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Carleson type measures on parabolic Bergman spaces

Dedicated to Professor Takahiko Nakazi on his sixtieth birthday

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Abstract. Let b^p_α, $0 < \alpha \leq 1$, be the parabolic Bergman space, the Banach space of solutions of parabolic equations $(\partial/\partial t + (-\Delta)^\alpha)u = 0$ on the upper half space $\boldsymbol{R}^{n+1}_+$ which have finite L^p norms. We study Carleson type measures on b^p_α and give a necessary and sufficient condition for a measure μ on $\boldsymbol{R}^{n+1}_+$ to be of Carleson type on b^p_α. As an application, we characterize bounded Toeplitz operators in the space b^2_α.

1. Introduction.

In a recent paper, Nishio, Shimomura, and Suzuki [**7**] have introduced parabolic Bergman spaces b^p_α on the upper half space $\boldsymbol{R}^{n+1}_+ = \{(x_1, \ldots, x_n, t); x \in \boldsymbol{R}^n,\ t > 0\}$ and proved many interesting and important properties of these spaces. Parabolic Bergman spaces are generalization of harmonic Bergman spaces introduced and studied by Ramey and Yi [**8**] and are defined as follows: For $0 < \alpha \leq 1$, let $L^{(\alpha)}$ be the parabolic operator

$$L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^\alpha, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

where, for $0 < \alpha < 1$, $(-\Delta)^\alpha$ is defined by

$$((-\Delta)^\alpha \varphi)(x,t) = -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y-x|>\delta} (\varphi(y,t) - \varphi(x,t))|y-x|^{-n-2\alpha} dy$$
$$\varphi \in C^\infty_0(\boldsymbol{R}^{n+1}_+),$$

with $C_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$ and $L^{(1)}$ is the standard heat operator. We say a continuous function $u(x,t)$ on $\boldsymbol{R}^{n+1}_+$ is $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, if $u \cdot \tilde{L}^{(\alpha)}\varphi \in L^1(\boldsymbol{R}^{n+1}_+, dV)$ and $\int u \cdot \tilde{L}^{(\alpha)}\varphi \, dV = 0$ for all $\varphi \in C^\infty_0(\boldsymbol{R}^{n+1}_+)$, where dV is the Lebesgue volume measure and

$$\big(\tilde{L}^{(\alpha)}\varphi\big)(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) + ((-\Delta)^\alpha \varphi)(x,t) \quad \varphi \in C^\infty_0(\boldsymbol{R}^{n+1}_+),$$

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is the adjoint of $L^{(\alpha)}$. The parabolic Bergman space b^p_α is the set of all $L^{(\alpha)}$-harmonic functions on $\boldsymbol{R}^{n+1}_+$ which belong to $L^p(\boldsymbol{R}^{n+1}_+, dV)$ and it is a Banach space with the L^p norm. It is known that $b^p_\alpha \subset C^\infty(\boldsymbol{R}^{n+1}_+)$ (see [**7**]). When $\alpha = 1/2$, $b^p_{1/2}$ coincide with harmonic Bergman spaces of Ramey and Yi [**8**].

In parabolic Bergman spaces the Huygens property is satisfied: If $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ (see section 2 for the definition)

$$u(y,s) = \int_{\boldsymbol{R}^n} u(x, s-t)\ W^{(\alpha)}(y-x,t)dx \tag{1.1}$$

for all $u \in b^p_\alpha$, and the authors of [**7**] have established the fundamental theory for parabolic Bergman spaces by using this property, generalizing the theory of harmonic Bergman spaces.

We say that a σ-finite positive Borel measure μ on $\boldsymbol{R}^{n+1}_+$ is a Carleson type measure on b^p_α if μ satisfies $|\nabla u| \in L^p(d\mu)$ whenever $u \in b^p_\alpha$. By the closed graph theorem this is equivalent to

$$\|\,|\nabla u|\,\|_{L^p(d\mu)} \le C\|u\|_{L^p(dV)}, \quad u \in b^p_\alpha \tag{1.2}$$

for a constant $C > 0$. The main purpose of this paper is to give a necessary and sufficient condition for a measure to be of Carleson type. Actually, we prove the following more general result (see Theorem 2): Let integers ℓ, $m \ge 0$, multi-index γ, $\lambda > -1$ and $1 \le p < \infty$ be such that $1 + \lambda + (\frac{|\gamma|}{2\alpha} + \ell - m)p > 0$. Then, μ satisfies

