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A time-change approach to Kotani's extension of Yor's formula

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Abstract. In [**3**], Kotani proved analytically that expectations for additive functionals of Brownian motion $\{B_t, t \geq 0\}$ of the form

$$E_0\left[f(B_t)g\left(\int_0^t \varphi(B_s)ds\right)\right]$$

have the asymptotics $t^{-3/2}$ as $t \to \infty$ for some suitable non-negative functions φ, f and g. This generalizes, in the asymptotic form, Yor's explicit formula [**10**] for exponential Brownian functionals.

In the present paper, we discuss this generalization probabilistically, by using a time-change argument. We may easily see from our argument that this asymptotics $t^{-3/2}$ comes from the transition probability of 3-dimensional Bessel process.

1. Introduction.

Let $(B = \{B_t, t \geq 0\}, P_x)$ be a one-dimensional Brownian motion starting from x: $P_x(B_0 = x) = 1$. Yor's formula for exponential additive functionals of Brownian motion states that, for all non-negative Borel-measurable functions f and g,

$$E_0\left[f(B_t)g\left(\int_0^t e^{-2B_s}ds\right)\right] = \int_{\boldsymbol{R}} dx \int_0^\infty \frac{dy}{y} f(x)g(y)\exp\left(-\frac{1+e^{2x}}{2y}\right)\theta\left(\frac{e^x}{y}, t\right). \quad (1.1)$$

See [**10**, formula (6.e)]; we also refer to [**1**]. Here, for fixed $z > 0$, $\theta(z, \cdot)$ denotes the density of the so-called Hartman-Watson distribution, whose integral representation is obtained in [**9**, Théorème (5.4)]. It is noted in [**1**] that $\lim_{t\to\infty} \sqrt{2\pi t^3}\theta(z,t) = K_0(z)$, the Macdonald function of order 0. From these, we may deduce that, for some suitable functions f and g, the expectation as on the left hand side of (1.1) has the asymptotics $t^{-3/2}$ as $t \to \infty$.

Later in [**3**], Kotani proved the same asymptotics for more general additive functionals, replacing e^{-2x} by $\varphi(x) \geq 0$ satisfying certain conditions. He employed an analytic approach, namely the Krein theory, in doing this.

In this paper, we deal with the same problem. Our approach employed here is a probabilistic one. Although we only discuss here the case where g is given by $g(x) = \exp(-x)$, we think that our approach provides us with a simpler way to understand why such an asymptotics appears even for general additive functionals, and that it is worthwhile to present it; we may easily deduce from our argument that the asymptotics $t^{-3/2}$ comes from the transition probability of 3-dimensional Bessel process:

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$$\sqrt{2\pi t^3}P_x^{(3)}(R_t \in dz) \to 2z^2 dz, \quad t \to \infty.$$

We assume $\varphi(x) \geq 0$ ($x \in \boldsymbol{R}$) is locally integrable and satisfies:

$$\text{(P1)} \quad \int^{\infty} x\varphi(x)dx < \infty, \qquad \text{(P2)} \quad \liminf_{x\to-\infty} \varphi(x) > 0.$$

We denote by f_0 the unique, strictly positive solution to the Sturm-Liouville equation

$$\frac{1}{2}f''(x) = \varphi(x)f(x) \tag{1.2}$$

with boundary conditions

$$f'(x) \to 1 \ (x \to \infty) \quad \text{and} \quad f(x) \to 0 \ (x \to -\infty). \tag{1.3}$$

The existence and uniqueness of such a solution is ensured by the above assumptions on φ.

Remark 1.1. By (P2), there exist constants $a < 0$ and $c, c' > 0$ such that

$$f_0(x) \leq c' e^{-c|x|} \quad \text{for all } x < a.$$

See Remark 2.1.

Let f be a non-negative function on $\boldsymbol{R}$ satisfying

$$\text{(A)} \quad \int_{\boldsymbol{R}} f(z)f_0(z)dz < \infty.$$

The purpose of this paper is to prove the following limit theorem: for every $x \in \boldsymbol{R}$,

$$\lim_{t\to\infty} \sqrt{2\pi t^3}E_x\left[f(B_t)\exp\left\{-\int_0^t \varphi(B_s)ds\right\}\right] = 2f_0(x)\int_{\boldsymbol{R}} f(z)f_0(z)dz. \tag{$*$}$$

We shall show that $(*)$ holds under some additional condition on f. Although we only discuss the simple case with $g(x) = \exp(-x)$, an assumption on f imposed in [**3**] is relaxed somewhat; indeed, in some case, we only need the minimal assumption (A) for $(*)$ to hold.

To state the result, we introduce the exponent $\gamma_0 \geq 0$ defined by:

$$\gamma_0 = \inf\left\{\gamma \geq 0; \liminf_{x\to-\infty} |x|^{-2\gamma}\varphi(x) > 0\right\}.$$

Theorem 1.1. (i) *The case* $\gamma_0 \leq 1$: *Assume* (A). *Moreover, we assume*

$$\text{(B)} \qquad \int_{-\infty} |z| f(z) f_0(z) dz < \infty.$$

Then $(*)$ *holds.*
(ii) *The case* $\gamma_0 > 1$: *Assume* (A). *Then* $(*)$ *holds.*

REMARK 1.2. In [**3**], it is assumed that, in the present setting,

$$\int_{-\infty} |z|^{3/2} f(z) f_0(z) dz < \infty$$

for both cases (i) and (ii).

REMARK 1.3. If, in particular, $\varphi(x) = O(|x|^\gamma)$ as $x \to -\infty$ for some $0 < \gamma \le 1$, then the condition (B) can be relaxed as:

$$\text{(B}'\text{)} \qquad \begin{cases} \int_{-\infty} |z|^{1-\gamma} f(z) f_0(z) dz < \infty & \text{for } \gamma < 1, \\ \int_{-\infty} (\log |z|) f(z) f_0(z) dz < \infty & \text{for } \gamma = 1. \end{cases}$$

We may easily deduce this from our argument used in the proof of Theorem 1.1. See, in particular, the proof of Lemma 3.6.

As a corollary to Theorem 1.1, we also see:

COROLLARY 1.1. *Under the same assumption as in Theorem* 1.1, *we have, for all* $x \in \boldsymbol{R}$,

$$\lim_{t\to\infty} \sqrt{t} \int_t^\infty ds\, E_x\left[f(B_s) \exp\left\{ -\int_0^s \varphi(B_u) du \right\}\right] = \frac{4}{\sqrt{2\pi}} f_0(x) \int_{\boldsymbol{R}} f(z) f_0(z) dz. \quad (1.4)$$

Note that the assertion is also a rewriting of Proposition 3.1. We give some remark on this corollary in Section 4.

As an application of Theorem 1.1, we give two examples; in both examples, we take $f(x) = e^{-\mu x}$ $(\mu > 0)$, which means, by the Cameron-Martin relation, that we may rewrite the assertions using the Brownian motion with drift $B^{(-\mu)} = \{B_t - \mu t, t \ge 0\}$ instead of the Brownian motion.

