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How can we escape Thomae's relations?

By Christian KRATTENTHALER† and Tanguy RIVOAL

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Abstract. In 1879, Thomae discussed the relations between two generic hypergeometric ${}_3F_2$-series with argument 1. It is well-known since then that, in combination with the trivial ones which come from permutations of the parameters of the hypergeometric series, Thomae had found a set of 120 relations. More recently, Rhin and Viola asked the following question (in a different, but equivalent language of integrals): If there exists a linear dependence relation over $\boldsymbol{Q}$ between two convergent ${}_3F_2$-series with argument 1, with integral parameters, and whose values are irrational numbers, is this relation a specialisation of one of the 120 Thomae relations? A few years later, Sato answered this question in the negative, by giving six examples of relations which cannot be explained by Thomae's relations. We show that Sato's counter-examples can be naturally embedded into two families of infinitely many ${}_3F_2$-relations, both parametrised by three independent parameters. Moreover, we find two more infinite families of the same nature. The families, which do not seem to have been recorded before, come from certain ${}_3F_2$-transformation formulae and contiguous relations. We also explain in detail the relationship between the integrals of Rhin and Viola and ${}_3F_2$-series.

1. Prelude: introduction and summary of the results.

In this article, we are interested in two families of two-term relationships between hypergeometric ${}_3F_2$-series with argument 1, and the possible links between them. The first family consists of 120 relations found by Thomae [**16**], which can be interpreted as the action of the symmetric group $\mathfrak{S}_5$ on five parameters related to the parameters of a generic ${}_3F_2$-series. This action has been discovered and rediscovered many times. We shall start our article by describing two of its seemingly different incarnations: one involving series (Thomae, Whipple, Hardy and others: see Section 2, in particular Theorem 3) and the other involving integrals (Dixon, Rhin–Viola: see Section 3, in particular Theorem 4), while in Section 4 we explain their equivalence.

Our main aim is to find a hypergeometric explanation of a second family of six "exotic" integral relations recently discovered by Sato [**13**] (see Theorem 5). The latter provide counter-examples to a conjecture of Rhin and Viola [**11**] (see Conjecture 1 in Section 3), which essentially predicted the universality of Thomae's relations in the case of integral parameters. As we shall show, this explanation is given by the following two identities in Theorems 1 and 2, respectively, which seemingly have not been stated explicitly before.

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The first one (with proof in Section 8) covers five of Sato's six original relations, and we will obtain from it infinitely many explicit counter-examples to the conjecture by Rhin and Viola (see Theorem 6 in Section 3).

Theorem 1. *Let α, β, γ be complex numbers such that $2\alpha + \beta + 1$ and $2\beta + \alpha + 1$ are not non-positive integers, and such that $\Re(2\alpha + 2\beta - \gamma) > 0$. Then*

$$ {}_3F_2\left[\begin{matrix} \alpha+1,\ \beta+1,\ \gamma \\ 2\alpha+\beta+1,\ 2\beta+\alpha+1 \end{matrix}; 1\right] = \frac{2(\alpha+\beta)}{2(\alpha+\beta)-\gamma}\, {}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma \\ 2\alpha+\beta+1,\ 2\beta+\alpha+1 \end{matrix}; 1\right]. \quad (1.1) $$

The second one (with proof in Section 10) covers the remaining counter-example of Sato. It implies another set of infinitely many counter-examples to the conjecture by Rhin and Viola (see Theorem 7 in Section 3).

Theorem 2. *For any complex numbers α, β, γ such that $\Re\big(2 - \beta - \frac{\alpha(\alpha-\gamma+1)}{\beta-1}\big) > 0$, and such that $\alpha + 1$ and $\gamma + \frac{\alpha(\alpha-\gamma+1)}{\beta-1}$ are not non-positive integers, we have the identity*

$$ {}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma \\ \alpha+1,\ \gamma+\frac{\alpha(\alpha-\gamma+1)}{\beta-1}+1 \end{matrix}; 1\right] $$
$$ = \frac{(\alpha-\beta+2)(\alpha+\alpha^2-\gamma-\alpha\gamma+\beta\gamma)}{(\alpha+1)(2\alpha+\alpha^2-\alpha\beta-\gamma-\alpha\gamma+\beta\gamma)}\, {}_3F_2\left[\begin{matrix} \alpha+1,\ \beta-1,\ \gamma \\ \alpha+2,\ \gamma+\frac{\alpha(\alpha-\gamma+1)}{\beta-1} \end{matrix}; 1\right]. \quad (1.2) $$

Clearly, since Sato's counter-examples are special cases of (1.1) and (1.2), but are not consequences of Thomae's relations (see Section 6), the two identities provide an answer to the question in the title. (Let us point out that Theorems 1 and 2 are "independent" of each other, that is, neither is it possible to derive Theorem 2 from a combination of Theorem 1 with Thomae's relations, nor is this possible in the other direction.)

Of course, there may exist many more ways of escaping Thomae's relations. For example, a rather simple-minded one consists in examining for which integral values of the parameters a ${}_3F_2$-series with argument 1 can be a rational number. In fact, a complete characterisation for the latter problem is available, see Theorem 8 in Section 5. Leaving this simple possibility aside, in our proofs of identities (1.1) and (1.2) in Sections 8 and 10, we make use of two fundamentally different ways to escape Thomae's relations:

(1) One applies a transformation formula transforming a ${}_3F_2$-series with argument 1 into a hypergeometric series with a larger number of parameters (in our case, this is the transformation formula (8.1) transforming a ${}_3F_2$-series into a very-well-poised ${}_7F_6$-series) in order to "exit" the "${}_3F_2$-domain," and then one "re-enters" the "${}_3F_2$-domain" in a different way (in our case, we use the same transformation formula in the other direction, but after a permutation of the parameters of the ${}_7F_6$-series has been carried out before).

(2) One starts with a ${}_3F_2$-series in which one lower parameter exceeds one upper parameter by a positive integer. Subsequently, one applies contiguous relations to obtain a sum of several series, in which for all but one the use of the contiguous relations has made these two parameters equal, and thus these ${}_3F_2$-series with argument 1 reduce

to a ${}_2F_1$-series (with argument 1), which can then be summed by means of the Gauß summation formula (10.3). The various results of these evaluations are then combined into one expression, thereby generating a (possibly huge) polynomial term, which is then equated to zero. In order to make this work, this polynomial must have integral solutions. (See the Remark after the proof of Proposition 1 in Section 10 for more precise explanations, and, in particular, for an explanation of the term "contiguous relation").

Whereas we failed to find results other than Theorem 1 by using recipe (1), we show in Section 12 that recipe (2) can be used in many more ways than the one yielding Theorem 2 (see Theorems 9 and 10), thus producing many more counter-examples to the conjecture by Rhin and Viola. In fact, there are certainly many more relations that can be found in that way. We report on a curious phenomenon in that context at the end of the "round-up" Section 13, where we indicate the ideas that we used to find the hypergeometric results in Theorems 1, 2, 9 and 10.

So, in summary, as disappointing as this may be, our results show that the conjecture of Rhin and Viola was over-optimistic. The counter-examples by Sato are not just rare exceptions, they even embed in infinite families of counter-examples, and there are others beyond that. In view of this, and since the data that we produced do not give much guidance, we better refrain from coming up with a modified conjecture towards a generating set of transformations for the relations between ${}_3F_2$-series that would correct the conjecture by Rhin and Viola. Nevertheless, finding one appears to be an interesting, and challenging, problem.

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2. Thomae's relations.

Hypergeometric series are defined by

$$ {}_{q+1}F_q\left[\begin{matrix}\alpha_0,\ \alpha_1,\ \dots,\ \alpha_q\\ \beta_1,\ \dots,\ \beta_q\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_0)_k\,(\alpha_1)_k\cdots(\alpha_q)_k}{k!\,(\beta_1)_k\cdots(\beta_q)_k}\,z^k, \tag{2.1}$$

where $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n\ge 1$. The series converges provided that the argument z is a complex number with $|z|<1$, $\alpha_j\in\boldsymbol{C}$ and $\beta_j\in\boldsymbol{C}\setminus\boldsymbol{Z}_{\le 0}$; it also converges for $z=1$ if in addition $\Re(\beta_1+\cdots+\beta_q)>\Re(\alpha_0+\cdots+\alpha_q)$. Any "permutation" in $\mathfrak{S}_{q+1}\times\mathfrak{S}_q$ acting on the upper parameters α_i, $i=0,1,\dots,q$, and the lower parameters β_i, $i=1,2,\dots,q$, on the left-hand side of (2.1) does not affect the value of the right-hand side: we use the term "trivial symmetries" to indicate this fact.

As mentioned in the introduction, the symmetric group $\mathfrak{S}_5$ acts classically on the hypergeometric series ${}_3F_2$-series with argument 1, which leads to exactly 120 formal relations between them. This group action is obtained using the following fundamental identity, due to Thomae [**16**, Equation (12)] (given as (3.2.2) in [**1**]), which is valid under certain conditions on the parameters to ensure convergence of the involved series,

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};1\right] = \frac{\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(e-a)\,\Gamma(d+e-b-c)}\,{}_3F_2\left[\begin{matrix} a,\ d-b,\ d-c \\ d,\ d+e-b-c \end{matrix};1\right]. \tag{2.2} $$

The iterative application of (2.2), together with the trivial symmetries, yields 120 relations, of which only 10 are inequivalent modulo the trivial symmetries. These were given by Thomae [**16**, Article 4] and put in a more suitable form by Whipple [**19**]. It is apparently Hardy [**6**, p. 499] who first gave a group theoretic interpretation: we state his observation in the striking form given in [**15**], [**17**].

THEOREM 3 (Hardy). *Let $s = s(x_1,x_2,x_3,x_4,x_5) = x_1+x_2+x_3-x_4-x_5$. The function*

$$ \frac{1}{\Gamma(s)\,\Gamma(2x_4)\,\Gamma(2x_5)}\,{}_3F_2\left[\begin{matrix} 2x_1-s,\ 2x_2-s,\ 2x_3-s \\ 2x_4,\ 2x_5 \end{matrix};1\right] \tag{2.3} $$

is a symmetric function of the five variables x_1,x_2,x_3,x_4,x_5.

Care is needed using this theorem, since s is not a symmetric function of x_1,x_2,x_3,x_4,x_5 and some of the ${}_3F_2$-series might not be convergent. This result is surprising since one could not expect *a priori* a much bigger invariance group than $\mathfrak{S}_3\times\mathfrak{S}_2$, obtained by the permutations of $\{x_1,x_2,x_3\}$ and $\{x_4,x_5\}$, which trivially leave (2.3) invariant.

