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Odd primary Steenrod algebra, additive formal group laws, and modular invariants

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Abstract. We give a conceptual clarification of Milnor's theorem, which tells us the Hopf algebra structure of the stable co-operations H_*H in the odd primary ordinary cohomology. Directly connecting H_*H with the quasi-strict automorphism group of some 1-dimensional additive formal group law and modular invariants, we give a new proof of this theorem of Milnor.

1. Introduction.

Suppose that p is an odd prime, and that H is the mod p Eilenberg-MacLane spectrum. Let $\mathscr{F}^*$ be the free associative graded algebra generated by the symbols $\beta, P^1, P^2, \dots$. Let S^* be the quotient algebra of $\mathscr{F}^*$ modulo the Adem relations. The Cartan formula gives a coalgebra structure of S^*. Therefore S^* is a Hopf algebra, and it is called the Steenrod algebra. As usual, we regard $\beta, P^1, P^2, \dots$ as elements in the stable operations H^*H. Then it is well known that S^* is isomorphic to H^*H as a Hopf algebra. Milnor [**7**] showed that S_*, the dual Hopf algebra of S^*, is isomorphic to the Hopf algebra $E(\tau_0, \tau_1, \dots) \otimes \boldsymbol{F}_p[\xi_1, \xi_2, \dots]$ whose coproduct is given by

$$\tau_n \mapsto \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i, \quad \xi_n \mapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i.$$

This induces the Hopf algebra structure of the stable co-operations H_*H.

Our aim is to reinforce and clarify this theorem of Milnor by introducing the quasi-strict automorphism group of a 1-dimensional additive formal group law and modular invariants. Our argument consists of two steps.

In the first step, we consider two functors $\mathrm{Op}(-)$ and $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ on the category of non-negative graded commutative algebras over $\boldsymbol{F}_p$. The functor $\mathrm{Op}(-)$ assigns $\mathrm{Op}(R_*)$, the set of all multiplicative operations

$$H^*(-) \longrightarrow H^*(-) \otimes R_*$$

which satisfy certain properties, to each R_*, a non-negatively graded commutative algebra over $\boldsymbol{F}_p$. The functor $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ assigns $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$, the set of all quasi-strict

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automorphisms of the 1-dimensional additive formal group law over the ring of dual numbers $R_*[\epsilon]/(\epsilon^2)$ to each R_*. Then $\mathrm{Op}(-)$ and $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ are represented by the graded algebras H_*H and $A_* = E(\bar{\tau}_0, \bar{\tau}_1, \dots) \otimes \boldsymbol{F}_p[\bar{\xi}_1, \bar{\xi}_2, \dots]$, respectively. In other words, we have natural isomorphisms

$$\lambda : \mathrm{Op}(-) \xrightarrow{\cong} \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(H_*H, -), \quad T : \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, -) \xrightarrow{\cong} \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-).$$

Moreover we can define a natural transformation

$$F : \mathrm{Op}(-) \to \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$$

which directly connects H_*H with the quasi-strict automorphism group of a 1-dimensional additive formal group law. These induce the following commutative diagram:

$$\begin{CD} \mathrm{Op}(-) @>F>> \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-) \\ @V{\lambda}V{\cong}V @V{\cong}V{T}V \\ \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(H_*H, -) @>N>> \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, -). \end{CD} \tag{1.1}$$

Here $N = T^{-1} \circ F \circ \lambda^{-1}$. In particular, we obtain the crucial homomorphism of algebras

$$N(\mathrm{id}_{H_*H}) : A_* \to H_*H.$$

The composite of two quasi-strict automorphisms is also a quasi-strict one. This means that $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ is a functor to the category of groups, which induces the Hopf algebra structure of A_*. Then we see $N(\mathrm{id}_{H_*H})$ is a Hopf algebra homomorphism.

In the second step, we show that $N(\mathrm{id}_{H_*H})$ is an isomorphism by the usage of a multiplicative operation

$$S_n : H^*(-) \longrightarrow H^*(-) \otimes D[n]_*,$$

where $D[n]_* = E(\tau[n]_0, \dots, \tau[n]_{n-1}) \otimes \boldsymbol{F}_p[\xi[n]_1, \dots, \xi[n]_n]$. The definition of S_n depends heavily upon Mùi's work on cohomology operations derived from modular invariants. Once such a multiplicative operation S_n is defined, we immediately obtain the following commutative diagram from (1.1):

$$\begin{array}{ccc} A_* & \xrightarrow{N(\mathrm{id}_{H_*H})} & H_*H \\ & \searrow^{T^{-1}\circ F(S_n)} & \downarrow{\scriptstyle \lambda(S_n)} \\ & & D[n]_*. \end{array}$$

Here $T^{-1} \circ F(S_n)$ is shown to be an isomorphism in some low range of homological degree, which becomes arbitrarily large as we choose sufficiently large n. This implies that $N(\mathrm{id}_{H_*H})$ is injective. Furthermore by the old work of Cartan [**2**], [**3**], the Poincaré series of A_* and that of H_*H are the same. Therefore $N(\mathrm{id}_{H_*H})$ is an isomorphism. This leads us to the Hopf algebra structure of H_*H, for we can easily obtain the Hopf algebra structure of A_*.

In [**6**], we showed a similar result in the mod 2 case, which tells us the Hopf algebra structure of the stable co-operations $H\boldsymbol{Z}/2_*H\boldsymbol{Z}/2$ in the mod 2 ordinary cohomology by using the strict automorphism group of a 1-dimensional additive formal group law and modular invariants. The approach in this paper is similar to the one we used in [**6**]. However there is a difference. The strict automorphism group of the 1-dimensional additive formal group law over R_* plays an important role in [**6**], whereas the quasi-strict one over $R_*[\epsilon]/(\epsilon^2)$ does it in this paper. The usage of the strict one over R_* in this paper determine the polynomial part $\boldsymbol{F}_p[\xi_1, \xi_2, \dots]$ of H_*H only.

This paper is divided into five sections and an appendix. In Section 2, we introduce the notion of multiplicative operations. We define a multiplicative operation ψ with good properties, which induces the natural isomorphism λ. In Section 3, we recall the definition of reduced power operations [**10**] and Mùi's results [**8**], [**9**], and introduce the multiplicative operation S_n by using these results. In Section 4, we study $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ and obtain the natural isomorphism T. In Section 5, we define the natural transformation F which relates $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ with $\mathrm{Op}(-)$, and then we show the main theorem (Theorem 5.2). In Appendix A, we define higher dimensional graded formal group laws and homomorphisms. Especially we study a certain 2-dimensional graded additive formal group law G_a and the quasi-strict automorphism group of G_a. Then we prove the main theorem by the usage of the quasi-strict automorphism group of G_a instead of $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$.

Throughout this paper, we use the following notations. Suppose that X and Y are spaces, and that p is an odd prime. We denote the mod p cohomology by $H^*(-)$. Let $e_1, \dots, e_n$ be the standard basis of $(\boldsymbol{Z}/p)^n$. Let

$$\epsilon_1, \dots, \epsilon_n \in H^1(B(\boldsymbol{Z}/p)^n) = \mathrm{Hom}((\boldsymbol{Z}/p)^n, \boldsymbol{Z}/p)$$

be the dual of $e_1, \dots, e_n$. Put $x_i = \beta\epsilon_i$, where β is the Bockstein homomorphism. Then we have

$$H^*(B(\boldsymbol{Z}/p)^n) = E(\epsilon_1, \dots, \epsilon_n) \otimes \boldsymbol{F}_p[x_1, \dots, x_n].$$

Any graded $\boldsymbol{F}_p$-algebra R_* is supposed to be non-negatively graded and commutative, that is to say, $R_n = 0$ for $n < 0$, and $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$.

We set degree as follows. For an element x in $H^n(X)$, we define the degree of x by $\deg x = n$. For a graded $\boldsymbol{F}_p$-algebra R_* and $r \in R_m$, we define the degree of r by $\deg r = -m$. Therefore $x \otimes r \in H^*(X) \otimes R_*$ is of degree $n - m$.

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2. Multiplicative operations.

We now define multiplicative operations in a way similar to Definition 2.1 in [**6**].

Definition 2.1. Let R_* be a graded $\boldsymbol{F}_p$-algebra. Consider the graded module whose degree k-part is $\prod_{n\geq 0} H^{k+n}(X)\otimes R_n$. By abuse of notation, we denote it by $H^*(X)\otimes R_*$. A natural operation $\gamma : H^*(X)\to H^*(X)\otimes R_*$ which preserves degree is said to be *multiplicative* if γ satisfies the following conditions:

(i) The following diagram is commutative:

$$\begin{array}{ccccc}
H^*(X)\otimes H^*(Y) & & \xrightarrow{\quad\times\quad} & & H^*(X\times Y) \\
\downarrow{\scriptstyle \gamma\otimes\gamma} & & & & \downarrow{\scriptstyle \gamma} \\
\begin{array}{c} H^*(X)\otimes R_* \\ \otimes H^*(Y)\otimes R_* \end{array} & \xrightarrow{1\otimes\mu\otimes 1} & \begin{array}{c} H^*(X)\otimes H^*(Y) \\ \otimes R_*\otimes R_* \end{array} & \xrightarrow{(\times)\otimes m} & H^*(X\times Y)\otimes R_*.
\end{array}$$

Here $\times$ is the cross product, m is the multiplication on R_*, and μ is defined by $\mu(x,y)=(-1)^{mn}(y,x)$ for $x\in R_m$ and $y\in H^n(Y)$.