EXAMPLE 1.1. For $\alpha > 0$, we take $\varphi(x) = \alpha e^{-2x}$. In this case f_0 is given by

$$f_0(x) = K_0\big(\sqrt{2\alpha} e^{-x}\big),$$

where K_0 denotes the Macdonald function of order 0. Using one of its integral representations (see, e.g., [**4**, formula (5.10.25)]), we may easily see:

$$\int_{\boldsymbol{R}} e^{-\mu x} K_0\big(\sqrt{2\alpha} e^{-x}\big) dx = 2^{\mu-2} \frac{1}{(\sqrt{2\alpha})^\mu} \left\{ \Gamma\left(\frac{\mu}{2}\right)\right\}^2.$$

Note that, in this case, we may apply (ii) of Theorem 1.1 and obtain

$$\lim_{t\to\infty}\sqrt{2\pi t^3}e^{\mu^2 t/2}E_x\left[\exp\left\{-\alpha\int_0^t e^{-2B_s^{(-\mu)}}ds\right\}\right]=2^{\mu-1}\left\{\Gamma\left(\frac{\mu}{2}\right)\right\}^2 e^{\mu x}\frac{K_0(\sqrt{2\alpha}e^{-x})}{(\sqrt{2\alpha})^\mu}.$$

This asymptotics has already been discussed in [**2**, Theorem 2.1], where Yor's formula was used.

Example 1.2. We take $\varphi(x)=\beta\mathbf{1}_{(-\infty,0)}(x)$ for $\beta>0$. In this case f_0 is given by

$$f_0(x)=\begin{cases} x+\dfrac{1}{\sqrt{2\beta}}, & x\geq 0,\\ \dfrac{1}{\sqrt{2\beta}}e^{-\sqrt{2\beta}|x|}, & x\leq 0.\end{cases}$$

Note that, if $\mu<\sqrt{2\beta}$, then

$$\int_{\boldsymbol{R}} e^{-\mu x}f_0(x)dx=\frac{\sqrt{2\beta}}{\mu^2(\sqrt{2\beta}-\mu)}<\infty,$$

and the assumption (B) is also fulfilled. Therefore, by (i) of Theorem 1.1, we have, for $\mu<\sqrt{2\beta}$,

$$\lim_{t\to\infty}\sqrt{2\pi t^3}e^{\mu^2 t/2}E_x\left[\exp\left\{-\beta\int_0^t \mathbf{1}_{(-\infty,0)}(B_s^{(-\mu)})ds\right\}\right]=\frac{2\sqrt{2\beta}}{\mu^2(\sqrt{2\beta}-\mu)}e^{\mu x}f_0(x).$$

The organization of this paper is as follows: in Section 2, we present some preliminaries; in Subsection 3.a, we prove Theorem 1.1; in Subsections 3.b and 3.c, we prove two propositions that are used in the proof of Theorem 1.1; in Section 4, we give some remark on a connection between our result and a related one in [**7**].

Throughout this paper, $R=\{R_t, t\geq 0\}$, together with a probability measure $P_x^{(3)}$, denotes a 3-dimensional Bessel process starting from x: $P_x^{(3)}(R_0=x)=1$, and $E_x^{(3)}$ denotes the expectation with respect to $P_x^{(3)}$. Other notation will be introduced as needed.

2. Preliminaries.

In this section, we prepare several preliminary results.

2.a. h-transform with respect to f_0.

Let X be the solution to the following SDE:

$$X_t=x+W_t+\int_0^t\frac{f_0'}{f_0}(X_s)ds,\quad t\geq 0,\ x\in\boldsymbol{R}, \tag{2.1}$$

where W is a standard one-dimensional Brownian motion. We denote by $\boldsymbol{P}_x$ the probability measure on the path space $C([0,\infty);\boldsymbol{R})$, induced by X. For every $t>0$ and every non-negative, measurable functional $F(w(s), s\le t)$ ($w\in C([0,\infty);\boldsymbol{R})$), it holds that, by the Girsanov theorem (see, e.g., [**5**]),

$$\boldsymbol{E}_x[F(X_s, s\le t)] = E_x\biggl[F(B_s, s\le t)\frac{f_0(B_t)}{f_0(x)}\exp\biggl\{-\int_0^t \varphi(B_s)ds\biggr\}\biggr].$$

Here we made the abuse of notation by letting X denote the canonical path in $C([0,\infty);\boldsymbol{R})$ under $\boldsymbol{P}_x$. From this relation, we have in particular

$$E_x\biggl[f(B_t)\exp\biggl\{-\int_0^t \varphi(B_s)ds\biggr\}\biggr] = f_0(x)\boldsymbol{E}_x\biggl[\frac{f}{f_0}(X_t)\biggr]. \tag{2.2}$$

2.b. Time-change.

Since $f_0'(x)\to 1$ as $x\to\infty$, the drift term $(f_0'/f_0)(x)$ of the SDE (2.1) behaves as $1/x$ when $x\to\infty$. So we may expect the solution X_t to behave asymptotically as 3-dimensional Bessel process as $t\to\infty$. To formulate this intuition mathematically, we shall consider expressing X as a time-change of a 3-dimensional Bessel process. For this purpose, we define the function g_0 by

$$g_0(x) = \biggl\{\int_x^\infty \frac{dy}{f_0(y)^2}\biggr\}^{-1}, \quad x\in\boldsymbol{R}.$$

By using the inverse function g_0^{-1} of g_0, X is expressed as:

$$X_t = g_0^{-1}(R_{a_t(R)}) \tag{2.3}$$

for some 3-dimensional Bessel process R starting from $y = g_0(x) > 0$. Here

$$a_t(R) = \inf\{s\ge 0; A_s(R) > t\},$$
$$A_s(R) = \int_0^s \bigl|(g_0^{-1})'(R_u)\bigr|^2\, du.$$

Since $(g_0^{-1})'(x)\ge 1$ and converges to 1 as $x\to\infty$ (see Lemma 2.1 below), we see that, $P_y^{(3)}$-a.s.,

$$A_s(R)\ge s \quad \text{for all } s\ge 0 \qquad \text{and} \qquad A_s(R)/s\to 1 \quad \text{as } s\to\infty. \tag{2.4}$$

The latter follows from L'Hospital's rule and the fact that R is transient. Since $a_t(R)$ is the inverse of $A_s(R)$, we also see that, $P_y^{(3)}$-a.s.,

$$a_t(R)\le t \quad \text{for all } t\ge 0 \qquad \text{and} \qquad a_t(R)/t\to 1 \quad \text{as } t\to\infty.$$

The latter property, in particular, combined with (2.3) and the fact that $(g_0^{-1})'(x) \to 1$ as $x \to \infty$, does indicate that X_t behaves as R_t as $t \to \infty$.

2.c. Key identity.
By (2.2), we are led to study the asymptotics of $\boldsymbol{E}_x[\frac{f}{f_0}(X_t)]$ instead of that of $E_x[f(B_t)\exp\{-\int_0^t \varphi(B_s)ds\}]$ itself. The key to doing this is the following identity:

$$\int_0^t \frac{f}{f_0}(X_s)ds = \int_0^{a_t(R)} \frac{f}{f_0}\big(g_0^{-1}(R_s)\big)\big|(g_0^{-1})'(R_s)\big|^2 ds. \tag{2.5}$$

To see that this relation holds, we differentiate the right hand side with respect to t, noting $\frac{d}{dt}a_t(R) = |(g_0^{-1})'(R_{a_t(R)})|^{-2}$:

$$\begin{aligned}\frac{d}{dt}(\text{right hand side of (2.5)}) &= \frac{f}{f_0}\big(g_0^{-1}(R_{a_t(R)})\big)\big|(g_0^{-1})'(R_{a_t(R)})\big|^2 \frac{d}{dt}a_t(R) \\ &= \frac{f}{f_0}\big(g_0^{-1}(R_{a_t(R)})\big) = \frac{f}{f_0}(X_t), \qquad \text{(by (2.3))}\end{aligned}$$

which implies (2.5).