3. The Rhin–Viola group for $\zeta(2)$.

In 1996, Rhin and Viola introduced in [**11**] the integral

$$ I(h,i,j,k,l) = \int_0^1\int_0^1 \frac{x^h(1-x)^i y^k(1-y)^j}{(1-xy)^{i+j-l+1}}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.1} $$

which is convergent under the assumption that h,i,j,k,l are non-negative integers, which will be the case throughout the rest of this article unless otherwise stated. Their motivation was to use the fact that $I(h,i,j,k,l)\in \boldsymbol{Q}+\boldsymbol{Q}\zeta(2)$ to get a good irrationality measure for $\zeta(2)=\sum_{n\geq 1}1/n^2=\pi^2/6$, as had been done in previous work using similar but less general integrals (see the bibliography in [**11**]). They developed a beautiful new algebraic method for handling the general case above and were rewarded with the best known irrationality measure for π^2. See also [**5**], [**23**] for related work.

From now on, we focus essentially on the hypergeometric structure underlying their method, which is made transparent by the identity (see Section 4, where we recall the proof for $z=1$)

$$
{}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};z\right] = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(a)\,\Gamma(d-a)\,\Gamma(b)\,\Gamma(e-b)} \times \int_0^1\int_0^1 \frac{x^{a-1}(1-x)^{d-a-1}y^{b-1}(1-y)^{e-b-1}}{(1-zxy)^c}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.2}
$$

which is valid provided $\Re(d) > \Re(a) > 0$ and $\Re(e) > \Re(b) > 0$ if $|z| < 1$, with the further assumption that $\Re(d+e-a-b-c) > 0$ if $z = 1$. To simplify, we set $B(h,i,j,k,l) = I(h,i,j,k,l)/(h!\,i!\,j!\,k!\,l!)$. The main new idea in [**11**] was to use the action of a group on the parameters of h,i,j,k,l leaving the value of $B(h,i,j,k,l)$ invariant. To do this, Rhin and Viola showed that, under the two changes of variables $\{X = y, Y = x\}$ and $\{X = (1-x)/(1-xy), Y = 1-xy\}$, the value of $I(h,i,j,k,l)$ (and hence also that of $B(h,i,j,k,l)$) is not changed if the parameters are permuted by the product of transpositions $\sigma = (h\ k)(i\ j)$ and the 5-cycle $\tau = (h\ i\ j\ k\ l)$. The group $\boldsymbol{T} = \langle\sigma,\tau\rangle$ generated by σ and τ is isomorphic to $\mathscr{D}_5$, the dihedral group of order 10: for a visual proof, place the letters h,i,j,k,l, in this order, at the vertices of a regular pentagon.

But a more important invariance group can be obtained by extending the action of σ and τ by linearity to the set $\mathscr{P} = \{h,i,j,k,l,j+k-h,k+l-i,l+h-j,h+i-k,i+j-l\}$ (by "linearity", we mean $\tau(h+i-k) = \tau(h)+\tau(i)-\tau(k) = i+j-l$, etc.). Provided the five values $j+k-h$, $k+l-i$, $l+h-j$, $h+i-k$, $i+j-l$ are non-negative (see Theorem 8 in Section 4 for the arithmetic meaning of this hypothesis), one can then use the apparent loss of the trivial symmetries in the parameters a,b,c and d,e on the right-hand side of (3.2) to prove that the value of $B(h,i,j,k,l)$ is invariant under the permutation on $\mathscr{P}$ defined by $\varphi = (h\ \ i+j-l)(i\ \ l+h-j)(j+k-h\ \ k+l-i)$. Rhin and Viola managed to prove that the group $\boldsymbol{\Phi} = \langle\varphi,\sigma,\tau\rangle$ acting on $\mathscr{P}$ and leaving the value of the associated integrals invariant can be viewed as the permutation group $\mathfrak{S}_5$ acting on the set $\{h+i,\ i+j,\ j+k,\ k+l,\ l+h\}$, and hence has cardinality 120. In fact, as they noticed, this remark was first made by Dixon [**4**], in an even more general form.

Theorem 4 (Dixon). *Assume that the complex numbers $h,i,j,k,l,j+k-h,k+l-i,l+h-j,h+i-k,i+j-l$ have real part > -1. Then the integral $B(h,i,j,k,l)$ (where $x!$ is assumed to mean $\Gamma(x+1)$ for complex x) is a symmetric function of the five parameters $h+i$, $i+j$, $j+k$, $k+l$, $l+h$.*

Finally, Rhin and Viola proposed the following conjecture.

Conjecture 1 (Rhin–Viola). *Let h,i,j,k,l,h',i',j',k',l' be non-negative integers.*

(i) *If $I(h,i,j,k,l) = I(h',i',j',k',l')$, then there exists $\rho \in \boldsymbol{T}$ such that $\rho(h) = h', \rho(i) = i', \rho(j) = j', \rho(k) = k'$ and $\rho(l) = l'$.*
(ii) *Suppose furthermore that the numbers*

$$
j+k-h,\quad k+l-i,\quad l+h-j,\quad h+i-k,\quad i+j-l \tag{3.3}
$$

$$
j'+k'-h',\quad k'+l'-i',\quad l'+h'-j',\quad h'+i'-k',\quad i'+j'-l' \tag{3.4}
$$

are all non-negative. If $I(h,i,j,k,l)/I(h',i',j',k',l') \in \boldsymbol{Q}$, then there exists $\rho \in \boldsymbol{\Phi}$

such that $\rho(h)=h', \rho(i)=i', \rho(j)=j', \rho(k)=k'$ *and* $\rho(l)=l'$.

The truth of (i) and (ii) would have shown that their method is optimal, but both have been shown to be false in 2001 by Susumu Sato [**13**], who found the following counter-examples, apparently by numerical inspection.

THEOREM 5 (Sato). *Both cases of Conjecture* 1 *are false, as shown by the following six counter-examples*:

$$I(1,1,1,1,1)=5-3\zeta(2)=I(3,1,1,2,0), \tag{3.5}$$

$$I(3,1,2,2,1)=79/4-12\zeta(2)=I(4,2,2,3,0), \tag{3.6}$$

$$I(3,1,2,1,1)=3\zeta(2)-59/12=I(3,3,1,3,0), \tag{3.7}$$

$$I(3,2,2,2,1)=10\zeta(2)-148/9=I(5,1,3,2,1), \tag{3.8}$$

$$I(3,0,3,1,1)=9\zeta(2)-59/4=9I(3,3,1,2,1), \tag{3.9}$$

$$I(3,1,3,1,0)=\zeta(2)-29/18=I(3,2,1,2,0). \tag{3.10}$$

(Sato mis-stated (3.9) as $I(3,0,3,1,1)=I(3,3,1,2,1)$. The reader should also note that (3.7) and (3.9) altogether relate four different integrals rationally.) Equation (3.5) is already a counter-example to both (i) and (ii). The following questions are natural, but were not considered by Sato:

- Are these counter-examples merely numerical accidents, or do they admit a theoretical explanation?
- Do there exist infinitely many counter-examples to the conjecture of Rhin and Viola?

We give a complete answer to both questions in the two theorems below which we prove in Sections 9 and 11, respectively.

THEOREM 6. (i) *Sato's counter-examples* (3.5) *up to* (3.9) *can be explained by purely hypergeometric means, i.e., there exists a general hypergeometric identity that generates them.*
(ii) *For each integer* $\alpha \geq 1$, *the equation*

$$I(2\alpha-1,2\alpha-1,\alpha,2\alpha-1,\alpha)=I(2\alpha+1,2\alpha-1,\alpha,2\alpha,\alpha-1) \tag{3.11}$$

provides a counter-example to the cases (i) *and* (ii) *of Conjecture* 1.

THEOREM 7. (i) *Sato's counter-example* (3.10) *can be explained by purely hypergeometric means, i.e., there exists a general hypergeometric identity that generates them.*
(ii) *For each integer* $\alpha \geq 2$, *the equation*

$$I(\alpha^2-1,\alpha-1,\alpha^2-\alpha+1,\alpha-1,0)=(\alpha-1)\,I(\alpha^2-1,\alpha,\alpha^2-\alpha-1,\alpha,0) \tag{3.12}$$

provides a counter-example to case (ii) *of Conjecture* 1, *and also to case* (i) *if* $\alpha=2$.

REMARKS. (1) A particularly elegant instance of (1.1) is the one where $\gamma = \alpha + \beta$: for any complex numbers α and β which are not non-positive integers and which satisfy $\Re(\alpha + \beta) > 0$, we have

$$ {}_3F_2\left[\begin{matrix}\alpha+1,\ \beta+1,\ \alpha+\beta\\ 2\alpha+\beta+1,\ 2\beta+\alpha+1\end{matrix};1\right] = 2\,{}_3F_2\left[\begin{matrix}\alpha,\quad \beta,\quad \alpha+\beta\\ 2\alpha+\beta+1,\ 2\beta+\alpha+1\end{matrix};1\right]. \tag{3.13} $$

The action of Thomae's relations on both sides of (3.13) independently provides ten variations of (3.13), up to trivial symmetries: one example, given by (9.5) in Section 9, will be used in the proof of Theorem 6. Equation (3.11) follows from the case of (3.13) where $\alpha = \beta$ is a positive integer. Furthermore, we shall show in Section 9 that $I(2\alpha - 1, 2\alpha - 1, \alpha, 2\alpha - 1, \alpha)$ tends to 0 as α tends to infinity: this fact implies that (3.11) provides infinitely many counter-examples to Conjecture 1.

(2) As we show in Section 7, the "general" hypergeometric identity that generates (3.5) up to (3.9) is exactly identity (1.1), via the translation between integrals and hypergeometric ${}_3F_2$-series given in (3.2). Equation (3.11) is a special case.

(3) Similarly, we show in Section 7 that the "general" hypergeometric identity that generates (3.10) is exactly identity (1.2), again via the translation (3.2). Equation (3.12) is a special case. Since we show there that the integral on the left-hand side of (3.12) tends to zero as α tends to infinity, also (3.12) provides infinitely many counter-examples to Conjecture 1.

(4) It would also be interesting to look at the analogous problem arising from the group action on the triple integral

$$ \int_0^1\int_0^1\int_0^1 \frac{u^h(1-u)^l v^k(1-v)^s w^j(1-w)^q}{(1-(1-uv)w)^{q+h-r+1}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \in \boldsymbol{Q} + \boldsymbol{Q}\zeta(3) $$

found by Rhin and Viola in [**12**]: do there exist exotic relationships between such integrals that are not described by this group action? Note that this action admits an interpretation in terms of very-well-poised ${}_7F_6$-series exactly in the style of Theorem 3 (see [**23**, Section 4] for the passage from the integrals to very-well-poised ${}_7F_6$-series, and [**17**, Proposition 5, $q \to 1$, p. 6698] for a particularly elegant formulation of the group structure). For very clear expositions of various group actions on (q-) hypergeometric series, see [**15**], [**17**] and the references therein.