$$\int_{\boldsymbol{R}^{n+1}_+} \left|\partial_x^\gamma \partial_t^\ell u\right|^p\, d\mu \le C \int_{\boldsymbol{R}^{n+1}_+} t^\lambda |\partial_t^m u|^p\, dV \quad \text{for all } u \in b^p_\alpha \tag{1.3}$$

with a constant $C > 0$ if and only if there exists a constant $K > 0$ such that

$$\mu\big(Q^{(\alpha)}(y,s)\big) \le K s^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p}$$

for all $(y,s) \in \boldsymbol{R}^{n+1}_+$, where $Q^{(\alpha)}(y,s)$ is a parabolic rectangle of order α with center (y,s) (see section 2 for the definition).

Carleson measures on the classical Hardy space are introduced by Carleson for studying the problem of interpolation by bounded analytic functions on the open unit disk in the complex plane (see [**2**]). Carleson type measures on the holomorphic Bergman space are first studied by Hastings [**4**], and further pursued by Stegenga [**9**], Luecking [**5**], and others. Carleson type measures have found its applications in some problems in Hardy or Bergman spaces. In this paper, we also study Carleson type measures on the parabolic Bergman spaces, and as an application of our result, we characterize bounded positive Toeplitz operators on these spaces.

We display here the plan of the paper. A fundamental solution of the parabolic operator $L^{(\alpha)}$ plays an important role for studying parabolic Bergman spaces and we present some estimates on the fundamental solution in section 2. In section 3, we give a sufficient condition for a measure μ to satisfy the estimate (1.3) and, in section 4, we prove that this is also a necessary condition. Analytic functions or harmonic functions satisfy the local submean inequality, which is very useful for studying Carleson type measures on usual Bergman spaces. This is also the case for $L^{(1)}$-harmonic functions. However, such an inequality is not available for $L^{(\alpha)}$-harmonic functions when $0 < \alpha < 1$ and, to overcome this difficulty, we use in these sections a Whitney decomposition of the upper half space by parabolic rectangles. In section 5, we study Toeplitz operators on b^2_α. The theory of Toeplitz operators on the Hardy space H^2 is classical now. Toeplitz operators are also defined on the holomorphic Bergman space, and several properties of positive Toeplitz operators are studied (see section 6 in [**12**]). As an application of the main theorem, we characterize bounded positive Toeplitz operators on parabolic Bergman spaces.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Upper and lower estimates of the fundamental solution.

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\boldsymbol{R}^n} \exp(-t|\xi|^{2\alpha} + i\ x \cdot \xi)\ d\xi & t > 0 \\ 0 & t \le 0, \end{cases} \tag{2.1}$$

where $x \cdot \xi$ is the inner product on $\boldsymbol{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. We note that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on $\boldsymbol{R}^{n+1}_+$. A fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ plays an important role for studying parabolic Bergman spaces, because $W^{(\alpha)}$ has the reproducing property

$$u(y,s) = -2 \int_{\boldsymbol{R}^{n+1}_+} u(x,t)\ \partial_t W^{(\alpha)}(x-y, t+s) dV(x,t) \tag{2.2}$$

for all $u \in b^p_\alpha$ and $(y,s) \in \boldsymbol{R}^{n+1}_+$ (see remark above Lemma 1 in §3). In case $\alpha = 1/2$, $W^{(1/2)}$ is the Poisson kernel for the upper half space, that is, $W^{(1/2)}(x,t) = \Gamma(\frac{n+1}{2}) t (t^2 + |x|^2)^{-(n+1)/2}$. When $\alpha = 1$, $W^{(1)}$ is the Gauss kernel, that is, $W^{(1)}(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$. In other case, any explicit forms are not known.