(ii) $\gamma(u)=u\otimes 1$ when u is a generator of $H^1(S^1)$.

Let $\tilde{H}^*(-)$ be the reduced mod p cohomology, and γ a multiplicative operation. Then γ induces the reduced operation $\tilde{\gamma} : \widetilde{H}^*(X)\to \widetilde{H}^*(X)\otimes R_*$, which satisfies the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{H}^*(X)\otimes \widetilde{H}^*(Y) & & \xrightarrow{\quad\wedge\quad} & & \widetilde{H}^*(X\wedge Y) \\
\downarrow{\scriptstyle \tilde{\gamma}\otimes\tilde{\gamma}} & & & & \downarrow{\scriptstyle \tilde{\gamma}} \\
\begin{array}{c} \widetilde{H}^*(X)\otimes R_* \\ \otimes \widetilde{H}^*(Y)\otimes R_* \end{array} & \xrightarrow{1\otimes\mu\otimes 1} & \begin{array}{c} \widetilde{H}^*(X)\otimes \widetilde{H}^*(Y) \\ \otimes R_*\otimes R_* \end{array} & \xrightarrow{(\wedge)\otimes m} & \widetilde{H}^*(X\wedge Y)\otimes R_*.
\end{array} \tag{2.1}$$

Here $\wedge$ is the smash product.

Lemma 2.2 (See [**6**, Lemma 2.2]). *Suppose that γ is a multiplicative operation. Then $\tilde{\gamma}$ is stable. That is, the following diagram is commutative*:

$$\begin{array}{ccc}
\widetilde{H}^n(X) & \xrightarrow{\quad\sigma\quad} & \widetilde{H}^{n+1}(\Sigma X) \\
\downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \tilde{\gamma}} \\
[\widetilde{H}^*(X)\otimes R_*]^n & \xrightarrow{\sigma\otimes 1} & [\widetilde{H}^*(\Sigma X)\otimes R_*]^{n+1}.
\end{array}$$

Here σ is the suspension isomorphism.

Proof. By the commutative diagram (2.1), we have the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{H}^*(S^1)\otimes\widetilde{H}^*(X) & & \xrightarrow{\quad\wedge\quad} & & \widetilde{H}^*(S^1\wedge X) \\
\downarrow{\scriptstyle \tilde{\gamma}\otimes\tilde{\gamma}} & & & & \downarrow{\scriptstyle \tilde{\gamma}} \\
\widetilde{H}^*(S^1)\otimes R_* \otimes \widetilde{H}^*(X)\otimes R_* & \xrightarrow{1\otimes\mu\otimes 1} & \widetilde{H}^*(S^1)\otimes\widetilde{H}^*(X) \otimes R_*\otimes R_* & \xrightarrow{\wedge\otimes m} & \widetilde{H}^*(S^1\wedge X)\otimes R_*.
\end{array}$$

For any element x in $\widetilde{H}^*(X)$, we have

$$\begin{aligned}
\tilde{\gamma}(\sigma(x)) = \tilde{\gamma}(u\wedge x) &= (\wedge\otimes m)\circ(1\otimes\mu\otimes 1)\circ(\tilde{\gamma}(u)\otimes\tilde{\gamma}(x)) \\
&= (\wedge\otimes m)\circ(1\otimes\mu\otimes 1)(u\otimes 1\otimes\tilde{\gamma}(x)) = u\wedge\tilde{\gamma}(x) = (\sigma\otimes 1)\circ\tilde{\gamma}(x).
\end{aligned}$$

This means that $\tilde{\gamma}$ is a stable operation. □

Let H be the mod p Eilenberg-MacLane spectrum. We want to introduce a multiplicative operation $\psi : H^*(X) \to H^*(X)\otimes H_*H$ with good properties. We define a map

$$\bar{\psi} : H^*(X) = \{X^+, H\}^* \to \{X^+, H\wedge H\}^*$$

by $\bar{\psi}(f) = i\wedge f \in \{S^0\wedge X^+, H\wedge H\}^*$, where $i : S^0 \to H$ is the unit map. Let $m : H\wedge H \to H$ be the multiplication on H. The map $\kappa : H^*(X)\otimes H_*H \to \{X^+, H\wedge H\}^*$ induced by $H\wedge(H\wedge H) \stackrel{m\wedge 1}{\longrightarrow} H\wedge H$ is an isomorphism since H_nH is finite dimensional for each n. (See [**6**, Lemma 2.3].) We set

$$\psi = \kappa^{-1}\circ\bar{\psi} : H^*(X) \to H^*(X)\otimes H_*H. \tag{2.2}$$

We see that ψ is a multiplicative operation by the same proof as [**6**, Lemma 2.4].

In the remainder of this section, we study properties of ψ. From now on, we assume that any graded algebra R_* over $\boldsymbol{F}_p$ is of finite type, that is, R_n is finite dimensional for each n. Since R_* is of finite type, $H^*(X)\otimes R_*$ satisfies the wedge axiom

$$H^*(\vee X_\alpha)\otimes R_* \cong \prod_\alpha H^*(X_\alpha)\otimes R_*.$$

Therefore $H^*(X)\otimes R_*$ is a cohomology theory, and we write HR_* for the spectrum representing it. The cohomology $H^*(\ )\otimes R_*$ has the products

$$H^*(X)\otimes R_*\otimes H^*(Y)\otimes R_* \longrightarrow H^*(X\times Y)\otimes R_*$$

$$(x\otimes r\otimes y\otimes r' \mapsto (-1)^{\deg r\cdot\deg y}(x\times y)\otimes r\cdot r'),$$

$$H^*(X)\otimes(H^*(Y)\otimes R_*) \longrightarrow H^*(X\times Y)\otimes R_* \qquad (x\otimes y\otimes r \mapsto (x\times y)\otimes r).$$

These imply that HR_* is a commutative ring spectrum and an H-module spectrum.

By Adams [**1**, III, 13.5], we have the isomorphism

$$\lambda : (HR_*)^*H \xrightarrow{\cong} \mathrm{Hom}^*_{\boldsymbol{F}_p}(H_*H, R_*).$$

Here this map is defined by $\lambda(x) = (H \wedge H \xrightarrow{1\wedge x} H \wedge HR_* \xrightarrow{\tau} HR_*)$ for $x \in \{H, HR_*\}^*$, where $\tau : H \wedge HR_* \to HR_*$ is the H-module map. It is easily seen that the following diagram is commutative:

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\psi} & H^*(X) \otimes H_*H \\ & \searrow^{\gamma} & \downarrow{\scriptstyle 1\otimes\lambda(\gamma)} \\ & & H^*(X) \otimes R_*. \end{array} \tag{2.3}$$

Let $\mathrm{Op}(R_*)$ be the set of all multiplicative operations over R_*. Given a graded algebra homomorphism $R_* \to R'_*$ and $\gamma \in \mathrm{Op}(R_*)$,

$$(1 \otimes r) \circ \gamma : H^*(X) \xrightarrow{\gamma} H^*(X) \otimes R_* \xrightarrow{1\otimes r} H^*(X) \otimes R'_*$$

is a multiplicative operation over R'_*. Therefore $\mathrm{Op}(-)$ is a covariant functor from the category of graded algebras to the category of sets. From Lemma 2.2, $\tilde{\gamma}$ is a stable cohomology operation. In conclusion, we can regard γ as an element in $(HR_*)^0H$, and hence we have $\mathrm{Op}(R_*) \subset (HR_*)^0H$. We denote the restriction $\mathrm{Op}(R_*) \to \mathrm{Hom}_{\boldsymbol{F}_p}(H_*H, R_*)$ of λ by the same symbol λ. Then we have the following theorem. Since the proof is the same as that of [**6**, Theorem 2.5], we omit it.

Theorem 2.3. *There is a one-to-one correspondence*

$$\lambda : \mathrm{Op}(R_*) \longrightarrow \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(H_*H, R_*).$$

Here λ is natural in R_*, and satisfies the commutativity of the diagram (2.3). *Especially $\lambda(\psi)$ is the identity map of H_*H.*

3. Steenrod's reduced power operations.

Let I be a finite ordered set. We denote by $\mathrm{Sym}(I)$ and $\mathrm{Alt}(I)$ the symmetric group and the alternating group of I, respectively. Let J be a finite ordered set, G a subgroup of $\mathrm{Alt}(I)$, and H a subgroup of $\mathrm{Alt}(J)$. Let $G \wr H = G \ltimes \prod_X H$, the wreath product of G and H. Then we have

$$G \times H \subset G \wr H \subset \mathrm{Alt}(I \times J), \tag{3.1}$$

where the first inclusion is given by the diagonal $H \to \prod_I H$.