4. From Dixon to Thomae.

In this section, we show more precisely how Rhin and Viola's integrals are related to hypergeometric series. To get a new expression for the integral $I(h, i, j, k, l)$, we transform the integrand of (3.1) by using the binomial series expansion

$$ \frac{1}{(1-xy)^{i+j-l+1}} = \sum_{n=0}^{\infty} \frac{(i+j-l+1)_n}{n!}\,(xy)^n, $$

and the beta integral evaluations

$$\int_0^1 x^{n+h}(1-x)^i \mathrm{d}x = \frac{(n+h)!\, i!}{(n+h+i+1)!}, \qquad \int_0^1 y^{n+k}(1-y)^j \mathrm{d}y = \frac{(n+k)!\, j!}{(n+k+j+1)!}.$$

We have

$$B(h,i,j,k,l) = \frac{1}{h!\,k!\,l!} \sum_{n=0}^{\infty} \frac{(n+h)!\,(n+k)!\,(i+j-l+1)_n}{n!\,(n+h+i+1)!\,(n+k+j+1)!} \tag{4.1}$$

$$= \frac{1}{l!\,(h+i+1)!\,(k+j+1)!}\, {}_3F_2\left[\begin{matrix} h+1,\ k+1,\ i+j-l+1 \\ h+i+2,\ k+j+2 \end{matrix}; 1\right], \tag{4.2}$$

since the interchange of summation and integral is justified by Fubini's theorem. The passage from (4.1) to (4.2) uses the trivial identity $(\alpha+n)! = \alpha!\,(\alpha+1)_n$.

Under this interpretation, it is not surprising that the group obtained by Dixon and Rhin-Viola should be a reformulation of Theorem 3, in terms of integrals rather than series. Indeed, if we define a bijection between the tuples (x_1,x_2,x_3,x_4,x_5) and (h,i,j,k,l) by

$$\begin{aligned} h+1 &= x_1 - x_2 - x_3 + x_4 + x_5, \\ i+1 &= -x_1 + x_2 + x_3 + x_4 - x_5, \\ j+1 &= x_1 - x_2 + x_3 - x_4 + x_5, \\ k+1 &= -x_1 + x_2 - x_3 + x_4 + x_5, \\ l+1 &= x_1 + x_2 + x_3 - x_4 - x_5 = s, \end{aligned}$$

then we see that $B(h,i,j,k,l)$, written as (4.2), perfectly matches (2.3) and the 120 possible series are all convergent if the ten integers in the set $\mathscr{P}$ are non-negative. Since

$$\begin{aligned} &2x_1 = l+h+2,\ 2x_2 = k+l+2,\ 2x_3 = i+j+2, \\ &2x_4 = h+i+2,\ 2x_5 = j+k+2, \end{aligned} \tag{4.3}$$

we also see that the symmetry of (2.3) in the variables x_1,x_2,x_3,x_4,x_5 is equivalent to the symmetry of $B(h,i,j,k,l)$ in the variables $h+i$, $i+j$, $j+k$, $k+l$, $l+h$.

5. When is $I(h,i,j,k,l)$ rational?

In this section we answer the question of "simple-minded" counter-examples to Conjecture 1 that was raised in the Introduction.

Theorem 8. *Let h,i,j,k,l be non-negative integers. Then the following assertions are equivalent*:

(a) *The integers $j+k-h$, $k+l-i$, $l+h-j$, $h+i-k$, $i+j-l$ are all non-negative.*
(b) *The integral $I(h,i,j,k,l)$ is an irrational number.*

Remarks. (1) We remarked earlier (see (3.2) and (4.2)) that $I(h,i,j,k,l)$ is essentially equal to a ${}_3F_2$-series. If we translate Theorem 8 into the analogous theorem for

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; 1\right], \tag{5.1} $$

via the relations $a = h+1$, $b = k+1$, $c = i+j-l+1$, $d = h+i+2$, $e = k+j+2$, we get the following necessary and sufficient condition for the irrationality of the series (5.1) for integral values of a, b, c, d, e:

$$ d+e \geq a+b+c+1, \quad a \geq 1, \quad b \geq 1, \quad c \geq 1, \tag{5.2} $$

$$ d \geq \max\{a, b, c\} + 1, \tag{5.3} $$

$$ e \geq \max\{a, b, c\} + 1. \tag{5.4} $$

(2) As a marginal consequence, Theorem 8 proves that the analogue of the case (ii) of Conjecture 1, where we now suppose that the non-negativity condition (3.3) is not true, cannot hold either. Indeed, if one of the integers in (3.3) were negative and none in (3.4) were negative, then the value of $I(h, i, j, k, l)/I(h', i', j', k', l')$ would be irrational and the conjecture would be empty. And if one of the integers in (3.3) and one in (3.4) were negative, then the conjecture would be trivially false because, although $I(h, i, j, k, l)$ and $I(h', i', j', k', l')$ are rational, there exist many $\boldsymbol{\Phi}$-unrelated choices for h, h', etc.: one may consider $I(1, 1, 1, 1, 3)$ and $I(1, 1, 1, 1, 4)$ for example.

Proof of Theorem 8. We first show the implication (b) $\Rightarrow$ (a). Since the parameters $j+k-h$, $k+l-i$, $l+h-j$, $h+i-k$, $i+j-l$ are cyclically permuted by $\tau \in \boldsymbol{T}$, if one of them were negative, then without loss of generality, we may assume that it is $i+j-l$. But $i+j-l \leq -1$ implies that the integrand of $I(h, i, j, k, l)$ is a polynomial with integral coefficients and hence that $I(h, i, j, k, l) \in \boldsymbol{Q}$.

The reverse implication (a) $\Rightarrow$ (b) is a little bit more complicated. Since i, j and $i+j-l$ are non-negative integers, we can write the expansion (4.1) in the equivalent form:

$$ I(h, i, j, k, l) = \frac{i!\, j!}{(i+j-l)!} \sum_{n=0}^{\infty} \frac{(n+1)_{i+j-l}}{(n+h+1)_{i+1}(n+k+1)_{j+1}}. \tag{5.5} $$

(We used trivial identities such as $(n+h+i)!/(n+h)! = (n+h+1)_{i+1}$.) We know that $I(h, i, j, k, l) \in \boldsymbol{Q} + \boldsymbol{Q}\zeta(2)$ and it will be enough to prove that the coefficient $p(h, i, j, k, l)$ of the irrational number $\zeta(2)$ is non-zero. A standard way to find an explicit expression for this coefficient is to expand the summand of (5.5), which is a rational function of n, in partial fractions (see the introduction of [**8**] for details in many similar cases and references). All computations done, one finds that

$$ p(h, i, j, k, l) = (-1)^{h+i+j+k+l} \sum_{s=\max(h,k,i+j-l)}^{\min(h+i,k+j)} \binom{i}{s-h}\binom{j}{s-k}\binom{s}{i+j-l}, $$

with the convention that the value of the sum is 0 if it is empty. The latter is the case if and only if $\min(h+i, k+j) < \max(h, k, i+j-l)$.

We now show that condition (a) ensures that the sum is non-empty and hence that

$$(-1)^{h+i+j+k+l} p(h, i, j, k, l) > 0,$$

because it is a sum of binomial coefficients. We have already used the fact that $i+j-l \geq 0$. Since the inequalities $h+i-k \geq 0$ and $k+j-h \geq 0$ imply that $\max(h, k) \leq \min(h+i, k+j)$, it only remains to show that $i+j-l \leq \min(h+i, k+j)$ to finally prove that $\min(h+i, k+j) \geq \max(h, k, i+j-l)$. But $\min(h+i, k+j) - (i+j-l) = \min(h+l-j, k+l-i) \geq 0$, which finishes the proof. □

6. Effective computation of Thomae relations.

In this section, we show how to compute the complete set of (generically) 120 Thomae relations (convergent or not) for any given ${}_3F_2$-series with argument 1. We need this to transform Sato's counter-examples into more suitable forms. The most effective way to do this is by using the parametrisation $2x_1 - s, 2x_2 - s, 2x_3 - s$ and $2x_4, 2x_5$ of the upper and lower parameters of the ${}_3F_2$-series from Theorem 3. If we denote the upper parameters by a, b, c and the lower parameters by d, e, then $x_1, x_2, x_3, x_4, x_5, s$ and a, b, c, d, e are related by

$$2x_1 = d+e-b-c, \quad 2x_2 = d+e-c-a, \quad 2x_3 = d+e-a-b, \quad 2x_4 = d, \quad 2x_5 = e,$$

and

$$s = x_1 + x_2 + x_3 - x_4 - x_5 = 2(d+e-a-b-c).$$

If one prefers the integral setting, then one gets the parametrisation of Theorem 3 of the integral $I(h, i, j, k, l)$ by the formulae (4.3) given in Section 4.

The following simple *Maple* commands compute all possible values of the arrays of parameters $[2x_{\rho(1)} - s_\rho, 2x_{\rho(2)} - s_\rho, 2x_{\rho(3)} - s_\rho; 2x_{\rho(4)}, 2x_{\rho(5)}]$ (with $s_\rho = x_{\rho(1)} + x_{\rho(2)} + x_{\rho(3)} - x_{\rho(4)} - x_{\rho(5)}$) over all permutations ρ of $\{1, 2, 3, 4, 5\}$, with the nice feature to output only the term-wise different arrays (viewed as 5-tuples by the program):

```
>with(combinat):
>p := (u,v,w,x,y)-> permute([u,v,w,x,y]):
>s := (u,v,w,x,y)-> u+v+w-x-y:
>A := (u,v,w,x,y)-> [2*u-s(u,v,w,x,y),2*v-s(u,v,w,x,y),2*w-s(u,v,w,x,y),2*x,2*y]:
>T := (u,v,w,x,y)-> seq(A(op(1,op(j,p(u,v,w,x,y))),op(2,op(j,p(u,v,w,x,y))),
  op(3,op(j,p(u,v,w,x,y))),op(4,op(j,p(u,v,w,x,y))),op(5,op(j,p(u,v,w,x,y)))),
  j = 1...nops(p(u,v,w,x,y))):
>F := (a,b,c,d,e)-> T((d+e-b-c)/2,(d+e -c-a)/2,(d+e-a-b)/2,d/2,e/2):
>I := (h,i,j,k,l)-> T((h+l+2)/2,(k+l+2)/2,(i+j+2)/2,(h+i+2)/2,(j+k+2)/2):
```

The function `T` computes all the different expressions for the value of the symmetric function in Theorem 3, `F` does the same for a ${}_3F_2[a,b,c;d,e]$ and `I` for ([1]) $I(h,i,j,k,l)$. Only the Gamma-factors are not computed, but this could be easily done. For example, we obtain

```
> I(1,1,1,1,1);
```

$$[2,2,2,4,4]$$

```
> I(3,1,1,2,0);
```

$$[4,3,3,6,5],\ [2,1,3,4,5],\ [3,2,3,6,4],\ [1,1,2,4,4],\ [2,4,2,5,5].$$

Maple outputs 25 other arrays for $I(3,1,1,2,0)$ but since they correspond to the five above by the trivial symmetries, we do not list them. We can also find the Thomae relations for both sides of counter-example (3.10):

```
> I(3,1,3,1,0);
```

$$[4,2,5,6,6],\ [1,2,2,6,3],\ [1,4,4,5,6],\ [1,1,1,3,5]$$

```
> I(3,2,1,2,0);
```

$$[4,3,4,7,5],\ [2,1,4,5,5],\ [1,1,3,4,5],\ [3,3,3,4,7].$$

This shows that the relations $I(1,1,1,1,1) = I(3,1,1,2,0)$ and $I(3,1,3,1,0) = I(3,2,1,2,0)$ are not consequences of Thomae relations. Similar computations provide a verification of the other counter-examples.

