\, t^{k-s-1}\, dt. \qquad (3.2)
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{1}{|z - (w', -(1+t)w_n)|^{n+k}} &\lesssim \frac{z_n + (1+t)w_n}{(z_n + (1+t)w_n)^{k+1} |z - (w', -(1+t)w_n)|^n} \\
&= \frac{nV(B)}{2(z_n + (1+t)w_n)^{k+1}} P(z, (w', (1+t)w_n)).
\end{aligned}$$

Therefore we know from (2.2) that (3.2) is less than or equal to some constant times

$$\int_0^\infty \int_0^\infty \frac{w_n^k}{(z_n + (1+t)w_n)^{k+1} (w_n + \delta)^{(n+\alpha)/p}}\, dw_n\, t^{k-s-1}\, dt. \qquad (3.3)$$

Choose $0 < a < s + 1$ satisfying $a < (n+\alpha)/p$. Then, after applying change of variable $(1+t)w_n \mapsto \eta$, we see that (3.3) becomes

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{\eta^k}{(z_n + \eta)^{k+1} (\eta + (1+t)\delta)^{(n+\alpha)/p}}\, d\eta\, (1+t)^{(n+\alpha)/p - (k+1)} t^{k-s-1}\, dt \\
&\quad \lesssim \int_0^\infty \int_0^\infty \frac{1}{(z_n + \eta)(\eta + \delta)^a}\, d\eta\, \frac{(1+t)^{(n+\alpha)/p - (k+1)}}{(1+t)^{(n+\alpha)/p - a}} t^{k-s-1}\, dt < \infty,
\end{aligned}$$

because $a > 0$, $k - s - 1 > -1$ and $s - a + 2 > 1$. This completes the proof. □

We prove the following lemma integrating by parts with respect to the w_n-variable and this plays an important role in proving Proposition 3.4.

LEMMA 3.3. *Let $\delta > 0$, $1 \le p < \infty$ and let $u \in b^p_\alpha(\boldsymbol{H}_\delta)$. Suppose that k and m are nonnegative integers. Then for every $z \in \boldsymbol{H}$ and for each $a, b > 0$,*

$$\int_{\boldsymbol{H}} [D^{k+1} P_z(w', aw_n)][D^m u(w', bw_n)] w_n^{m+k} dw = \frac{(-1)^{m+k+1}(m+k)!}{(a+b)^{m+k+1}} u(z).$$

PROOF. Let $z \in \boldsymbol{H}$ and let $a, b > 0$. Because $u \in b^p_\alpha(\boldsymbol{H}_\delta)$, we know from (2.4) that for each nonnegative integer l, $D^l u$ is a continuous bounded harmonic function on $\overline{\boldsymbol{H}}$. Thus, we have from (2.5) that for each nonnegative integer l,

$$\begin{aligned}
&\int_{\boldsymbol{H}} P_z(w', aw_n)[D^{l+1}u(w', bw_n)]w_n^l \, dw' \, dw_n \\
&\quad = \int_0^\infty \int_{\partial \boldsymbol{H}} P\big((z', z_n + aw_n), (w', 0)\big)[D^{l+1}u(w', bw_n)] \, dw' \, w_n^l \, dw_n \\
&\quad = \int_0^\infty \big[D^{l+1}u\big(z', z_n + (a+b)w_n\big)\big] \, w_n^l \, dw_n. \qquad (3.4)
\end{aligned}$$

We see from (2.5) and (3.4) with $l = 0$ case, after integrating by parts with respect to the last component, that

$$\begin{aligned}
&\int_{\boldsymbol{H}} [DP_z(w', aw_n)]u(w', bw_n) \, dw \\
&\quad = -\frac{1}{a} \int_{\partial \boldsymbol{H}} P_z(w', 0)u(w', 0) \, dw' - \frac{b}{a} \int_{\boldsymbol{H}} P_z(w', aw_n)Du(w', bw_n) \, dw \\
&\quad = -\frac{1}{a} u(z) - \frac{b}{a} \int_0^\infty Du\big(z', z_n + (a+b)w_n\big) \, dw_n \\
&\quad = \frac{-u(z)}{a+b}. \qquad (3.5)
\end{aligned}$$