Consider the vector space $E^n = E_1 \times \cdots \times E_n$, where $E_i = \boldsymbol{Z}/p$. Let Σ_{p^n} and A_{p^n} be the symmetric group $\mathrm{Sym}(E^n)$ and the alternating group $\mathrm{Alt}(E^n)$ on the point set E^n, respectively. The vector space E^n acts on itself: $g \in E^n$ sends $h \in E^n$ to $g+h$. Thereby we can regard E^n as a permutation group on E^n. This implies the inclusion $E^n \subset \Sigma_{p^n}$. We define a Sylow p-subgroup $\Sigma_{p^n,p}$ of Σ_{p^n} by $\Sigma_{p^n,p} = E_1 \int \cdots \int E_n$. Using (3.1) repeatedly, we have

$$A_{p^n} \supset \Sigma_{p^n,p} \supset E_1 \times \cdots \times E_n = E^n.$$

We recall reduced power operations in [**10**]. Let G be a subgroup of $\mathrm{Alt}(I)$, where I is an ordered set of m elements. Given a space X and $X_i = X$ for any $i \in I$, we put $X^I = \prod_{i \in I} X_i$ and $E_G(X) = EG \times_G X^I$. Steenrod defined the power operation

$$P_G : H^q(X) \longrightarrow H^{mq}(E_G(X)).$$

Let $d_G : BG \times X \to E_G(X)$ be the diagonal map. Then we see

$$d_G^* P_G : H^q(X) \to H^{mq}(BG \times X).$$

Let G' be a subgroup of G. The inclusion $i_{G,G'} : G' \hookrightarrow G$ induces $BG' \to BG$ and $E_{G'}(X) \to E_G(X)$, which are denoted by the same symbol $i_{G,G'}$. It is known in [**10**, VII] that $i_{G,G'}^* P_G = P_{G'}$. Moreover d_G and $i_{G,G'}$ are continuous. Hence we have the following equalities for $E^n \subset \Sigma_{p^n,p} \subset A_{p^n}$:

$$i_{A_{p^n},E^n}^* d_{A_{p^n}}^* P_{A_{p^n}} = i_{\Sigma_{p^n,p},E^n}^* d_{\Sigma_{p^n,p}}^* P_{\Sigma_{p^n,p}} = d_n^* P_n : H^q(X) \to H^{p^n q}(BE^n \times X). \quad (3.2)$$

Here $d_n = d_{E^n}$ and $P_n = P_{E^n}$.

$d_n^* P_n$ has the following fundamental properties.

Lemma 3.1. *Put $h = (p-1)/2$. Then we have*

(i) $d_n^* P_n = d_1^* P_1 d_{n-1}^* P_{n-1}$.
(ii) $d_n^* P_n(uv) = (-1)^{nhqr} d_n^* P_n(u) \cdot d_n^* P_n(v)$, *where $q = \deg u$ and $r = \deg v$.*

Put $GL_n = GL_n(\boldsymbol{F}_p)$, $SL_n = \{w \in GL_n | \det w = 1\}$ and $\widetilde{SL}_n = \{w \in GL_n | (\det w)^h = 1\}$. Consider the graded algebras

$$\boldsymbol{F}_p[x_1, \ldots, x_n], \quad E(\epsilon_1, \ldots, \epsilon_n) \otimes \boldsymbol{F}_p[x_1, \ldots, x_n]$$

with $\deg \epsilon_i = 1$ and $\deg x_i = 2$. Here $E(\ )$ is an exterior algebra over $\boldsymbol{F}_p$. Any subgroup K of GL_n operates naturally on them. Let

$$\boldsymbol{F}_p[x_1, \ldots, x_n]^K, \quad (E(\epsilon_1, \ldots, \epsilon_n) \otimes \boldsymbol{F}_p[x_1, \ldots, x_n])^K$$

be the subalgebras of the K-invariants. Recall the K-invariants for the case of $K = GL_n$,

SL_n or $\widetilde{SL}_n$ from Dickson [**4**] and Mùi [**8**], [**9**]. These are needed to describe $d_n^* P_n(u)$ for $u \in H^*(X)$, which leads us to the definition of a multiplicative operation S_n.

Put $[e_1, \ldots, e_n] = \det(x_i^{p^{e_j}})$ for any sequence of non-negative integers $(e_1, \ldots, e_n)$. In particular, we set $L_{n,s} = [0, \ldots, \hat{s}, \ldots, n]$ for $0 \leq s \leq n$, and $L_n = L_{n,n}$. By the definition, we have $\deg L_{n,s} = 2(p^{n+1} - p^s)/(p-1)$ and $\deg L_n = 2(p^n - 1)/(p-1)$. According to [**4**], [**8**, I.4.15, I.4.16] and [**9**, 2.1], $L_{n,s}/L_n$ is an element in $\boldsymbol{F}_p[x_1, \ldots, x_n]$, and it is denoted by $Q_{n,s}$. Note that $Q_{n,n} = 1$. By the definition, we see $\deg Q_{n,s} = 2(p^n - p^s)$. From Dickson [**4**], we have

$$\boldsymbol{F}_p[x_1, \ldots, x_n]^{SL_n} = \boldsymbol{F}_p[L_n, Q_{n,1}, \ldots, Q_{n,n-1}].$$

Let (a_{ij}) be a matrix of type (n,n) over a graded algebra. Then the determinant $\det(a_{ij})$ is defined as follows:

$$\det(a_{ij}) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

We set

$$[k; e_{k+1}, \ldots, e_n] = \frac{1}{k!} \det \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ \vdots & \ddots & \vdots \\ \epsilon_1 & \cdots & \epsilon_n \\ x_1^{p^{e_{k+1}}} & \cdots & x_n^{p^{e_{k+1}}} \\ \vdots & \ddots & \vdots \\ x_1^{p^{e_n}} & \cdots & x_n^{p^{e_n}} \end{pmatrix},$$

for any sequence of non-negative integers $(e_{k+1}, \ldots, e_n)$. For $0 \leq s_1 < \cdots < s_k \leq n-1$, we put

$$M_{n,s_1,\ldots,s_k} = [k; 0, \ldots, \hat{s}_1, \ldots, \hat{s}_k, \ldots, n-1].$$

Then we obtain $\deg M_{n,s_1,\ldots,s_k} = k + 2(p^n - 1)/(p-1) - 2(p^{s_1} + \cdots + p^{s_k})$. Here are results of Mùi.

THEOREM 3.2 ([**8**, I.4.8]). *We have the direct sum decomposition*

$$\begin{aligned} &(E(\epsilon_1, \ldots, \epsilon_n) \otimes \boldsymbol{F}_p[x_1, \ldots, x_n])^{SL_n} \\ &\quad \cong \boldsymbol{F}_p[L_n, Q_{n,1}, \ldots, Q_{n,n-1}] \oplus \bigoplus_{0 \leq s_1 < \cdots < s_k \leq n-1} \boldsymbol{F}_p[L_n, Q_{n,1}, \ldots, Q_{n,n-1}] \cdot M_{n,s_1,\cdots,s_k} \end{aligned}$$

as an $\boldsymbol{F}_p[L_n, Q_{n,1}, \ldots, Q_{n,n-1}]$-module. The multiplicative structure is given by the relations

$$M_{n,s}^2 = 0, \quad M_{n,s_1} \cdots M_{n,s_k} = (-1)^{k(k-1)/2} M_{n,s_1,\ldots,s_k} L_n^{k-1}$$

for $0 \le s_1 < \cdots < s_k \le n-1$.

From [**9**, Lemma 2.1] and this theorem, we obtain the following corollary.

COROLLARY 3.3 ([**9**, 2.5]). *We have the direct sum decomposition*

$$\begin{aligned}&(E(\epsilon_1,\dots,\epsilon_n)\otimes \boldsymbol{F}_p[x_1,\dots,x_n])^{\widetilde{SL}_n}\\ &\quad\cong \boldsymbol{F}_p[\tilde{L}_n,Q_{n,1},\dots,Q_{n,n-1}]\oplus \bigoplus_{0\le s_1<\cdots<s_k\le n-1}\boldsymbol{F}_p[\tilde{L}_n,Q_{n,1},\dots,Q_{n,n-1}]\cdot \tilde{M}_{n,s_1,\cdots,s_k}\end{aligned}$$

as an $\boldsymbol{F}_p[\tilde{L}_n,Q_{n,1},\dots,Q_{n,n-1}]$*-module. Here* $\tilde{L}_n = L_n^h$ *and* $\tilde{M}_{n,s_1,\dots,s_k} = M_{n,s_1,\dots,s_k}L_n^{h-1}$. *Then* $\deg \tilde{L}_n = p^n - 1$ *and* $\deg \tilde{M}_{n,s_1,\dots,s_k} = k-1+p^n-2(p^{s_1}+\cdots+p^{s_k})$.