7. The pattern behind Sato's counter-examples.

With the interpretation given in Section 4, the case (ii) of Conjecture 1 can be reformulated as follows:

If there exists a linear dependence relation over $\boldsymbol{Q}$ between two convergent ${}_3F_2$-series with argument 1, with integral parameters, and whose values are irrational numbers, then this relation is a specialisation of one of the 120 Thomae relations.

Sato's counter-examples destroy this hope. We can formulate his counter-examples (3.5) and (3.6) in hypergeometric form (with simplification of the Gamma-factors) as follows:

$$ {}_3F_2\left[\begin{matrix}2,2,2\\4,4\end{matrix};1\right] = \frac{3}{20}\,{}_3F_2\left[\begin{matrix}4,3,3\\6,5\end{matrix};1\right] \quad\text{and}\quad {}_3F_2\left[\begin{matrix}4,3,3\\6,6\end{matrix};1\right] = \frac{2}{21}\,{}_3F_2\left[\begin{matrix}5,4,5\\8,7\end{matrix};1\right]. \qquad (7.1)$$

Under this form, the parameters on the left-hand sides and those on the right-hand sides seem still rather unrelated, and it is thus still unclear whether we face numerical accidents or if there is something deeper behind.

However, a natural thing to do here is to seek new numerical relations by applying Thomae's transformations (using the *Maple* commands of the previous section) to each of the four ${}_3F_2$-series in (7.1), independently. We find that we are trying to prove that

[1] The letter "`I`" denotes the complex number i in *Maple* and one must use another symbol. But since LaTeX is not *Maple*, there is no problem here.

$$ {}_3F_2\left[\begin{matrix}2,2,2\\4,4\end{matrix};1\right]=2\,{}_3F_2\left[\begin{matrix}1,1,2\\4,4\end{matrix};1\right]\quad\text{and}\quad {}_3F_2\left[\begin{matrix}3,2,3\\6,5\end{matrix};1\right]=2\,{}_3F_2\left[\begin{matrix}2,1,3\\6,5\end{matrix};1\right], \tag{7.2}$$

where a pattern now emerges, explained by the earlier identity (3.13).

The hypergeometric forms of the three counter-examples (3.7), (3.8) and (3.9) are

$$\begin{gathered} {}_3F_2\left[\begin{matrix}4,2,3\\6,5\end{matrix};1\right]=\frac{3}{35}\,{}_3F_2\left[\begin{matrix}4,4,5\\8,6\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}4,3,4\\7,6\end{matrix};1\right]=\frac{5}{7}\,{}_3F_2\left[\begin{matrix}6,3,4\\8,7\end{matrix};1\right],\\ {}_3F_2\left[\begin{matrix}4,2,3\\5,6\end{matrix};1\right]=\frac{1}{21}\,{}_3F_2\left[\begin{matrix}4,3,4\\8,5\end{matrix};1\right], \end{gathered}\tag{7.3}$$

which become much more illuminating when rewritten as

$$\begin{gathered} {}_3F_2\left[\begin{matrix}3,2,4\\6,5\end{matrix};1\right]=\frac{1}{3}\,{}_3F_2\left[\begin{matrix}2,1,4\\6,5\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}3,2,2\\6,5\end{matrix};1\right]=\frac{2}{3}\,{}_3F_2\left[\begin{matrix}2,1,2\\6,5\end{matrix};1\right],\\ {}_3F_2\left[\begin{matrix}3,2,4\\6,5\end{matrix};1\right]=\frac{1}{3}\,{}_3F_2\left[\begin{matrix}2,1,4\\6,5\end{matrix};1\right], \end{gathered}$$

by using Thomae's relations. (In particular, (3.7) and (3.9) are consequences of the same identity). The connexion with Theorem 1 is now clear.

Finally, the hypergeometric form of the counter-example (3.10) is

$$ {}_3F_2\left[\begin{matrix}4,2,5\\6,6\end{matrix};1\right]=\frac{5}{9}\,{}_3F_2\left[\begin{matrix}4,3,4\\7,5\end{matrix};1\right], \tag{7.4}$$

which is obviously the special case $\alpha=4$, $\beta=3$, $\gamma=4$ of (1.2).

8. Proof of Theorem 1.

For the proof of (1.1) we need the following transformation formula due to Verma and Jain (see [**3**, (3.5.10), $q\to 1$, reversed], being implied by [**18**, (4.1)]) between a ${}_3F_2$-series and a very-well-poised ${}_7F_6$-series:

$$\begin{aligned} {}_3F_2\left[\begin{matrix}b,c,d\\a,a-b+c\end{matrix};1\right]&=\frac{\Gamma(2a)\,\Gamma(2a-2b-d)\,\Gamma(a-b+c)\,\Gamma(a-d+c)}{\Gamma(2a-2b)\,\Gamma(2a-d)\,\Gamma(a+c)\,\Gamma(a-b-d+c)}\\ &\quad\times{}_7F_6\left[\begin{matrix}a-\frac12,\frac a2+\frac34,b,\frac d2,\frac d2+\frac12,\frac a2-\frac c2,\frac a2-\frac c2+\frac12\\ \frac a2-\frac14,a-b+\frac12,a-\frac d2+\frac12,a-\frac d2,\frac a2+\frac c2+\frac12,\frac a2+\frac c2\end{matrix};1\right]. \end{aligned}\tag{8.1}$$

If we apply this transformation to the ${}_3F_2$-series on the left-hand side of (1.1), then we obtain

$$\frac{\Gamma(\alpha+2\beta+1)\,\Gamma(4\alpha+2\beta+2)\,\Gamma(2\alpha+2\beta-\gamma)\,\Gamma(2\alpha+2\beta-\gamma+2)}{\Gamma(2\alpha+2\beta)\,\Gamma(2\alpha+2\beta+2)\,\Gamma(\alpha+2\beta-\gamma+1)\,\Gamma(4\alpha+2\beta-\gamma+2)}$$
$$\times\,{}_7F_6\left[\begin{matrix}2\alpha+\beta+\frac{1}{2},\alpha+\frac{\beta}{2}+\frac{5}{4},\alpha+1,\frac{\gamma}{2},\frac{\gamma}{2}+\frac{1}{2},\alpha,\alpha+\frac{1}{2}\\ \alpha+\frac{\beta}{2}+\frac{1}{4},\alpha+\beta+\frac{1}{2},2\alpha+\beta-\frac{\gamma}{2}+\frac{3}{2},2\alpha+\beta-\frac{\gamma}{2}+1,\alpha+\beta+\frac{3}{2},\alpha+\beta+1\end{matrix};1\right]. \tag{8.2}$$

We permute the parameters in the ${}_7F_6$-series to get the equivalent expression

$$\frac{\Gamma(\alpha+2\beta+1)\,\Gamma(4\alpha+2\beta+2)\,\Gamma(2\alpha+2\beta-\gamma)\,\Gamma(2\alpha+2\beta-\gamma+2)}{\Gamma(2\alpha+2\beta)\,\Gamma(2\alpha+2\beta+2)\,\Gamma(\alpha+2\beta-\gamma+1)\,\Gamma(4\alpha+2\beta-\gamma+2)}$$
$$\times\,{}_7F_6\left[\begin{matrix}2\alpha+\beta+\frac{1}{2},\alpha+\frac{\beta}{2}+\frac{5}{4},\alpha,\frac{\gamma}{2},\frac{\gamma}{2}+\frac{1}{2},\alpha+\frac{1}{2},\alpha+1\\ \alpha+\frac{\beta}{2}+\frac{1}{4},\alpha+\beta+\frac{3}{2},2\alpha+\beta-\frac{\gamma}{2}+\frac{3}{2},2\alpha+\beta-\frac{\gamma}{2}+1,\alpha+\beta+1,\alpha+\beta+\frac{1}{2}\end{matrix};1\right]. \tag{8.3}$$

To this ${}_7F_6$-series, we apply the transformation (8.1) in the backward direction, that is we apply the transformation

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}a,\ \frac{a}{2}+1,\ b,\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2}\\ \frac{a}{2},\ a-b+1,\ a-c+1,\ a-c+\frac{1}{2},\ a-d+1,\ a-d+\frac{1}{2}\end{matrix};1\right]\\ &\quad=\frac{\Gamma(2a-2b+1)\,\Gamma(2a-2c+1)\,\Gamma(2a-2d+1)\,\Gamma(2a-b-2c-2d+1)}{\Gamma(2a+1)\,\Gamma(2a-2b-2c+1)\,\Gamma(2a-b-2d+1)\,\Gamma(2a-2c-2d+1)}\\ &\qquad\times\,{}_3F_2\left[\begin{matrix}2c,\ b,\ a-2d+\frac{1}{2}\\ 2a-b-2d+1,\ a+\frac{1}{2}\end{matrix};1\right].\end{aligned}$$

Thus we directly arrive at the right-hand side of (1.1).

9. Proof of Theorem 6.

As already mentioned in Section 7, the cases $(\alpha,\beta,\gamma)=(1,1,2)$, $(2,1,3)$, $(2,1,4)$, $(2,1,2)$, $(2,1,4)$ of identity (1.1) are simply reformulations of Sato's counter-examples (3.5) up to (3.9) and (i) is proved.

For (ii), the idea is to prove that no specialisation of both sides of (3.13) can follow from the 120 Thomae relations, at least when $\alpha=\beta$. One may note that (3.13) cannot formally be a consequence of any of Thomae's relations since two (one would be enough) of its specialisation are not such consequences. But this does not rule out the possibility that some other specialisations would follow from Thomae's relations. However, we show that this is never the case when $\alpha=\beta$ is a positive integer.

