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Topological proof of Bott periodicity and characterization of *BR*

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Abstract. We give another proof of $(1,1)$-periodicity of M. F. Atiyah's KR-theory and the characterization of the classifying space of KR-theory.

1. Introduction.

Recall that a *real* space is a space with an involution and that a *real* vector bundle E is a complex vector bundle over a *real* space X equipped with an involutive conjugate linear automorphism of E over the given involution of the base real space X ([**2**]). M. F. Atiyah defined KR-theory as the Grothendieck group of the monoid of *real* vector bundles. By its nature there are natural transformations from KR-theory to KU-theory and KO-theory. KR-theory can be regarded as $\boldsymbol{Z}_2$-equivariant K-theory with action of $\boldsymbol{Z}_2$ given by a conjugate linear automorphism. KR-theory is known to be representable as are KU-theory and KO-theory. The classifying space, BR, of KR-theory is a real space BU with involution the conjugation of BU. As BU and BO have the periodicity, so does BR. The $\boldsymbol{Z}_2$-equivariant periodicity of BR,

$$\boldsymbol{Z} \times BR \simeq_{\boldsymbol{Z}_2} \Omega^{1,1}(\boldsymbol{Z} \times BR),$$

is called the $(1,1)$-periodicity.

The purpose of this paper is to give another proof of the $(1,1)$-periodicity of BR and to characterize BR by a topological way. This paper is also the generalization of [**4**] to the $\boldsymbol{Z}_2$-equivariant case. We prove the $(1,1)$-periodicity of certain spaces in Theorem 1 and of BR in Theorem 2, and characterize BR in Theorem 3.

2. τ-space.

According to [**1**] we call a space with an involution by a τ-space, which is a $\boldsymbol{Z}_2$-equivariant space and also a *real* space in [**2**]. The involution of a τ-space is denoted by τ. A pointed τ-space is a τ-space with a base point which is a fixed point of τ. A $\boldsymbol{Z}_2$-equivariant map between τ-spaces is called a τ-map. Let Top_0 be the category of pointed spaces and base point preserving maps, and Top_0^τ be the category of pointed τ-spaces and base point preserving τ-maps. A homotopy and a Hopf space in Top_0^τ are called a τ-homotopy and a Hopf τ-space. A τ-homotopy and the set of τ-homotopy classes are denoted by $\simeq_\tau$ and $[\ ,\ ]^\tau$.

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We consider two functors from Top_0^τ to Top_0 called the forgetful functor and the fixed point functor. The forgetful functor

$$\psi : Top_0^\tau \to Top_0$$

is to forget involutions and the fixed point functor

$$\phi : Top_0^\tau \to Top_0$$

is to restrict to fixed point sets of τ. We denote the natural inclusion $\phi(X) \hookrightarrow \psi(X)$ by i_X. A τ-complex X is a CW-complex in Top_0^τ such that $\phi(X)$ is a subcomplex of $\psi(X)$.

Let $\boldsymbol{R}^{p,q}$ be the τ-space of $\boldsymbol{R}^{p+q}$ with the involution

$$\tau(x_1, \ldots, x_p, x_{p+1}, \ldots\ldots, x_{p+q}) = (-x_1, \ldots, -x_p, x_{p+1}, \ldots, x_{p+q}),$$

and $\Sigma^{p,q}$ be the pointed τ-space of the one point compactification of $\boldsymbol{R}^{p,q}$ with the base point ∞. Let $\Omega^{p,q}X$ be $\mathrm{Hom}_{Top_0}((\Sigma^{p,q}, \infty), (X, x_0))$ for a pointed τ-space X with a base point x_0, then $\Omega^{p,q}X$ comes to be a pointed τ-space with the constant map as a base point and

$$\tau(f)(x) = \tau(f(\tau(x)))$$

as an involution for $f \in \Omega^{p,q}X, x \in X$. Let X, Y be pointed τ-spaces, then we have the canonical isomorphism as follows.

$$[\Sigma^{p,q} \wedge X, Y]^\tau \cong [X, \Omega^{p,q}Y]^\tau$$

By the isomorphism above we have the adjoint map of a base point preserving τ-map $f : \Sigma^{p,q} \wedge X \to Y$ denoted by $\mathrm{Ad}^{p,q}f : X \to \Omega^{p,q}Y$, which is a base point preserving τ-map.

Lemma 2.1. *Let X, Y be pointed τ-spaces and $f : \Sigma^{1,0} \wedge X \to Y$ be a base point preserving τ-map, then we have the following.*

1. *$\phi(i) : \phi(X) \to \phi(\Sigma^{1,0} \wedge X)$ is a homeomorphism, where $i : X \to \Sigma^{1,0} \wedge X$ is the natural inclusion.*
2. *We have the following commutative diagram, where $\mathrm{ev}_0 : \phi(\Omega^{1,0}X) \to \phi(X)$ is the evaluation at 0.*

$$\begin{array}{ccc} \phi(X) & \xrightarrow{\phi(i)} & \phi(\Sigma^{1,0} \wedge X) \\ {\scriptstyle \phi(\mathrm{Ad}^{1,0}f)}\downarrow & & \downarrow{\scriptstyle \phi(f)} \\ \phi(\Omega^{1,0}Y) & \xrightarrow[\mathrm{ev}_0]{} & \phi(Y) \end{array}$$