As in [**8**], we put $V_k = \prod_{a_i\in \boldsymbol{Z}/p}(a_1x_1+\cdots+a_{k-1}x_{k-1}+x_k)$. Then we obtain the relations

$$L_n = V_1V_2\cdots V_n, \quad Q_{n,s} = Q_{n-1,s}V_n^{p-1}+Q_{n-1,s-1}^p.$$

PROPOSITION 3.4 ([**9**, 2.6]). *Suppose* $U_k = M_{k,k-1}L_{k-1}^{h-1}$. *Then we have*

$$\begin{aligned}V_{k+1} &= (-1)^k\sum_{s=0}^{k}(-1)^sQ_{k,s}x_{k+1}^{p^s},\\ U_{k+1} &= (-1)^k\left(\tilde{L}_k\epsilon_{k+1}+\sum_{s=0}^{k-1}(-1)^{s+1}\tilde{M}_{k,s}x_{k+1}^{p^s}\right),\end{aligned}$$

where $\deg V_{k+1} = 2p^k$ *and* $\deg U_{k+1} = p^k$.

We identify $H^*(BE^n)$ with the above algebra $E(\epsilon_1,\dots,\epsilon_n)\otimes \boldsymbol{F}_p[x_1,\dots,x_n]$. Using the invariants, we describe the images of

$$i^*_{\Sigma_{p^n,p},E^n} : H^*(B\Sigma_{p^n,p})\to H^*(BE^n), \quad i^*_{A_{p^n},E^n} : H^*(BA_{p^n})\to H^*(BE^n)$$

as follows.

THEOREM 3.5 ([**8**, II Theorem 5.2], [**9**, Theorem 3.10]).

$$\operatorname{Im} i^*_{\Sigma_{p^n,p},E^n} = E(U_1,\dots,U_n)\otimes \boldsymbol{F}_p[V_1,\dots,V_n].$$

In the proof of [**9**, Theorem 3.10], it is shown that

$$\operatorname{Im} i^*_{A_{p^n},E^n} = \operatorname{Im} i^*_{\Sigma_{p^n,p},E^n}\cap [H^*(BE^n)]^{\widetilde{SL}_n}.$$

From [**9**, Lemma 3.11] and Corollary 3.3, we see

$$\operatorname{Im} i^*_{\Sigma_{p^n},p,E^n} \cap [H^*(BE^n)]^{\widetilde{SL}_n} = E(\tilde{M}_{n,0},\ldots,\tilde{M}_{n,n-1}) \otimes \boldsymbol{F}_p[\tilde{L}_n, Q_{n,1},\ldots,Q_{n,n-1}].$$

Therefore we have the following theorem.

Theorem 3.6 ([**9**, Theorem 3.10]).

$$\operatorname{Im} i^*_{A_{p^n},E^n} = E(\tilde{M}_{n,0},\ldots,\tilde{M}_{n,n-1}) \otimes \boldsymbol{F}_p[\tilde{L}_n, Q_{n,1},\ldots,Q_{n,n-1}].$$

Since we have $d_n^* P_n = i^*_{A_{p^n},E^n} d^*_{A_{p^n}} P_{A_{p^n}}$ from the equality (3.2), we obtain

$$\operatorname{Im} d_n^* P_n \subset (E(\tilde{M}_{n,0},\ldots,\tilde{M}_{n,n-1}) \otimes \boldsymbol{F}_p[\tilde{L}_n, Q_{n,1},\ldots,Q_{n,n-1}]) \otimes H^*(X).$$

Hence the following is well defined.

Definition 3.7 ([**9**, Definition 4.1]). For every $u \in H^q(X)$, we write

$$d_n^* P_n(u) = \sum \tilde{M}_{n,s_1} \cdots \tilde{M}_{n,s_k} \tilde{L}_n^{r_0} Q_{n,1}^{r_1} \cdots Q_{n,n-1}^{r_{n-1}} \otimes \mathscr{D}_{S,R}(u),$$

where the summation runs over all sequences $S = (s_1,\ldots,s_k)$ with $0 \le s_1 < \cdots < s_k \le n-1$ and all sequences of non-negative integers $R = (r_0,\ldots,r_{n-1})$. This formula defines the maps

$$\mathscr{D}_{S,R} : H^q(X) \longrightarrow H^{p^n q - |S,R|}(X),$$

where $|S,R| = kp^n + r_0(p^n-1) + 2\big(\sum_{j=1}^{n-1} r_j(p^n - p^j) - \sum_{i=1}^{k} p^{s_i}\big)$.

Mùi proved the following lemma about $\mathscr{D}_{S,R}$.

Lemma 3.8 ([**9**, Lemma 4.2]). *If $q-k-r_0$ is not even or $q < k + r_0 + 2(r_1 + \cdots + r_{n-1})$, then $\mathscr{D}_{S,R}(u) = 0$.*

Let us introduce a multiplicative operation S_n. We set $\mu(q) = (h!)^q(-1)^{hq(q-1)/2}$, and

$$\Gamma[n]_* = E(\tilde{M}_{n,0},\ldots,\tilde{M}_{n,n-1}) \otimes \boldsymbol{F}_p[\tilde{L}_n^{\pm}, Q_{n,1},\ldots,Q_{n,n-1}].$$

Then we define an operation $\bar{S}_n : H^*(X) \longrightarrow \Gamma[n]_* \otimes H^*(X)$ by

$$x \mapsto \mu(\deg x)^{-n} \tilde{L}_n^{-\deg x} d_n^* P_n(x).$$

Since $\deg \tilde{L}_n^{-\deg x} = -(\deg x)\cdot(p^n-1)$ and $\deg d_n^* P_n(x) = (\deg x)\cdot p^n$, we see $\bar{S}_n$ preserves degree. Put $\tau[n]_i = (-1)^{i+1}\tilde{M}_{n,i}/\tilde{L}_n$ for $0 \le i \le n-1$, and $\xi[n]_i = (-1)^i Q_{n,i}/\tilde{L}_n^2$ for $1 \le i \le n$. Denote by $D[n]_*$ the subalgebra generated by $\tau[n]_0,\ldots,\tau[n]_{n-1}$ and $\xi[n]_1,\ldots,\xi[n]_n$ in $\Gamma[n]_*$. Then we can easily see

$$D[n]_* = E(\tau[n]_0, \dots, \tau[n]_{n-1}) \otimes \boldsymbol{F}_p[\xi[n]_1, \dots, \xi[n]_n].$$

Here $\deg \tau[n]_i = -(2p^i - 1)$ and $\deg \xi[n]_i = -2(p^i - 1)$, i.e., $\tau[n]_i \in D[n]_{2p^i-1}$ and $\xi[n]_i \in D[n]_{2(p^i-1)}$. By Lemma 3.8, we have the following lemma.

LEMMA 3.9. $\operatorname{Im}(\bar{S}_n) \subset D[n]_* \otimes H^*(X)$.

We denote by S_n the composite operation

$$H^*(X) \xrightarrow{\bar{S}_n} D[n]_* \otimes H^*(X) \xrightarrow{\mu} H^*(X) \otimes D[n]_*,$$

where μ is the interchange map $\sum a_i \otimes b_i \mapsto \sum (-1)^{\deg a_i \cdot \deg b_i} b_i \otimes a_i$.

LEMMA 3.10. *The cohomology operation S_n is multiplicative.*

PROOF. For $u \in H^q(X)$ and $v \in H^r(X)$, we have

$$\begin{aligned} \bar{S}_n(uv) &= \mu(q+r)^{-n} \tilde{L}_n^{-(q+r)} d_n^* P_n(uv) \\ &= \mu(q+r)^{-n} \tilde{L}_n^{-(q+r)} (-1)^{nhqr} d_n^* P_n(u) \cdot d_n^* P_n(v) \\ &= \bar{S}_n(u) \cdot \bar{S}_n(v). \end{aligned}$$

Here the second equality follows from Lemma 3.1, and the third is obvious since $\mu(q+r) = \mu(q) \cdot \mu(r) \cdot (-1)^{hqr}$. Therefore S_n satisfies the condition (i) in Definition 2.1.

It remains to prove that S_n satisfies the condition (ii) in Definition 2.1. Let u be a generator of $H^1(S^1)$. Since

$$d_1^* P_1(u) = \mu(1)(y^h \otimes u) = \mu(1)(\tilde{L}_1 \otimes u),$$

we have $S_n(u) = u \otimes 1$. □

For $H^*(BE_{k+1}) = E(\epsilon_{k+1}) \otimes \boldsymbol{F}_p[x_{k+1}]$, we consider

$$d_k^* P_k : H^*(BE_{k+1}) \longrightarrow H^*(B(E_1 \times \cdots \times E_k) \times BE_{k+1}) = H^*(BE^{k+1}).$$

Then the following theorem is known in Mùi [**8**].

THEOREM 3.11 ([**8**], [**9**, Theorem 3.8], [**11**, Proposition 1.1(iii)]). *We have*

$$d_k^* P_k(\epsilon_{k+1}) = (-h!)^k U_{k+1}, \quad d_k^* P_k(x_{k+1}) = V_{k+1}.$$

This implies the following corollary. It is used in the proof of Theorem 5.2.

