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Critical points of the symmetric functions of the eigenvalues of the Laplace operator and overdetermined problems

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Abstract. We consider the Dirichlet and the Neumann eigenvalue problem for the Laplace operator on a variable nonsmooth domain, and we prove that the elementary symmetric functions of the eigenvalues splitting from a given eigenvalue upon domain deformation have a critical point at a domain with the shape of a ball. Correspondingly, we formulate overdetermined boundary value problems of the type of the Schiffer conjecture.

1. Introduction.

By the celebrated Rayleigh-Krahn-Faber Theorem, a ball in $\boldsymbol{R}^n$ minimizes the first eigenvalue of the Laplace operator under Dirichlet boundary conditions among the bounded connected subsets of $\boldsymbol{R}^n$ with a prescribed volume. Thus in a sense, the ball is a critical point for the functional which takes a bounded connected domain to the first eigenvalue of the Dirichlet Laplacian, under the constraint of constancy of the volume.

Instead, less known seem to be corresponding properties for higher order eigenvalues, and for the eigenvalues of the Neumann Laplacian (see Ashbaugh [**1**]). This paper concerns eigenvalues of all orders and multiplicity, both for the Dirichlet and the Neumann problem for the Laplace operator.

We first illustrate our work for the Dirichlet problem. We consider an open connected subset Ω of $\boldsymbol{R}^n$ of finite measure satisfying the condition

$$W_0^{1,2}(\Omega) \text{ is compactly imbedded in } L^2(\Omega), \tag{1.1}$$

and we deform Ω by a Lipschitz continuous homeomorphism ϕ of a class $\mathscr{A}_\Omega$ which we introduce in (2.3) below, and we consider the weak formulation of the Dirichlet eigenvalue problem for the operator $-\Delta$ in the deformed domain $\phi(\Omega)$. Namely, we consider the problem

$$\int_{\phi(\Omega)} DvDw^t\,dy = \lambda \int_{\phi(\Omega)} vw\,dy \qquad \forall w \in W_0^{1,2}(\phi(\Omega)) \tag{1.2}$$

in the unknowns $v \in W_0^{1,2}(\phi(\Omega))$ (the Dirichlet eigenfunctions), $\lambda \in \boldsymbol{R}$ (the Dirichlet eigenvalues). Under our assumptions on Ω and ϕ, such problem is well known to have a sequence of eigenvalues

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$$0 < \lambda_1[\phi] < \lambda_2[\phi] \leq \dots, \tag{1.3}$$

which we write as many times as their multiplicity. For each $\phi \in \mathscr{A}_\Omega$, the volume of the deformed domain $\phi(\Omega)$ is given by the functional

$$\mathscr{V}[\phi] \equiv \int_\Omega |\det D\phi| dx. \tag{1.4}$$

Now we fix $\tilde{\phi} \in \mathscr{A}_\Omega$. As is well known, simple eigenvalues can be shown to depend real analytically on $\phi \in \mathscr{A}_\Omega$ (cf. *e.g.*, Prodi [**13**]). Now, if both Ω and $\tilde{\phi}$ were regular enough, the condition that a simple eigenvalue $\lambda_j[\cdot]$ be critical at $\tilde{\phi}$ on a level set of the functional $\mathscr{V}[\cdot]$ can be rewritten, by exploiting the Hadamard variational formulas for $\lambda_j[\cdot]$, as the following well known overdetermined problem for the first eigenvalue of $-\Delta$

$$\begin{cases} -\Delta v = \lambda_j[\tilde{\phi}]v & \text{in } \tilde{\phi}(\Omega), \\ v = 0 & \text{on } \partial\tilde{\phi}(\Omega), \\ \left(\frac{\partial v}{\partial \nu}\right)^2 & \text{is constant on } \partial\tilde{\phi}(\Omega), \end{cases} \tag{1.5}$$

where ν denotes the exterior unit normal to $\partial\tilde{\phi}(\Omega)$ (cf. *e.g.* Chatelain [**2**], Henry [**5**]). Now problem (1.5) is satisfied for some j if $\tilde{\phi}(\Omega)$ is a ball. Not only, it is also known that if $\tilde{\phi}(\Omega)$ is bounded problem (1.5) can have a solution for $j = 1$ if and only if $\tilde{\phi}(\Omega)$ is a ball (see Henry [**5**]).

For eigenvalues of higher multiplicity, the situation is more complicated. To begin with, higher order eigenvalues are not differentiable functions of $\phi \in \mathscr{A}_\Omega$. However, one can prove that if we consider a finite nonempty subset F of $\boldsymbol{N} \setminus \{0\}$ of indices, and if we consider the set of ϕ's for which $\lambda_j[\phi]$ for $j \in F$ does not equal any of the $\lambda_l[\phi]$ for $l \in \boldsymbol{N} \setminus (F \cup \{0\})$, then the elementary symmetric functionals

$$\Lambda_{F,s}[\phi] \equiv \sum_{j_1,\dots,j_s \in F \ j_1 < \dots < j_s} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi], \tag{1.6}$$

depend real analytically on ϕ, for all $s = 1, \dots, |F|$. Here $|F|$ denotes the number of elements of F. Now let $\tilde{\phi} \in \mathscr{A}_\Omega$ be such that $\tilde{\phi}(\Omega)$ is regular enough. Also, assume that the eigenvalues $\lambda_j[\tilde{\phi}]$ have a common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$. By imposing the condition that $\Lambda_{F,s}[\cdot]$ has a critical point at $\tilde{\phi} \in \mathscr{A}_\Omega$ at a level set for the volume functional $\mathscr{V}$, one obtains the following problem

$$\begin{cases} -\Delta \tilde{v}_j = \lambda_F[\tilde{\phi}]\tilde{v}_j & \text{in } \tilde{\phi}(\Omega), \\ \tilde{v}_j = 0 & \text{on } \partial\tilde{\phi}(\Omega), \\ \sum_{j=1}^{|F|} \left(\frac{\partial \tilde{v}_j}{\partial \nu}\right)^2 & \text{is constant on } \partial\tilde{\phi}(\Omega), \end{cases} \tag{1.7}$$

for all orthonormal bases $\tilde{v}_1, \dots, \tilde{v}_{|F|}$ of eigenfunctions corresponding to the eigenvalue

$\lambda_F[\tilde{\phi}]$. Actually, we can formulate problem (1.7) under very weak regularity assumptions, even when there is no exterior normal at the boundary points (cf. Theorems 2.15, 2.25, Proposition 2.21, Remark 2.28). Our formulation can be regarded as a weak formulation of problem (1.7). Then we verify that (1.7) is satisfied if $\tilde{\phi}(\Omega)$ is a ball (cf. Theorem 2.30), and we cast a new problem, which could be regarded as a variant of the Schiffer conjecture for the Dirichlet problem.