We describe some properties of $W^{(\alpha)}$. Let $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$ and $\boldsymbol{N}_0^n = \boldsymbol{N}_0 \times \cdots \times \boldsymbol{N}_0$ (n factors). For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{N}_0^n$, ∂_x^γ denotes the differential monomial $\partial^{|\gamma|}/\partial_{x_1}^{\gamma_1} \ldots \partial_{x_n}^{\gamma_n}$; and let $\partial_t = \partial/\partial_t$. Making a change of variable, we have $W^{(\alpha)}(x,t) = t^{-n/2\alpha} W^{(\alpha)}(t^{-1/2\alpha}x, 1)$. By (2.1), the inductive method implies that

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = t^{-\frac{n+|\beta|}{2\alpha} - k} \big(\partial_x^\beta \partial_t^k W^{(\alpha)}\big)(t^{-1/2\alpha}x, 1). \tag{2.3}$$

When $0 < \alpha < 1$,

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,1) = O(|x|^{-n-2\alpha-|\beta|}) \quad (|x| \to \infty), \tag{2.4}$$

and $\partial_x^\beta \partial_t^k W^{(\alpha)}(x,1)$ is bounded for $|x| \le 1$ (see (2.8) in [**7**]). In fact, for $x_0 = (1,0,\cdots,0) \in \boldsymbol{R}^n$ we put $\psi_\alpha(t) = W^{(\alpha)}(x_0,t)$, then we have $\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = \partial_x^\beta\left[|x|^{-n-2\alpha k}\psi_\alpha^{(k)}(|x|^{-2\alpha}t)\right]$. The Leibnitz rule and the boundedness of $\psi_\alpha^{(k)}(t)$ imply (2.4). When $\alpha = 1$, as in the proof of Lemma 1.4 in [**10**], we have $\partial_x^\beta \partial_t^k W^{(1)}(x,1) = p(x)\exp(-|x|^2/4)$, where $p(x)$ is a polynomial. Therefore, we also have $\partial_x^\beta \partial_t^k W^{(1)}(x,1) = O(|x|^{-n-2-|\beta|})$ $(|x| \to \infty)$ and $\partial_x^\beta \partial_t^k W^{(1)}(x,1)$ is bounded for $|x| \le 1$. We give upper and lower estimates of $W^{(\alpha)}$.

Proposition 1. *Let $\beta = (\beta_1,\ldots,\beta_n) \in \boldsymbol{N}_0^n$ be a multi-index and $k \in \boldsymbol{N}_0$. Then, the following estimates hold.*

(1) *There is a constant $C > 0$ such that*

$$\left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)\right| \le C\frac{t^{-k+1}}{(t+|x|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+1}} \tag{2.5}$$

for all $(x,t) \in \boldsymbol{R}_+^{n+1}$.

(2) *Let $t > 0$. If each β_j is even, then there are constants σ, $C > 0$ such that*

$$\inf\left\{\left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)\right|; |x| \le \sigma t^{1/2\alpha}\right\} \ge C t^{-\frac{n+|\beta|}{2\alpha}-k}, \tag{2.6}$$

where σ and C depend on $n,\alpha,\beta,$ and k. Otherwise,

$$\inf\left\{\left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)\right|; |x| \le \sigma t^{1/2\alpha}\right\} = 0, \tag{2.7}$$

for all $\sigma > 0$.

Proof. (1) Since $\partial_x^\beta \partial_t^k W^{(\alpha)}(x,1) = O(|x|^{-n-2\alpha-|\beta|})$ $(|x| \to \infty)$, if $|t^{-1/2\alpha}x| \ge 1$ then we have

$$\left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)\right| = t^{-\frac{n+|\beta|}{2\alpha}-k}\left|\left(\partial_x^\beta \partial_t^k W^{(\alpha)}\right)(t^{-1/2\alpha}x,1)\right| \le C\frac{t^{-k+1}}{|x|^{n+|\beta|+2\alpha}}.$$

The condition $|x| \ge t^{1/2\alpha}$ implies that $|x|^{2\alpha} = 2^{-1}|x|^{2\alpha} + 2^{-1}|x|^{2\alpha} \ge 2^{-1}(t+|x|^{2\alpha})$.

If $|t^{-1/2\alpha}x| \le 1$, then the boundedness of $|(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/2\alpha}x,1)|$ implies that

$$\left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)\right| \le C\frac{t^{-k+1}}{t^{\frac{n+|\beta|}{2\alpha}+1}}.$$

Since $t \ge 2^{-1}(t+|x|^{2\alpha})$, we have the estimate (2.5).