COROLLARY 3.12. *For $\epsilon \in H^1(B\boldsymbol{Z}/p)$ and $x \in H^2(B\boldsymbol{Z}/p)$, we have*

$$S_n(\epsilon) = \epsilon \otimes 1 + \sum_{s=0}^{n-1} x^{p^s} \otimes \tau[n]_s, \quad S_n(x) = x \otimes 1 + \sum_{s=1}^{n} x^{p^s} \otimes \xi[n]_s.$$

Proof. By Lemma 3.4 and Theorem 3.11, we obtain

$$\begin{aligned}\bar{S}_n(\epsilon) &= \mu(1)^{-n}\tilde{L}_n^{-1}d_n^*P_n(\epsilon)\\ &= (h!)^{-n}\tilde{L}_n^{-1}(-h!)^n(-1)^n\left(\tilde{L}_n\otimes\epsilon+\sum_{s=0}^{n-1}(-1)^{s+1}\tilde{M}_{n,s}\otimes x^{p^s}\right)\\ &= 1\otimes\epsilon+\sum_{s=0}^{n-1}\tau[n]_s\otimes x^{p^s}.\end{aligned}$$

This induces $S_n(\epsilon)=\epsilon\otimes 1+\sum_{s=0}^{n-1}x^{p^s}\otimes\tau[n]_s$.

We see

$$\begin{aligned}\mu(2) = (h!)^2(-1)^h &= (1\cdot 2\cdots(p-1)/2)^2(-1)^h\\ &= (1\cdot 2\cdots(p-1)/2)\cdot\{(-1)\cdot(-2)\cdots(-(p-1)/2)\}\\ &= 1\cdot 2\cdots p-1=-1.\end{aligned}$$

Therefore by Lemma 3.4 and Theorem 3.11, we have

$$\bar{S}_n(x)=\mu(2)^{-n}\tilde{L}_n^{-2}d_n^*P_n(x)=(-1)^{-n}\tilde{L}_n^{-2}(-1)^n\sum_{s=0}^{n}(-1)^sQ_{n,s}x^{p^s}=\sum_{s=0}^{n}\xi[n]_s\otimes x^{p^s},$$

where $\xi[n]_0=1$. In consequence, we have $S_n(x)=x\otimes 1+\sum_{s=1}^{n}x^{p^s}\otimes\xi[n]_s$. □

4. Some 1-dimensional additive formal group law.

Let g_a be the 1-dimensional additive formal group law. For each graded algebra R_*, we consider the graded algebra $R_*[\epsilon]/(\epsilon^2)$, the ring of dual numbers of R_*. Here $\deg\epsilon=1$.

Definition 4.1. We set $\deg x=2$. A power series $f(x)=\sum_{i=1}^{\infty}(\epsilon m_i+n_i)x^i$ in $R_*[\epsilon]/(\epsilon^2)[[x]]$ is called a *quasi-strict* automorphism of g_a over $R_*[\epsilon]/(\epsilon^2)$ if it satisfies the following three conditions:

(i) $f(x+y)=f(x)+f(y)$
(ii) $n_1=1$
(iii) $m_i\in R_{2i-1}$ and $n_i\in R_{2i-2}$.

Remark. The condition (iii) is equivalent to $\deg\epsilon m_ix^i=\deg n_ix^i=2$ for any i.

The condition (i) in this definition implies $m_i=0$ and $n_i=0$ for $i\neq p^\alpha$, and thereby we can express a quasi-strict automorphism $f(x)$ as

$$f(x)=\sum_{k=0}^{\infty}(\epsilon a_k+b_k)x^{p^k},\quad a_k\in R_{2p^k-1},\ b_k\in R_{2p^k-2},\ b_0=1.$$

We write $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$ for the set of all quasi-strict automorphisms over $R_*[\epsilon]/(\epsilon^2)$. Then $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ is a functor from the category of graded algebras over $\boldsymbol{F}_p$ to the category of sets. We put $A_* = E(\bar{\tau}_0, \bar{\tau}_1, \dots) \otimes \boldsymbol{F}_p[\bar{\xi}_1, \bar{\xi}_2, \dots]$, where $\bar{\tau}_i \in A_{2p^i-1}$ and $\bar{\xi}_i \in A_{2p^i-2}$. We have a natural isomorphism of sets

$$T : \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, R_*) \longrightarrow \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*), \qquad h \mapsto \sum_{k=0}^{\infty}(\epsilon h(\bar{\tau}_k) + h(\bar{\xi}_k))x^{p^k}, \tag{4.1}$$

where $\bar{\xi}_0 = 1$. We define a product of $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$ by the composition $(g \cdot f)(x) = f(g(x))$. Then $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$ is a group, and therefore $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ is a functor to the category of groups. Furthermore $\mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, -)$ is also a functor to the category of groups via (4.1), and this induces the coproduct $\Delta : A_* \to A_* \otimes A_*$. Given a couple of elements in $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$:

$$f(x) = \sum_{j=0}^{\infty}(\epsilon a'_j + b'_j)x^{p^j}, \quad a'_j, b'_j \in R_*, \quad b'_0 = 1;$$

$$g(x) = \sum_{k=0}^{\infty}(\epsilon a''_k + b''_k)x^{p^k}, \quad a''_j, b''_j \in R_*, \quad b''_0 = 1,$$

we obtain the product

$$\begin{aligned}
(f \cdot g)(x) = \sum_{i=0}^{\infty}(\epsilon a_i + b_i)x^{p^i} &= \sum_{k=0}^{\infty}\left(\epsilon a''_k + b''_k\right)\left(\sum_{j=0}^{\infty}\left(\epsilon a'_j + b'_j\right)x^{p^j}\right)^{p^k} \\
&= \sum_{k=0}^{\infty}\left(\epsilon a''_k + b''_k\right)\left(\sum_{j=0}^{\infty}\left(\epsilon a'_j + b'_j\right)^{p^k} x^{p^{j+k}}\right) \\
&= \left(\epsilon a''_0 + b''_0\right)\sum_{j=0}^{\infty}\left(\epsilon a'_j + b'_j\right)x^{p^j} + \sum_{k=1}^{\infty}\left(\epsilon a''_k + b''_k\right)\sum_{j=0}^{\infty} {b'}_j^{p^k} x^{p^{j+k}} \\
&= \sum_{i=0}^{\infty}\left(\epsilon\left(a'_i + \sum_{k=0}^{i} {b'}_{i-k}^{p^k} a''_k\right) + \sum_{k=0}^{i} {b'}_{i-k}^{p^k} b''_k\right)x^{p^i}.
\end{aligned}$$

Therefore the coproduct Δ is given by

$$\Delta(\bar{\tau}_i) = \bar{\tau}_i \otimes 1 + \sum_{k=0}^{i} \bar{\xi}_{i-k}^{p^k} \otimes \bar{\tau}_k, \quad \Delta(\bar{\xi}_i) = \sum_{k=0}^{i} \bar{\xi}_{i-k}^{p^k} \otimes \bar{\xi}_k.$$

Therefore we have the following theorem.

THEOREM 4.2. *Let A_* be the Hopf algebra $E(\bar{\tau}_0, \bar{\tau}_1, \dots) \otimes \boldsymbol{F}_p[\bar{\xi}_1, \bar{\xi}_2, \dots]$ whose coproduct is given by*

$$\Delta(\bar{\tau}_i) = \bar{\tau}_i \otimes 1 + \sum_{k=0}^{i} \bar{\xi}_{i-k}^{p^k} \otimes \bar{\tau}_k, \quad \Delta(\bar{\xi}_i) = \sum_{k=0}^{i} \bar{\xi}_{i-k}^{p^k} \otimes \bar{\xi}_k.$$

Then T is a natural isomorphism of groups.

5. A relation between H_*H and $\mathrm{AUT}_{F_p}(g_a)$.

The product $a : B\boldsymbol{Z}/p \times B\boldsymbol{Z}/p \to B\boldsymbol{Z}/p$ induces the coproduct map

$$a^* : H^*(B\boldsymbol{Z}/p) \cong E(\epsilon) \otimes \boldsymbol{F}_p[x] \longrightarrow H^*(B\boldsymbol{Z}/p \times B\boldsymbol{Z}/p) \cong E(\epsilon_1, \epsilon_2) \otimes \boldsymbol{F}_p[x_1, x_2],$$

and we see that $a^*(\epsilon) = \epsilon_1 + \epsilon_2$ and $a^*(x) = x_1 + x_2$. Consider a multiplicative operation $\gamma : H^*(X) \to H^*(X) \otimes R_*$. If $X = B\boldsymbol{Z}/p$, we get the following isomorphisms

$$H^*(B\boldsymbol{Z}/p) \otimes R_* \cong E(\epsilon) \otimes R_*[[x]] \cong R_*[\epsilon]/(\epsilon^2)[[x]].$$

For $\gamma(\epsilon) \in [H^*(B\boldsymbol{Z}/p) \otimes R_*]^1$ and $\gamma(x) \in [H^*(B\boldsymbol{Z}/p) \otimes R_*]^2$, we define an element $f_\gamma(x)$ in $R_*[\epsilon]/(\epsilon^2)[[x]]$ as $f_\gamma(x) = \epsilon\gamma(\epsilon) + \gamma(x)$.