In case of smooth domains, the problem consists in classifying all subsets Ω of $\boldsymbol{R}^n$ for which there exists a Dirichlet eigenvalue $\tilde{\lambda}$ corresponding to a finite subset F of indices, and an orthonormal basis $\tilde{v}_1, \ldots, \tilde{v}_{|F|}$ of eigenfunctions corresponding to the eigenvalue $\tilde{\lambda}$, for which $\sum_{j=1}^{|F|} \left(\frac{\partial \tilde{v}_j}{\partial \nu}\right)^2$ is constant on the boundary. For the statement for nonsmooth domains, we refer to Problem 2.33 below.

Then we develop the same approach for the Neumann problem, and by imposing the condition that the elementary symmetric functions of the eigenvalues corresponding to a finite set of indices F have a critical point with constant volume constraint, we obtain the formulation of an overdetermined problem.

In case of smooth domains, the problem consists in classifying all subsets Ω of $\boldsymbol{R}^n$ for which there exists a Neumann eigenvalue $\tilde{\gamma}$ corresponding to a finite subset F of indices, and an orthonormal basis $\tilde{v}_1, \ldots, \tilde{v}_{|F|}$ of Neumann eigenfunctions corresponding to the eigenvalue $\tilde{\gamma}$, for which $\sum_{j=1}^{|F|} |D\tilde{v}_j|^2 - \tilde{\gamma}\tilde{v}_j^2$ is constant on the boundary. For the statement for nonsmooth domains, we refer to Problem 3.17 below.

The paper is organized as follows. Section 2 is devoted to the Dirichlet problem, and section 3 is devoted to the Neumann problem.

2. Critical deformations for the symmetric functions of the Dirichlet eigenvalues.

We first introduce some notation. Let $\mathscr{X}$, $\mathscr{Y}$ be real Banach spaces. We say that the space $\mathscr{X}$ is continuously imbedded in the space $\mathscr{Y}$ provided that $\mathscr{X}$ is a linear subspace of $\mathscr{Y}$, and that the inclusion map is continuous. We denote by $\boldsymbol{N}$ the set of natural numbers including 0. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. If $A \equiv (a_{rs})_{r,s=1,\ldots,n}$ is an $n \times n$ matrix with real entries, we denote by A^t the transpose matrix of A. All elements of $\boldsymbol{R}^n$ are thought of as row vectors.

Let Ω be an open subset of $\boldsymbol{R}^n$. Throughout this paper, we shall consider only case $n \geq 2$. We denote by $\mathrm{cl}\Omega$, and by $\partial\Omega$, and by $|\Omega|$, the closure, the boundary and the measure of Ω, respectively. We denote by $L^2(\Omega)$ the space of square summable real valued measurable functions defined on Ω, and by $W_0^{1,2}(\Omega)$ the Sobolev space obtained by taking the closure of the space $\mathscr{D}(\Omega)$ of the C^∞ functions with compact support in Ω in the Sobolev space $W^{1,2}(\Omega)$ of distributions in Ω which have weak derivatives up to the first order in $L^2(\Omega)$, endowed with the norm defined by

$$\|u\|_{W^{1,2}(\Omega)} \equiv \left\{ \|u\|_{L^2(\Omega)}^2 + \sum_{l=1}^{n} \|u_{x_l}\|_{L^2(\Omega)}^2 \right\}^{1/2} \qquad \forall u \in W^{1,2}(\Omega). \tag{2.1}$$

Now, we are interested in open connected subsets Ω of $\boldsymbol{R}^n$ such that condition (1.1) holds. It is interesting to note that (1.1) certainly holds if Ω has finite measure (cf. *e.g.*, Tartar [**14**, p. 45]). As is well known, if (1.1) holds, then the Poincaré inequality holds in Ω (cf. *e.g.*, Evans [**3**, Proof of Theorem 1, p. 275]). Then we find convenient to introduce in $W_0^{1,2}(\Omega)$ its usual 'energy' scalar product

$$\langle u_1, u_2 \rangle \equiv \int_\Omega Du_1 Du_2^t \, dx \qquad \forall u_1, u_2 \in W_0^{1,2}(\Omega). \tag{2.2}$$

We denote by $w_0^{1,2}(\Omega)$ the Hilbert space $W_0^{1,2}(\Omega)$ endowed with the scalar product of (2.2). Then we deform Ω by a Lipschitz continuous homeomorphism of the class $\mathscr{A}_\Omega$ which we now introduce. We denote by $\mathrm{Lip}(\Omega)$ the set of Lipschitz continuous functions of Ω to $\boldsymbol{R}$, and we set

$$\mathscr{A}_\Omega \equiv \left\{ \phi \in (\mathrm{Lip}(\Omega))^n : l_\Omega[\phi] \equiv \inf \left\{ \frac{|\phi(x) - \phi(y)|}{|x-y|} : x, y \in \Omega, x \neq y \right\} > 0 \right\}. \tag{2.3}$$

We note that

$$l_\Omega[\phi] \leq |\det D\phi(x)|^{1/n}, \tag{2.4}$$

for almost all $x \in \Omega$ (cf. [**12**, Lemma 4.22]). Now it can be verified that if Ω satisfies (1.1) and if $\phi \in \mathscr{A}_\Omega$, then $\phi(\Omega)$ also satisfies (1.1) (cf. [**6**, Proposition 3.7 (vii)]). Accordingly, the Dirichlet eigenvalue problem (1.2) has a sequence of eigenvalues as in (1.3). As usual, we introduce the seminorm

$$|f|_1 \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|} : x, y \in \Omega, \ x \neq y \right\} \qquad \forall f \in \mathrm{Lip}(\Omega),$$

on $\mathrm{Lip}(\Omega)$. It is easily seen that $\mathscr{A}_\Omega$ is open in $(\mathrm{Lip}(\Omega))^n$ (cf. [**12**, Proposition 4.29], [**9**, Theorem 3.11]). As is well known, $(\mathrm{Lip}(\Omega), |\cdot|_1)$ is a complete seminormed space. However, we prefer to deal with a normed space, rather than with a seminormed space. Then we will state our results for an arbitrary normed space $\mathscr{X}_\Omega$, continuously imbedded in $(\mathrm{Lip}(\Omega), |\cdot|_1)$. Alternatively, one could also endow $\mathrm{Lip}(\Omega)$ with a norm which renders $\mathrm{Lip}(\Omega)$ a Banach space continuously imbedded in $(\mathrm{Lip}(\Omega), |\cdot|_1)$, and take $\mathscr{X}_\Omega$ equal to such Banach space.

We now find convenient to introduce the following notation. We set

$$\begin{aligned} \Upsilon[\phi, v_1, v_2, \lambda, \psi] \equiv & \int_{\phi(\Omega)} \big[Dv_1 Dv_2^t - \lambda_F[\phi] v_1 v_2 \big] \mathrm{div}(\psi \circ \phi^{(-1)}) \, dy \\ & - \int_{\phi(\Omega)} Dv_1 \big[D(\psi \circ \phi^{(-1)}) + D(\psi \circ \phi^{(-1)})^t \big] Dv_2^t \, dy, \end{aligned} \tag{2.5}$$

for all $\phi \in \mathscr{A}_\Omega$, v_1, $v_2 \in W^{1,2}(\phi(\Omega))$, $\lambda \in \boldsymbol{R}$, $\psi \in (\mathrm{Lip}(\Omega))^n$. Then we introduce the following result of [**9**, Theorems 3.21, 3.38] concerning the real analyticity of the symmetric functions of the eigenvalues of the Dirichlet Laplacian on $\phi(\Omega)$ upon variation of ϕ and the corresponding Hadamard formulas.