(2) We show that if each β_j is even then $\partial_x^\beta \partial_t^k W^{(\alpha)}(0,1) \ne 0$, and otherwise $\partial_x^\beta \partial_t^k W^{(\alpha)}(0,1) = 0$, by the induction of k. When $k = 0$, elementary calculations show that

$$\partial_x^\beta W^{(\alpha)}(x,t) = i^{|\beta|} t^{-\frac{n+|\beta|}{2\alpha}} \int_{\boldsymbol{R}^n} e^{-|\xi|^{2\alpha}+i(t^{-1/2\alpha}x\cdot\xi)}\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}\, d\xi.$$

Therefore, we have $\partial_x^\beta W^{(\alpha)}(0,1) = i^{|\beta|}\int_{\boldsymbol{R}^n} e^{-|\xi|^{2\alpha}}\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}\,d\xi$. If each β_j is even, then $\int_{\boldsymbol{R}^n} e^{-|\xi|^{2\alpha}}\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}\,d\xi > 0$. It follows that $\partial_x^\beta W^{(\alpha)}(0,1) \neq 0$. If there exits $1 \le j \le n$ such that β_j is odd, then $\int_{-\infty}^{\infty} e^{-|\xi|^{2\alpha}}\xi_j^{\beta_j}\,d\xi_j = 0$. It follows that $\partial_x^\beta W^{(\alpha)}(0,1) = 0$. Suppose that the inductive assumption holds for k. Then,

$$\begin{aligned}\partial_t\big[\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)\big] &= \partial_t\Big[t^{-\frac{n+|\beta|}{2\alpha}-k}\big(\partial_x^\beta\partial_t^k W^{(\alpha)}\big)(t^{-1/2\alpha}x,1)\Big]\\ &= \Big(-\frac{n+|\beta|}{2\alpha}-k\Big)t^{-\frac{n+|\beta|}{2\alpha}-k-1}\big(\partial_x^\beta\partial_t^k W^{(\alpha)}\big)(t^{-1/2\alpha}x,1)\\ &\quad + t^{-\frac{n+|\beta|}{2\alpha}-k}\sum_{j=1}^n\Big(-\frac{1}{2\alpha}\Big)t^{-\frac{1}{2\alpha}-1}x_j\,\Big(\frac{\partial}{\partial_{x_j}}\partial_x^\beta\partial_t^k W^{(\alpha)}\Big)(t^{-1/2\alpha}x,1).\end{aligned}$$

Thus, we have $\partial_x^\beta\partial_t^{k+1}W^{(\alpha)}(0,1) = \big(-\frac{n+|\beta|}{2\alpha}-k\big)\partial_x^\beta\partial_t^k W^{(\alpha)}(0,1)$. Therefore, if each β_j is even then $\partial_x^\beta\partial_t^{k+1}W^{(\alpha)}(0,1) \neq 0$, and otherwise $\partial_x^\beta\partial_t^{k+1}W^{(\alpha)}(0,1) = 0$.

We show the estimate (2.6). Suppose that each β_j is even. Since $|\partial_x^\beta\partial_t^k W^{(\alpha)}(0,1)| > 0$ and $|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,1)|$ are continuous on $\boldsymbol{R}^n$, there exist constants $\sigma, C > 0$ such that $|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,1)| \ge C$ for $0 \le |x| \le \sigma$. Therefore, if $0 \le |t^{-1/2\alpha}x| \le \sigma$, then we have

$$\big|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)\big| = t^{-\frac{n+|\beta|}{2\alpha}-k}\big|\big(\partial_x^\beta\partial_t^k W^{(\alpha)}\big)(t^{-1/2\alpha}x,1)\big| \ge Ct^{-\frac{n+|\beta|}{2\alpha}-k}.$$

Otherwise, the assertion is clear. Thus, we have the proposition. □

For $(y,s) = (y_1,\dots,y_n,s) \in \boldsymbol{R}_+^{n+1}$, let

$$Q^{(\alpha)}(y,s) = \{(x,t) \in \boldsymbol{R}^{n+1};\, |x_j - y_j| < 2^{-1}s^{1/2\alpha}\ (1 \le j \le n),\ s \le t \le 2s\}.$$

We call them parabolic rectangles of order α with center (y,s). Clearly, $V(Q^{(\alpha)}(y,s)) = s^{\frac{n}{2\alpha}+1}$.