3. *$\psi(\Omega^{1,0}X) \to \phi(\Omega^{1,0}X) \xrightarrow{\mathrm{ev}_0} \phi(X)$ is the fibration which is the pullback of the path fibration over $\psi(X)$ by i_X.*

Proof. 1, 2 is trivial and the following commutative diagram shows 3, where $*$ is the base point of X.

$$\begin{array}{ccccc}
\psi(\Omega^{1,0}X) & \longrightarrow & \phi(\Omega^{1,0}X) & \xrightarrow{\mathrm{ev}_0} & \phi(X) \\
\wr\downarrow & & \wr\downarrow & & \| \\
\Omega X & \longrightarrow & \mathrm{Map}(([0,1],0,1),(\psi(X),\phi(X),*)) & \xrightarrow{\mathrm{ev}_0} & \phi(X) \\
\| & & \downarrow & & \downarrow i_X \\
\Omega X & \longrightarrow & \mathrm{Map}(([0,1],0,1),(\psi(X),\psi(X),*)) & \xrightarrow{\mathrm{ev}_0} & \psi(X)
\end{array}$$ □

3. (1,1)-periodic τ-space.

In this section we prove the $(1,1)$-periodicity of the special kind of τ-spaces satisfying some common topological properties with BR.

Let $\boldsymbol{CP}_\tau^n$ be the τ-space of $\boldsymbol{CP}^n$ with the conjugation as the involution and $\boldsymbol{Z}$ be the pointed τ-space with 0 as the base point and the trivial involution. It is obvious that $\phi(\boldsymbol{CP}_\tau^n) = \boldsymbol{R}P^n$ and $\boldsymbol{CP}_\tau^1 = \Sigma^{1,1}$. The pointed τ-space of $\boldsymbol{CP}_\tau^n$ with a disjoint point as the base point is denoted by $\boldsymbol{CP}_{\tau\,+}^n$.

Theorem 1. *Let X be a Hopf τ-space which is a τ-complex of finite type and let*

$$\lambda : \Sigma^{1,1} \wedge (\boldsymbol{Z} \times X) \to \boldsymbol{Z} \times X,\ j_+ : \boldsymbol{CP}_{\tau\,+}^\infty \to \boldsymbol{Z} \times X \text{ and } \alpha : \psi(X) \to \phi(X)$$

be base point preserving τ-maps such that $j_+(\boldsymbol{CP}_\tau^\infty) \subset 1 \times X$ and a continuous map respectively. Suppose X, λ, j_+, α satisfy the following.

1. *$H^*(\psi(X); \boldsymbol{Z}) = \boldsymbol{Z}[x_1, x_2, x_3, \ldots]$ and $H^*(\phi(X); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_1, y_2, y_3, \ldots]$, where $|x_i| = 2i, |y_i| = i$ and $i_X^*(x_i) = y_i^2$.*
2. *$\psi(\lambda(1 \wedge j_+))^* : H^*(\psi(\boldsymbol{Z} \times X); \boldsymbol{Z}) \to H^*(\psi(\Sigma^{1,1} \wedge \boldsymbol{CP}_{\tau\,+}^\infty); \boldsymbol{Z})$ and $\phi(\lambda(1 \wedge j_+))^* : H^*(\phi(\boldsymbol{Z} \times X); \boldsymbol{Z}_2) \to H^*(\phi(\Sigma^{1,1} \wedge \boldsymbol{CP}_{\tau\,+}^\infty); \boldsymbol{Z}_2)$ are epic.*
3. *$\alpha i_X \simeq \phi(\mu)\triangle$ and $i_X\alpha \simeq \psi(\mu)(1 \times \tau)\triangle$, where μ is the multiplication of X and $\triangle$ is the diagonal map.*
4. *$\mathrm{Ad}^{1,1}\lambda$ is a Hopf τ-map.*

Then we have:

$$\mathrm{Ad}^{1,1}\lambda : \boldsymbol{Z} \times X \simeq_\tau \Omega^{1,1}(\boldsymbol{Z} \times X).$$

For the rest of this paper the notations X, λ, j_+, α are fixed to those in Theorem 1.

In order to show the periodicity of $\psi(\boldsymbol{Z} \times X)$, we need the following proposition [**4**, Theorem 2.1].

Proposition 3.1. *Let Y be a Hopf space which is a CW-complex of finite type such that*

$$H^*(Y; \boldsymbol{Z}) = \boldsymbol{Z}[c_1, c_2, c_3, \ldots],\ |c_i| = 2i.$$

Suppose that we have maps

$$j : \boldsymbol{C}P^\infty \to Y \text{ and } \kappa : \Sigma^2 Y \to Y$$

which satisfy the following.