Lemma 5.1. *$f_\gamma(x)$ is a quasi-strict automorphism of g_a over $R_*[\epsilon]/(\epsilon^2)$. In other words, $f_\gamma(x)$ is an element in $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$.*

Proof. Since a multiplicative operation γ preserves degree, $f_\gamma(x)$ satisfies the condition (iii) in Definition 4.1.

From the commutative diagram

$$\begin{array}{ccc} H^*(B\boldsymbol{Z}/p) \otimes H^*(B\boldsymbol{Z}/p) & \xrightarrow{\gamma\otimes\gamma} & H^*(B\boldsymbol{Z}/p) \otimes R_* \otimes H^*(B\boldsymbol{Z}/p) \otimes R_* \\ \times\downarrow & & \downarrow((\times)\otimes m)\circ(1\otimes\mu\otimes 1) \\ H^*(B\boldsymbol{Z}/p \times B\boldsymbol{Z}/p) & \xrightarrow{\gamma} & H^*(B\boldsymbol{Z}/p \times B\boldsymbol{Z}/p) \otimes R_*, \end{array}$$

we have

$$\begin{aligned} \gamma(\epsilon_1 + \epsilon_2) &= \gamma(\epsilon \times 1 + 1 \times \epsilon) \\ &= ((\times) \otimes m) \circ (1 \otimes \mu \otimes 1)(\gamma(\epsilon) \otimes \gamma(1) + \gamma(1) \otimes \gamma(\epsilon)) = \gamma(\epsilon_1) + \gamma(\epsilon_2), \\ \gamma(x_1 + x_2) &= \gamma(x \times 1 + 1 \times x) \\ &= ((\times) \otimes m) \circ (1 \otimes \mu \otimes 1)(\gamma(x) \otimes \gamma(1) + \gamma(1) \otimes \gamma(x)) = \gamma(x_1) + \gamma(x_2). \end{aligned}$$

It follows from the above equalities that $f_\gamma(x_1 + x_2) = f_\gamma(x_1) + f_\gamma(x_2)$, and thereby $f_\gamma(x)$ satisfies the condition (i) in Definition 4.1.

It remains to show that $f_\gamma(x)$ satisfies the condition (ii) in Definition 4.1. Since $f_\gamma(x)$ satisfies the conditions (i) and (iii) in Definition 4.1, we obtain the form

$$f_\gamma(x) = \sum_{k=0}^{\infty} (\epsilon a_k + b_k) x^{p^k}, \qquad a_k \in R_{2p^k-1},\ b_k \in R_{2p^k-2}.$$

It is enough to prove $b_0 = 1$. Let z be the element in $H^2(BS^1)$ which satisfies $j^*(z) = x$ for the inclusion $j : B\boldsymbol{Z}/p \to BS^1$. Then we see that $H^*(BS^1) \cong \boldsymbol{F}_p[z]$, and that j^* is injective. Moreover we can write $\gamma(z)$ as $\gamma(z) = \sum_{k=0}^{\infty} c_k z^{p^k}$. From the commutative diagram

$$\begin{array}{ccc} H^*(BS^1) & \xrightarrow{\gamma} & H^*(BS^1) \otimes R_* \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*\otimes 1} \\ H^*(B\boldsymbol{Z}/p) & \xrightarrow{\gamma} & H^*(B\boldsymbol{Z}/p) \otimes R_*, \end{array}$$

we have

$$\sum_{k=0}^{\infty} c_k x^{p^k} = (j^* \otimes 1) \circ \gamma(z) = \gamma \circ j^*(z) = \gamma(x).$$

By the definition of $f_\gamma(x)$ and the preceding equality, we see that $b_0 = 1$ is equivalent to $c_0 = 1$.

Let $l : S^2 \to BS^1$ be the inclusion, and u the element in $H^2(S^2)$ which satisfies $l^*(z) = u$. By Definition 2.1 (ii) and Lemma 2.2, we have $\gamma(u) = u \otimes 1$. From the commutative diagram

$$\begin{array}{ccc} H^2(BS^1) & \xrightarrow{\gamma} & [H^*(BS^1) \otimes R_*]^2 \\ {\scriptstyle l^*}\downarrow & & \downarrow{\scriptstyle l^*\otimes 1} \\ H^2(S^2) & \xrightarrow{\gamma} & [H^*(S^2) \otimes R_*]^2, \end{array}$$

we obtain

$$u \otimes c_0 = (l^* \otimes 1) \circ \gamma(z) = \gamma \circ l^*(z) = \gamma(u) = u \otimes 1.$$

Hence $c_0 = 1$. This completes the proof of the lemma. $\square$

By this lemma, we can define a natural transformation $F : \mathrm{Op}(-) \to \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ by $F(\gamma) = f_\gamma(x)$. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Op}(-) & \xrightarrow{F} & \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-) \\ {\scriptstyle \lambda}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle T} \\ \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(H_*H, -) & \xrightarrow{N} & \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, -). \end{array} \tag{5.1}$$

Here $N = T^{-1} \circ F \circ \lambda^{-1}$. We write χ_γ for $T^{-1} \circ F(\gamma) \in \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*, R_*)$. We obtain two algebra homomorphisms $\chi_\psi : A_* \to H_*H$ and $\chi_{S_n} : A_* \to D[n]_*$ from the multiplicative operations $\psi : H^*(X) \to H^*(X) \otimes H_*H$ in (2.2) and $S_n : H^*(X) \to H^*(X) \otimes D[n]_*$ in

Lemma 3.10, respectively. From Theorem 2.3, we see $N(\mathrm{id}_{H_*H}) = \chi_\psi$, where id_{H_*H} is the identity map of H_*H. Since N is a natural transformation, we have

$$N(l) = N(l \circ \mathrm{id}_{H_*H}) = l \circ N(\mathrm{id}_{H_*H}) = l \circ \chi_\psi$$

for any graded algebra homomorphism $l : H_*H \to R_*$. From the commutative diagram (5.1) and the above equality, we see

$$\chi_{S_n} = T^{-1} \circ F(S_n) = N \circ \lambda(S_n) = N(\lambda(S_n)) = \lambda(S_n) \circ \chi_\psi,$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} A_* & \xrightarrow{\chi_\psi} & H_*H \\ & \searrow^{\chi_{S_n}} & \downarrow{\scriptstyle \lambda(S_n)} \\ & & D[n]_*. \end{array} \tag{5.2}$$

The map $H \wedge S^0 \wedge H \overset{1\wedge i\wedge 1}{\to} H \wedge H \wedge H$ induces the coproduct map

$$\delta : H_*H = \{S^0, H \wedge H\}_* \longrightarrow \{S^0, H \wedge H \wedge H\}_* \cong H_*H \otimes H_*H.$$

Then H_*H is a Hopf algebra and $H^*(X)$ is an H_*H-comodule with $\psi : H^*(X) \to H^*(X) \otimes H_*H$.

The following is the main theorem.

Theorem 5.2. *$\chi_\psi = N(\mathrm{id}_{H_*H}) : A_* \longrightarrow H_*H$ is a Hopf algebra isomorphism.*

Proof. By Corollary 3.12, we have

$$f_{S_n}(x) = \epsilon S_n(\epsilon) + S_n(x) = \sum_{i=0}^{n-1} \epsilon\tau[n]_i x^{p^i} + \left(x + \sum_{j=1}^{n} \xi[n]_i x^{p^i}\right) = \sum_{i=0}^{n} (\epsilon\tau[n]_i + \xi[n]_i) x^{p^i},$$

where $\xi[n]_0 = 1$ and $\tau[n]_n = 0$. From the definition of χ_{S_n} and T, we see

$$\begin{aligned} &\chi_{S_n}(\bar\tau_i) = \tau[n]_i \quad (0 \le i \le n-1), && \chi_{S_n}(\bar\tau_i) = 0 \quad (i \ge n), \\ &\chi_{S_n}(\bar\xi_i) = \xi[n]_i \quad (1 \le i \le n), && \chi_{S_n}(\bar\xi_i) = 0 \quad (i > n). \end{aligned}$$

In consequence, $\chi_{S_n} : H_*H \to D[n]_*$ is an isomorphism for $* \le 2p^n - 2$, which becomes arbitrarily large. This and the commutative diagram (5.2) imply χ_ψ is injective. Cartan [**3**] showed that the Poincaré series of H_*H is equal to

$$\prod_{i=1}^{\infty} \frac{1 + t^{2p^{i-1}-1}}{1 - t^{2p^i - 2}}.$$

The Poincaré series of A_* and that of H_*H are the same, and hence χ_ψ is bijective.

We need to show that χ_ψ is a Hopf algebra homomorphism. Since ψ is an H_*H-comodule map, we have

$$(\psi \otimes 1) \circ \psi = (1 \otimes \delta) \circ \psi : H^*(X) \longrightarrow H^*(X) \otimes H_*H \otimes H_*H.$$

It is not difficult to see that χ_ψ is a Hopf algebra homomorphism. (See [**6**, Theorem 4.1] for details.) □

Appendix A. Higher dimensional graded formal group laws.