Corollary 1. *Let $\beta \in \boldsymbol{N}_0^n$ be a multi-index, $k \in \boldsymbol{N}_0$, and $(y,s) \in \boldsymbol{R}_+^{n+1}$. If each β_j is even, then there are constants ρ, $C > 0$ such that*

$$C^{-1}s^{-\frac{n+|\beta|}{2\alpha}-k} \le \big|\partial_x^\beta\partial_t^k W^{(\alpha)}(x-y,t+s)\big| \le Cs^{-\frac{n+|\beta|}{2\alpha}-k} \tag{2.8}$$

for all $(x,t) \in Q^{(\alpha)}(y,\rho s)$, where ρ and C depend on n, α, β, and k.

Proof. Let $(y,s) \in \boldsymbol{R}_+^{n+1}$ and σ be the constant in (2) of Proposition 1, then we can choose a constant $\rho > 0$ such that $2^{-1}\rho^{1/2\alpha}n^{1/2} \le \sigma(\rho+1)^{1/2\alpha}$. If $(x,t) \in Q^{(\alpha)}(y,\rho s)$, then $|x-y| \le 2^{-1}(\rho s)^{1/2\alpha}n^{1/2} \le \sigma(\rho s+s)^{1/2\alpha} \le \sigma(t+s)^{1/2\alpha}$. Therefore, (2) of Proposition 1 and the definition of $Q^{(\alpha)}(y,\rho s)$ imply that

$$\big|\partial_x^\beta\partial_t^k W^{(\alpha)}(x-y,t+s)\big| \ge C(t+s)^{-\frac{n+|\beta|}{2\alpha}-k} \ge C(2\rho s+s)^{-\frac{n+|\beta|}{2\alpha}-k} = C's^{-\frac{n+|\beta|}{2\alpha}-k}.$$

A consequence of (1) of Proposition 1 and the definition of $Q^{(\alpha)}(y, \rho s)$ imply the second inequality of (2.8). $\square$

The following theorem is important in this paper.

THEOREM 1. *Let* $1 \le r < \infty$, $\beta \in \boldsymbol{N}_0^n$ *be a multi-index,* $k \in \boldsymbol{N}$, *and* $\delta \in \boldsymbol{R}$. *If there exist constants* $\varepsilon, K > 0$ *such that* $(\frac{n+|\beta|}{2\alpha} + k)r - \varepsilon > \delta > \frac{n}{2\alpha} - \varepsilon$ *and* $\mu(Q^{(\alpha)}(\xi, \tau)) \le K\tau^\varepsilon$ *for all* $(\xi, \tau) \in \boldsymbol{R}_+^{n+1}$, *then there exists a constant* $C > 0$ *such that*

$$\int_{\boldsymbol{R}_+^{n+1}} t^\delta \left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t+s)\right|^r \, d\mu(x,t) \le C s^{\delta - (\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon} \tag{2.9}$$

for all $(y, s) \in \boldsymbol{R}_+^{n+1}$.

PROOF. Let $(y, s) \in \boldsymbol{R}_+^{n+1}$. For a multi-index $\nu = (\nu_1, \ldots, \nu_n) \in \boldsymbol{Z}^n$ and $m \in \boldsymbol{Z}$, put

$$\begin{aligned} Q_{\nu,m} = \big\{(x,t)\ ;\ & \nu_j (2^m s)^{1/2\alpha} \le x_j - y_j \le (\nu_j + 1)(2^m s)^{1/2\alpha}\ (1 \le j \le n), \\ & 2^m s \le t \le 2 \cdot 2^m s\big\}. \end{aligned}$$

Then, $\{Q_{\nu,m}\}$ is a set of parabolic rectangles of order α, and $\boldsymbol{R}_+^{n+1} = \cup Q_{\nu,m}$. Therefore, (1) of Proposition 1 and the hypothesis in Theorem 1 imply that