We first determine the 120 Thomae relations for the left-hand side of (3.13) when $\alpha=\beta$ and to do this painlessly, we express

$${}_3F_2\left[\begin{matrix}\alpha+1,\ \alpha+1,\ 2\alpha\\ 3\alpha+1,\ 3\alpha+1\end{matrix};1\right] \tag{9.1}$$

in the symmetric form (2.3) in Theorem 3, which gives $2x_1=2x_2=2x_4=2x_5=3\alpha+1$

and $2x_3 = 4\alpha$. The permutations of x_1, x_2, x_3, x_4 and x_5 show that (9.1) is related only to the ${}_3F_2$-series

$$ {}_3F_2\left[\begin{matrix}\alpha+1,\ \alpha+1,\ 2\alpha\\ 3\alpha+1,\ 3\alpha+1\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}2\alpha,\ 2\alpha,\ 2\alpha\\ 3\alpha+1,\ 4\alpha\end{matrix};1\right], \tag{9.2} $$

and those obtained by the trivial symmetries. The same process applied to the right-hand side of (3.13) (for $\alpha=\beta$),

$$ {}_3F_2\left[\begin{matrix}\alpha,\ \alpha,\ 2\alpha\\ 3\alpha+1,\ 3\alpha+1\end{matrix};1\right], \tag{9.3} $$

gives $2x_1 = 2x_2 = 3\alpha+2$, $2x_3 = 4\alpha+2$ and $2x_4 = 2x_5 = 3\alpha+1$. The permutations of x_1, x_2, x_3, x_4 and x_5 show that (9.3) is related to the five ${}_3F_2$-series

$$ \begin{gathered} {}_3F_2\left[\begin{matrix}\alpha,\ \alpha,\ 2\alpha\\ 3\alpha+1,\ 3\alpha+1\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}2\alpha+2,\ 2\alpha+1,\ 2\alpha+1\\ 4\alpha+2,\ 3\alpha+2\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}2\alpha+1,\ \alpha+1,\ \alpha\\ 3\alpha+2,\ 3\alpha+1\end{matrix};1\right],\\ {}_3F_2\left[\begin{matrix}2\alpha+1,\ 2\alpha+1,\ 2\alpha\\ 4\alpha+2,\ 3\alpha+1\end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix}2\alpha+2,\ \alpha+1,\ \alpha+1\\ 3\alpha+2,\ 3\alpha+2\end{matrix};1\right], \end{gathered} \tag{9.4} $$

and those obtained by the trivial symmetries.

Inspection quickly reveals the impossibility of any numerical coincidence between one of the two arrays of parameters in (9.2) and one of the five arrays in (9.4), even with trivial symmetries. However, each such coupling provides a variation of (3.13) and, for example, we have that

$$ {}_3F_2\left[\begin{matrix}2\alpha,\ 2\alpha,\ 2\alpha\\ 4\alpha,\ 3\alpha+1\end{matrix};1\right] = \frac{\alpha(2\alpha+1)}{(3\alpha+1)(4\alpha+1)}\,{}_3F_2\left[\begin{matrix}2\alpha+2,\ 2\alpha+1,\ 2\alpha+1\\ 4\alpha+2,\ 3\alpha+2\end{matrix};1\right], \tag{9.5} $$

which will be used below.

We are now in a position to prove the claim about the infinity of counter-examples to the cases (i) and (ii) of Conjecture 1. First, thanks to (4.2), we have that

$$ I(2\alpha-1,2\alpha-1,\alpha,2\alpha-1,\alpha) = \frac{(2\alpha-1)!^3\,\alpha!}{(4\alpha-1)!\,(3\alpha)!}\,{}_3F_2\left[\begin{matrix}2\alpha,\ 2\alpha,\ 2\alpha\\ 4\alpha,\ 3\alpha+1\end{matrix};1\right] $$

and

$$ \begin{aligned} &I(2\alpha+1,2\alpha-1,\alpha,2\alpha,\alpha-1)\\ &\qquad = \frac{(2\alpha+1)!\,(2\alpha)!\,(2\alpha-1)!\,\alpha!}{(4\alpha+1)!\,(3\alpha+1)!}\,{}_3F_2\left[\begin{matrix}2\alpha+2,\ 2\alpha+1,\ 2\alpha+1\\ 4\alpha+2,\ 3\alpha+2\end{matrix};1\right]. \end{aligned} $$

We can relate these two equations by (9.5), and the simplification of the Gamma-factors yields

$$I(2\alpha-1,2\alpha-1,\alpha,2\alpha-1,\alpha)=I(2\alpha+1,2\alpha-1,\alpha,2\alpha,\alpha-1),$$

which is exactly the identity (3.11) we are looking for. For both integrals, the non-negativity conditions (3.3) and (3.4) in case (ii) of Conjecture 1 are verified and the above discussion proves that there exists no permutation ρ in the group $\boldsymbol{\Phi}$ (and, *a fortiori*, also none in $\boldsymbol{T}$) such that $\rho(2\alpha-1)=2\alpha+1$, $\rho(2\alpha-1)=2\alpha-1$, $\rho(\alpha)=\alpha$, $\rho(2\alpha-1)=2\alpha$, $\rho(\alpha)=\alpha-1$. Thus, for each value of the positive integer α, we obtain a counter-example to the cases (i) and (ii) of Conjecture 1 at the same time. That this provides infinitely many counter-examples is a consequence of the fact that $I(2\alpha-1,2\alpha-1,\alpha,2\alpha-1,\alpha)$ tends to 0 as α tends to infinity, because

$$\begin{aligned}&\lim_{\alpha\to+\infty} I(2\alpha-1,2\alpha-1,\alpha,2\alpha-1,\alpha)^{1/\alpha}\\&\quad=\max_{(x,y)\in[0,1]^2}\left(\frac{x^2(1-x)^2y^2(1-y)}{(1-xy)^2}\right)=17-12\sqrt{2}<1.\end{aligned}$$

REMARK. We could do the same thing with α not necessarily equal to β. To find all Thomae relations for the left-hand side of (3.13), one should use Theorem 3 with

$$2x_1=2x_4=2\alpha+\beta+1,\quad 2x_2=2x_5=2\beta+\alpha+1,\quad 2x_3=2\alpha+2\beta,$$

leading to five different arrays up to trivial symmetries, and for the right-hand side with

$$\begin{aligned}&2x_1=2\alpha+\beta+2,\quad 2x_2=2\beta+\alpha+2,\quad 2x_3=2\alpha+2\beta+2,\\&2x_4=2\alpha+\beta+1,\quad 2x_5=2\beta+\alpha+1,\end{aligned}$$

leading to a complete set of 120 different arrays for generic α and β (in fact, only 10 arrays, up to trivial symmetries). This explains why we consider only the case $\alpha=\beta$, which is much simpler to deal with.

10. Proof of Theorem 2.

In order to derive Theorem 2, we require the following proposition, relating two "contiguous" ${}_3F_2$-series in a way that the "rest" is a closed form expression. (See the Remark after the proof of the proposition for an explanation of the term "contiguous.")

PROPOSITION 1. *For any complex numbers a,b,c such that $\Re(d-b-c+1)>0$, and such that $a+1$ and d are not non-positive integers, we have the identity*

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,b,c\\a+1,\ d\end{matrix};1\right]&=\frac{(a-b+1)(a-b+2)(a-c+1)(d-1)}{(a+1)(b-1)(a-d+2)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a+1,\ b-1,\ c\\a+2,\ d-1\end{matrix};1\right]\\&\quad+\frac{(1-a-a^2-b+c+ac-bc-d+bd)}{(b-1)(a-d+2)(a-d+1)}\,\frac{\Gamma(d)\,\Gamma(d-b-c+1)}{\Gamma(d-b)\,\Gamma(d-c)}.\end{aligned}\tag{10.1}$$

Proof. We start by applying the contiguous relation

$$
{}_3F_2\left[\begin{matrix}A_1,\ A_2,\ A_3\\ B_1,\ B_2\end{matrix};z\right]=\frac{(1-A_1+A_2)(B_1-1)}{(A_1-1)(1+A_2-B_1)}\,{}_3F_2\left[\begin{matrix}A_1-1,\ A_2,\ A_3\\ B_1-1,\ B_2\end{matrix};z\right]
$$
$$
+\frac{A_2(B_1-A_1)}{(A_1-1)(B_1-A_2-1)}\,{}_3F_2\left[\begin{matrix}A_1-1,\ A_2+1,\ A_3\\ B_1,\ B_2\end{matrix};z\right],\tag{10.2}
$$

with $A_1=b$, $A_2=a$, and $B_1=d$, to the ${}_3F_2$-series on the left-hand side. This yields the expression

$$
\frac{(a-b+1)(d-1)}{(b-1)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a,\ b-1,\ c\\ d-1,\ a+1\end{matrix};1\right]-\frac{a(d-b)}{(b-1)(a-d+1)}\,{}_2F_1\left[\begin{matrix}b-1,\ c\\ d\end{matrix};1\right].
$$

We sum the ${}_2F_1$-series by means of the Gauß summation formula (see [**14**, (1.7.6); Appendix (III.3)])

$$
{}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.\tag{10.3}
$$

Thus, we obtain

$$
\frac{(a-b+1)(d-1)}{(b-1)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a,\ b-1,\ c\\ a+1,\ d-1\end{matrix};1\right]-\frac{a\,\Gamma(d)\,\Gamma(d-b-c+1)}{(b-1)(a-d+1)\,\Gamma(d-b)\,\Gamma(d-c)}.
$$

Next we apply the contiguous relation

$$
{}_3F_2\left[\begin{matrix}A_1,\ A_2,\ A_3\\ B_1,\ B_2\end{matrix};z\right]={}_3F_2\left[\begin{matrix}A_1+1,\ A_2,\ A_3\\ B_1,\ B_2\end{matrix};z\right]-z\frac{A_2A_3}{B_1B_2}\,{}_3F_2\left[\begin{matrix}A_1+1,\ A_2+1,\ A_3+1\\ B_1+1,\ B_2+1\end{matrix};z\right],\tag{10.4}
$$

with $A_1=a$. This gives

$$
\frac{(a-b+1)(d-1)}{(b-1)(a-d+1)}\,{}_2F_1\left[\begin{matrix}b-1,\ c\\ d-1\end{matrix};1\right]-\frac{(a-b+1)c}{(a+1)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a+1,\ b,\ c+1\\ a+2,\ d\end{matrix};1\right]
$$
$$
+\frac{a\,\Gamma(d)\,\Gamma(d-b-c+1)}{(b-1)(-1-a+d)\,\Gamma(d-b)\,\Gamma(d-c)}.
$$

Of course, the ${}_2F_1$-series can be summed by means of the Gauß summation formula (10.3). After some simplification, we arrive at

$$
-\frac{(a-b+1)c}{(a+1)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a+1,\ b,\ c+1\\ 2+a,\ d\end{matrix};1\right]+\frac{(1+a+c-d)\,\Gamma(d)\,\Gamma(d-b-c)}{(a-d+1)\,\Gamma(d-b)\,\Gamma(d-c)}.\tag{10.5}
$$

Now we apply another time the contiguous relation (10.2), this time with $A_1 = b$, $A_2 = a+1$, and $B_1 = d$. We obtain

$$
\begin{aligned}
&- \frac{(a-b+1)(a-b+2)c(d-1)}{(a+1)(b-1)(a-d+1)(a-d+2)}\, {}_3F_2\left[\begin{matrix} a+1,\ b-1,\ c+1 \\ a+2,\ d-1 \end{matrix}; 1\right] \\
&+ \frac{(a-b+1)c(d-b)}{(b-1)(a-d+1)(a-d+2)}\, {}_2F_1\left[\begin{matrix} b-1,\ c+1 \\ d \end{matrix}; 1\right] \\
&+ \frac{(a+c-d+1)\,\Gamma(d)\,\Gamma(d-b-c)}{(a-d+1)\,\Gamma(d-b)\,\Gamma(d-c)},
\end{aligned}
$$

and after evaluation of the ${}_2F_1$-series by means of Gauß' summation formula (10.3),

$$
\begin{aligned}
&- \frac{(a-b+1)(a-b+2)c(d-1)}{(a+1)(b-1)(a-d+1)(a-d+2)}\, {}_3F_2\left[\begin{matrix} a+1,\ b-1,\ c+1 \\ a+2,\ d-1 \end{matrix}; 1\right] \\
&+ \frac{P(a,b,c,d)\,\Gamma(d)\,\Gamma(d-b-c)}{(b-1)(a-d+1)(a-d+2)\,\Gamma(d-b)\,\Gamma(d-c)},
\end{aligned}
$$

where

$$
\begin{aligned}
P(a,b,c,d) = &-2 - 3a - a^2 + 2b + 3ab + a^2b - 3c - 2ac + 3bc + abc - c^2 - ac^2 \\
&+ bc^2 + 3d + 2ad - 3bd - 2abd + 2cd + acd - 2bcd - d^2 + bd^2.
\end{aligned}
$$