1. $(\kappa(1 \wedge j))^* : H^*(Y;\boldsymbol{Z}) \to H^*(\Sigma^2\boldsymbol{C}P^\infty;\boldsymbol{Z})$ *is epic.*
2. $\mathrm{Ad}^2\tilde{\kappa} : Y \to \Omega^2(Y\langle 2\rangle)$ *is a Hopf map, where* $Y\langle 2\rangle$ *is the* 2*-connected fibre space of* Y *and* $\tilde{\kappa}$ *is the lift of* κ*.*

Then we have

$$\mathrm{Ad}^2\tilde{\kappa} : Y \xrightarrow{\sim} \Omega^2(Y\langle 2\rangle).$$

Lemma 3.1.

$$\psi(\mathrm{Ad}^{1,1}\lambda) : \psi(\boldsymbol{Z} \times X) \xrightarrow{\sim} \psi(\Omega^{1,1}(\boldsymbol{Z} \times X))$$

Proof. Since $\psi(X)$, $\psi(\lambda|_{\Sigma^{1,1}\wedge X})$ and $\psi(j_+|_{\boldsymbol{C}P^\infty_\tau})$ satisfy the conditions in Proposition 3.1, we have:

$$\psi(\mathrm{Ad}^{1,1}\lambda) : \psi(X) \xrightarrow{\sim} \psi(\Omega^{1,1}(X))_0,$$

where $\psi(\Omega^{1,1}(X))_0$ is the path component of the constant maps. By computing the Leray-Serre spectral sequence of the path fibration over $\psi(\boldsymbol{Z} \times X)$ and by Theorem 1.2, we have:

$$\psi(\mathrm{Ad}^{0,1}\lambda(1 \wedge j_+))^* : H^1(\psi(\Omega^{0,1}(\boldsymbol{Z} \times X));\boldsymbol{Z}) \xrightarrow{\sim} H^1(\psi(\Sigma^{1,0} \wedge \boldsymbol{C}P^\infty_{\tau\,+});\boldsymbol{Z}).$$

Since $\psi(\mathrm{Ad}^{1,1}\lambda)$ is a Hopf map and $j_+(\boldsymbol{C}P^\infty) \subset 1 \times X$, we have:

$$\psi(\mathrm{Ad}^{1,1}\lambda)_* : \pi_0(\psi(\boldsymbol{Z} \times X)) \xrightarrow{\sim} \pi_0(\psi(\Omega^{1,1}(\boldsymbol{Z} \times X))). \qquad \square$$

Let $\lambda_n : \Sigma^{1,1} \wedge \ldots \wedge \Sigma^{1,1} = \Sigma^{n,n} \to X$ be the following base point preserving τ-map.

$$\lambda(1 \wedge \lambda) \cdots (1 \wedge \ldots \wedge \lambda)(1 \wedge \ldots \wedge j_+|_{\boldsymbol{C}P^1_\tau})$$

Corollary 3.1. $\pi_{2n-1}(\psi(X)) = 0$ *and* $\pi_{2n}(\psi(X)) \cong \boldsymbol{Z}$ *which is generated by* $\psi(\lambda_n)$ $(n > 0)$*.*

Next we show the periodicity of $\phi(X)$. Recall the class $\mathscr{C}$ theory of abelian groups ([**7**]). We first show the periodicity of $\phi(X)$ modulo $\mathscr{C}_2$, where $\mathscr{C}_2$ is the class of odd order finite abelian groups.

Proposition 3.2.

$$\phi(\mathrm{Ad}^{1,1}\lambda) : \phi(X) \xrightarrow{\sim} \phi(\Omega^{1,1}(X))_0 \quad \mathrm{mod}\ \mathscr{C}_2.$$

PROOF. We compute the Leray-Serre spectral sequence of the fibration

$$\psi(\Omega^{1,0}X) \to \phi(\Omega^{1,0}X) \xrightarrow{p} \phi(X)$$

by making use of 3 in Lemma 2.1 and $i_X^*(x_i) = y_i^2$, where $p : \phi(\Omega^{1,0}X) \to \phi(X)$ is the evaluation at 0. By applying 2, 3 in Lemma 2.1 to $\lambda(1 \wedge j_+) : \Sigma^{1,1}\boldsymbol{C}P_\tau^\infty{}_+ \to \boldsymbol{Z} \times X$ and by that $\phi(\mathrm{Ad}^{1,0}\lambda(1 \wedge j_+))^*$ is epic, we see that $\{\phi(\mathrm{Ad}^{1,0}\lambda(1 \wedge j_+))^* p^*(y_i)\}$ form a basis of generators of $H^*(\phi(\Omega^{1,0}X); \boldsymbol{Z}_2)$. Then, by the same way of computation of $H^*(U/O; \boldsymbol{Z}_2)$ from the fibration $U \to U/O \to BO$, we have:

$$H^*(\phi(\Omega^{1,0}X); \boldsymbol{Z}_2) \cong \bigwedge(p^*(y_1), p^*(y_2), p^*(y_3), \ldots).$$