2.d. Properties of g_0.
We summarize here several properties of g_0 in a lemma. Some of them were already referred to above.

Lemma 2.1.

(i) $\lim_{x\to\infty} g_0'(x) = 1$, $\lim_{x\to-\infty} g_0(x) = 0$.
(ii) g_0 *is convex.*
(iii) $(g_0^{-1})'(x) \geq 1$, *is non-increasing, and converges to* 1 *as* $x \to \infty$.
(iv) $g_0 \geq f_0 f_0'$.
(v) $\limsup_{x\downarrow 0} x(g_0^{-1})'(x) < \infty$.

Before giving a proof, we give an example:

Example 2.1 (recall Example 1.2). In the case $\varphi(x) = \beta\mathbf{1}_{(-\infty,0)}(x)$ for $\beta > 0$, g_0 and $(g_0^{-1})'$ are given respectively by:

$$g_0(x) = \begin{cases} x + \dfrac{1}{\sqrt{2\beta}}, & x \geq 0, \\[2ex] \dfrac{2}{\sqrt{2\beta}}\dfrac{1}{1+\exp(-2\sqrt{2\beta}x)}, & x \leq 0; \end{cases}$$

$$(g_0^{-1})'(x) = \begin{cases} \dfrac{1}{\sqrt{2\beta}}\dfrac{1}{x(2-\sqrt{2\beta}x)}, & 0 < x \leq \dfrac{1}{\sqrt{2\beta}}, \\[2ex] 1, & x \geq \dfrac{1}{\sqrt{2\beta}}. \end{cases}$$

Note that $x(g_0^{-1})'(x) \to 1/(2\sqrt{2\beta})$ as $x \downarrow 0$.

Proof of Lemma 2.1. The latter assertion of (i) is obvious. For the former, note that $g_0' = (g_0/f_0)^2$. So it suffices to check $f_0(x)/g_0(x) \to 1$ as $x \to \infty$, which is immediate from L'Hospital's rule:

$$\lim_{x\to\infty} \frac{f_0(x)}{g_0(x)} = \lim_{x\to\infty} \frac{\left(\int_x^\infty \frac{dy}{f_0(y)^2}\right)'}{\left(\frac{1}{f_0(x)}\right)'} = \lim_{x\to\infty} \frac{1}{f_0'(x)} = 1.$$

Now we set $h_0 = f_0/g_0$. We have just seen $h_0(x) \to 1$ as $x \to \infty$. Note that h_0 also satisfies $(1/2)h_0'' = \varphi h_0$ (in fact, h_0 gives a solution to (1.2) linearly independent of f_0). This indicates, in particular, that h_0 is convex. Combining these, we see that $h_0 \geq 1$ and is non-increasing. Properties (ii)–(iv) are variants of this fact on h_0, so we omit the proof. For (v), first note that, by the condition (P2) on φ, there exist $a < 0, c > 0$ such that $\varphi \geq c$ on $(-\infty, a)$. Therefore $f_0'' = 2\varphi f_0 \geq 2cf_0$ on $(-\infty, a)$. Multiplying both sides by $f_0' > 0$ and integrating over $(-\infty, x)$ for $x < a$, we get $f_0'(x)^2 \geq 2cf_0(x)^2$, hence

$$\frac{f_0'(x)}{f_0(x)} \geq \sqrt{2c} \quad \text{for all } x < a. \tag{2.6}$$

Noting $(g_0^{-1})'(x) = 1/g_0'(g_0^{-1}(x)) = f_0(g_0^{-1}(x))^2/x^2$, we see that

$$\limsup_{x\downarrow 0} x(g_0^{-1})'(x) = \limsup_{y\to-\infty} \frac{f_0(y)^2}{g_0(y)} \leq \limsup_{y\to-\infty} \frac{f_0(y)}{f_0'(y)} \leq \frac{1}{\sqrt{2c}},$$

where we used the property (iv) for the first inequality and (2.6) for the second. This shows (v). ☐

Remark 2.1. From (2.6), we may see that, as $x \to -\infty$, f_0 decays exponentially or faster; indeed, by (2.6),

$$\log \frac{f_0(a)}{f_0(x)} = \int_x^a \frac{f_0'(y)}{f_0(y)} dy \geq \sqrt{2c}(a - x), \quad x < a,$$

which is rewritten as

$$f_0(x) \leq f_0(a)e^{\sqrt{2c}(x-a)}, \quad x < a.$$

2.e. Proof of (2.3).

Before closing this section, we prove the time-change relation (2.3) for the sake of completeness of the paper.

By definition, it is easily checked that

$$\frac{1}{2}g_0''(x) + \frac{f_0'}{f_0}(x)g_0'(x) = \frac{g_0'(x)^2}{g_0(x)}.$$

So, by Itô's formula,

$$g_0(X_t) = y + \int_0^t g_0'(X_s)dW_s + \int_0^t \frac{g_0'(X_s)^2}{g_0(X_s)}ds, \tag{2.7}$$

where, as before, we write $y = g_0(x)$. Since the second term on the right hand side is a martingale, there exists a Brownian motion $\widetilde{W}$ such that

$$\int_0^t g_0'(X_s)dW_s = \widetilde{W}_{G_t(X)}, \quad G_t(X) = \int_0^t g_0'(X_s)^2\, ds.$$

Now we prepare the 3-dimensional Bessel process R that is given as the strong solution to the following SDE driven by $\widetilde{W}$:

$$R_t = y + \widetilde{W}_t + \int_0^t \frac{ds}{R_s}.$$

Note that $R_{G_t(X)}$ satisfies:

$$\begin{aligned} R_{G_t(X)} &= y + \widetilde{W}_{G_t(X)} + \int_0^{G_t(X)} \frac{ds}{R_s} \\ &= y + \int_0^t g_0'(X_s)dW_s + \int_0^t \frac{g_0'(X_s)^2}{R_{G_s(X)}}ds. \end{aligned}$$

Comparing this with (2.7), we conclude the following relation:

$$g_0(X_t) = R_{G_t(X)}. \tag{2.8}$$

We remark that (2.8) is a Feller-type representation of X in terms of 3-dimensional Bessel process. It now remains to prove $G_t(X) = a_t(R)$. Since $a_t(R)$ is the inverse of $A_s(R)$, it suffices to check $A_{G_t(X)}(R) = t$. To this end, we compute:

$$\begin{aligned} \frac{d}{dt}A_{G_t(X)}(R) &= \left|(g_0^{-1})'(R_{G_t(R)})\right|^2 \frac{d}{dt}G_t(X) && \text{(by definition)} \\ &= \left|g_0'(g_0^{-1}(R_{G_t(R)}))\right|^{-2} g_0'(X_t)^2 \\ &= g_0'(X_t)^{-2}g_0'(X_t)^2 && \text{(by (2.8))} \\ &= 1, \end{aligned}$$

which implies $A_{G_t(X)}(R) = t$. Here, for the second line, we used the relation $(g_0^{-1})' =$

$1/g_0'(g_0^{-1})$. Now (2.3) is proved.

3. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1.

3.a. Proof of Theorem 1.1.

We begin with the following lemma.