Similarly we see that for each nonnegative integer l,

$$\begin{aligned}
&\int_{\boldsymbol{H}} [DP_z(w', aw_n)][D^{l+1}u(w', bw_n)]w_n^{l+1} \, dw \\
&\quad = -\frac{b}{a} \int_{\boldsymbol{H}} P_z(w', aw_n)[D^{l+2}u(w', bw_n)]w_n^{l+1} \, dw \\
&\qquad - \frac{l+1}{a} \int_{\boldsymbol{H}} P_z(w', aw_n)[D^{l+1}u(w', bw_n)]w_n^l \, dw.
\end{aligned}$$

We also know from (3.4) and integration by parts that the above becomes

$$\begin{aligned}
&-\frac{b}{a} \int_0^\infty \big[D^{l+2}u\big(z', z_n + (a+b)w_n\big)\big] w_n^{l+1} \, dw_n \\
&\qquad - \frac{l+1}{a} \int_0^\infty \big[D^{l+1}u\big(z', z_n + (a+b)w_n\big)\big] {w_n}^l \, dw_n \\
&\quad = -\frac{l+1}{a+b} \int_0^\infty \big[D^{l+1}u\big(z', z_n + (a+b)w_n\big)\big] w_n^l \, dw_n. \qquad (3.6)
\end{aligned}$$

After applying integration by parts l-times to (3.6), we see that (3.6) equals

$$\frac{(-1)^l(l+1)!}{(a+b)^{l+2}}u(z). \tag{3.7}$$

Thus (3.5) and (3.7) imply that for each nonnegative integer l,

$$\int_{\boldsymbol{H}}[DP_z(w',aw_n)][D^l u(w',bw_n)]\,w_n^l\,dw = \frac{(-1)^{l+1}l!}{(a+b)^{l+1}}\,u(z). \tag{3.8}$$

Hence,

$$\begin{aligned} a^k &\int_{\boldsymbol{H}}[D^{k+1}P_z(w',aw_n)][D^m u(w',bw_n)]w_n^{m+k}\,dw \\ &= (-1)^k\int_{\boldsymbol{H}}[DP_z(w',aw_n)]D^k\big([D^m u(w',bw_n)]w_n^{m+k}\big)\,dw \\ &= (-1)^k\sum_{j=0}^{k}C(k,j)\left\{\frac{b^j(m+k)!}{(m+j)!}\int_{\boldsymbol{H}}[DP_z(w',aw_n)][D^{(m+j)}u(w',bw_n)]w_n^{m+j}\,dw\right\}, \end{aligned} \tag{3.9}$$

where $C(k,j)=k!/j!(k-j)!$. Thus we see from (3.8) that (3.9) equals

$$\frac{(-1)^{m+k+1}(m+k)!}{(a+b)^{m+1}}\sum_{j=0}^{k}C(k,j)\left(\frac{-b}{a+b}\right)^j u(z) = \frac{(-1)^{m+k+1}a^k(m+k)!}{(a+b)^{m+k+1}}u(z).$$

This completes the proof. □

The following reproducing property of integral operators with its kernel through the fractional derivative of the extended Poisson kernel is a main tool in finding the Bergman kernel.

PROPOSITION 3.4. *Let $\delta > 0$, $\alpha > -1$, $1 \le p < \infty$ and let $s > -1$. Then for every $u \in b^p_\alpha(\boldsymbol{H}_\delta)$ and for each $z \in \boldsymbol{H}$,*

$$u(z) = C_s\int_{\boldsymbol{H}}[\mathscr{D}^{s+1}P_z(w)]u(w)\,dV_s(w),$$

where

$$C_s = \frac{(-1)^{\lceil s\rceil+1}2^{s+1}}{\Gamma(s+1)}. \tag{3.10}$$

PROOF. Let $u \in b^p_\alpha(\boldsymbol{H}_\delta)$ and let $z \in \boldsymbol{H}$. If s is a nonnegative integer, then C_s in (3.10) is $(-1)^{s+1}2^{s+1}/s!$. Thus we get the desired result by taking $m=0$, $k=s$ and $a=b=1$ in Lemma 3.3.