Then we see that $\phi(\mathrm{Ad}^{1,0}\lambda(1 \wedge j_+))^* p^*(y_i)$ are the basis of the $\boldsymbol{Z}_2$-module $H^*(\phi(\Sigma^{0,1} \wedge \boldsymbol{C}P_\tau^\infty{}_+); \boldsymbol{Z}_2)$, by 1, 2 in Lemma 2.1 and we have

$$H_*(\phi(\Omega^{1,0}(\boldsymbol{Z} \times X)); \boldsymbol{Z}_2) \cong \Delta(\mathrm{Im}\, \phi(\mathrm{Ad}^{1,0}\lambda(1 \wedge j_+))_*),$$

where $\Delta(x_1, x_2, \ldots)$ is the $\boldsymbol{Z}_2$-algebra whose $\boldsymbol{Z}_2$-module basis are $x_{i_1} \cdots x_{i_k}$ $(i_1 < \ldots < x_{i_k})$. By computing the Leray-Serre spectral sequence of the path fibration over $\phi(\Omega^{1,0}(X))\langle 1 \rangle$, we obtain

$$H_*(\phi(\Omega^{1,1}X)_0; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[\mathrm{Im}\, \phi(\mathrm{Ad}^{1,1}\lambda j_+|_{\boldsymbol{C}P_\tau^\infty})_*].$$

Since $\boldsymbol{Z}_2[\mathrm{Im}\, \phi(\mathrm{Ad}^{1,1}\lambda j_+|_{\boldsymbol{C}P_\tau^\infty})_*]$ has algebra generators in each positive dimension and $\phi(\mathrm{Ad}^{1,1}\lambda)$ is a Hopf map, we have

$$\phi(\mathrm{Ad}^{1,1}\lambda)_* : H_*(\phi(X); \boldsymbol{Z}_2) \xrightarrow{\sim} H_*(\phi(\Omega^{1,1}X)_0; \boldsymbol{Z}_2).$$

Therefore the proof is completed by the mod $\mathscr{C}_2$ J.H.C. Whitehead theorem. □

PROPOSITION 3.3.

$$\pi_n(\phi(X)) \otimes \boldsymbol{Z}[1/2] \cong \begin{cases} \boldsymbol{Z}[1/2] & n = 4k\,(k > 0) \\ 0 & \textit{otherwise}, \end{cases}$$

$i_{X*} : \pi_{4k}(\phi(X)) \otimes \boldsymbol{Z}[1/2] \to \pi_{4k}(\psi(X)) \otimes \boldsymbol{Z}[1/2]$ is an isomorphism and

$$\pi_n(\phi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2] \cong \begin{cases} \boldsymbol{Z}[1/2] & n = 0, 4k\,(k \geq 0) \\ 0 & \textit{otherwise}, \end{cases}$$

$i_{\Omega^{1,1}X*} : \pi_{4k}(\phi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2] \to \pi_{4k}(\psi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2]$ is an isomorphism. Especially we have:

$$(i_{\Omega^{1,1}X})_* : \pi_0(\phi(\Omega^{1,1}X) \xrightarrow{\sim} \pi_0(\psi(\Omega^{1,1}X)) \cong \boldsymbol{Z}.$$

PROOF. By 3 in Theorem 1 we see that

$$(\alpha i_X)_* = 2 : \pi_*(\phi(X)) \to \pi_*(\phi(X))$$

and

$$(i_X \alpha)_* = 1 + \tau_* : \pi_*(\psi(X)) \to \pi_*(\psi(X)).$$

Then $i_{X*} : \pi_*(\phi(X)) \otimes \boldsymbol{Z}[1/2] \to \pi_*(\psi(X)) \otimes \boldsymbol{Z}[1/2]$ is split monic. By Corollary 3.1 generators of $\pi_{2k}(\psi(X)) \cong \boldsymbol{Z}$ are represented by the τ-map λ_k. Then $\tau_* : \pi_{2k}(\psi(X)) \to \pi_{2k}(\psi(X)) \cong \boldsymbol{Z}$ is the mapping degree of $\tau : \psi(\Sigma^{k,k}) \to \psi(\Sigma^{k,k})$. Hence we have

$$(i_X \alpha)_* = 1 + (-1)^k : \pi_{2k}(\psi(X)) \to \pi_{2k}(\psi(X))$$

and complete the proof of the first part.

Consider the fibration

$$\psi(\Omega^{1,1}X) \xrightarrow{\alpha'} \phi(\Omega^{1,1}X) \to \phi(\Omega^{0,1}X)$$

as the one of 3 in Lemma 2.1, then we have the following exact sequence:

$$\cdots \to \pi_n(\psi(\Omega^{1,1}X)) \xrightarrow{\alpha'_*} \pi_n(\phi(\Omega^{1,1}X)) \to \pi_{n+1}(\phi(X)) \to \pi_{n-1}(\psi(\Omega^{1,1}X)) \to \cdots .$$

It is easily seen by 3 in Lemma 2.1 that:

$$(i_{\Omega^{1,1}X}\alpha')_* = 1 - \tau_* : \pi_*(\psi(\Omega^{1,1}X)) \to \pi_*(\psi(\Omega^{1,1}X)).$$

Since $\tau_* : \pi_{2k+2}(\psi(\Omega^{1,1}X)) \to \pi_{2k+2}(\psi(\Omega^{1,1}X)) = \pi_{2k+4}(\psi(X)) \cong \boldsymbol{Z}$ is the mapping degree of $\tau : \Sigma^{k+1,k+1} \to \Sigma^{k+1,k+1}$, we have:

$$(i_{\Omega^{1,1}X}\alpha')_* = 1 - (-1)^{k+1} : \pi_{2k+2}(\psi(\Omega^{1,1}X)) \to \pi_{2k+2}(\psi(\Omega^{1,1}X)).$$

Then we complete the proof of the second part.