LEMMA 3.1. *Let $k(\xi)$ $(\xi > 0)$ be a non-negative, locally integrable function satisfying*

$$\int_{0+} \xi^2 k(\xi) d\xi < \infty \quad \text{and} \quad \int^{\infty} \xi k(\xi) d\xi < \infty.$$

Then it holds that, for all $y > 0$,

$$E_y^{(3)}\left[\int_0^{\infty} k(R_s) ds\right] < \infty.$$

PROOF. The assertion is immediate from Fubini's theorem and the fact that

$$\int_0^{\infty} ds P_y^{(3)}(R_s \in d\xi) = \frac{2\xi}{y}(\xi \wedge y) d\xi. \qquad \square$$

Now we take $k(\xi) = \frac{f}{f_0}(g_0^{-1}(\xi))|(g_0^{-1})'(\xi)|^2$. Then the assumption of Lemma 3.1 is fulfilled; indeed, by making the change of variables with $\xi = g_0(z)$,

$$\int_0^{\infty} \xi^2 k(\xi) d\xi = \int_{\boldsymbol{R}} f(z) f_0(z) dz, \tag{3.1}$$

which is finite by (A). Applying Lemma 3.1 to this k, we see in particular that, for each $y > 0$,

$$E_y^{(3)}\left[\int_{a_t(R)}^{\infty} k(R_s) ds\right] < \infty, \quad t \geq 0.$$

Note that, since $a_t(R) \to \infty$ as $t \to \infty$ $P_y^{(3)}$-a.s., the left hand side converges to 0 as $t \to \infty$.

PROPOSITION 3.1. *Under the same assumption as in Theorem* 1.1, *it holds that, as $t \to \infty$,*

$$\sqrt{t} E_y^{(3)}\left[\int_{a_t(R)}^{\infty} k(R_s) ds\right] \to \frac{4}{\sqrt{2\pi}} \int_{\boldsymbol{R}} f(z) f_0(z) dz.$$

A key step to showing Proposition 3.1 is:

Lemma 3.2. *We have the following decomposition*:

$$E_y^{(3)}\left[\int_{a_t(R)}^{\infty} k(R_s)ds\right] = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_t^{\infty} ds\, E_y^{(3)}[k(R_s)], \qquad I_2(t) = \int_0^t ds\, E_y^{(3)}[\mathbf{1}_{\{A_s(R)>t\}}k(R_s)].$$

Proof. By the definition of $a_t(R)$ and by Fubini's theorem,

$$\begin{aligned} E_y^{(3)}\left[\int_{a_t(R)}^{\infty} k(R_s)ds\right] &= E_y^{(3)}\left[\int_{\{s;A_s(R)>t\}} k(R_s)ds\right] \\ &= \int_0^{\infty} ds\, E_y^{(3)}[\mathbf{1}_{\{A_s(R)>t\}}k(R_s)]. \end{aligned}$$

Now the assertion follows from the fact that $A_s(R) \geq s$ for all $s \geq 0$ (recall (2.4)). □

We have the following two propositions concerning this decomposition:

Proposition 3.2. *Under the assumption* (A),

$$\sqrt{t}I_1(t) \to \frac{4}{\sqrt{2\pi}}\int_{\boldsymbol{R}} f(z)f_0(z)dz \quad \text{as } t \to \infty.$$

Proposition 3.3. *Under the same assumption as in Theorem* 1.1,

$$\sqrt{t}I_2(t) \to 0 \quad \text{as } t \to \infty.$$

Proofs are given in Subsections 3.b and 3.c, respectively. We now easily see Proposition 3.1 follows from these:

Proof of Proposition 3.1. The assertion is an immediate consequence of Lemma 3.2, Propositions 3.2 and 3.3. □

Using Proposition 3.1, we prove Theorem 1.1:

Proof of Theorem 1.1. By the relation (2.5), we have, for each $x \in \boldsymbol{R}$,

$$\int_t^{\infty} \boldsymbol{E}_x\left[\frac{f}{f_0}(X_s)\right]ds = E_y^{(3)}\left[\int_{a_t(R)}^{\infty} k(R_s)ds\right], \quad t \geq 0.$$

Here, as before, $y = g_0(x)$. Then, by Proposition 3.1, we have

$$\int_t^\infty \boldsymbol{E}_x\left[\frac{f}{f_0}(X_s)\right]ds \sim t^{-1/2} \times \frac{4}{\sqrt{2\pi}}\int_{\boldsymbol{R}} f(z)f_0(z)dz \quad \text{as } t \to \infty.$$

Here and below, for positive functions $\alpha(t), \beta(t)$ $(t > 0)$, we use the notation $\alpha(t) \sim \beta(t)$ as $t \to \infty$ to mean $\lim_{t\to\infty} \alpha(t)/\beta(t) = 1$. Since the convergence of the left hand side to 0 is monotone, we may differentiate both sides with respect to t to get

$$\boldsymbol{E}_x\left[\frac{f}{f_0}(X_t)\right] \sim t^{-3/2} \times \frac{2}{\sqrt{2\pi}}\int_{\boldsymbol{R}} f(z)f_0(z)dz \quad \text{as } t \to \infty.$$

Now the theorem follows from this and the relation (2.2). □

The rest of the section is devoted to proving Propositions 3.2 and 3.3. In the following, every argument is done for an arbitrarily fixed $y > 0$, which means it is not necessary to relate y to the starting point of the Brownian motion B in such a way as $y = g_0(x)$. So we use below x to denote a variable, not the starting point.

3.b. Proof of Proposition 3.2.

Here we prove Proposition 3.2.

Proof. By changing the variables with $s = tu$ in the definition of $I_1(t)$,

$$\begin{aligned}\sqrt{t}I_1(t) &= \sqrt{t} \times t\int_1^\infty du\, E_y^{(3)}[k(R_{tu})]\\ &= t^{3/2}\int_1^\infty du \int_0^\infty d\xi\, p^{(3)}(tu; y, \xi)k(\xi),\end{aligned}$$

where $p^{(3)}$ denotes the transition density of 3-dimensional Bessel process:

$$p^{(3)}(s; x, z) = \frac{1}{\sqrt{2\pi s}}\frac{z}{x}\exp\left\{-\frac{(z-x)^2}{2s}\right\}\left\{1 - \exp\left(-\frac{2xz}{s}\right)\right\}, \quad s > 0,\ x, z > 0.$$

Noting the function $(1 - e^{-x})/x$ $(x > 0)$ is dominated by 1 and converges to 1 as $x \downarrow 0$, we easily see that, for each fixed u and ξ,

$$t^{3/2}p^{(3)}(tu; y, \xi) \le \frac{2\xi^2}{\sqrt{2\pi u^3}} \quad \text{for all } t > 0, \quad t^{3/2}p^{(3)}(tu; y, \xi) \to \frac{2\xi^2}{\sqrt{2\pi u^3}} \quad \text{as } t \to \infty. \tag{3.2}$$

Moreover,

$$\begin{aligned}\int_1^\infty du \int_0^\infty d\xi\, \frac{2\xi^2}{\sqrt{2\pi u^3}}k(\xi) &= \frac{2}{\sqrt{2\pi}}\int_1^\infty \frac{du}{\sqrt{u^3}}\int_0^\infty d\xi\, \xi^2 k(\xi)\\ &= \frac{4}{\sqrt{2\pi}}\int_{\boldsymbol{R}} dz\, f(z)f_0(z) < \infty\end{aligned}$$

by (A). The second equality follows from the relation (3.1). Now the assertion is immediate from the dominated convergence theorem. □

3.c. Proof of Proposition 3.3.