THEOREM 2.6. *Let Ω be a connected open subset of $\boldsymbol{R}^n$ such that* (1.1) *holds. Let $\mathscr{X}_\Omega$ be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let F be a finite nonempty subset of $\boldsymbol{N}\setminus\{0\}$. Let*

$$\mathscr{A}_\Omega^{\mathscr{D}}[F] \equiv \big\{\phi\in\mathscr{A}_\Omega\cap\mathscr{X}_\Omega^n : \lambda_l[\phi]\notin\{\lambda_j[\phi] : j\in F\}\forall l\in\boldsymbol{N}\setminus(F\cup\{0\})\big\}.$$

Then the following statements hold.

(i) *The set $\mathscr{A}_\Omega^{\mathscr{D}}[F]$ is open in $\mathscr{X}_\Omega^n$.*
(ii) *Let $s\in\{1,\dots,|F|\}$. The function $\Lambda_{F,s}$ of $\mathscr{A}_\Omega^{\mathscr{D}}[F]$ to $\boldsymbol{R}$ defined by*

$$\Lambda_{F,s}[\phi]\equiv\sum_{j_1,\dots,j_s\in F\ j_1<\dots<j_s}\lambda_{j_1}[\phi]\cdots\lambda_{j_s}[\phi]\qquad\forall\phi\in\mathscr{A}_\Omega^{\mathscr{D}}[F]\tag{2.7}$$

is real analytic.
(iii) *Let $\tilde\phi\in\mathscr{A}_\Omega^{\mathscr{D}}[F]$ be such that the eigenvalues $\lambda_j[\tilde\phi]$ assume a common value $\lambda_F[\tilde\phi]$ for all $j\in F$. Let $\tilde v_1,\dots,\tilde v_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\lambda_F[\tilde\phi]$ of $-\Delta$ in $W_0^{1,2}(\tilde\phi(\Omega))$, where the orthonormality is taken with respect to the scalar product of $w_0^{1,2}(\tilde\phi(\Omega))$ (cf.* (2.2)). *Then we have*

$$d_{|\phi=\tilde\phi}(\Lambda_{F,s})[\psi]=\lambda_F^s[\tilde\phi]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\Upsilon[\phi,\tilde v_l,\tilde v_l,\lambda_F[\tilde\phi],\psi],\tag{2.8}$$

for all $\psi\in\mathscr{X}_\Omega^n$. If we further assume that $\tilde v_l\in W^{2,2}(\tilde\phi(\Omega))$ for $l=1,\dots,|F|$, then

$$\Upsilon[\phi,\tilde v_l,\tilde v_l,\lambda_F[\tilde\phi],\psi]=-\int_{\tilde\phi(\Omega)}\mathrm{div}\big[\big(\psi\circ\tilde\phi^{(-1)}\big)|D\tilde v_l|^2\big]\,dy,\tag{2.9}$$

for all $\psi\in(\mathrm{Lip}(\Omega)\cap L^\infty(\Omega))^n$, $l=1,\dots,|F|$.

As we have said in the introduction, we are interested in the critical points of the symmetric functions (2.7) on the level sets of the volume functional defined in (1.4). In order to give a precise definition and a characterization of such points (cf. Definition 2.12 and Theorem 2.15), we introduce the following proposition.

PROPOSITION 2.10. *Let Ω be a connected open nonempty subset of $\boldsymbol{R}^n$ of finite measure. Let $\mathscr{X}_\Omega$ be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Then the following statements hold*

(i) *The map $\mathscr{V}$ of $\mathscr{A}_\Omega\cap\mathscr{X}_\Omega^n$ to $\boldsymbol{R}$ defined by* (1.4) *is real analytic. If $\tilde\phi\in\mathscr{A}_\Omega\cap\mathscr{X}_\Omega^n$, then the differential of $\mathscr{V}$ at $\tilde\phi$ is delivered by the formula*

$$d\mathscr{V}[\tilde{\phi}](\psi) = \int_{\tilde{\phi}(\Omega)} \operatorname{div}\bigl(\psi \circ \tilde{\phi}^{(-1)}\bigr) dy \qquad \forall \psi \in \mathscr{X}_{\Omega}^{n}. \tag{2.11}$$

(ii) *If* $\mathscr{V}_0 \in]0, +\infty[$,

$$V[\mathscr{V}_0] \equiv \bigl\{\phi \in \mathscr{A}_{\Omega} \cap \mathscr{X}_{\Omega}^{n} : \mathscr{V}[\phi] = \mathscr{V}_0\bigr\},$$

and if $V[\mathscr{V}_0] \neq \varnothing$, *then* $V[\mathscr{V}_0]$ *is a real analytic manifold of* $\mathscr{X}_{\Omega}^{n}$ *of codimension* 1.

PROOF. Statement (i) follows by standard calculus in Banach space, and by inequality (2.4) (see also [**9**, Proof of Lemma 3.26]). For statement (ii), it suffices to note that if $\tilde{\phi} \in V[\mathscr{V}_0]$ and $\psi = \tilde{\phi}$, then $d\mathscr{V}[\tilde{\phi}](\psi) = n|\tilde{\phi}(\Omega)|$, and that accordingly $d\mathscr{V}[\tilde{\phi}]$ is surjective. □

Then we have the following well known Definition.

DEFINITION 2.12. Let Ω be a connected open nonempty subset of $\boldsymbol{R}^n$ of finite measure. Let $\mathscr{X}_{\Omega}$ be a Banach space continuously imbedded in $\mathrm{Lip}(\Omega)$. Let $\mathscr{V}_0 \in]0, +\infty[$. Let $\tilde{\phi}$ be an element of $V[\mathscr{V}_0]$. Let $\mathscr{F}$ be a differentiable function of a neighborhood of $\tilde{\phi}$ in $\mathscr{X}_{\Omega}^{n}$ to $\boldsymbol{R}$. Then $\tilde{\phi}$ is said to be critical for $\mathscr{F}$ on $V[\mathscr{V}_0]$ provided that

$$\operatorname{Ker} d\mathscr{V}[\tilde{\phi}] \leq \operatorname{Ker} d\mathscr{F}[\tilde{\phi}]. \tag{2.13}$$

It is also well known that (2.13) holds if and only if there exists $c \in \boldsymbol{R}$ (a Lagrange multiplier) such that

$$d\mathscr{F}[\tilde{\phi}] + c\, d\mathscr{V}[\tilde{\phi}] = 0. \tag{2.14}$$

Then we have the following characterization.