The final contiguous relation that we apply is

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} A_1,\ A_2,\ A_3 \\ B_1,\ B_2 \end{matrix}; z\right] = &\frac{A_1-A_2-1}{A_1-1}\, {}_3F_2\left[\begin{matrix} A_1-1,\ A_2,\ A_3 \\ B_1,\ B_2 \end{matrix}; z\right] \\
&+ \frac{A_2}{A_1-1}\, {}_3F_2\left[\begin{matrix} A_1-1,\ A_2+1,\ A_3 \\ B_1,\ B_2 \end{matrix}; z\right],
\end{aligned}
\tag{10.6}
$$

with $A_1 = c+1$ and $A_2 = a+1$. The result is

$$
\begin{aligned}
&\frac{(a-b+1)(a-b+2)(a-c+1)(d-1)}{(a+1)(b-1)(a-d+1)(a-d+2)}\, {}_3F_2\left[\begin{matrix} a+1,\ b-1,\ c \\ a+2,\ d-1 \end{matrix}; 1\right] \\
&- \frac{(a-b+1)(a-b+2)(d-1)}{(b-1)(a-d+1)(a-d+2)}\, {}_2F_1\left[\begin{matrix} b-1,\ c \\ d-1 \end{matrix}; 1\right] \\
&+ \frac{P(a,b,c,d)\,\Gamma(d)\,\Gamma(d-b-c)}{(b-1)(a-d+1)(a-d+2)\,\Gamma(d-b)\,\Gamma(d-c)}.
\end{aligned}
$$

A last use of Gauß' summation formula and some simplification then leads to (10.1). □

Remark. Since we shall re-use it in Section 12, it will be beneficial if we briefly summarise the idea of the proof of the above proposition: it is crucially based on the fact that the ${}_3F_2$-series on the left of (10.1) has the parameter a on top and the parameter $a+1$ at the bottom. Now we apply elementary contiguous relations (such as the one in (10.2)). In principle, it expresses our ${}_3F_2$-series as a sum of two other ${}_3F_2$-series in which the parameters are "contiguous" to the original ${}_3F_2$-series, meaning that they differ from the parameters of the original series by small integer amounts. (In (10.2), these differences are 0 and ± 1.) However, in one of the two ${}_3F_2$-series on the right-hand side of the relation, the top parameter $A_2 = a$ is raised by 1, while the bottom parameter $B_2 = a+1$ is left invariant. Thus, the two $(a+1)$'s cancel, and the ${}_3F_2$-series reduces to a ${}_2F_1$-series, to which the Gauß summation formula (10.3) can be applied to express it in closed form. This partial simplification happens as well when we apply the contiguous relations (10.4) and (10.6). Thus, each time, we obtain a ${}_3F_2$-series plus an additional expression in closed form. These additional expressions are put together, and they finally form the expression containing the gamma functions on the right-hand side of (10.1). However, since several similar, but not identical, such expressions were put together, when factoring the resulting term, a polynomial factor built up. Hence, in order to obtain a relation between two ${}_3F_2$-series without any additional term, this polynomial factor must vanish. While, normally (i.e., if one plays the above described game in a random fashion), equating this polynomial factor to zero will not have any nice solutions (in particular, no *integral* solutions, which we would however need to construct counter-examples to the conjecture by Rhin and Viola, in the cases of Propositions 1–3), the contiguous relations have been carefully selected so that at least one of the variables a, b, c, d is contained only linearly in the polynomial factor. This makes it possible to have many non-trivial solutions when equating the polynomial factor to zero.

In view of the above remark, the proof of Theorem 2 is now straight-forward.

Proof of Theorem 2. If we now choose d such that the polynomial factor on the right-hand side of (10.1) vanishes, that is,

$$d = c + \frac{a(a-c+1)}{b-1} + 1,$$

and subsequently do the replacements $a \to \alpha$, $b \to \beta$, $c \to \gamma$, then we obtain exactly (1.2). $\square$

11. Proof of Theorem 7.

As already mentioned in Section 7, the case $(\alpha, \beta, \gamma) = (4, 3, 4)$ of identity (1.2) is a simple reformulation of Sato's counter-example (3.10). Thus, (i) is proved.

For (ii), we proceed in a similar fashion as in Section 9. First of all, we observe that identity (3.12) is the special case of (1.2) in which α is replaced by α^2, and in which $\beta = \alpha + 1$ and $\gamma = \alpha^2$, again via the translation (3.2). To wit, this is

$$ {}_3F_2\left[\begin{matrix} \alpha^2,\ \alpha^2,\ \alpha+1 \\ \alpha^2+1,\ \alpha^2+\alpha+1 \end{matrix};1\right] = \frac{\alpha^3+1}{\alpha^2+1}\, {}_3F_2\left[\begin{matrix} \alpha^2+1,\ \alpha^2,\ \alpha \\ \alpha^2+2,\ \alpha^2+\alpha \end{matrix};1\right]. \tag{11.1} $$

In the sequel we concentrate on this special case, always assuming that α is a positive integer strictly greater than 1.

Using Thomae's relations, we can generate three other series which are related to the ${}_3F_2$-series on the left-hand side of (11.1), namely

$$ {}_3F_2\left[\begin{matrix} \alpha+1,\ \alpha+1,\ \alpha+1 \\ \alpha+2,\ \alpha^2+\alpha+1 \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} 1,\ 1,\ \alpha+1 \\ \alpha+2,\ \alpha^2+1 \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} 1,\ \alpha^2,\ \alpha^2-\alpha \\ \alpha^2+1,\ \alpha^2+1 \end{matrix};1\right]. \tag{11.2} $$

On the other hand, there are six series related to the ${}_3F_2$-series on the right-hand side of (11.1),

$$ {}_3F_2\left[\begin{matrix} \alpha-1,\ \alpha^2,\ \alpha^2 \\ \alpha^2+1,\ \alpha^2+\alpha \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} \alpha-1,\ \alpha,\ \alpha \\ \alpha+1,\ \alpha^2+\alpha \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} 2,\ \alpha^2+1,\ \alpha^2-\alpha+2 \\ \alpha^2+2,\ \alpha^2+2 \end{matrix};1\right], $$
$$ {}_3F_2\left[\begin{matrix} 1,\ \alpha^2,\ \alpha^2-\alpha+2 \\ \alpha^2+1,\ \alpha^2+2 \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} 1,\ 2,\ \alpha \\ \alpha+1,\ \alpha^2+2 \end{matrix};1\right],\quad {}_3F_2\left[\begin{matrix} 1,\ 1,\ \alpha-1 \\ \alpha+1,\ \alpha^2+1 \end{matrix};1\right]. \tag{11.3} $$

None of these match with the series on the left-hand side of (11.1) or with one of the series in (11.2). Thus, indeed, for any positive integer α, (3.12) is a counter-example to the conjecture by Rhin and Viola.

Finally, in order to see that (3.12) produces infinitely many counter-examples, we show again that the involved integral tends to zero when α tends to infinity. Indeed, for $\alpha \ge 1$, we have

$$ \begin{aligned} &I(\alpha^2-1,\alpha-1,\alpha^2-\alpha+1,\alpha-1,0) \\ &\quad= \int_0^1\int_0^1 \frac{x^{\alpha^2-1}(1-x)^{\alpha-1}y^{\alpha-1}(1-y)^{\alpha^2-\alpha+1}}{(1-xy)^{\alpha^2+1}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\le \int_0^1\int_0^1 x^{\alpha-1}y^{\alpha-1}\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = \sum_{k=0}^{\infty}\frac{1}{(k+\alpha)^2}, \end{aligned} $$

from which the claim follows. (In the second line, we used the trivial facts that $x^{\alpha^2} \le x^{\alpha}$ and $(1-x)^{\alpha-1}(1-y)^{\alpha^2-\alpha+1} \le (1-xy)^{\alpha^2}$ for $0 \le x, y \le 1$.) This completes the proof of Theorem 7.

Remark. It is obvious that Theorem 2 will generate many more counter-examples to the conjecture by Rhin and Viola, by choosing the parameters α,β,γ to be positive integers in other ways such that $\alpha(\alpha-\gamma+1)/(\beta-1)$ is as well a positive integer (and such that the conditions (5.2)–(5.4) are satisfied). To have a convenient parametrisation, one would replace γ by $\alpha+1-\gamma$, subsequently α by a_1a_2, γ by c_1c_2, and β by a_1c_1+1. The resulting relation is

$$
{}_3F_2\left[\begin{matrix}a_1a_2,\ a_1c_1+1,\ a_1a_2-c_1c_2+1\\ a_1a_2+1,\ a_1a_2+a_2c_2-c_1c_2+2\end{matrix};1\right]=\frac{(a_1a_2-a_1c_1+1)(a_1a_2+a_2c_2-c_1c_2+1)}{(a_1a_2+1)(a_2c_2-c_1c_2+1)}\\
\times{}_3F_2\left[\begin{matrix}a_1a_2+1,\ a_1c_1,\ a_1a_2-c_1c_2+1\\ a_1a_2+2,\ a_1a_2+a_2c_2-c_1c_2+1\end{matrix};1\right].
\tag{11.4}
$$

12. More exotic contiguous relations.

In this section, we present two more relations of the kind of Theorem 2 (which itself followed from the more general Proposition 1), see Theorems 9 and 10. These are obtained along the lines described in the Remark after the proof of Proposition 1. The two theorems imply further counter-examples to the conjecture by Rhin and Viola.