Similarly to the proof of Proposition 3.2, we rewrite $\sqrt{t}I_2(t)$ as:

$$\begin{aligned}\sqrt{t}I_2(t) &= t^{3/2}\int_0^1 du\int_0^\infty P_y^{(3)}(R_{tu}\in d\xi)\,k(\xi)P^{(3)}_{y,tu,\xi}(A_{tu}(r)>t)\\ &= \int_0^1 du\int_0^\infty d\xi\,k(\xi)\psi_y(u,\xi,t), \end{aligned}\tag{3.3}$$

where we set

$$\psi_y(u,\xi,t) = t^{3/2}p^{(3)}(tu;y,\xi)P^{(3)}_{y,tu,\xi}(A_{tu}(r)>t) \tag{3.4}$$

and, for $s>0$ and $x,z>0$, we denote by the pair $(r=\{r_u, 0\le u\le s\}, P^{(3)}_{x,s,z})$ a pinned 3-dimensional Bessel process over $[0,s]$ such that $P^{(3)}_{x,s,z}(r_0=x, r_s=z)=1$. We prove Proposition 3.3 in four steps.

Step 1. We start with the following proposition:

Proposition 3.4. *For each fixed $0<u<1$ and $\xi>0$,*

$$\psi_y(u,\xi,t)\to 0 \quad \text{as } t\to\infty.$$

As was already seen in (3.2), $t^{3/2}p^{(3)}(tu;y,\xi)$ is dominated by a quantity independent of t. Therefore, rewriting the set $\{A_{tu}(r)>t\}=\{\frac{1}{tu}A_{tu}(r)>\frac{1}{u}\}$, we see the proof of Proposition 3.4 is reduced to showing the following proposition:

Proposition 3.4′. *For each $\varepsilon>0$ and $\xi>0$,*

$$P^{(3)}_{y,T,\xi}\left(\frac{1}{T}A_T(r)>1+\varepsilon\right)\to 0 \quad \text{as } T\to\infty.$$

The proof given here relies on the fact that the FKG inequality is applicable to the laws of pinned 3-dimensional Bessel processes (see the appendix).

Lemma 3.3. *For each $\varepsilon>0$ and $x,z>0$,*

$$P^{(3)}_{x,T,\sqrt{T}z}\left(\frac{1}{T}A_T(r)\le 1+\varepsilon\right)\to 1 \quad \text{as } T\to\infty.$$

In the following proof, we say that a function F defined on the path space $C([0,T];\boldsymbol{R})$ is non-decreasing (resp. non-increasing) if $F(w_1)\le F(w_2)$ (resp. $F(w_1)\ge F(w_2)$) for all $w_1,w_2\in C([0,T];\boldsymbol{R})$ satisfying $w_1(t)\le w_2(t)$ for all $0\le t\le T$.

PROOF OF LEMMA 3.3. Since $(g_0^{-1})'$ is non-increasing, $A_T(r)$ is non-increasing in r, hence the indicator function of the set $\{\frac{1}{T}A_T(r) \le 1+\varepsilon\}$ is non-decreasing in r. So, by the FKG inequality, we see $P^{(3)}_{x,T,\eta}(\frac{1}{T}A_T(r) \le 1+\varepsilon)$ is non-decreasing in η. By using this, we have

$$\begin{aligned}
&P_x^{(3)}\left(\frac{1}{T}A_T(R) \le 1+\varepsilon, R_T \le \sqrt{T}z\right) \\
&= \int_0^{\sqrt{T}z} P_x^{(3)}(R_t \in d\eta) P^{(3)}_{x,T,\eta}\left(\frac{1}{T}A_T(r) \le 1+\varepsilon\right) \\
&\le P^{(3)}_{x,T,\sqrt{T}z}\left(\frac{1}{T}A_T(r) \le 1+\varepsilon\right) P_x^{(3)}(R_T \le \sqrt{T}z).
\end{aligned}$$

Dividing both sides by $P_x^{(3)}(R_T \le \sqrt{T}z)$, we obtain:

$$\frac{P_x^{(3)}(\frac{1}{T}A_T(R) \le 1+\varepsilon, R_T \le \sqrt{T}z)}{P_x^{(3)}(R_T \le \sqrt{T}z)} \le P^{(3)}_{x,T,\sqrt{T}z}\left(\frac{1}{T}A_T(r) \le 1+\varepsilon\right). \tag{3.5}$$

Since, as $T \to \infty$, $A_T(R)/T \to 1$ $P_x^{(3)}$-a.s. (recall (2.4)), the convergence in probability is implied:

$$\lim_{T\to\infty} P_x^{(3)}\left(\frac{1}{T}A_T(R) \le 1+\varepsilon\right) = 1.$$

We also note that, by the scaling property,

$$\lim_{T\to\infty} P_x^{(3)}(R_T \le \sqrt{T}z) = P_0^{(3)}(R_1 \le z) > 0.$$

Combining these, we see that the left hand side of (3.5) converges to 1 as $T \to \infty$, and so does the right hand side. This shows the lemma. □

By using this lemma, we prove Proposition 3.4′:

PROOF OF PROPOSITION 3.4′. Conditionally on $r_{T/2} = \eta$, the process $\{r_t, 0 \le t \le T\}$ is identical in law with the process $r^1 \bullet r^2$ defined by:

$$(r^1 \bullet r^2)(t) = \begin{cases} r^1(t), & 0 \le t \le \dfrac{T}{2}, \\ r^2(T-t), & \dfrac{T}{2} \le t \le T, \end{cases}$$

where r^1 (resp. r^2) is a pinned 3-dimensional Bessel process over $[0, T/2]$ with $r^1(0) = y, r^1(T/2) = \eta$ (resp. with $r^2(0) = \xi, r^2(T/2) = \eta$), and r^1 and r^2 are taken to be independent. It then holds that

$$P^{(3)}_{y,T,\xi}\left(\frac{1}{T}A_T(r) > 1+\varepsilon\right)$$
$$= \int_0^\infty P^{(3)}_{y,T,\xi}\bigl(r_{\frac{T}{2}} \in d\eta\bigr) P^{(3)}_{y,\frac{T}{2},\eta} \otimes P^{(3)}_{\xi,\frac{T}{2},\eta}\left(\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon\right). \tag{3.6}$$

Note that the integrand on the right hand side is non-increasing in η by the FKG inequality (recall the argument in the proof of Lemma 3.3). Therefore, using the FKG inequality again, we see that (3.6) is dominated by

$$\int_0^\infty P^{(3)}_{0,T,0}\bigl(r_{\frac{T}{2}} \in d\eta\bigr) P^{(3)}_{y,\frac{T}{2},\eta} \otimes P^{(3)}_{\xi,\frac{T}{2},\eta}\left(\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon\right). \tag{3.7}$$

Changing the variables with $\eta = \sqrt{T}z$, and noting

$$\left\{\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon\right\} \subset \left\{\frac{2}{T}A_{\frac{T}{2}}(r^1) > 1+\varepsilon\right\} \cup \left\{\frac{2}{T}A_{\frac{T}{2}}(r^2) > 1+\varepsilon\right\},$$

we see further that (3.7) is dominated by

$$\int_0^\infty P^{(3)}_{0,1,0}\bigl(r_{\frac{1}{2}} \in dz\bigr)$$
$$\times \left\{1 - P^{(3)}_{y,\frac{T}{2},\sqrt{T}z}\left(\frac{2}{T}A_{\frac{T}{2}}(r^1) \le 1+\varepsilon\right) P^{(3)}_{\xi,\frac{T}{2},\sqrt{T}z}\left(\frac{2}{T}A_{\frac{T}{2}}(r^2) \le 1+\varepsilon\right)\right\},$$

which converges to 0 as $T \to \infty$ by Lemma 3.3. So the proposition is proved. □

Step 2. First we introduce the cut-off of $|(g_0^{-1})'|^2$:

$$\theta_y(x) = \bigl|(g_0^{-1})'(x \wedge y)\bigr|^2 - \bigl|(g_0^{-1})'(y)\bigr|^2, \quad x > 0.$$