THEOREM 2.15. *Let* Ω *be a connected open nonempty subset of* $\boldsymbol{R}^n$ *of finite measure. Let* $\mathscr{X}_{\Omega}$ *be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let* F *be a finite nonempty subset of* $\boldsymbol{N} \setminus \{0\}$. *Let* $\mathscr{V}_0 \in]0, +\infty[$. *Let* $\tilde{\phi} \in V[\mathscr{V}_0]$ *be such that* $\lambda_j[\tilde{\phi}]$ *assume a common value* $\lambda_F[\tilde{\phi}]$ *for all* $j \in F$ *and such that* $\lambda_l[\tilde{\phi}] \neq \lambda_F[\tilde{\phi}]$ *for all* $l \in \boldsymbol{N} \setminus (F \cup \{0\})$. *Let* $s = 1, \dots, |F|$. *The function* $\tilde{\phi}$ *is a critical point for* $\Lambda_{F,s}$ *on* $V[\mathscr{V}_0]$ *if and only if there exists an orthonormal basis* $\tilde{v}_1, \dots, \tilde{v}_{|F|}$ *of the eigenspace corresponding to the eigenvalue* $\lambda_F[\tilde{\phi}]$ *of* $-\Delta$ *in* $W_0^{1,2}(\tilde{\phi}(\Omega))$, *where the orthonormality is taken with respect to the scalar product of* $w_0^{1,2}(\tilde{\phi}(\Omega))$ (*cf.* (2.2)), *and a constant* $c \in \boldsymbol{R}$ *such that*

$$\sum_{l=1}^{|F|} \Upsilon[\tilde{\phi}, \tilde{v}_l, \tilde{v}_l, \lambda_F[\tilde{\phi}], \psi] + c \int_{\tilde{\phi}(\Omega)} \operatorname{div}\bigl(\psi \circ \tilde{\phi}^{(-1)}\bigr) dy = 0 \tag{2.16}$$

for all $\psi \in \mathscr{X}_{\Omega}^{n}$.

Now our goal is to express condition (2.16) in a convenient way. To do so we need some preliminaries. We set

$$\mathscr{D}_\Omega \equiv \{\eta \in \boldsymbol{R}^\Omega : \text{there exists } \varphi \in \mathscr{D}(\boldsymbol{R}^n) \text{ such that } \varphi_{|\Omega} = \eta\}.$$

If u is a function of Ω to $\boldsymbol{R}$, we denote by u_Ω the function of $\boldsymbol{R}^n$ to $\boldsymbol{R}$ defined by

$$u_\Omega(x) \equiv u(x) \quad \text{if } x \in \Omega, \qquad u_\Omega(x) \equiv 0 \quad \text{if } x \in \boldsymbol{R}^n \setminus \Omega.$$

We note that the symbol u_Ω should not be confused with the symbol $u_{|\Omega}$, which denotes the restriction of u to Ω.

As is well known, the space $L^\infty(\Omega)$ is canonically isometric to the strong dual of $L^1(\Omega)$. Accordingly, if $\{f_l\}_{l\in\boldsymbol{N}}$ is a sequence in $L^\infty(\Omega)$, we shall say that $\{f_l\}_{l\in\boldsymbol{N}}$ has a weak* limit f in $L^\infty(\Omega)$, if

$$\lim_{l\to\infty} \int_\Omega f_l g \, dx = \int_\Omega f g \, dx \qquad \forall g \in L^1(\Omega).$$

As is well known, all bounded sequences in $L^\infty(\Omega)$ have weakly* convergent subsequences. Similarly, we shall speak about weakly* convergent subsequences of $(L^\infty(\Omega))^n$. Then we have the following technical Lemma of [**11**, §5].

LEMMA 2.17. *Let Ω be a connected open subset of $\boldsymbol{R}^n$. Let $m \in \boldsymbol{N}$. Let $\tilde{\phi} \in \mathscr{A}_\Omega$. Let $\mathscr{X}_\Omega$ be a linear subspace of* $\mathrm{Lip}(\Omega)$ *containing $\mathscr{D}_\Omega$. Let Ξ be a linear map of* $(\mathrm{Lip}(\Omega))^n$ *to $\boldsymbol{R}^m$. Assume that $\lim_{l\to\infty} \Xi[\psi_l] = \Xi[\psi]$ in $\boldsymbol{R}^m$ whenever $\{\psi_l\}_{l\in\boldsymbol{N}}$ is a sequence in* $(\mathrm{Lip}(\Omega))^n$, *$\psi \in (\mathrm{Lip}(\Omega))^n$, and*

$$\lim_{l\to\infty} D\big(\psi_l \circ \tilde{\phi}^{(-1)}\big) = D\big(\psi \circ \tilde{\phi}^{(-1)}\big) \qquad \text{weakly* in } \big(L^\infty(\tilde{\phi}(\Omega))\big)^{n^2}. \tag{2.18}$$

Then the following equality holds

$$\{\Xi[\psi] : \psi \in \mathscr{X}_\Omega^n\} = \{\Xi[\xi \circ \tilde{\phi}] : \xi \in \big(\mathscr{D}_{\tilde{\phi}(\Omega)}\big)^n\}. \tag{2.19}$$

Then we have the following technical statement.

PROPOSITION 2.20. *Let the same assumptions of Theorem* 2.15 *hold. Let $\mathscr{X}_\Omega$ contain $\mathscr{D}_\Omega$. Then condition* (2.16) *holds for all $\psi \in \mathscr{X}_\Omega^n$ if and only if it holds for all $\psi = \xi \circ \tilde{\phi}$ with $\xi \in \big(\mathscr{D}_{\tilde{\phi}(\Omega)}\big)^n$.*

PROOF. By the membership of $\tilde{v}_h$ in $W^{1,2}(\tilde{\phi}(\Omega))$, and by the Hölder inequality, and by the definition (2.5) of Υ, and by Lemma 2.17 applied with Ξ equal to the operator in the variable ψ defined by the left hand side of (2.16), we conclude that the image of the left hand side of (2.16) for $\psi \in \mathscr{X}_\Omega^n$ coincides with the image of the left hand side of (2.16) for $\psi = \xi \circ \tilde{\phi}$ with $\xi \in \big(\mathscr{D}_{\tilde{\phi}(\Omega)}\big)^n$. □

Since the functions $\xi \circ \tilde{\phi}$ for $\xi \in \big(\mathscr{D}_{\tilde{\phi}(\Omega)}\big)^n$ are bounded, we can invoke equality (2.9), and obtain the following.