Since $H^1(\phi(X); \boldsymbol{Z}_2) \neq 0$, we have $\pi_1(\phi(X)) \neq 0$. By the above we have:

$$(i_{\Omega^{1,1}X}\alpha')_* = 2 : \pi_0(\psi(\Omega^{1,1}X)) \to \pi_0(\psi(\Omega^{1,1}X))$$

Thus we obtain that:

$$i_{\Omega^{1,1}X*} : \pi_0(\phi(\Omega^{1,1}X)) \xrightarrow{\sim} \pi_0(\psi(\Omega^{1,1}X)) \cong \boldsymbol{Z}.$$

This completes the proof of the last part. $\square$

LEMMA 3.2.

$$\phi(\mathrm{Ad}^{1,1}\lambda) : \phi(X) \xrightarrow{\simeq} \phi(\Omega^{1,1}(X))_0$$

PROOF. Consider the following commutative diagram, where $(i_X)_*, (i_{\Omega^{1,1}X})_*$, $\psi(\mathrm{Ad}^{1,1}\lambda)_*$ are isomorphisms for $n > 0$ by Lemma 3.1 and Proposition 3.3.

$$\begin{array}{ccc} \pi_{4n}(\phi(X)) \otimes \boldsymbol{Z}[1/2] & \xrightarrow{\phi(\mathrm{Ad}^{1,1}\lambda)_*} & \pi_{4n}(\phi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2] \\ {\scriptstyle (i_X)_*}\downarrow & & \downarrow{\scriptstyle (i_{\Omega^{1,1}X})_*} \\ \pi_{4n}(\psi(X)) \otimes \boldsymbol{Z}[1/2] & \xrightarrow{\psi(\mathrm{Ad}^{1,1}\lambda)_*} & \pi_{4n}(\psi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2] \end{array}$$

Then we see that:

$$\phi(\mathrm{Ad}^{1,1}\lambda)_* : \pi_{4n}(\phi(X)) \otimes \boldsymbol{Z}[1/2] \xrightarrow{\simeq} \pi_{4n}(\phi(\Omega^{1,1}X)) \otimes \boldsymbol{Z}[1/2].$$

By Proposition 3.2, the class $\mathscr{C}$ theory and J.H.C. Whitehead theorem, we obtain

$$\phi(\mathrm{Ad}^{1,1}\lambda) : \phi(X) \xrightarrow{\simeq} \phi(\Omega^{1,1}X)_0.$$

We have the following commutative diagram, where $(i_{\boldsymbol{Z}\times X})_*, (i_{\Omega^{1,1}(\boldsymbol{Z}\times X)})_*, \psi(\mathrm{Ad}^{1,1}\lambda)_*$ are isomorphisms by Proposition 3.3. We complete the proof by J.H.C.Whitehead theorem.

$$\begin{array}{ccc} \pi_0(\phi(\boldsymbol{Z} \times X)) & \xrightarrow{\phi(\mathrm{Ad}^{1,1}\lambda)_*} & \pi_0(\phi(\Omega^{1,1}(\boldsymbol{Z} \times X))) \\ {\scriptstyle (i_{\boldsymbol{Z}\times X})_*}\downarrow & & \downarrow{\scriptstyle (i_{\Omega^{1,1}(\boldsymbol{Z}\times X)})_*} \\ \pi_0(\psi(\boldsymbol{Z} \times X)) & \xrightarrow{\psi(\mathrm{Ad}^{1,1}\lambda)_*} & \pi_0(\psi(\Omega^{1,1}(\boldsymbol{Z} \times X))) \end{array}$$

□

PROOF OF THEOREM 1. Let Y, Z be τ-spaces and $f : Y \to Z$ be a τ-map. We see that f is a τ-homotopy equivalence if and only if $\psi(f)$ and $\phi(f)$ are homotopy equivalences by [**1**]. By Lemmas 3.1 and 3.2 we have

$$\psi(\mathrm{Ad}^{1,1}\lambda) : \psi(\boldsymbol{Z} \times X) \xrightarrow{\simeq} \psi(\Omega^{1,1}(\boldsymbol{Z} \times X))$$

and

$$\phi(\mathrm{Ad}^{1,1}\lambda) : \phi(\boldsymbol{Z} \times X) \xrightarrow{\simeq} \phi(\Omega^{1,1}(\boldsymbol{Z} \times X)).$$

□

4. (1,1)-periodicity of *BR*.

We prove the $(1,1)$-periodicity of KR-theory.