Proposition 2. *For any complex numbers a,b,c such that $\Re(d-b-c+1)>0$, and such that $a+1$ and d are not non-positive integers, we have the identity*

$$
{}_3F_2\left[\begin{matrix}a,b,c\\ a+1,\ d\end{matrix};1\right]=\frac{bc(a-d-1)(a-d)}{(a-b)(a-c)d(d+1)}\,{}_3F_2\left[\begin{matrix}a,\ b+1,\ c+1\\ a+1,\ d+2\end{matrix};1\right]\\
+\frac{a(bc+ad-bd-cd)}{(a-b)(a-c)}\frac{\Gamma(d)\,\Gamma(d-b-c+1)}{\Gamma(d-b+1)\,\Gamma(d-c+1)}.
\tag{12.1}
$$

Proof. To the left-hand side, we apply the contiguous relation

$$
{}_3F_2\left[\begin{matrix}A_1,\ A_2,\ A_3\\ B_1,\ B_2\end{matrix};z\right]=\frac{A_2(B_1-A_1)}{(A_2-A_1)B_1}\,{}_3F_2\left[\begin{matrix}A_1,\ A_2+1,\ A_3\\ B_1+1,\ B_2\end{matrix};z\right]\\
+\frac{A_1(B_1-A_2)}{(A_1-A_2)B_1}\,{}_3F_2\left[\begin{matrix}A_1+1,\ A_2,\ A_3\\ B_1+1,\ B_2\end{matrix};z\right]
\tag{12.2}
$$

with $A_1=b$, $A_2=c$, and $B_1=d$. As a result we obtain

$$
\frac{c(d-b)}{(c-b)d}\,{}_3F_2\left[\begin{matrix}a,\ b,\ c+1\\ a+1,\ d+1\end{matrix};1\right]+\frac{b(d-c)}{(b-c)d}\,{}_3F_2\left[\begin{matrix}a,\ b+1,\ c\\ a+1,\ d+1\end{matrix};1\right].
$$

We apply the contiguous relation (12.2) again, to the first series with $A_1=a$, $A_2=b$, and $B_1=d+1$, to the second with $A_1=a$, $A_2=c$, and $B_1=d+1$. After some simplification, this leads to the expression

$$
-\frac{bc(a-d-1)(a-d)}{(-a+b)(a-c)d(d+1)}\,{}_3F_2\left[\begin{matrix}a,\ b+1,\ c+1\\ a+1,\ d+2\end{matrix};1\right]\\
+\frac{ac(d-b)(d-b+1)}{(a-b)(c-b)d(d+1)}\,{}_2F_1\left[\begin{matrix}b,\ c+1\\ d+2\end{matrix};1\right]+\frac{ab(d-c)(d-c+1)}{(a-c)(b-c)d(d+1)}\,{}_2F_1\left[\begin{matrix}b+1,\ c\\ d+2\end{matrix};1\right].
$$

Finally, we use Gauß' summation formula (10.3) to evaluate the two ${}_2F_1$-series. Some manipulation then yields the claimed result on the right-hand side of (12.1). □

We may now choose a so that the second term on the right-hand side of (12.1) vanishes, that is, we choose

$$a = b + c - \frac{bc}{d}.$$

After the additional replacements of b by β, of c by γ, and of d by $\beta\gamma/\delta$, we arrive at the following result.

THEOREM 9. *For any complex numbers α, β, γ such that $\Re\big(\frac{\beta\gamma}{\delta} - \beta - \gamma + 1\big) > 0$, and such that $\beta + \gamma - \delta + 1$ and $\frac{\beta\gamma}{\delta}$ are not non-positive integers, we have the identity*

$$ {}_3F_2\left[\begin{matrix} \beta+\gamma-\delta,\ \beta,\ \gamma \\ \beta+\gamma-\delta+1,\ \frac{\beta\gamma}{\delta}\end{matrix};1\right] = \frac{\beta\gamma+\delta-\beta\delta-\gamma\delta+\delta^2}{\beta\gamma+\delta}\,{}_3F_2\left[\begin{matrix} \beta+\gamma-\delta,\ \beta+1,\ \gamma+1 \\ \beta+\gamma-\delta+1,\ \frac{\beta\gamma}{\delta}+2\end{matrix};1\right]. \tag{12.3}$$

Again, if one wants a more convenient parametrisation for generating counter-examples to the conjecture by Rhin and Viola, then one would replace β by b_1b_2, γ by c_1c_2, and δ by b_1c_1. The resulting relation then is

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} b_1b_2+c_1c_2-b_1c_1,\ b_1b_2,\ c_1c_2 \\ b_1b_2+c_1c_2-b_1c_1+1,\ b_2c_2\end{matrix};1\right] \\ &\quad= \frac{b_1b_2c_1c_2+b_1c_1-b_1b_2b_1c_1-c_1c_2b_1c_1+b_1^2c_1^2}{b_1b_2c_1c_2+b_1c_1} \\ &\qquad\times {}_3F_2\left[\begin{matrix} b_1b_2+c_1c_2-b_1c_1,\ b_1b_2+1,\ c_1c_2+1 \\ b_1b_2+c_1c_2-b_1c_1+1,\ b_2c_2+2\end{matrix};1\right]. \end{aligned}\tag{12.4}$$

PROPOSITION 3. *For any complex numbers a, b, c such that $\Re(d - b - c + 1) > 0$, and such that $a + 1$ and d are not non-positive integers, we have the identity*

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a,b,c \\ a+1,\ d\end{matrix};1\right] &= \frac{(a-b+1)(a-c+1)}{(a+1)(a-d+1)}\,{}_3F_2\left[\begin{matrix} a+1,\ b,\ c \\ a+2,\ d\end{matrix};1\right] \\ &\quad - \frac{\Gamma(d)\,\Gamma(d-b-c+1)}{(a-d+1)\,\Gamma(d-b)\,\Gamma(d-c)}. \end{aligned}\tag{12.5}$$

PROOF. The first few steps of this proof are identical with the one of Proposition 1. More precisely, we use that the series on the left-hand side is equal to the expression (10.5). There, we apply now instead the contiguous relation (10.6) with $A_1 = c + 1$ and $A_2 = a + 1$. As a result, we obtain

$$\frac{(a-b+1)(a-c+1)}{(a+1)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a+1,\ b,\ c\\ a+2,\ d\end{matrix};1\right]$$
$$-\frac{a-b+1}{a-d+1}\,{}_2F_1\left[\begin{matrix}b,\ c\\ d\end{matrix};1\right]+\frac{(a+c-d+1)\,\Gamma(d)\,\Gamma(d-b-c)}{(a-d+1)\,\Gamma(d-b)\,\Gamma(d-c)},$$

which, by another use of the Gauß summation formula (10.3) and some simplification, turns out to be equal to the right-hand side of (12.5). □

An iterative use of Proposition 3 produces the following formula.

COROLLARY 1. *For any complex numbers a,b,c such that $\Re(d-b-c+1)>0$, and such that $a+1$ and d are not non-positive integers, we have the identity*

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,\ b,\ c\\ a+1,\ d\end{matrix};1\right]&=\frac{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}{(a+1)(a+2)(a-d+2)(a-d+1)}\,{}_3F_2\left[\begin{matrix}a+2,\ b,\ c\\ a+3,\ d\end{matrix};1\right]\\&\quad-\frac{(3+5a+2a^2-b-ab-c-ac+bc-d-ad)}{(a+1)(a-d+2)(a-d+1)}\\&\quad\times\frac{\Gamma(d)\,\Gamma(d-b-c+1)}{\Gamma(d-b)\,\Gamma(d-c)}.\end{aligned}\tag{12.6}$$

If we now choose d such that the polynomial factor on the right-hand side of (12.6) vanishes, that is,

$$d=2a-b-c+3+\frac{bc}{a+1},$$

then we obtain the following theorem.

THEOREM 10. *For any complex numbers α,β,γ such that $\Re\big(2\alpha-2\beta-2\gamma+\frac{\beta}{\alpha+1}+4\big)>0$, and such that $\alpha+1$ and $2\alpha-\beta-\gamma+\frac{\beta\gamma}{\alpha+1}+3$ are not non-positive integers, we have the identity*

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ \gamma\\ \alpha+1,\ 2\alpha-\beta-\gamma+\frac{\beta\gamma}{\alpha+1}+3\end{matrix};1\right]\\&\quad=\frac{(\alpha+1)(\alpha-\beta+2)(\alpha-\gamma+2)}{(\alpha+2)(3\alpha+\alpha^2-\beta-\alpha\beta-\gamma-\alpha\gamma+\beta\gamma+2)}\\&\qquad\times{}_3F_2\left[\begin{matrix}\alpha+2,\ \beta,\ \gamma\\ \alpha+3,\ 2\alpha-\beta-\gamma+\frac{\beta\gamma}{\alpha+1}+3\end{matrix};1\right].\end{aligned}\tag{12.7}$$

If one wants a more convenient parametrisation for generating counter-examples to the conjecture by Rhin and Viola, then one would replace β by b_1b_2, γ by c_1c_2, and α by b_1c_1-1. The resulting relation then is

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} b_1c_1-1,\ b_1b_2,\ c_1c_2 \\ b_1c_1,\ 2b_1c_1-b_1b_2+b_2c_2-c_1c_2+1 \end{matrix};1\right] \\
&\quad = \frac{(b_1c_1-b_1b_2+1)(b_1c_1-c_1c_2+1)}{(b_1c_1+1)(b_1c_1-b_1b_2+b_2c_2-c_1c_2+1)} \\
&\qquad \times {}_3F_2\left[\begin{matrix} b_1c_1+1,\ b_1b_2,\ c_1c_2 \\ b_1c_1+2,\ 2b_1c_1-b_1b_2+b_2c_2-c_1c_2+1 \end{matrix};1\right]. \qquad (12.8)
\end{aligned}
$$

13. Postlude: how were these identities found?

The reader may wonder how we found the identities in Theorem 1 and Propositions 1–3 (the latter implying Theorems 2, 9 and 10) and their proofs. This section describes some of the ideas that led us to their discovery, with some of them being interesting in their own right, as we believe. Since we shall make reference to it several times, we mention right away that all the hypergeometric calculations were carried out using the first author's *Mathematica* package HYP [**7**].

The counter-examples (3.5)–(3.10) of Sato, in their original form, do not give any hints for a general result that may be behind them. However, as we explain in Section 7, if we bring them into different, but equivalent, forms using Thomae's relations, patterns emerge. More precisely, by staring at the forms (7.2) and (7.3) of (3.5)–(3.9), we extracted the wild guess that (1.2) should hold. The first proof that we found (which is not presented here) showed first the special case $\gamma = \alpha + \beta$ of (1.2), given in (3.13), by using elementary contiguous relations. A somewhat involved analytic continuation argument, using the Gosper-Zeilberger algorithm (see below) and Carlson's theorem then extended (3.13) to (1.1).

However, it was "obvious" to us that one should be able to prove (1.2) by a combination of several classical transformation formulae for hypergeometric series. Clearly, since we know that (1.2) is not a consequence of Thomae's relations, the classical ${}_3F_2$-transformations are not of any use. So we asked HYP to tell us which (of the built-in) transformations can be applied to the left-hand side of (1.2). (This is done by using `TListe`; see [**7**].) The only "non-standard" transformation that HYP came up with was (8.1). (This is `T3240` in HYP.) So we applied it and quickly realized that we could exchange α and $\alpha+1$ in the obtained ${}_7F_6$-series (cf. (8.2) and (8.3)) and apply (8.1) in the other direction, in order to obtain a result different from the original ${}_3F_2$-series, which then turned out to be exactly the right-hand side of (1.2).