We recall higher dimensional commutative formal group laws in Hazewinkel [**5**]. For convenience, we abbreviate commutative formal group laws to formal group laws.

Definition A.1. An *n-dimensional formal group law* over a ring A is an n-tuple of power series $F(X,Y) = (F(1)(X,Y), \dots, F(n)(X,Y))$ in $2n$ indeterminates $X_1, \dots, X_n; Y_1, \dots, Y_n$ such that

(i) $F(i)(X,Y) \equiv X_i + Y_i \mod (X_1, \dots, X_n, Y_1, \dots, Y_n)^2, \quad i = 1, \dots, n;$
(ii) $F(i)(F(i)(X,Y), Z) = F(i)(X, F(i)(Y,Z)), \quad i = 1, \dots, n;$
(iii) $F(i)(X,Y) = F(i)(Y,X), \quad i = 1, \dots, n.$

Definition A.2. Let $F(X,Y)$ be an n-dimensional formal group law over a ring A and $G(X',Y')$ an m-dimensional formal group law over A. A *homomorphism* over A, $F(X,Y) \to G(X',Y')$ is an m-tuple of formal power series $\alpha(X)$ in n indeterminates $X_1, \dots, X_n$ such that $\alpha(X) \equiv 0 \mod (X_1, \dots, X_n)$ and $\alpha(F(X,Y)) = G(\alpha(X), \alpha(Y))$.

We introduce higher dimensional graded formal group laws over a graded $\boldsymbol{F}_p$-algebra and homomorphisms between them.

Definition A.3. Suppose that α_i is odd for $0 \le i \le s$, and that α_j is even for $s+1 \le j \le n$. Let X_j and Y_j be indeterminates of degree α_j. Then an n-tuple of elements $F(X,Y) = (F(1)(X,Y), \dots, F(n)(X,Y))$ in

$$E(X_1, \dots, X_s; Y_1, \dots, Y_s) \otimes R_*[[X_{s+1}, \dots, X_n; Y_{s+1}, \dots, Y_n]]$$

is called an *n-dimensional graded formal group law* over a graded F_p-algebra R_* if it satisfies the following conditions:

(i) $F(i)$ is a homogeneous formal power series of degree α_i, i.e., $\deg t_{I,I'} X^I Y^{I'} = \alpha_i$ if $F(i) = \sum t_{I,I'} X^I Y^{I'}$, where $X^I = X_1^{k_1} \cdots X_n^{k_n}$, $Y^{I'} = Y_1^{k'_1} \cdots Y_n^{k'_n}$ and $t_{I,I'} \in R_*$ for $I = (k_1, \dots, k_n)$ and $I' = (k'_1, \dots, k'_n)$;
(ii) $F(i)(X,Y) \equiv X_i + Y_i \mod (X_1, \dots, X_n, Y_1, \dots, Y_n)^2$;
(iii) $F(i)(F(i)(X,Y), Z) = F(i)(X, F(i)(Y,Z))$;
(iv) $F(i)(X,Y) = F(i)(Y,X)$.

In particular, we define a graded formal group law $G_a(X,Y)$ by $G_a(i)(X,Y) = X_i + Y_i$, which is called a *graded additive formal group law*.

DEFINITION A.4. Suppose that α_i is odd for $1 \leq i \leq s$, and that α_i is even for $s+1 \leq i \leq n$. Suppose that β_j is odd for $1 \leq j \leq s'$, and that β_j is even for $s'+1 \leq j \leq m$. Given indeterminates X_i, Y_i and X'_j, Y'_j such that $\deg X_i = \deg Y_i = \alpha_i$ and $\deg X'_j = \deg Y'_j = \beta_j$, let $F(X,Y)$ be an n-dimensional graded formal group law with X_i, Y_i over a graded $\boldsymbol{F}_p$-algebra R_*, and $G(X', Y')$ an m-dimensional graded formal group law with X'_j, Y'_j over R_*. Then a *homomorphism* $f(X) : F(X,Y) \to G(X',Y')$ is an m-tuple of elements $f(X) = (f(1)(X), \dots, f(m)(X))$ in

$$E(X_1, \dots, X_s) \otimes R_*[[X_{s+1}, \dots, X_n]]$$

which satisfies the following conditions:

(i) $f(i)(X)$ is a homogeneous formal power series of degree β_i, i.e., $\deg t_I X^I = \beta_i$ if $F(i) = \sum t_I X^I$, where $X^I = X_1^{k_1} \cdots X_n^{k_n}$, and $t_I \in R_*$ for $I = (k_1, \cdots, k_n)$;
(ii) $f(X) \equiv 0 \mod (X_1, \dots, X_n)$;
(iii) $f(F(X,Y)) = G(f(X), f(Y))$.

A homomorphism $f(X) : F \to G$ is called an *isomorphism* if there exists a homomorphism $g(X') : G \to F$ such that $f(g(X')) = X'$ and $g(f(X)) = X$. Let $J(f)$ be the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

where $f(i)(X) \equiv a_{i1}X_1 + \cdots + a_{in}X_n \mod (X_1, \dots, X_n)^2$. Note that $J(f)$ is a matrix over a graded algebra. We can easily see that $f(X)$ is an isomorphism if and only if $J(f)$ is invertible. Suppose that $J(f)$ is an upper triangular matrix with all diagonal entries 1, i.e.

$$\begin{pmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Then we see that $J(f)$ is invertible, and $f(X)$ is called a *quasi-strict* isomorphism.

We now consider a 2-dimensional graded additive formal group law $G_a(\epsilon_1, x_1; \epsilon_2, x_2)$ with $\deg \epsilon_i = 1$ and $\deg x_i = 2$. Write $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$ for the set of all quasi-strict automorphisms of $G_a(\epsilon_1, x_1; \epsilon_2, x_2)$ over a graded $\boldsymbol{F}_p$-algebra R_*. Obviously $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ is a functor from the category of graded algebras to the category of sets. By the definition of quasi-strict automorphisms, an element $f(X) = (f(1)(\epsilon, x), f(2)(\epsilon, x))$ in $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$ satisfies the following conditions:

$$f(1)(\epsilon, x),\ f(2)(\epsilon, x) \in E(\epsilon) \otimes R_*[[x]]; \tag{A.1}$$

$$\deg \epsilon = 1,\ \deg x = 2,\ \deg f(1) = 1,\ \deg f(2) = 2; \tag{A.2}$$

$$f(1)(\epsilon_1+\epsilon_2, x_1+x_2) = f(1)(\epsilon_1, x_1) + f(1)(\epsilon_2, x_2); \tag{A.3}$$

$$f(2)(\epsilon_1+\epsilon_2, x_1+x_2) = f(2)(\epsilon_1, x_1) + f(2)(\epsilon_2, x_2); \tag{A.4}$$

$$f(1)(\epsilon, x) \equiv \epsilon + a_0 x,\ f(2)(\epsilon, x) \equiv x \mod (\epsilon, x)^2. \tag{A.5}$$

We express a quasi-strict automorphism $f(X) = (f(1)(\epsilon, x), f(2)(\epsilon, x))$ as

$$f(1)(\epsilon, x) = \sum_{i=0}^{\infty} \big(\epsilon m_i + m_i'\big) x^i, \quad f(2)(\epsilon, x) = \sum_{i=1}^{\infty} \big(\epsilon n_i + n_i'\big) x^i,$$

where $m_0 = 1$, $m_0' = 0$, $n_1' = 1$, $m_i \in R_{2i}$, $m_i' \in R_{2i-1}$, $n_i \in R_{2i-1}$, and $n_i' \in R_{2i-2}$. From the conditions (A.3) and (A.4), we see

$$\sum_{i=0}^{\infty} \big((\epsilon_1+\epsilon_2)m_i + m_i'\big)(x_1+x_2)^i = \sum_{i=0}^{\infty} \big(\epsilon_1 m_i + m_i'\big) x_1^i + \sum_{i=0}^{\infty} \big(\epsilon_2 m_i + m_i'\big) x_2^i,$$

$$\sum_{i=1}^{\infty} \big((\epsilon_1+\epsilon_2)n_i + n_i'\big)(x_1+x_2)^i = \sum_{i=1}^{\infty} \big(\epsilon_1 n_i + n_i'\big) x_1^i + \sum_{i=1}^{\infty} \big(\epsilon_2 n_i + n_i'\big) x_2^i.$$