Let BR be the τ-space of BU with the conjugation as the involution. As in section 1, $\boldsymbol{Z} \times BR$ is the classifying space of KR-theory. It is obvious that

$$\psi(BR) = BU \text{ and } \phi(BR) = BO.$$

Let ξ_n, η_n and $\boldsymbol{n}$ be the universal bundle of $BU(n), \boldsymbol{C}P^n$ and the trivial complex bundle of rank n respectively, then $\xi_n, \eta_n, \boldsymbol{n}$ are the *real* vector bundle. We denote the virtual *real* vector bundle $\lim_{\rightarrow}(\xi_n - \boldsymbol{n})$ on BR by ξ. We regard ξ, η_n as the virtual *real* vector bundles on $\boldsymbol{Z} \times BR$ and $\boldsymbol{C}P^n_{\tau\ +}$. The Bott map $\beta : \Sigma^{1,1}(\boldsymbol{Z} \times BR) \to \boldsymbol{Z} \times BR$ is defined as the classifying map of the virtual *real* vector bundle $(\eta_1 - \boldsymbol{1})\hat{\otimes}_{\boldsymbol{C}}\xi$ on $\boldsymbol{C}P^1_\tau \wedge (\boldsymbol{Z} \times BR) = \Sigma^{1,1}(\boldsymbol{Z} \times BR)$.

Lemma 4.1. *Let*

$$\beta : \Sigma^{1,1}(\boldsymbol{Z} \times BR) \to \boldsymbol{Z} \times BR,\ i_+ : \boldsymbol{C}P^\infty_{\tau\ +} \hookrightarrow \boldsymbol{Z} \times BR \text{ and } \boldsymbol{r} : \psi(BR) \to \phi(BR)$$

be the Bott map, the natural inclusion such that $i_+(\boldsymbol{C}P^\infty_\tau) \subset 1 \times BR$ *and the realization map respectively. Then we have the following.*

1. $H^*(\psi(BR); \boldsymbol{Z}) = \boldsymbol{Z}[c_1, c_2, c_3, \ldots]$ *and* $H^*(\phi(BR); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_1, w_2, w_3, \ldots]$, *where* $|c_i| = 2i, |w_i| = i$ *and* $i^*_{BR}(c_i) = w_i^2$.
2. $\psi(\beta(1 \wedge i_+))^* : H^*(\psi(\boldsymbol{Z} \times BR); \boldsymbol{Z}) \to H^*(\psi(\Sigma^{1,1} \wedge \boldsymbol{C}P^\infty_{\tau\ +}); \boldsymbol{Z})$ *and* $\phi(\beta(1 \wedge i_+))^* : H^*(\phi(\boldsymbol{Z} \times BR); \boldsymbol{Z}_2) \to H^*(\phi(\Sigma^{1,1} \wedge \boldsymbol{C}P^\infty_{\tau\ +}); \boldsymbol{Z}_2)$ *are epic.*
3. $\boldsymbol{r}i_{BR} \simeq \phi(\mu')\triangle$, $i_{BR}\boldsymbol{r} \simeq \psi(\mu')(1 \times \tau)\triangle$, *where* μ' *is the natural multiplication of* BR *and* $\triangle$ *is the diagonal map.*
4. $\mathrm{Ad}^{1,1}\beta$ *is a Hopf* τ*-map.*

Proof.

1. Well-known.

2. By computing the Chern classes of $\psi((\eta_1 - \boldsymbol{1})\hat{\otimes}_{\boldsymbol{C}}\xi)$, [**4**, Proposition 3.1] shows that

$$\psi(\beta(1 \wedge i_+))^* : H^*(\psi(\boldsymbol{Z} \times BR); \boldsymbol{Z}) \to H^*(\psi(\Sigma^{1,1} \wedge \boldsymbol{C}P^\infty_{\tau\ +}); \boldsymbol{Z}) \text{ is epic.}$$

Let $w(\phi(\xi)) = 1 + w_1(\phi(\xi)) + w_2(\phi(\xi)) + \ldots$. Then we have the following.

$$\begin{aligned}
&\phi(\beta(1 \wedge i_+))^* w(\phi(\xi)) \\
&\quad = w(\phi(\beta(1 \wedge i_+))^{-1}\phi(\xi)) \\
&\quad = w(\phi(\eta_1 - \boldsymbol{1})\hat{\otimes}_{\boldsymbol{R}}\phi(\eta_\infty - \boldsymbol{1})) \\
&\quad = w(\phi(\eta_1)\hat{\otimes}_{\boldsymbol{R}}\phi(\eta_\infty) - \phi(\eta_1)\hat{\otimes}_{\boldsymbol{R}}\phi(\boldsymbol{1}) - \phi(\boldsymbol{1})\hat{\otimes}_{\boldsymbol{R}}\phi(\eta_\infty) + \phi(\boldsymbol{1})\hat{\otimes}_{\boldsymbol{R}}\phi(\boldsymbol{1})) \\
&\quad = w(\phi(\eta_1)\hat{\otimes}_{\boldsymbol{R}}\phi(\eta_\infty))w(\phi(\eta_1)\hat{\otimes}_{\boldsymbol{R}}\phi(\boldsymbol{1}))^{-1}w(\phi(\boldsymbol{1})\hat{\otimes}_{\boldsymbol{R}}\phi(\eta_\infty))^{-1} \\
&\quad = (1 + w_1(\phi(\eta_1)) + w_1(\phi(\eta_\infty)))(1 + w_1(\phi(\eta_1)))^{-1}(1 + w_1(\phi(\eta_\infty)))^{-1} \\
&\quad = 1 + \sum_{i=1}^{\infty} w_1(\phi(\eta_1))w_1(\phi(\eta_\infty))^i
\end{aligned}$$