Having found an explanation for 83.33333... percent of Sato's counter-examples did not completely satisfy us. We also wanted an explanation for (3.10). Since this is just one single identity, there is only very little guidance where to look for. What caught our eyes was that, in the hypergeometric form (7.4), both ${}_3F_2$-series were balanced (that is, the sum of the lower parameters exceeds the sum of the upper parameters by exactly 1). Not only that, in both series there is a lower parameter which exceeds an upper parameter by exactly 1. So, we made our computer work out the values of all series of the form

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ a+1,\ b+c \end{matrix};1\right]
$$

for $1 \le a, b, c \le 40$, and then compared which series were rational multiples of each other. By staring at the results, we extracted identities such as

$$ {}_3F_2\left[\begin{matrix} \alpha^2,\ \alpha+1,\ \alpha^2 \\ \alpha^2+1,\ \alpha^2+\alpha+1 \end{matrix}; 1\right] = \frac{\alpha^3+1}{\alpha^2+1}\, {}_3F_2\left[\begin{matrix} \alpha^2+1,\ \alpha,\ \alpha^2 \\ \alpha^2+2,\ \alpha^2+\alpha \end{matrix}; 1\right] \tag{13.1} $$

(this is identity (11.1), the special case $\alpha \to \alpha^2$, $\beta = \alpha+1$, $\gamma = \alpha^2$ of (1.2)), or

$$ {}_3F_2\left[\begin{matrix} \alpha^2-\alpha+1,\ \alpha,\ \alpha^2-\alpha \\ \alpha^2-\alpha+2,\ \alpha^2 \end{matrix}; 1\right] = \frac{\alpha}{\alpha^2+1}\, {}_3F_2\left[\begin{matrix} \alpha^2-\alpha+1,\ \alpha+1,\ \alpha^2-\alpha+1 \\ \alpha^2-\alpha+2,\ \alpha^2+2 \end{matrix}; 1\right] \tag{13.2} $$

(this is the special case $\beta = \alpha$, $\gamma = \alpha^2-\alpha$, $\delta = \alpha - 1$ of (12.3)) or

$$ {}_3F_2\left[\begin{matrix} 6\alpha+1,\ 4\alpha+2,\ 3\alpha+1 \\ 6\alpha+2,\ 7\alpha+3 \end{matrix}; 1\right] = \frac{3\alpha+2}{3\alpha+3}\, {}_3F_2\left[\begin{matrix} 6\alpha+3,\ 4\alpha+2,\ 3\alpha+1 \\ 6\alpha+4,\ 7\alpha+3 \end{matrix}; 1\right] \tag{13.3} $$

(this is the special case $\alpha \to 6\alpha+1$, $\beta = 4\alpha+2$, $\gamma = 3\alpha+1$ of (12.7)).

We then attempted to prove these identities. It seems sort of "obvious" that one should be able to prove them by using known contiguous relations. Indeed, in HYP there are approximately 100 such contiguous relations built-in. We played with those, but we were not able to arrive at the right-hand sides of the conjectured identities. At some point, we had the idea to "cheat" and to make recourse to the "modern" way of treating hypergeometric series, namely applying the Gosper-Zeilberger algorithm (see [**2**], [**10**], [**20**], [**21**], [**22**]; what we do below is in the spirit of [**9**]). For example, aiming to prove (a generalisation of) (13.1), we considered the series

$$ {}_3F_2\left[\begin{matrix} a+n,\ b-n,\ c \\ a+n+1,\ b+c-n \end{matrix}; 1\right] \tag{13.4} $$

and tried to find a first-order recurrence for it (which is what (1.2) is). Thus, we put the summand of this series,

$$ F(n,k) = \frac{(a+n)_k\,(b-n)_k\,(c)_k}{(a+n+1)_k\,(b+c-n)_k\,k!} $$

into the Gosper-Zeilberger algorithm, and we got

$$ \begin{aligned} &(a+n+1)(b-n-1)(a-b-c+2n+1)(a-b-c+2n+2)F(n,k) \\ &\qquad + (a-b+2n+2)(a-b+2n+1)(a-c+n+1)(b+c-n-1)F(n+1,k) \\ &\quad = \Delta_k F(n,k)R(n,k), \end{aligned} \tag{13.5} $$

where

$$\begin{aligned}R(n,k) = {} & \frac{k(b+c+k-n-1)}{(a+n)(b+c-n-1)(b+k-n-1)} \\ & \times (k(b+c-n-1)(a+n+1) \\ & \qquad \times (1-a-a^2-2b+b^2+ac+n-2an-2bn+cn) \\ & \qquad + \text{terms not containing } k),\end{aligned}$$

and where Δ_k is the forward difference operator, $(\Delta_k f)(k) = f(k+1) - f(k)$. If we now sum both sides of (13.5) over k from 0 to N, then we obtain

$$\begin{aligned}&(a+n+1)(b-n-1)(a-b-c+2n+1)(a-b-c+2n+2)\sum_{k=0}^{N} F(n,k) \\ &\quad + (a-b+2n+2)(a-b+2n+1)(a-c+n+1)(b+c-n-1)\sum_{k=0}^{N} F(n+1,k) \\ &\quad = F(n,N+1)R(n,N+1),\end{aligned}$$

since the terms on the right-hand side telescope. Subsequently, the limit $N \to \infty$ yields

$$\begin{aligned}&(a+n+1)(b-n-1)(a-b-c+2n+2)(a-b-c+2n+1) \\ &\quad \times {}_3F_2\left[\begin{matrix} a+n,\ b-n,\ c \\ a+n+1,\ b+c-n\end{matrix}; 1\right] \\ &\quad = (a-b+2n+1)(a-b+2n+2)(a-c+n+1)(b+c-n-1) \\ &\quad \times {}_3F_2\left[\begin{matrix} a+n+1,\ b-n-1,\ c \\ a+n+2,\ b+c-n-1\end{matrix}; 1\right] \\ &\quad + (a+n+1)(1-a-a^2-b+c+ac-bc-d+bd+n-2an-bn+2cn-dn) \\ &\quad \times \frac{\Gamma(b+c-n)}{\Gamma(c)\,\Gamma(b-n)}. \end{aligned} \tag{13.6}$$

(The reader should notice that this is (10.1) with a replaced by $a+n$, b replaced by $b-n$, and d replaced by $b+c-n$.)

At this point, we became greedy. Why should this be something special for balanced series? So, we replaced the bottom parameter $b+c-n$ in (13.4) by $d-n$, —and we were disappointed to learn that the Gosper-Zeilberger algorithm is unable to find a two-term recurrence for this more general series. (It finds only a three-term recurrence.) However, it *does* find a two-term recurrence for every d of the form $d = b+c+m$, where m is a non-negative integer. From the data for $m = 0$ (given in (13.6)) and for $m = 1, 2, 3$, one is then easily able to work out a (at this point, conjectural) formula for the output of the algorithm, namely if

$$F(n,k)=\frac{(a+n)_k\,(b-n)_k\,(c)_k}{(a+n+1)_k\,(b+c+m-n)_k\,k!},$$

then

$$\begin{aligned}(b-n)_m\,(c)_m\,&(a+n+1)(b-n-1)\\ &\times(a-b-c-m+2n+1)(a-b-c-m+2n+2)F(n,k)\\ &+(b-n)_m\,(c)_m\,(a-b+2n+2)(a-b+2n+1)\\ &\times(a-c+n+1)(b+c+m-n-1)F(n+1,k)=\Delta_kF(n,k)R(n,k),\end{aligned}\tag{13.7}$$

where

$$\begin{aligned}R(n,k)=&\frac{k(b+c+m+k-n-1)}{(a+n)(b+c+m-n-1)(b+k-n-1)}\\ &\times(k^{m+1}(b+c+m-n-1)(a+n+1)\\ &\quad\cdot(1-a-a^2-2b+b^2+ac+n-2an-2bn+cn-m(n+1-b))\\ &\quad+\text{terms with lower powers in }k).\end{aligned}$$

If one, for simplicity, replaces $b+c+m$ by d in (13.7), sums both sides over k from 0 to N, and finally lets N tend to infinity, one arrives exactly at (10.1), with a replaced by $a+n$, b replaced by $b-n$, and d replaced by $d-n$.

As we pointed out, this is at best a half rigorous derivation of (10.1) in the case that the difference $d-b-c$ is a non-negative integer, but there is no guarantee at all that this formula should also hold for *any* d. (To explain two of the possible pitfalls: first, there are always two ways to translate expressions such as $(c)_m$ into gamma functions: $(c)_m=\Gamma(c+m)/\Gamma(c)=(-1)^m\Gamma(1-c)/\Gamma(1-c-m)$. These lead to different formulae if m is replaced by $d-b-c$, where d is arbitrary. Second, sometimes one may even miss whole additional terms in a formula, which one does not see if some parameter is specialised to a non-negative integer because this additional term happens to vanish for this specialisation.) However, one can now *prove* (10.1) continuing along the above lines: first, one verifies that (10.1) is valid for $d=b$ by using Gauß' summation formula (10.3). Next, one replaces d by $b+n$ in (10.1), and one uses the Gosper-Zeilberger algorithm to find recurrences in n for the left-hand and the right-hand sides of (10.1). Thus, one knows that (10.1) holds with $d=b+n$ for any non-negative integer n. Since both sides of (10.1) are analytic in d in a neighbourhood of ∞, one can use the principle of analytic continuation to deduce that (10.1) holds for any complex d where both sides are defined.

We did that, but finally we did succeed to work out a proof using known contiguous relations. Since this is completely elementary and, as we believe, more instructive, this is the proof that we have included in Section 10. For obtaining the general identities which are behind (13.2) and (13.3), given in Propositions 2 and 3, we proceeded similarly. In fact, the contiguous relations (10.2), (10.4), (10.6), (12.2), which we used in the proofs, are `C55`, `C15`, `C27`, and `C54`, respectively, in HYP.

Our computer experiments suggest that the above procedure produces a relation of the type (10.1) for *any* series

$$ {}_3F_2\left[\begin{matrix} a+a_1n,\ b+b_1n,\ c+c_1n \\ a+a_1n+a_2,\ d+d_1n \end{matrix};1\right], $$

as long as a_1, a_2, b_1, c_1, d_1 are integers, a_2 a positive integer, and $b_1 + c_1 = d_1$. However, most of the time none of a, b, c, d appears linearly in the big polynomial factor on the right-hand side. This makes it difficult to extract a general solution of the Diophantine equation which arises when one equates the polynomial factor to zero. Nevertheless, experimentally, there are many solutions for various choices of a_1, b_1, c_1, d_1.

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Christian KRATTENTHALER
Institut Camille Jordan
Université Claude Bernard Lyon-I
21, avenue Claude Bernard
F-69622 Villeurbanne Cedex
France
E-mail: kratt@euler.univ-lyon1.fr

Tanguy RIVOAL
Institut Fourier
CNRS UMR 5582/Université Grenoble 1
100 rue des Maths, BP 74
38402 Saint-Martin d'Hères cedex
France
E-mail: rivoal@ujf-grenoble.fr