Now, assume that s is not an integer. Let $k=\lceil s\rceil$. Then $k-s>0$. Thus,

$$\begin{aligned}
&\int_{\boldsymbol{H}} [D^{s+1}P_z(w)]u(w)\,dV_s(w)\\
&= \int_{\boldsymbol{H}} [\mathscr{D}^{-(k-s)}D^{k+1}P_z(w)]u(w)\,dV_s(w)\\
&= \int_{\boldsymbol{H}} \left(\frac{1}{\Gamma(k-s)}\int_0^\infty [D^{k+1}P_z(w', w_n+t)]\,t^{k-s-1}dt\right)u(w)\,dV_s(w)\\
&= \frac{1}{\Gamma(k-s)}\int_{\boldsymbol{H}} \left(\int_0^\infty [D^{k+1}P_z(w', (1+t)w_n)]\,t^{k-s-1}\,dt\right)u(w)\,w_n^k\,dw, \qquad (3.11)
\end{aligned}$$

where we used change of variable $t \mapsto tw_n$. Then we know from Lemma 3.2 that we can switch the order of integrations in (3.11). Thus, we see from Lemma 3.3 that (3.11) becomes

$$\begin{aligned}
&\frac{1}{\Gamma(k-s)}\int_0^\infty \left(\int_{\boldsymbol{H}} [D^{k+1}P_z(w', (1+t)w_n)]\,u(w)\,w_n^k\,dw\right)t^{k-s-1}\,dt\\
&\quad= \left(\frac{(-1)^{k+1}k!}{\Gamma(k-s)}\int_0^\infty \frac{t^{k-s-1}}{(2+t)^{k+1}}\,dt\right)u(z). \qquad (3.12)
\end{aligned}$$

We see, after using the change of variable $2/(2+t) \mapsto t$, that the quantity in parenthesis of (3.12) becomes

$$\frac{(-1)^{k+1}k!}{\Gamma(k-s)2^{s+1}}\int_0^1 t^s(1-t)^{k-s-1}\,dt = \frac{(-1)^{k+1}}{2^{s+1}}\Gamma(s+1),$$

where we used the relation between the beta function and gamma function. Therefore we get the desired result and the proof is complete. ☐

The proof of the following proposition is very similar to that of Theorem 2.1 in [**6**], where they proved it for $u \in b^p$. Thus we omit the proof.

Proposition 3.5. *Let $\alpha > -1$ and let $1 \le p < \infty$. If $u \in b^p_\alpha$, then the integral*

$$\int_{\partial\boldsymbol{H}} |u(w', w_n)|^p dw'$$

increases as w_n decreases.

For a function u on $\boldsymbol{H}$, define $u_\delta(z) = u(z', z_n+\delta)$ for $\delta > 0$. Then we easily get the following result from Proposition 3.5.

Proposition 3.6. *Let $\alpha > -1$, $1 \le p < \infty$ and let $u \in b^p_\alpha$. Then*

$$\lim_{\delta\to 0^+} \|u_\delta - u\|_{L^p_\alpha(\boldsymbol{H})} = 0.$$

Proof. We know from Proposition 3.5 that $\|u_\delta\|_{b^p_\alpha(\boldsymbol{H})} \le \|u\|_{b^p_\alpha(\boldsymbol{H})}$. Thus $u_\delta \in b^p_\alpha$.

We also know from Proposition 3.5 that for $0<\epsilon<R$,

$$\begin{aligned}\|u_\delta-u\|^p_{L^p_\alpha(\boldsymbol{H})}&\lesssim\left(\int_0^\epsilon+\int_R^\infty\right)\int_{\partial\boldsymbol{H}}\big(|u_\delta(z',z_n)|^p+|u(z',z_n)|^p\big)\,dz'\,z_n^\alpha\,dz_n\\&\quad+\int_\epsilon^R\int_{\partial\boldsymbol{H}}|u_\delta(z',\epsilon)-u(z',\epsilon)|^p\,dz'\,z_n^\alpha\,dz_n\\&\lesssim\left(\int_0^\epsilon+\int_R^\infty\right)\int_{\partial\boldsymbol{H}}|u(z',z_n)|^p\,dz'\,z_n^\alpha\,dz_n\\&\quad+\frac{R^{\alpha+1}}{\alpha+1}\int_{\partial\boldsymbol{H}}|u_\delta(z',\epsilon)-u(z',\epsilon)|^p\,dz'.\end{aligned}\tag{3.13}$$