Here $\wedge$ means the minimum. We fix $u_0 \in (0,1)$ in such a way that $u_0 < 1/|(g_0^{-1})'(y)|^2$. We divide the strip $\{(u,\xi); 0 < u < 1, \xi > 0\}$ into three regions:

$$D_1 = (0,u_0) \times (0,y), \qquad D_2 = (0,u_0) \times [y,\infty), \qquad D_3 = [u_0,1) \times (0,\infty).$$

In this step, we prove:

Proposition 3.5. *For each fixed* $0 < u < 1$, $\xi > 0$,

$$\psi_y(u,\xi,t) \le \Psi_y(u,\xi) \quad \textit{for all } t > 0,$$

where

$$\Psi_y(u,\xi)=\begin{cases} C_1\dfrac{\xi}{\sqrt{u}}\Big(\displaystyle\int_0^{\xi} z^2\theta_y(z)dz+\xi\int_{\xi}^{y} z\theta_y(z)dz\Big) & \textit{on } D_1, \\ C_2\dfrac{\xi}{\sqrt{u}} & \textit{on } D_2, \\ \dfrac{2\xi^2}{\sqrt{2\pi u^3}} & \textit{on } D_3, \end{cases}$$

with constants C_1, C_2 independent of u and ξ:

$$C_1=8/\{\sqrt{2\pi}y^2(1-u_0|(g_0^{-1})'(y)|^2)\},\qquad C_2=C_1\int_0^y z^2\theta_y(z)dz.$$

REMARK 3.1. The constant C_2 above is finite; to see this, we only have to check, by the definition of θ, $\int_{0+} z^2|(g_0^{-1})'(z)|^2\,dz<\infty$, which is immediate from (v) of Lemma 2.1.

The bound on D_3 is obvious (recall (3.2)). So we keep $u<u_0$ for a while and will not indicate this unless it is necessary. Since y is fixed, we often suppress it from the notation; e.g., we write θ for θ_y. Put $tu=T$.

LEMMA 3.4. *It holds that*

$$P^{(3)}_{y,T,\xi}\Big(\frac{1}{T}A_T(r)>\frac{1}{u}\Big)\le C_3 u E^{(3)}_{y,T,\xi}\Big[\frac{1}{T}\int_0^T\theta(r_s)ds\Big].$$

Here $C_3=1/(1-u_0|(g_0^{-1})'(y)|^2)$.

PROOF. Note that the following inclusions hold:

$$\begin{aligned}\Big\{\frac{1}{T}A_T(r)>\frac{1}{u}\Big\}&\subset\Big\{\frac{1}{T}\int_0^T\big|(g_0^{-1})'(r_s\wedge y)\big|^2\,ds>\frac{1}{u}\Big\}\\ &=\Big\{\frac{1}{T}\int_0^T\theta(r_s)ds>\frac{1-u|(g_0^{-1})'(y)|^2}{u}\Big\}\\ &\subset\Big\{\frac{1}{T}\int_0^T\theta(r_s)ds>\frac{1-u_0|(g_0^{-1})'(y)|^2}{u}\Big\},\end{aligned}$$

Here, for the first line, we used the fact that $(g_0^{-1})'$ is non-increasing (Lemma 2.1 (iii)), and the definition of θ for the second. Now the assertion follows from Chebyshev's inequality. □

By using this lemma, we shall prove:

LEMMA 3.5. *$\psi(u,\xi,t)$ is dominated by*

$$C_3\frac{\xi}{y\sqrt{2\pi u}}\int_0^y dz\,\theta(z)\int_{|z-\xi|}^{z+\xi}da\biggl(\exp\biggl\{-\frac{(a+y-z)^2}{2T}\biggr\}-\exp\biggl\{-\frac{(a+y+z)^2}{2T}\biggr\}\biggr).$$

Proof. By Lemma 3.4, and by the definition (3.4) of $\psi(u,\xi,t)$,

$$\psi(u,\xi,t)\le C_3t^{1/2}p^{(3)}(T;y,\xi)E^{(3)}_{y,T,\xi}\biggl[\int_0^T\theta(r_s)ds\biggr].$$

Using the law of r at time s, we see:

$$E^{(3)}_{y,T,\xi}\biggl[\int_0^T\theta(r_s)ds\biggr]=\int_0^T ds\int_0^y dz\,\theta(z)\frac{p^{(3)}(s;y,z)p^{(3)}(T-s;z,\xi)}{p^{(3)}(T;y,\xi)}.$$

The second integral is taken only over $(0,y)$ because, by definition, $\theta(z)=0$ for $z\ge y$. We also note that

$$\begin{aligned}&\int_0^T ds\,p^{(3)}(s;y,z)p^{(3)}(T-s;z,\xi)\\&=\frac{\xi}{y}\int_{|z-y|}^{z+y}db\int_{|z-\xi|}^{z+\xi}da\frac{a+b}{\sqrt{2\pi T^3}}\exp\biggl\{-\frac{(a+b)^2}{2T}\biggr\}\\&=\frac{\xi}{y}\int_{|z-\xi|}^{z+\xi}\frac{da}{\sqrt{2\pi T}}\biggl(\exp\biggl\{-\frac{(a+y-z)^2}{2T}\biggr\}-\exp\biggl\{-\frac{(a+y+z)^2}{2T}\biggr\}\biggr)\end{aligned}$$

for $z<y$. Combining these yields the lemma. □

Now we are prepared to prove Proposition 3.5.

Proof of Proposition 3.5. The bound for the case $(u,\xi)\in D_3$ follows from the former of (3.2). For the other two cases, we use the following fact: for $0<\alpha<\beta$, the function $e^{-\alpha x}-e^{-\beta x}$ $(x\ge 0)$ is bounded from above by $1-(\alpha/\beta)$. Using this, we easily see that, for each $a>0$ and $z<y$,

$$\exp\biggl\{-\frac{(a+y-z)^2}{2T}\biggr\}-\exp\biggl\{-\frac{(a+y+z)^2}{2T}\biggr\}\le\frac{4z}{a+y+z}\quad\text{for all } T>0.$$

Combining this with Lemma 3.5, we have, for all $t>0$,

$$\psi(u,\xi,t)\le 4C_3\frac{\xi}{y\sqrt{2\pi u}}\int_0^y dz\,z\theta(z)\int_{|z-\xi|}^{z+\xi}\frac{da}{a+y+z}.$$

Note that the integral with respect to da above is dominated by $2(z\wedge\xi)/y$; indeed,

$$\int_{|z-\xi|}^{z+\xi} \frac{da}{a+y+z} = \log\left(1 + \frac{z+\xi-|z-\xi|}{|z-\xi|+y+z}\right)$$
$$\leq \frac{z+\xi-|z-\xi|}{|z-\xi|+y+z} \leq \frac{z+\xi-|z-\xi|}{y}.$$

Now the bounds for the cases D_1 and D_2 follow from these. □

Step 3. The purpose of this step is to show the following:

PROPOSITION 3.6. *Under the same assumption as in Theorem* 1.1*,*

$$\int_0^1 du \int_0^\infty d\xi\, k(\xi)\Psi(u,\xi) < \infty.$$

Once this proposition is shown, then, combining this with Propositions 3.4 and 3.5, we see Proposition 3.3 follows immediately from the dominated convergence theorem.