PROPOSITION 2.21. *Let the same assumptions of Theorem* 2.15 *hold. Let* $\mathscr{X}_\Omega$ *contain* $\mathscr{D}_\Omega$. *Let* $\tilde{v}_l \in W^{2,2}(\tilde{\phi}(\Omega))$ *for* $l = 1, \ldots, |F|$. *Then condition* (2.16) *holds for all* $\psi \in \mathscr{X}_\Omega^n$ *if and only if*

$$\int_{\tilde{\phi}(\Omega)} \operatorname{div}\left\{ \left[\sum_{l=1}^{|F|} |D\tilde{v}_l|^2 - c \right] \xi \right\} dy = 0 \qquad \forall \xi \in \left(\mathscr{D}_{\tilde{\phi}(\Omega)}\right)^n. \tag{2.22}$$

In order to gain a better understanding of condition (2.22), we set

$$\overset{o}{W}{}^{1,1}(\Omega) \equiv \left\{ u \in W^{1,1}(\Omega) : u_\Omega \in W^{1,1}(\boldsymbol{R}^n) \right\}.$$

Clearly,

$$W_0^{1,1}(\Omega) \subseteq \overset{o}{W}{}^{1,1}(\Omega),$$

and equality holds if Ω is of class C^1. Then we have the following variant of a known technical statement (see [**11**, §5]).

LEMMA 2.23. *Let* Ω *be an open subset of* $\boldsymbol{R}^n$ *such that* $\partial\Omega$ *has zero* n*-dimensional Lebesgue measure. Let* $r \in \boldsymbol{N}\setminus\{0\}$. *Let* $f_1, \ldots, f_r \in W^{1,1}(\Omega)$. *Let* q *denote the canonical projection of* $W^{1,1}(\Omega)$ *onto the quotient space* $W^{1,1}(\Omega)/\overset{o}{W}{}^{1,1}(\Omega)$. *The dimension of the space generated by* $\{q(f_l)\}_{l=1,\ldots,r}$ *in* $W^{1,1}(\Omega)/\overset{o}{W}{}^{1,1}(\Omega)$ *equals the dimension of the space*

$$\left\{ \left(\int_\Omega \operatorname{div}(\xi f_1) dx, \ldots, \int_\Omega \operatorname{div}(\xi f_r) dx \right) : \xi \in (\mathscr{D}_\Omega)^n \right\}.$$

Now, it can be easily verified that if $\phi \in \mathscr{A}_\Omega$, then ϕ can be extended uniquely to a Lipschitz continuous homeomorphism of $\mathrm{cl}\Omega$ onto $\mathrm{cl}\phi(\Omega)$, which we still denote by ϕ, and

$$\phi(\partial\Omega) = \partial\phi(\Omega), \tag{2.24}$$

(see [**11**, §5]). Then we have the following.

THEOREM 2.25. *Let the same assumptions of Theorem* 2.15 *hold. Let* $\partial\Omega$ *have zero* n*-dimensional Lebesgue measure. Let* $\mathscr{X}_\Omega$ *contain* $\mathscr{D}_\Omega$. *Let* $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ *for all* $h = 1, \ldots, |F|$. *Then condition* (2.22), *or equivalently condition* (2.16), *holds if and only if*

$$\sum_{l=1}^{|F|} |D\tilde{v}_l|^2 - c \in \overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega)). \tag{2.26}$$

REMARK 2.27. We note that if $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$, then by standard elliptic regularity theory, we have $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ for $h = 1, \dots, |F|$ (cf. *e.g.*, Troianiello [**15**, Theorem 3.29, p. 195]). Of course, the same may happen also under weaker regularity assumptions.

Furthermore, we note that if we assume that Ω is of class $C^{1,1}$, and that $\tilde{\phi} \in \mathscr{A}_\Omega$ has continuous partial derivatives in Ω satisfying a Lipschitz condition in Ω, then $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$ (cf. *e.g.*, [**8**, Lemma 2.4]).

REMARK 2.28. If we know that $\tilde{v}_h \in C^1(\mathrm{cl}\tilde{\phi}(\Omega))$ for $h = 1, \dots, |F|$, and that $\tilde{\phi}(\Omega)$ is of class C^1, and if we denote by ν the exterior normal to $\partial\tilde{\phi}(\Omega)$, then condition (2.26) takes the more familiar form

$$\sum_{l=1}^{|F|} \left(\frac{\partial \tilde{v}_l}{\partial \nu} \right)^2 = c \qquad \text{on } \partial\tilde{\phi}(\Omega). \tag{2.29}$$

By standard Elliptic Theory, condition $\tilde{v}_h \in C^1(\mathrm{cl}\tilde{\phi}(\Omega))$ for $h = 1, \dots, |F|$ holds if $\tilde{\phi}(\Omega)$ is of class $C^{1,\alpha}$ for some $\alpha \in]0, 1[$ (cf. *e.g.*, Gilbarg and Trudinger [**4**, Theorem 8.33, p. 210]).

As the following Proposition shows, condition (2.29) holds if $\tilde{\phi}(\Omega)$ is a ball. It clearly suffices to consider the unit ball $\boldsymbol{B}_n \equiv \{x \in \boldsymbol{R}^n : |x| < 1\}$.

PROPOSITION 2.30. *Let $\tilde{\lambda}$ be a Dirichlet eigenvalue of $-\Delta$ in $\boldsymbol{B}_n$. Let F be the set of $j \in \boldsymbol{N} \setminus \{0\}$ such that the j-th Dirichlet eigenvalue of $-\Delta$ in $\boldsymbol{B}_n$ coincides with $\tilde{\lambda}$. Let $\tilde{v}_1, \dots, \tilde{v}_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\tilde{\lambda}$ of $-\Delta$ in $W_0^{1,2}(\boldsymbol{B}_n)$, where the orthonormality is taken with respect to the scalar product in $w_0^{1,2}(\boldsymbol{B}_n)$ (cf. (2.2)). Let ν denote the exterior unit normal to $\partial\boldsymbol{B}_n$. Then $\sum_{j=1}^{|F|} \tilde{v}_j^2$ is a radial function and $\sum_{j=1}^{|F|} \bigl(\frac{\partial \tilde{v}_j}{\partial \nu}\bigr)^2$ is constant on $\partial\boldsymbol{B}_n$.*

PROOF. Let $O_n(\boldsymbol{R})$ denote the group of the orthogonal linear transformations in $\boldsymbol{R}^n$. By rotation invariance of the Laplace operator, $\tilde{v}_j \circ A$ is an eigenfunction corresponding to $\tilde{\lambda}$ for all $j = 1, \dots, |F|$ and for all $A \in O_n(\boldsymbol{R})$. If $A \in O_n(\boldsymbol{R})$, a straightforward computation shows that $\{\tilde{v}_j \circ A : j = 1, \dots, |F|\}$ is an orthonormal system in $w_0^{1,2}(\boldsymbol{B}_n)$. Since both $\{\tilde{v}_j : j = 1, \dots, |F|\}$ and $\{\tilde{v}_j \circ A : j = 1, \dots, |F|\}$ are orthonormal bases, then there exists $R[A] \in O_n(\boldsymbol{R})$ with matrix $(R_{ij}[A])_{ij=1,\dots,|F|}$ such that $\tilde{v}_j \circ A = \sum_{l=1}^{|F|} R_{jl}[A]\tilde{v}_l$. Then we have

$$\sum_{j=1}^{|F|} \tilde{v}_j^2 \circ A = \sum_{j=1}^{|F|} \tilde{v}_j^2. \tag{2.31}$$