If $i \geq 1$, then

$$(\epsilon_1+\epsilon_2)(x_1+y_1)^i = \epsilon_1 x_1^i + \epsilon_2 x_2^i + \epsilon_1 x_2^i + \epsilon_2 x_1^i + A,$$

where A is a polynomial. This implies $m_i = n_i = 0$ for $i \geq 1$. If $i = p^\alpha$, then $(x_1+x_2)^i = x_1^i + x_2^i$, and if $i \neq p^\alpha$, then $(x_1+x_2)^i = x^i + y^i + xyB$, where B is a non-zero polynomial. Therefore we have $m_i = 0$ and $n_i = 0$ for $i \neq p^\alpha$. These show that the conditions (A.1)–(A.5) are equivalent to

$$\begin{aligned} f(1)(\epsilon, x) &= \epsilon + a_0 x + a_1 x^p + \cdots + a_n x^{p^n} + \cdots \\ f(2)(\epsilon, x) &= \qquad\quad x + b_1 x^p + \cdots + b_n x^{p^n} + \cdots, \end{aligned}$$

where $a_i \in R_{2p^i-1}$ and $b_i \in R_{2p^i-2}$. We put $\hat{A}_* = E(\hat{\tau}_0, \hat{\tau}_1, \dots) \otimes \boldsymbol{F}_p[\hat{\xi}_1, \hat{\xi}_2, \dots]$, where $\hat{\tau}_i \in \hat{A}_{2p^i-1}$ and $\hat{\xi}_i \in \hat{A}_{2p^i-2}$. Define a natural map

$$\hat{T} : \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(\hat{A}_*, R_*) \longrightarrow \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$$

by

$$h \mapsto f(X) = (f(1)(\epsilon, x), f(2)(\epsilon, x)) = \left(\epsilon + \sum_{i=0}^{\infty} h(\hat{\tau}_i) x^{p^i}, x + \sum_{i=1}^{\infty} h(\hat{\xi}_i) x^{p^i}\right).$$

Obviously $\hat{T}$ is an isomorphism of sets. A product of $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$ is defined by the composition $(f \cdot g)(X) = g(f(X))$, i.e.,

$$(f \cdot g)(X) = ((f \cdot g)(1)(X), (f \cdot g)(2)(X))$$
$$= (g(1)(f(1)(\epsilon, x), f(2)(\epsilon, x)), g(2)(f(1)(\epsilon, x), f(2)(\epsilon, x))).$$

We see that $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$ is a group, and therefore $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ is a functor to the category of groups. Then there exists a unique coproduct $\hat{\Delta} : \hat{A}_* \to \hat{A}_* \otimes \hat{A}_*$ such that $\hat{T}$ is a group isomorphism. We express a couple of elements $f(X) = (f(1)(X), f(2)(X))$ and $g(X) = (g(1)(X), g(2)(X))$ in $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$ as

$$f(1)(X) = \epsilon + \sum_{j=0}^{\infty} a'_j x^{p^j}, \quad f(2)(X) = \sum_{j=0}^{\infty} b'_j x^{p^j}, \qquad b'_0 = 1;$$
$$g(1)(X) = \epsilon + \sum_{k=0}^{\infty} a''_k x^{p^k}, \quad g(2)(X) = \sum_{k=0}^{\infty} b''_k x^{p^k}, \qquad b''_0 = 1.$$

Then we can describe the product $(f \cdot g)(X)$ as

$$(f \cdot g)(1)(X) = \epsilon + \sum_{i=0}^{\infty} a_i x^{p^i} = \left(\epsilon + \sum_{j=0}^{\infty} a'_j x^{p^j}\right) + \sum_{k=0}^{\infty} a''_k \left(\sum_{j=0}^{\infty} b'_j x^{p^j}\right)^{p^k}$$
$$= \epsilon + \sum_{i=0}^{\infty} \left(a'_i + \sum_{k=0}^{i} {b'}^{p^k}_{i-k} a''_k\right) x^{p^i},$$
$$(f \cdot g)(2)(X) = \sum_{i=0}^{\infty} b_i x^{p^i} = \sum_{k=0}^{\infty} b''_k \left(\sum_{j=0}^{\infty} b'_j x^{p^j}\right)^{p^k} = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} {b'_{i-k}}^{p^k} b''_k\right) x^{p^i}.$$

These imply that

$$\hat{\Delta}(\hat{\tau}_i) = \hat{\tau}_i \otimes 1 + \sum_{k=0}^{i} \hat{\xi}_{i-k}^{p^k} \otimes \hat{\tau}_k, \quad \hat{\Delta}(\hat{\xi}_i) = \sum_{k=0}^{i} \hat{\xi}_{i-k}^{p^k} \otimes \hat{\xi}_k.$$

Therefore we have the following theorem.

Theorem A.5. *Let $\hat{A}_*$ be the Hopf algebra $E(\hat{\tau}_0, \hat{\tau}_1, \dots) \otimes \boldsymbol{F}_p[\hat{\xi}_1, \hat{\xi}_2, \dots]$ whose coproduct is given by*

$$\hat{\Delta}(\hat{\tau}_i) = \hat{\tau}_i \otimes 1 + \sum_{k=0}^{i} \hat{\xi}_{i-k}^{p^k} \otimes \hat{\tau}_k, \quad \hat{\Delta}(\hat{\xi}_i) = \sum_{k=0}^{i} \hat{\xi}_{i-k}^{p^k} \otimes \hat{\xi}_k.$$

Then $\hat{T}$ is a natural isomorphism of groups.

We can prove the main theorem by the usage of $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ instead of $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$ as follows. Let $\gamma : H^*(X) \to H^*(X) \otimes R_*$ be a multiplicative operation. For $X = B\boldsymbol{Z}/p$, we have

$$\gamma : H^*(B\boldsymbol{Z}/p) \cong E(\epsilon) \otimes \boldsymbol{F}_p[x] \longrightarrow H^*(B\boldsymbol{Z}/p) \otimes R_* \cong E(\epsilon) \otimes R_*[[x]].$$

We define $\hat{f}_\gamma(X) = (\hat{f}_\gamma(1)(\epsilon, x),\ \hat{f}_\gamma(2)(\epsilon, x))$ by $\hat{f}_\gamma(1)(\epsilon, x) = \gamma(\epsilon)$ and $\hat{f}_\gamma(2)(\epsilon, x) = \gamma(x)$. We can prove that $\hat{f}_\gamma(X)$ is a quasi-strict automorphism of G_a over R_* in a way similar to the proof of Lemma 5.1. Let $\hat{F} : \mathrm{Op}(-) \to \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ be the natural transformation which sends γ to $\hat{f}_\gamma(X)$. As in Section 5, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Op}(-) & \xrightarrow{\hat{F}} & \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-) \\
\lambda \big\downarrow \cong & & \cong \big\uparrow \hat{T} \\
\mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(H_*H, -) & \xrightarrow{\hat{N}} & \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(\hat{A}_*, -).
\end{array} \tag{A.6}$$

Here $\hat{N} = \hat{T}^{-1} \circ \hat{F} \circ \lambda^{-1}$. We put

$$\hat{\chi}_\psi = \hat{N}(\mathrm{id}_{H_*H}) : \hat{A}_* \to H_*H,$$

where id_{H_*H} is the identity map of H_*H. In a way similar to the proof of Theorem 5.2, we can show the following theorem.

THEOREM A.6. *$\hat{\chi}_\psi$ is a Hopf algebra isomorphism.*

Now we have the three natural operations $\mathrm{Op}(-)$, $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ and $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$. The Hopf algebras H_*H, $\hat{A}_*$ and A_* represent them, respectively. We want to investigate relations among them. First we construct a natural transformation

$$W : \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-) \to \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-),$$

and see relations among $\mathrm{Op}(-)$, $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-)$ and $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-)$. Given an element

$$f(X) = (f(1)(X), f(2)(X))$$

in $\mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(R_*)$, we put $W(f(X)) = \epsilon f(1)(X) + f(2)(X)$. It is well defined since $\epsilon f(1)(X) + f(2)(X)$ is an element in $\mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(R_*)$. Moreover W is an isomorphism. By the definition of F in Section 5, the following diagram is commutative:

$$\begin{array}{ccc}
 & & \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-) \\
 & \overset{F}{\nearrow} & \big\uparrow \\
\mathrm{Op}(-) & & \big| \; W \\
 & \underset{\hat{F}}{\searrow} & \big| \\
 & & \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-).
\end{array} \tag{A.7}$$

Next we study relations among H_*H, A_* and $\hat{A}_*$. Consider the Hopf algebra isomorphism $W' : A_* \to \hat{A}_*$ given by $\bar{\tau}_i \mapsto \hat{\tau}_i$ and $\bar{\xi}_i \mapsto \hat{\xi}_i$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Aut}_{\boldsymbol{F}_p}(G_a)(-) & \xrightarrow{W} & \mathrm{AUT}_{\boldsymbol{F}_p}(g_a)(-) \\ {\scriptstyle \hat{T}}\uparrow\ \cong & & \cong\ \uparrow{\scriptstyle T} \\ \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(\hat{A}_*,-) & \xrightarrow{W'^*} & \mathrm{Hom}_{\boldsymbol{F}_p\text{-alg}}(A_*,-). \end{array} \tag{A.8}$$

From the commutative diagrams (5.1), (A.6), (A.7) and (A.8), we obtain a commutative diagram of isomorphisms

$$\begin{array}{ccc} A_* & & \\ & \searrow^{\chi_\psi} & \\ {\scriptstyle W'}\downarrow & & H_*H \\ & \nearrow_{\hat{\chi}_\psi} & \\ \hat{A}_*. & & \end{array}$$

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