Let I and II denote, respectively, the two summands of (3.13). Then for a given $\epsilon_1>0$, we can choose ϵ sufficiently small and R sufficiently large so that $I<\epsilon_1^p$. This is possible because $u\in b^p_\alpha$. We know from Proposition 3.5 that $u_\delta\in h^p(\boldsymbol{H})$ where $h^p(\boldsymbol{H})$ is the harmonic Hardy space on $\boldsymbol{H}$. (See [**1**] for details.) Therefore Theorem 7.8 in [**1**] implies that $\lim_{\delta\to0^+}II=0$ for each fixed $\epsilon>0$ and $R>0$. This completes the proof. □

For $\alpha>-1$, define $R_\alpha(z,w)$ by

$$R_\alpha(z,w)=C_\alpha\mathscr{D}^{\alpha+1}P_z(w),$$

where C_α is the constant given in (3.10). In Corollary 3.9 below, we show that the Bergman kernel for b^2_α is $R_\alpha(z,w)$. For this purpose, we first estimate the size of derivatives of $R_\alpha(z,w)$.

THEOREM 3.7. *Let $\alpha>-1$, $s>-n-\alpha$ and let β be a multi-index. Then*

$$\left|D_z^\beta\mathscr{D}^s_{z_n}R_\alpha(z,w)\right|\lesssim\frac{1}{|z-\overline{w}|^{n+\alpha+|\beta|+s}}$$

for $z,w\in\boldsymbol{H}$.

PROOF. We first estimate the size of $|R_\alpha(z,w)|$. If α is a nonnegative integer, then we get the desired result from (2.3). Assume that α is not an integer. Note that $\big|z-\big(w',-(w_n+t)\big)\big|\approx|z-\overline{w}|+t$ for $z,w\in\boldsymbol{H}$, $t>0$. Then (2.3) implies that

$$\begin{aligned}|R_\alpha(z,w)|&\lesssim\int_0^\infty\left|D^{\lceil\alpha\rceil+1}P\big(z,(w',w_n+t)\big)\right|\,t^{\lceil\alpha\rceil-\alpha-1}\,dt\\&\lesssim\int_0^\infty\frac{t^{\lceil\alpha\rceil-\alpha-1}}{(|z-\overline{w}|+t)^{n+\lceil\alpha\rceil}}\,dt\\&\approx\frac{1}{|z-\overline{w}|^{n+\alpha}},\end{aligned}$$

where we used change of variable $t\mapsto|z-\overline{w}|t$. This shows that for each fixed $w\in\boldsymbol{H}$,

$R_\alpha(\cdot,w)\in\mathscr{F}_{n+\alpha}$. Thus $\mathscr{D}^s_{z_n}R_\alpha(z,w)$ is well defined for $s>-n-\alpha$.

Now, let's estimate the size of $|D^\beta_z\mathscr{D}^s_{z_n}R_\alpha(z,w)|$. The case that both s and α are integers is proved by (2.3). Assume that both α and s are not integers. Let $k=\lceil\alpha\rceil$. Set $l=\lceil s\rceil$ if $s>-1$ and $l=0$ if $s\le -1$. Then we see from the definition of fractional derivative that $|D^\beta_z\mathscr{D}^s_{z_n}R_\alpha(z,w)|$ becomes

$$\begin{aligned}&\left|C_\alpha D^\beta_z\mathscr{D}^s_{z_n}\int_0^\infty\left[D^{k+1}_{w_n}P_z(w',w_n+t)\right]t^{k-\alpha-1}\,dt\right|\\&\quad\lesssim\int_0^\infty\int_0^\infty\left|D^\beta_z D^l_{z_n}D^{k+1}_{w_n}P\big((z',z_n+r),(w',w_n+t)\big)\right|t^{k-\alpha-1}r^{l-s-1}\,dt\,dr.\end{aligned}$$

After applying (2.3) once again, we see that the above is less than or equal to some constant times

$$\begin{aligned}\int_0^\infty\int_0^\infty\frac{t^{k-\alpha-1}r^{l-s-1}}{(|z-\overline{w}|+r+t)^{n+k+|\beta|+l}}\,dt\,dr&\approx\int_0^\infty\frac{r^{l-s-1}}{(|z-\overline{w}|+r)^{n+\alpha+|\beta|+l}}\,dr\\&\approx\frac{1}{|z-\overline{w}|^{n+\alpha+|\beta|+s}}.\end{aligned}$$

Here we use the change of variable a couple of times, i.e., $t\mapsto(|z-\overline{w}|+r)t$ and $r\mapsto|z-\overline{w}|r$.