Since (2.31) holds for all $A \in O_n(\boldsymbol{R})$, we conclude that $\sum_{j=1}^{|F|} \tilde{v}_j^2$ is radial. Since the Laplace operator is rotation invariant, then $\Delta\bigl\{ \sum_{j=1}^{|F|} \tilde{v}_j^2 \bigr\}$ is also radial. Next we note that

$$\sum_{j=1}^{|F|} \Delta\big(\tilde{v}_j^2\big) = -2\tilde{\lambda}\sum_{j=1}^{|F|} \tilde{v}_j^2 + 2\sum_{j=1}^{|F|} |D\tilde{v}_j|^2.$$

Hence, $\sum_{j=1}^{|F|} |D\tilde{v}_j|^2$ is radial. Since $\boldsymbol{B}_n$ is an open set of class C^∞, standard elliptic regularity theory implies that $\tilde{v}_j \in C^\infty(\mathrm{cl}\boldsymbol{B}_n)$. Since $\tilde{v}_j$ vanishes on $\partial\boldsymbol{B}_n$ for $j = 1, \ldots, |F|$, we have $\sum_{j=1}^{|F|} |D\tilde{v}_j|^2 = \sum_{j=1}^{|F|} \big(\frac{\partial \tilde{v}_j}{\partial \nu}\big)^2$ on $\partial\boldsymbol{B}_n$, and accordingly $\sum_{j=1}^{|F|} \big(\frac{\partial \tilde{v}_j}{\partial \nu}\big)^2$ is constant on $\partial\boldsymbol{B}_n$. □

Now, if we know that Ω is an open connected subset of $\boldsymbol{R}^n$, and that $\tilde{\phi} \in \mathscr{A}_\Omega$, and that $\tilde{\phi}(\Omega)$ is a ball, then $\tilde{\phi}^{(-1)} \in \mathscr{A}_{\tilde{\phi}(\Omega)}$, and Ω must have finite measure, and $\partial\Omega$ must have zero n-dimensional Lebesgue measure (cf. (2.24)) and (1.1) holds (cf. [**6**, Proposition 3.7 (vii)]). Then by combining Theorem 2.15, Theorem 2.25, and Remark 2.28, we deduce that the following theorem holds.

Theorem 2.32. *Let Ω be a connected open nonempty subset of $\boldsymbol{R}^n$. Let $\mathscr{X}_\Omega$ be a Banach space continuously imbedded in* Lip(Ω) *and containing $\mathscr{D}_\Omega$. If $\tilde{\phi} \in \mathscr{A}_\Omega$, and if $\tilde{\phi}(\Omega)$ is a ball, and if $\tilde{\lambda}$ is a Dirichlet eigenvalue of $-\Delta$ in $\tilde{\phi}(\Omega)$, and if F is the set of $j \in \boldsymbol{N} \setminus \{0\}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$, then $\Lambda_{F,s}[\cdot]$ has a critical point at $\tilde{\phi}$ on $V[\mathscr{V}[\tilde{\phi}]]$, for all $s = 1, \ldots, |F|$.*

It would be interesting to know whether there are other ϕ's which are critical for the symmetric functions $\Lambda_{F,s}[\cdot]$ on the level set of the volume function $\mathscr{V}$, and for which $\tilde{\phi}(\Omega)$ is not a ball. In terms of boundary value problems, one can state the following problem.

Problem 2.33. *Classify all open connected subsets Ω of $\boldsymbol{R}^n$ of finite measure with $\partial\Omega$ of zero n-dimensional Lebesgue measure, and for which there exists a Dirichlet eigenvalue $\tilde{\lambda}$ corresponding to the set of indices $F \subseteq \boldsymbol{N} \setminus \{0\}$ and an orthonormal basis $\tilde{v}_1, \ldots, \tilde{v}_{|F|}$ of the eigenspace associated to the eigenvalue $\tilde{\lambda}$, where the orthonormality is taken with respect to the scalar product in $w_0^{1,2}(\Omega)$ (cf. (2.2)), for which $\sum_{j=1}^{|F|} |D\tilde{v}_j|^2$ is, up to an additive constant, an element of $\overset{o}{W}{}^{1,1}(\Omega)$.*

3. Critical deformations for the symmetric functions of the Neumann eigenvalues.

For the Neumann problem, we are interested in open connected subsets Ω of $\boldsymbol{R}^n$ of finite measure $|\Omega|$ such that

$$W^{1,2}(\Omega) \text{ is compactly imbedded in } L^2(\Omega). \tag{3.1}$$

As is well known, if (3.1) holds, then the Poincaré-Wirtinger inequality holds in Ω (cf. *e.g.*, Evans [**3**, Proof of Theorem 1, p. 275]). Now it can be verified that if Ω satisfies (3.1) and if $\phi \in \mathscr{A}_\Omega$, then $\phi(\Omega)$ also satisfies (3.1) (cf. *e.g.*, [**7**, Proposition 2.6 (ii)]). Accordingly, the Neumann eigenvalue problem

$$\int_{\phi(\Omega)} DvDw^t\,dy = \gamma \int_{\phi(\Omega)} vw\,dy \qquad \forall w \in W^{1,2}(\phi(\Omega)) \tag{3.2}$$

in the unknowns $v \in W^{1,2}(\phi(\Omega))$ (the Neumann eigenfunctions), $\gamma \in \boldsymbol{R}$ (the Neumann eigenvalues) has a sequence of eigenvalues

$$0 = \gamma_0[\phi] < \gamma_1[\phi] \le \gamma_2[\phi] \le \dots,$$

which we write as many times as their multiplicity. The eigenvalue $\gamma_0[\phi]$ corresponds to the constant eigenfunction. We find convenient to introduce the space

$$W^{1,2,0}(\Omega) \equiv \left\{ u \in W^{1,2}(\Omega) : \int_\Omega u\,dx = 0 \right\}.$$

We denote by $w^{1,2,0}(\Omega)$ the space $W^{1,2,0}(\Omega)$ endowed with the energy scalar product

$$\langle u_1, u_2 \rangle \equiv \int_\Omega Du_1 Du_2^t\,dx \qquad \forall u_1, u_2 \in W^{1,2}(\Omega). \tag{3.3}$$

Then all the nonconstant eigenfunctions of problem (3.2) belong to $w^{1,2,0}(\phi(\Omega))$. Then we have the following version of Theorem 2.6 for the Neumann problem (cf. [**10**, §2]).

THEOREM 3.4. *Let Ω be a connected open subset of $\boldsymbol{R}^n$ of finite measure such that* (3.1) *holds. Let $\mathscr{X}_\Omega$ be a normed space continuously imbedded in* Lip(Ω). *Let F be a finite nonempty subset of $\boldsymbol{N} \setminus \{0\}$. Let*

$$\mathscr{A}_\Omega^{\mathscr{N}}[F] \equiv \left\{ \phi \in \mathscr{A}_\Omega \cap \mathscr{X}_\Omega^n : \gamma_l[\phi] \notin \{\gamma_j[\phi] : j \in F\} \forall l \in \boldsymbol{N} \setminus (F \cup \{0\}) \right\}.$$

Then the following statements hold.