The integrability of $k(\xi)\Psi(u,\xi)$ on D_2 and D_3 is obvious; indeed, by definition,

$$\int_{D_i} dud\xi\, k(\xi)\Psi(u,\xi) = \begin{cases} C_2 \displaystyle\int_0^{u_0} \frac{du}{\sqrt{u}} \int_y^\infty d\xi\, \xi k(\xi), & i=2, \\ \displaystyle\frac{2}{\sqrt{2\pi}} \int_{u_0}^1 \frac{du}{\sqrt{u^3}} \int_0^\infty d\xi\, \xi^2 k(\xi), & i=3, \end{cases}$$

both of which are finite by the relation (3.1) and the assumption (A). So we need only to prove the integrability on D_1. For this purpose, we prove the following proposition first.

PROPOSITION 3.7. *Under the same assumption as in Theorem* 1.1*, it holds that*

$$\int_{0+} d\xi\, \xi^2 k(\xi) \int_\xi^y dz\, z\left|(g_0^{-1})'(z)\right|^2 < \infty. \tag{3.8}$$

To see this proposition holds, first note that, by changing the variables, the left hand side of (3.8) is rewritten as:

$$\int_{-\infty} d\eta\, f(\eta) f_0(\eta) \int_\eta^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)}. \tag{3.9}$$

Here we write $x^* = g_0^{-1}(y)$. Recall $\gamma_0 = \sup\{\gamma \geq 0; \liminf_{x\to-\infty} |x|^{-2\gamma}\varphi(x) > 0\}$.

LEMMA 3.6. (i) *If $\gamma_0 \leq 1$, then there exists a constant $c > 0$ such that*

$$\int_\eta^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)} \leq c(1+|\eta|) \quad \textit{for all } \eta \leq x^*.$$

(ii) *If* $\gamma_0 > 1$, *then*

$$\int_{-\infty}^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)} < \infty.$$

Once this lemma is shown, then Proposition 3.7 follows immediately:

Proof of Proposition 3.7. Using (i) of Lemma 3.6, we see that, in the case $\gamma_0 \le 1$, (3.9) is dominated by

$$c \int_{-\infty} d\eta\, f(\eta) f_0(\eta)(1 + |\eta|).$$

Note that this is finite by the assumption (B). For the case $\gamma_0 > 1$, we may bound (3.9) from above by

$$\int_{-\infty} d\eta\, f(\eta) f_0(\eta) \times \int_{-\infty}^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)}.$$

Note that this is also finite by the assumption (A) and (ii) of Lemma 3.6. So the proposition is proved. □

With the help of Proposition 3.7, we give a proof of Proposition 3.6, the main objective of this step:

Proof of Proposition 3.6. We have already seen above that $k(\xi)\Psi(u, \xi)$ is integrable on $D_2 \cup D_3$. For the integrability on $D_1 = (0, u_0) \times (0, y)$, it suffices to prove, by the definition of Ψ,

$$\int_0^y d\xi\, \xi k(\xi) \int_0^\xi dz\, z^2 \theta(z) < \infty, \tag{3.10}$$

$$\int_0^y d\xi\, \xi^2 k(\xi) \int_\xi^y dz\, z\theta(z) < \infty. \tag{3.11}$$

Note that, by (v) of Lemma 2.1, we may find a constant $c > 0$ such that

$$\int_0^\xi z^2 \left|(g_0^{-1})'(z)\right|^2 dz \le c\xi$$

for every sufficiently small ξ. Therefore

$$\int_{0+} d\xi\, \xi k(\xi) \int_0^\xi dz\, z^2 \left|(g_0^{-1})'(z)\right|^2 \le c \int_{0+} d\xi\, \xi^2 k(\xi),$$

which is finite by the relation (3.1) and the assumption (A). From this and the definition of θ, (3.10) follows. (3.11) is a consequence of Proposition 3.7 and the definition of θ. □

It now remains to prove Lemma 3.6. To this end, we prepare the following lemma:

LEMMA 3.7. *Suppose that there exists a $\gamma \geq 0$ such that*

$$\liminf_{x \to -\infty} |x|^{-2\gamma} \varphi(x) > 0.$$

Then there exist constants $a < 0$ and $c > 0$ such that

$$\frac{f_0(x)^2}{g_0(x)} \leq c|x|^{-\gamma} \quad \textit{for all } x < a.$$

PROOF. By the assumption, there exist $a < 0, c > 0$ such that $\varphi(z) \geq c|z|^{2\gamma}$ for all $z < a$. Combining this with $f_0'' = 2\varphi f_0$, we see that, for all $z < a$, $f_0''(z) \geq 2c|z|^{2\gamma} f_0(z)$. Multiplying both sides by $f_0' > 0$, we have

$$f_0''(z) f_0'(z) \geq 2c|z|^{2\gamma} f_0(z) f_0'(z) \quad \text{for all } z < a.$$

Integrating both sides over $(-\infty, x)$ for $x < a$, we see:

$$\begin{aligned}\frac{1}{2} f_0'(x)^2 &\geq 2c \int_{-\infty}^{x} |z|^{2\gamma} f_0(z) f_0'(z)\, dz \\ &= c|x|^{2\gamma} f_0(x)^2 + 2c\gamma \int_{-\infty}^{x} |z|^{2\gamma - 1} f_0(z)^2\, dz \\ &\geq c|x|^{2\gamma} f_0(x)^2.\end{aligned}$$

Here we used integration by parts formula for the equality. (As was seen in Remark 2.1, f_0 decays exponentially or faster at $-\infty$, provided that $\liminf_{x \to -\infty} \varphi(x) > 0$. So the assumption here also ensures $\lim_{x \to -\infty} |x|^{2\gamma} f_0(x)^2 = 0$.) We thus obtain $f_0(x)/f_0'(x) \leq \sqrt{2c}|x|^{-\gamma}$ for all $x < a$. Note that, by (iv) of Lemma 2.1, $f_0^2/g_0 \leq f_0/f_0'$. Combining these ends the proof. $\square$

Using this lemma, we prove Lemma 3.6:

PROOF OF LEMMA 3.6. For the case (i), we may apply Lemma 3.7 with $\gamma = 0$ and get

$$\int_{\eta}^{a} dz\, \frac{f_0(z)^2}{g_0(z)} \leq c(a + |\eta|) \quad \text{for all } \eta < a,$$

for some $a < 0$ and $c > 0$. This implies (i). For the case (ii), we may take $1 < \gamma < \gamma_0$ so that $\liminf_{x \to -\infty} |x|^{-2\gamma} \varphi(x) > 0$. Applying Lemma 3.7 to this γ yields, in particular,

$$\int_{-\infty} dz\, \frac{f_0(z)^2}{g_0(z)} < \infty;$$

indeed, by Lemma 3.7, for some $a < 0$ and $c > 0$,

$$\int_{-\infty}^{a} dz\, \frac{f_0(z)^2}{g_0(z)} \le c \int_{-\infty}^{a} \frac{dz}{|z|^{\gamma}} < \infty.$$

So the assertion (ii) is also proved. □

Step 4. We are now in a position to prove Proposition 3.3:

Proof of Proposition 3.3. Recall the expression (3.3) of $\sqrt{t} I_2(t)$. We then see that the proposition is a consequence of Propositions 3.4, 3.5 and 3.6, and the dominated convergence theorem. □