Since $w_1(\phi(\eta_1))w_1(\phi(\eta_\infty))^i$ is the basis of $H^*(\phi(\Sigma^{1,1} \wedge \boldsymbol{C}P^\infty_{\tau\ +}); \boldsymbol{Z}_2)$, we have

$$\phi(\beta(1\wedge i_+))^* : H^*(\phi(\boldsymbol{Z}\times BR);\boldsymbol{Z}_2)\to H^*(\phi(\Sigma^{1,1}\wedge \boldsymbol{C}P^\infty_\tau{}_+);\boldsymbol{Z}_2)\text{is epic.}$$

3. Well-known.
4. We show the following diagram is homotopy commutative.

$$\begin{array}{ccc} \Sigma^{1,1}\wedge(BR\times BR) & \xrightarrow{1\wedge\mu'} & \Sigma^{1,1}\wedge BR \\ {\scriptstyle\beta\times\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ BR\times BR & \xrightarrow{\mu'} & BR \end{array}$$

Since both $(1\wedge\mu')\beta$ and $(\beta\times\beta)\mu'$ are the classifying map of the same virtual *real* vector bundle $(\eta_1-\mathbf{1})\hat{\otimes}_{\boldsymbol{C}}(\xi\times\xi)$, then we have $(1\wedge\mu')\beta\simeq(\beta\times\beta)\mu'$. □

By Lemma 4.1 we apply Theorem 1 to BR and obtain the $(1,1)$-periodicity of BR.

THEOREM 2.

$$\mathrm{Ad}^{1,1}\beta : \boldsymbol{Z}\times BR\simeq_\tau \Omega^{1,1}(\boldsymbol{Z}\times BR)$$

5. Characterization of BR.

We prove that $X\simeq_\tau BR$, which is the characterization of BR by X.

Let Y be a pointed τ-space and $Q(Y)$ be $\lim_\to \Omega^{n,n}\Sigma^{n,n}Y$. We call a τ-space Y an infinite loop τ-space if there is a pointed τ-space Z_n for any n such that $Y\simeq_\tau\Omega^{n,n}Z_n$. Note that BR and X are the infinite loop τ-spaces by Theorems 1 and 2. For the infinite loop τ-space Y we define the infinite loop τ-map $\xi_Y : Q(Y)\to Y$ as follows.

$$\xi_Y=\lim_{\to}\nu_n^{-1}\Omega^{n,n}((\mathrm{Ad}^{n,n})^{-1}\nu_n) : Q(Y)\to Y,$$

where $\nu_n : Y\simeq_\tau\Omega^{n,n}Z_n$ for a pointed τ-space Z. According to [**3**] and [**5**] we have the Segal-Becker τ-splitting

$$\epsilon : \boldsymbol{Z}\times BR\to Q(\boldsymbol{C}P^\infty_\tau{}_+)$$

such that ϵ is a pointed τ-map and

$$\xi_{\boldsymbol{Z}\times BR}Q(i_+)\epsilon\simeq_\tau \mathrm{id}_{\boldsymbol{Z}\times BR},$$

where i_+ is as in Lemma 4.1. It is shown in [**6**] that

$$\psi(\xi_{\boldsymbol{Z}\times BR}Q(i_+)) : \psi(Q(\boldsymbol{C}P^\infty_\tau{}_+))\to\psi(\boldsymbol{Z}\times BR)$$

splits by ϵ such that

$$\psi(Q(\boldsymbol{C}P^\infty_\tau{}_+))\simeq\psi(\boldsymbol{Z}\times BR)\times F,$$

where $\pi_n(F)$ is finite for any n.

Theorem 3.

$$\xi_{\boldsymbol{Z}\times X}Q(j_+)\epsilon : \boldsymbol{Z}\times BR \simeq_\tau \boldsymbol{Z}\times X$$

Proof. We denote $\xi_{\boldsymbol{Z}\times X}Q(j_+)\epsilon$ by f. As in the proof of Theorem 2, we show that $\psi(f)$ and $\phi(f)$ are homotopy equivalences.