(i) *The set $\mathscr{A}_\Omega^{\mathscr{N}}[F]$ is open in $\mathscr{X}_\Omega^n$.*

(ii) *Let $s \in \{1, \dots, |F|\}$. The function $\Gamma_{F,s}$ of $\mathscr{A}_\Omega^{\mathscr{N}}[F]$ to $\boldsymbol{R}$ defined by*

$$\Gamma_{F,s}[\phi] \equiv \sum_{j_1,\dots,j_s \in F\ j_1<\dots<j_s} \gamma_{j_1}[\phi] \cdots \gamma_{j_s}[\phi] \qquad \forall \phi \in \mathscr{A}_\Omega^{\mathscr{N}}[F]$$

is real analytic.

(iii) *Let $\tilde\phi \in \mathscr{A}_\Omega^{\mathscr{N}}[F]$ be such that the eigenvalues $\gamma_j[\tilde\phi]$ assume a common value $\gamma_F[\tilde\phi]$ for all $j \in F$. Let $\tilde v_1,\dots,\tilde v_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\gamma_F[\tilde\phi]$ of $-\Delta$ in $W^{1,2,0}(\tilde\phi(\Omega))$, where the orthonormality is taken with respect to the scalar product of $w^{1,2,0}(\tilde\phi(\Omega))$ (cf.* (3.3)). *Then we have*

$$d_{|\phi=\tilde\phi}(\Gamma_{F,s})[\psi] = \gamma_F^s[\tilde\phi] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \Upsilon[\phi, \tilde v_l, \tilde v_l, \gamma_F[\tilde\phi], \psi], \tag{3.5}$$

for all $\psi \in \mathscr{X}_\Omega^n$. *If we further assume that* $\tilde{v}_l \in W^{2,2}(\tilde{\phi}(\Omega))$ *for* $l = 1, \dots, |F|$, *then*

$$\Upsilon[\phi, \tilde{v}_l, \tilde{v}_l, \gamma_F[\tilde{\phi}], \psi] = \int_{\tilde{\phi}(\Omega)} \mathrm{div}\big\{\big[|D\tilde{v}_l|^2 - \gamma_F[\tilde{\phi}]\tilde{v}_l^2\big]\big(\psi \circ \tilde{\phi}^{(-1)}\big)\big\}dy, \tag{3.6}$$

for all $\psi \in (\mathrm{Lip}(\Omega) \cap L^\infty(\Omega))^n$, $l = 1, \dots, |F|$.

Then we have the following characterization.

Theorem 3.7. *Let* Ω *be a connected open nonempty subset of* $\boldsymbol{R}^n$ *of finite measure such that* (3.1) *holds. Let* $\mathscr{X}_\Omega$ *be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let* F *be a finite nonempty subset of* $\boldsymbol{N} \setminus \{0\}$. *Let* $\mathscr{V}_0 \in]0, +\infty[$. *Let* $\tilde{\phi} \in V[\mathscr{V}_0]$ *be such that* $\gamma_j[\tilde{\phi}]$ *assume a common value* $\gamma_F[\tilde{\phi}]$ *for all* $j \in F$ *and such that* $\gamma_l[\tilde{\phi}] \neq \gamma_F[\tilde{\phi}]$ *for all* $l \in \boldsymbol{N} \setminus (F \cup \{0\})$. *Let* $s = 1, \dots, |F|$. *The function* $\tilde{\phi}$ *is a critical point for* $\Gamma_{F,s}$ *on* $V[\mathscr{V}_0]$ *if and only if there exists an orthonormal basis* $\tilde{v}_1, \dots, \tilde{v}_{|F|}$ *of the eigenspace corresponding to the eigenvalue* $\gamma_F[\tilde{\phi}]$ *of* $-\Delta$ *in* $W^{1,2,0}(\tilde{\phi}(\Omega))$, *where the orthonormality is taken with respect to the scalar product of* $w^{1,2,0}(\tilde{\phi}(\Omega))$ (*cf.* (3.3)), *and a constant* $c \in \boldsymbol{R}$ *such that*

$$\sum_{l=1}^{|F|} \Upsilon[\tilde{\phi}, \tilde{v}_l, \tilde{v}_l, \gamma_F[\tilde{\phi}], \psi] + c \int_{\tilde{\phi}(\Omega)} \mathrm{div}\big(\psi \circ \tilde{\phi}^{(-1)}\big)dy = 0 \tag{3.8}$$

for all $\psi \in \mathscr{X}_\Omega^n$.

It is interesting to note that although here we are dealing with the Neumann problem, the critical point condition (3.8) has the same form of the critical point condition (2.16) for the Dirichlet problem.

By arguing exactly as for the Dirichlet problem, we can prove the following technical statement.

Proposition 3.9. *Let the same assumptions of Theorem* 3.7 *hold. Let* $\mathscr{X}_\Omega$ *contain* $\mathscr{D}_\Omega$. *Let* $\tilde{v}_l \in W^{2,2}(\tilde{\phi}(\Omega))$ *for* $l = 1, \dots, |F|$. *Then condition* (3.8) *holds for all* $\psi \in \mathscr{X}_\Omega^n$ *if and only if*

$$\int_{\tilde{\phi}(\Omega)} \mathrm{div}\left\{\left[\sum_{l=1}^{|F|} \big(|D\tilde{v}_l|^2 - \gamma_F[\tilde{\phi}]\tilde{v}_l^2\big) + c\right]\xi\right\}dy = 0 \qquad \forall \xi \in \big(\mathscr{D}_{\tilde{\phi}(\Omega)}\big)^n. \tag{3.10}$$

Then we have the following.

Theorem 3.11. *Let the same assumptions of Theorem* 3.7 *hold. Let* $\partial\Omega$ *have zero* n*-dimensional Lebesgue measure. Let* $\mathscr{X}_\Omega$ *contain* $\mathscr{D}_\Omega$. *Let* $\tilde{v}_h \in W^{2,2}(\tilde{\phi}(\Omega))$ *for all* $h = 1, \dots, |F|$. *Then condition* (3.10), *or equivalently condition* (3.8), *holds if and only if*

$$\sum_{l=1}^{|F|}\left(|D\tilde{v}_l|^2-\gamma_F[\tilde{\phi}]\tilde{v}_l^2\right)+c\in\overset{o}{W}{}^{1,1}(\tilde{\phi}(\Omega)). \tag{3.12}$$

As for the Dirichlet problem, we note that if $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$, then by standard elliptic regularity theory, we have $\tilde{v}_r\in W^{2,2}(\tilde{\phi}(\Omega))$ for $r=1,\dots,|F|$ (cf. *e.g.*, Troianiello [**15**, Theorem 3.17, p. 179]). Of course, the same may happen also under weaker regularity assumptions.