The remaining cases can be proved similarly and the proof is complete. □

We see from Theorem 3.7 that $|R_\alpha(z,w)|\lesssim|z-\overline{w}|^{-(n+\alpha)}$. Thus Proposition 3.1 with $a=(n+\alpha)q-n$ and $b=\alpha-a$ implies that for $1<q<\infty$,

$$\|R_\alpha(z,\cdot)\|_{L^q_\alpha(\boldsymbol{H})}\lesssim z_n^{(n+\alpha)(1/q-1)}.\tag{3.14}$$

With this estimate, we can show easily that $R_\alpha(z,\cdot)$ reproduces every b^p_α-function.

Theorem 3.8. *Let $\alpha>-1$ and let $1\le p<\infty$. If $u\in b^p_\alpha$, then for every $z\in\boldsymbol{H}$,*

$$u(z)=\int_{\boldsymbol{H}}u(w)R_\alpha(z,w)\,dV_\alpha(w).$$

Proof. Fix $z\in\boldsymbol{H}$. Note that $u_\delta\in b^p_\alpha(\boldsymbol{H}_\delta)$ for $\delta>0$. Therefore we have from Proposition 3.4 that for $\delta>0$,

$$\begin{aligned}&\left|\int_{\boldsymbol{H}}u(w)R_\alpha(z,w)\,dV_\alpha(w)-u(z)\right|\\&\quad\le\left|\int_{\boldsymbol{H}}\big(u(w)-u_\delta(w)\big)R_\alpha(z,w)\,dV_\alpha(w)\right|+|u_\delta(z)-u(z)|.\end{aligned}\tag{3.15}$$

If $1<p<\infty$, then Hölder's inequality implies that (3.15) is less than or equal to

$$\|u - u_\delta\|_{L^p_\alpha(\boldsymbol{H})} \|R_\alpha(z,\cdot)\|_{L^q_\alpha(\boldsymbol{H})} + |u_\delta(z) - u(z)|,$$

where q denotes the index conjugate to p. Thus we see by letting $\delta \to 0^+$ that

$$u(z) = \int_{\boldsymbol{H}} u(w) R_\alpha(z,w)\, dV_\alpha(w)$$

from Proposition 3.6 and (3.14).

If $p = 1$, then we get the desired result from the estimate,

$$\|R_\alpha(z,\cdot)\|_\infty \lesssim z_n^{-(n+\alpha)}.$$

This completes the proof. □

If α is a nonnegative integer, then $R_\alpha(z,\cdot)$ is harmonic on $\boldsymbol{H}$ for each fixed $z \in \boldsymbol{H}$, because it is a partial derivative of a harmonic function. If α is not an integer, then for z, $w \in \boldsymbol{H}$

$$R_\alpha(z,w) = C_\alpha \int_0^\infty t^{\lceil \alpha \rceil - \alpha - 1} D^{\lceil \alpha \rceil + 1} P\bigl(z, (w', w_n + t)\bigr)\, dt.$$

Therefore by passing the Laplacian through the integral above, we see that it is harmonic on $\boldsymbol{H}$. Thus the next result follows directly from Theorem 3.8 and (3.14).

COROLLARY 3.9. *For $\alpha > -1$, $R_\alpha(z,\cdot)$ is the Bergman kernel for b^2_α.*

References

[1] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, New York, 1992.

[2] F. Beatrous, Behavior of holomorphic functions near weakly pseudoconvex boundary points, Indiana Univ. Math. J., **40** (1991), 915–966.

[3] R. R. Coifman and R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, Astérisque, **77** (1980), 11–66.

[4] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman spaces, Springer-Verlag, New York, 2000.

[5] H. Kang and H. Koo, Estimates of the harmonic Bergman kernel on smooth domains, J. Funct. Anal., **185** (2001), 220–239.

[6] W. Ramey and H. Yi, Harmonic Bergman functions on half-spaces, Trans. Amer. Math. Soc., **348** (1996), 633–660.

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