4. A remark on Corollary 1.1.

We shall consider taking φ as f in Corollary 1.1. Then we see every assumption in Theorem 1.1 is fulfilled; indeed, by the equation $(1/2)f_0'' = \varphi f_0$,

$$\begin{aligned} \int_{\boldsymbol{R}} \varphi(z) f_0(z)\, dz &= \frac{1}{2} \int_{\boldsymbol{R}} f_0''(z)\, dz \\ &= \frac{1}{2}\{f_0'(+\infty) - f_0'(-\infty)\} = \frac{1}{2} < \infty, \end{aligned}$$

and, from integration by parts, it is also seen that, for all $a < 0$,

$$\begin{aligned} \int_{-\infty}^{a} |z| \varphi(z) f_0(z)\, dz &= \frac{1}{2} \int_{-\infty}^{a} |z| f_0''(z)\, dz \\ &= \frac{1}{2}\big(|a| f_0'(a) + f_0(a)\big) < \infty. \end{aligned}$$

As a consequence, (1.4) holds with $f = \varphi$:

$$\lim_{t\to\infty} \sqrt{t} \int_t^{\infty} ds\, E_x\left[\varphi(B_s) \exp\left\{ - \int_0^s \varphi(B_u)\, du \right\}\right] = \sqrt{\frac{2}{\pi}} f_0(x).$$

Note that

$$E_x\left[\varphi(B_s) \exp\left\{ - \int_0^s \varphi(B_u)\, du \right\}\right] = -\frac{d}{ds} E_x\left[\exp\left\{ - \int_0^s \varphi(B_u)\, du \right\}\right].$$

Moreover, since φ can be bounded from below by $c\mathbf{1}_{(-\infty,a)}$ for some $a < 0$ and $c > 0$ by the condition (P2), it can be easily checked that

$$\limsup_{s\to\infty} E_x\left[\exp\left\{ - \int_0^s \varphi(B_u)\, du \right\}\right] \le \limsup_{s\to\infty} E_x\left[\exp\left\{ - c \int_0^s \mathbf{1}_{(-\infty,a)}(B_u)\, du \right\}\right] = 0,$$

with the help of the scaling property of Brownian motion. Combining these, we have

$$\lim_{t\to\infty} \sqrt{t} E_x \left[\exp\left\{ -\int_0^t \varphi(B_s)\, ds \right\} \right] = \sqrt{\frac{2}{\pi}} f_0(x),$$

which partly recovers the result of [**7**, Section 3].

Appendix.

In this appendix, we prove the FKG inequality is applicable to the laws of pinned 3-dimensional Bessel processes (or, more precisely, to their finite-dimensional marginals). For the formulation of the FKG inequality, we refer to [**6**], [**8**].

For $t > 0$ and $x, y > 0$, let $q(t; x, y)$ denote the transition density function of absorbing Brownian motion:

$$q(t; x, y) = \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{x^2 + y^2}{2t} \right) \sinh\left(\frac{xy}{t} \right).$$

Note that

$$p^{(3)}(t; x, y) = \frac{y}{x} q(t; x, y). \tag{A.1}$$

Lemma A.1. *For each fixed $t > 0$, it holds that*

$$q(t; x_1 \vee y_1, x_2 \vee y_2) q(t; x_1 \wedge y_1, x_2 \wedge y_2) \geq q(t; x_1, x_2) q(t; y_1, y_2) \tag{A.2}$$

for all $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$. Here $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Proof. We divide the case into four cases: (i) $x_1 \geq y_1$, $x_2 \geq y_2$; (ii) $x_1 \leq y_1$, $x_2 \leq y_2$; (iii) $x_1 \geq y_1$, $x_2 \leq y_2$; (iv) $x_1 \leq y_1$, $x_2 \geq y_2$. In both cases (i) and (ii), (A.2) holds as an equality. So, by symmetry, we only need to consider either (iii) or (iv). Here we give a proof in the case (iii). By the definition of $q(t; x, y)$, the proof is reduced to showing the following: for $x_1 \geq y_1$ and $x_2 \leq y_2$,

$$\sinh\left(\frac{x_1 y_2}{t} \right) \sinh\left(\frac{x_2 y_1}{t} \right) \geq \sinh\left(\frac{x_1 x_2}{t} \right) \sinh\left(\frac{y_1 y_2}{t} \right). \tag{A.3}$$

Rewriting (A.3) as

$$\frac{\sinh(\frac{y_2}{t} x_1)}{\sinh(\frac{x_2}{t} x_1)} \geq \frac{\sinh(\frac{y_2}{t} y_1)}{\sinh(\frac{x_2}{t} y_1)},$$

we see that it suffices to prove, for $\beta > \alpha > 0$,

$$\frac{\sinh(\beta x)}{\sinh(\alpha x)} \text{ is non-decreasing in } x > 0.$$

This can be easily checked as:

$$\frac{d}{dx}\left\{\frac{\sinh(\beta x)}{\sinh(\alpha x)}\right\} = \frac{(\beta^2-\alpha^2)x}{2\{\sinh(\alpha x)\}^2}\left(\frac{\sinh\{(\beta+\alpha)x\}}{(\beta+\alpha)x} - \frac{\sinh\{(\beta-\alpha)x\}}{(\beta-\alpha)x}\right) \geq 0,$$

where the last inequality follows from the fact that $\sinh(y)/y$ is increasing in $y > 0$. So the lemma is proved. □

For $T > 0$, let $\Delta = \{0 < t_1 < \cdots < t_n < T\}$ be a partition of the interval $[0, T]$. For $a, b > 0$, we denote by $\Phi_\Delta(x; a, b)$ $(x = (x_i)_{1\leq i\leq n})$ the finite-dimensional distribution function of the pinned 3-dimensional Bessel process $P^{(3)}_{a,T,b}$ taken at the time sequence $(t_i)_{1\leq i\leq n}$:

$$\Phi_\Delta(x; a, b) = \frac{p^{(3)}(t_1; a, x_1)p^{(3)}(t_2 - t_1; x_1, x_2) \times \cdots \times p^{(3)}(T - t_n; x_n, b)}{p^{(3)}(T; a, b)}.$$

The next lemma shows $\Phi_\Delta(\cdot; a, b)$ fulfills the assumption of [**6**, Theorem 3]:

Lemma A.2. *For $a \geq a' > 0$ and $b \geq b' > 0$, it holds that*

$$\Phi_\Delta(x \vee y; a, b)\Phi_\Delta(x \wedge y; a', b') \geq \Phi_\Delta(x; a, b)\Phi_\Delta(y; a', b')$$

for all $x = (x_i)_{1\leq i\leq n} \in (0,\infty)^n$ and $y = (y_i)_{1\leq i\leq n} \in (0,\infty)^n$. Here $x \vee y = (x_i \vee y_i)_{1\leq i\leq n}$ and $x \wedge y = (x_i \wedge y_i)_{1\leq i\leq n}$.

Proof. Note that, by the relation (A.1), $\Phi_\Delta(x; a, b)$ is rewritten as

$$\Phi_\Delta(x; a, b) = \frac{q(t_1; a, x_1)q(t_2 - t_1; x_1, x_2) \times \cdots \times q(T - t_n; x_n, b)}{q(T; a, b)}.$$

Therefore the assertion follows immediately from Lemma A.1. □

Remark A.1. It is easily checked that the assertion of this lemma still holds even if either a' or b' is (or, both of them are) equal to 0; in that case, $\Phi_\Delta(x; a', b')$ should be replaced by, say, if $a' = 0$,

$$\Phi_\Delta(x; 0, b') = \frac{\widetilde{q}(t_1; x_1)q(t_2 - t_1; x_1, x_2) \times \cdots \times q(T - t_n; x_n, b')}{\widetilde{q}(T; b')},$$

where

$$\widetilde{q}(t; x) = \frac{x}{\sqrt{2\pi t^3}}\exp\left(-\frac{x^2}{2t}\right).$$

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