Remark 3.13. If we know that $\tilde{v}_r\in C^1(\mathrm{cl}\tilde{\phi}(\Omega))$ for $r=1,\dots,|F|$, and that $\tilde{\phi}(\Omega)$ is of class C^1, then condition (3.12) takes the more familiar form

$$\sum_{l=1}^{|F|}\left(|D\tilde{v}_l|^2-\gamma_F[\tilde{\phi}]\tilde{v}_l^2\right)+c=0 \qquad \text{on } \partial\tilde{\phi}(\Omega). \tag{3.14}$$

As the following shows, condition (3.14) holds if $\tilde{\phi}(\Omega)$ is a ball. As for the Dirichlet problem, it suffices to consider the unit ball.

Proposition 3.15. *Let $\tilde{\gamma}>0$ be a Neumann eigenvalue of* (3.2) *in $\boldsymbol{B}_n$. Let F be the set of $j\in\boldsymbol{N}\setminus\{0\}$ such that the j-th Neumann eigenvalue of $-\Delta$ in $\boldsymbol{B}_n$ coincides with $\tilde{\gamma}$. Let $\tilde{v}_1,\dots,\tilde{v}_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\tilde{\gamma}$ of $-\Delta$ in $W^{1,2,0}(\boldsymbol{B}_n)$, where the orthonormality is taken with respect to the scalar product in $w^{1,2,0}(\boldsymbol{B}_n)$ (cf.* (3.3)*). Let ν denote the exterior unit normal to $\partial\boldsymbol{B}_n$. Then $\sum_{j=1}^{|F|}\tilde{v}_j^2$ and $\sum_{j=1}^{|F|}|D\tilde{v}_j|^2$ are radial functions, and $\sum_{l=1}^{|F|}\left(|D\tilde{v}_l|^2-\tilde{\gamma}\tilde{v}_l^2\right)$ is constant on $\partial\boldsymbol{B}_n$.*

Proof. By proceeding exactly as in the proof of Proposition 2.30, we can prove that $\sum_{j=1}^{|F|}\tilde{v}_j^2$ and $\sum_{j=1}^{|F|}|D\tilde{v}_j|^2$ are radial. Hence we deduce the constancy on the boundary requested by the statement. □

Then by combining Theorem 3.7, Theorem 3.11, and Remark 3.13, we deduce the validity of the following.

Theorem 3.16. *Let Ω be a connected open nonempty subset of $\boldsymbol{R}^n$ of finite measure such that* (3.1) *holds. Let $\mathscr{X}_\Omega$ be a Banach space continuously imbedded in* $\mathrm{Lip}(\Omega)$ *and containing $\mathscr{D}_\Omega$. If $\tilde{\phi}\in\mathscr{A}_\Omega$, and if $\tilde{\phi}(\Omega)$ is a ball, and if $\tilde{\gamma}>0$ is a Neumann eigenvalue of $-\Delta$ in $\tilde{\phi}(\Omega)$, and if F is the set of $j\in\boldsymbol{N}\setminus\{0\}$ such that $\gamma_j[\tilde{\phi}]=\tilde{\gamma}$, then $\Gamma_{F,s}[\cdot]$ has a critical point at $\tilde{\phi}$ on $V[\mathscr{V}[\tilde{\phi}]]$, for all $s=1,\dots,|F|$.*

It would be interesting to know whether there are other ϕ's which are critical for the symmetric functions $\Gamma_{F,s}[\cdot]$ on the level set of the volume function $\mathscr{V}$, and for which $\tilde{\phi}(\Omega)$ is not a ball. In terms of boundary value problems, one can state the following problem.

Problem 3.17. *Classify all open connected subsets Ω of $\boldsymbol{R}^n$ of finite measure for which* (3.1) *holds, and for which there exists a Neumann eigenvalue $\tilde{\gamma}>0$ corresponding to the set of indices $F\subseteq\boldsymbol{N}\setminus\{0\}$ and an orthonormal basis $\tilde{v}_1,\dots,\tilde{v}_{|F|}$ of the eigenspace* *associated to the eigenvalue $\tilde{\gamma}$, where the orthonormality is taken with respect to the scalar product in $w^{1,2,0}(\Omega)$ (cf. (3.3)), for which $\sum_{l=1}^{|F|}\left(|D\tilde{v}_l|^2 - \tilde{\gamma}\tilde{v}_l^2\right)$ is, up to an additive constant, an element of $\overset{o}{W}{}^{1,1}(\Omega)$.*

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References

[1] M. S. Ashbaugh, Open problems on eigenvalues of the Laplacian, Analytic and geometric inequalities and applications, Math. Appl., **478**, Kluwer Acad. Publ., Dordrecht, 1999, pp. 13–28.

[2] T. Chatelain, A new approach to two overdetermined eigenvalue problems of Pompeiu type, Élasticité, viscoélasticité et contrôle optimal, Lyon, 1995, 235–242: (electronic), ESAIM Proc., **2**, Soc. Math. Appl. Indust., Paris, 1997.

[3] L. C. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.

[4] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1983.

[5] D. Henry, Topics in nonlinear analysis, Trabalho de Matemática, **192**, Univ. Brasilia, Março 1982.

[6] P. D. Lamberti and M. Lanza de Cristoforis, A global Lipschitz continuity result for a domain dependent Dirichlet eigenvalue problem for the Laplace operator, Zeitschrift für Analysis und ihre Anwendungen, **24** (2005), pp. 277–304.

[7] P. D. Lamberti and M. Lanza de Cristoforis, A global Lipschitz continuity result for a domain dependent Neumann eigenvalue problem for the Laplace operator, J. Differential Equations, **216** (2005), pp. 109–133.

[8] P. D. Lamberti and M. Lanza de Cristoforis, An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain, Glasg. Math. J., **44** (2002), pp. 29–43.

[9] P. D. Lamberti and M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator, J. Nonlinear Convex Anal., **5** (2004), pp. 19–42.

[10] P. D. Lamberti and M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Neumann problem for the Laplace operator, submitted, 2003.

[11] P. D. Lamberti and M. Lanza de Cristoforis, Persistence of eigenvalues and multiplicity in the Dirichlet problem for the Laplace operator on nonsmooth domains, to appear on Math. Phys. Anal. Geom., 2003.

[12] M. Lanza de Cristoforis, Properties and pathologies of the composition and inversion operator in Schauder spaces, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl., **15** (1991), pp. 93–109.

[13] G. Prodi, Dipendenza dal dominio degli autovalori dell'operatore di Laplace, Istituto Lombardo, Rend. Sc., **128** (1994), pp. 3–18.

[14] L. Tartar, An introduction to Sobolev spaces and Interpolation spaces, Carnegie-Mellon Univ., Pittsburgh, 2000.

[15] G. M. Troianiello, Elliptic differential equations and obstacle problems, Plenum Press, New York and London, 1987.

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