As in the proof of [**4**, Theorem 4.1] we consider the following commutative diagram, where $i : \boldsymbol{CP}^\infty_\tau{}_+ \hookrightarrow Q(\boldsymbol{CP}^\infty_\tau{}_+)$ is the natural inclusion.

$$\begin{array}{ccccc}
 & & \psi(\boldsymbol{CP}^\infty_\tau{}_+) & \xrightarrow{\psi(j_+)} & \psi(\boldsymbol{Z}\times X) \\
 & & {\scriptstyle\psi(i)}\downarrow & & \| \\
\psi(\boldsymbol{Z}\times BR) & \xrightarrow{\epsilon} & \psi(\xi_{\boldsymbol{Z}\times X}Q(\boldsymbol{CP}^\infty_\tau{}_+)) & \xrightarrow{\psi(Q(j_+))} & \psi(\boldsymbol{Z}\times X)
\end{array}$$

Then we have the following commutative diagram.

$$\begin{array}{ccccc}
 & & H_*(\psi(\boldsymbol{CP}^\infty_\tau{}_+);\boldsymbol{Z}) & \xrightarrow{\psi(j_+)_*} & H_*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z}) \\
 & & {\scriptstyle\psi(i)_*}\downarrow & & \| \\
H_*(\psi(\boldsymbol{Z}\times BR);\boldsymbol{Z}) & \xrightarrow{\epsilon_*} & H_*(\psi(Q(\boldsymbol{CP}^\infty_\tau{}_+));\boldsymbol{Z}) & \xrightarrow{\psi(\xi_{\boldsymbol{Z}\times X}Q(j_+))_*} & H_*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z}) \\
\| & & {\scriptstyle\text{projection}}\downarrow & & \| \\
H_*(\psi(\boldsymbol{Z}\times BR);\boldsymbol{Z}) & \xrightarrow{\sim} & H_*(\psi(Q(\boldsymbol{CP}^\infty_\tau{}_+));\boldsymbol{Z})/\text{torsion} & \xrightarrow{\varphi} & H_*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z})
\end{array}$$

We see that $\operatorname{Im}\psi(j_+)_* \subset \operatorname{Im}\varphi$. It is shown in the proof of [**4**, Theorem 2.1] that $\operatorname{Im}\psi(j_+)_*$ generates the algebra $H_*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z})$. Since $\psi(\xi_{\boldsymbol{Z}\times X})$ and $\psi(Q(j_+))$ are loop maps, $\psi(\xi_{\boldsymbol{Z}\times X}Q(j_+))_*$ is an algebra map. Hence we obtain that φ is an isomorphism. Therefore we obtain $\psi(f) : \psi(\boldsymbol{Z}\times BR) \simeq \psi(\boldsymbol{Z}\times X)$.

For any $x \in H^*(\psi(\boldsymbol{Z}\times BR);\boldsymbol{Z}_2)$ there exists a unique $y \in H^*(\phi(\boldsymbol{Z}\times BR);\boldsymbol{Z}_2)$ such that $i_{\boldsymbol{Z}\times BR}{}^*(x) = y^2$. Since $(i_{\boldsymbol{Z}\times X})^*(x_i) = y_i^2$ for $x_i \in H^*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z}_2)$ and $y_i \in H^*(\phi(\boldsymbol{Z}\times X);\boldsymbol{Z}_2)$, we obtain that $\phi(f)$ is a homotopy equivalence $\bmod \mathscr{C}_2$ by the following commutative diagram.

$$\begin{array}{ccc}
H^*(\psi(\boldsymbol{Z}\times X);\boldsymbol{Z}_2) & \xrightarrow{\psi(f)^*} & H^*(\psi(\boldsymbol{Z}\times BR);\boldsymbol{Z}_2) \\
{\scriptstyle(i_{\boldsymbol{Z}\times X})^*}\downarrow & & \downarrow{\scriptstyle(i_{\boldsymbol{Z}\times BR})^*} \\
H^*(\phi(\boldsymbol{Z}\times X);\boldsymbol{Z}_2) & \xrightarrow{\phi(f)^*} & H^*(\phi(\boldsymbol{Z}\times BR);\boldsymbol{Z}_2)
\end{array}$$

Consider the following commutative diagram, where $(i_{\boldsymbol{Z}\times BR})_*, (i_{\boldsymbol{Z}\times X})_*, \psi(f)_*$ are split monic by the above and Proposition 3.3.

$$\begin{array}{ccc} \pi_*(\phi(\boldsymbol{Z}\times BR))\otimes \boldsymbol{Z}[1/2] & \xrightarrow{\phi(f)_*} & \pi_*(\phi(\boldsymbol{Z}\times X))\otimes \boldsymbol{Z}[1/2] \\ {\scriptstyle (i_{\boldsymbol{Z}\times BR})_*}\downarrow & & \downarrow{\scriptstyle (i_{\boldsymbol{Z}\times X})_*} \\ \pi_*(\psi(\boldsymbol{Z}\times BR))\otimes \boldsymbol{Z}[1/2] & \xrightarrow{\psi(f)_*} & \pi_*(\psi(\boldsymbol{Z}\times X))\otimes \boldsymbol{Z}[1/2] \end{array}$$

Then we see that $\phi(f)_* : \pi_*(\phi(\boldsymbol{Z}\times BR))\times \boldsymbol{Z}[1/2] \to \pi_*(\phi(\boldsymbol{Z}\times X))\times \boldsymbol{Z}[1/2]$ is an isomorphism. Hence the proof is completed. □

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