$$\begin{aligned}
&\int_{\boldsymbol{R}_+^{n+1}} t^\delta \left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t+s)\right|^r \, d\mu(x,t) \\
&\le C \int_{\boldsymbol{R}_+^{n+1}} \frac{t^\delta (t+s)^{(-k+1)r}}{(t+s+|x-y|^{2\alpha})^{(\frac{n+|\beta|}{2\alpha}+1)r}} \, d\mu(x,t) \\
&= C \sum_{\nu \in \boldsymbol{Z}^n, m \in \boldsymbol{Z}} \int_{Q_{\nu,m}} \frac{t^\delta (t+s)^{(-k+1)r}}{(t+s+|x-y|^{2\alpha})^{(\frac{n+|\beta|}{2\alpha}+1)r}} \, d\mu(x,t) \\
&\le C \sum_{\nu \in \boldsymbol{Z}^n, m \in \boldsymbol{Z}} \frac{(2^m s)^\delta (2^m s + s)^{(-k+1)r}}{\left\{2^m s + s + 2^m s(|\nu_1|^2 + \cdots + |\nu_n|^2)^\alpha\right\}^{(\frac{n+|\beta|}{2\alpha}+1)r}} (2^m s)^\varepsilon \\
&= C s^{\delta - (\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon} \sum_{m \in \boldsymbol{Z}} \left\{2^{\delta - (\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\right\}^m \\
&\quad \times \left\{ \sum_{\nu \in \boldsymbol{Z}^n} \frac{(1+2^{-m})^{(-k+1)r}}{\left\{1 + 2^{-m} + (|\nu_1|^2 + \cdots + |\nu_n|^2)^\alpha\right\}^{(\frac{n+|\beta|}{2\alpha}+1)r}} \right\} \\
&\le C s^{\delta - (\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon} \sum_{m \in \boldsymbol{Z}} \left\{2^{\delta - (\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\right\}^m \int_{\boldsymbol{R}^n} \frac{(1+2^{-m})^{(-k+1)r}}{(1+2^{-m}+|x|^{2\alpha})^{(\frac{n+|\beta|}{2\alpha}+1)r}} dx.
\end{aligned}$$

For each $a > 0$, elementary calculations show that

$$\int_{\boldsymbol{R}^n} \frac{1}{(a+|x|^{2\alpha})^{(\frac{n+|\beta|}{2\alpha}+1)r}}dx = C\int_0^\infty \frac{\eta^{n-1}}{(a+\eta^{2\alpha})^{(\frac{n+|\beta|}{2\alpha}+1)r}}d\eta$$
$$\le C\int_0^\infty \frac{\eta^{n-1}}{(a^{\frac{1}{2\alpha}}+\eta)^{(n+|\beta|+2\alpha)r}}d\eta = Ca^{\frac{n}{2\alpha}-(\frac{n+|\beta|}{2\alpha}+1)r}.$$

Since $1+2^{-m}\ge 1$ $(m\ge 0)$ and $1+2^{-m}\ge 2^{-m}$ $(m<0)$, we have

$$\int_{\boldsymbol{R}_+^{n+1}} t^\delta \left|\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y,t+s)\right|^r d\mu(x,t)$$
$$\le Cs^{\delta-(\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\sum_{m\in\boldsymbol{Z}}\left\{2^{\delta-(\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\right\}^m(1+2^{-m})^{\frac{n}{2\alpha}-(\frac{n+|\beta|}{2\alpha}+k)r}$$
$$= Cs^{\delta-(\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\left[\sum_{m\ge 0}\left\{2^{\delta-(\frac{n+|\beta|}{2\alpha}+k)r+\varepsilon}\right\}^m+\sum_{m<0}\left\{2^{\delta-\frac{n}{2\alpha}+\varepsilon}\right\}^m\right].$$

Thus, we have the theorem. □

3. Some inequalities for derivatives.

Let $c_k=\frac{(-2)^k}{k!}$. We give a sufficient condition for a measure μ to satisfy the estimate (1.3). The following lemma is Theorem 6.7 of [**7**]. Lemma 1 follows from the Huygens property and the induction of m, k. Particularly, the inductive arguments are the same method as in the proof of Lemma 4.6 of [**8**].

LEMMA 1. *Let $u\in b_\alpha^p$ and $(y,s)\in\boldsymbol{R}_+^{n+1}$. If $1\le p<\infty$, then*

$$u(y,s)=-2c_{m+j}\int_{\boldsymbol{R}_+^{n+1}}\partial_t^m u(x,t)\; t^{m+j}\partial_t^{j+1}W^{(\alpha)}(x-y,t+s)dV(x,t) \tag{3.1}$$

for all $m,\ j\in\boldsymbol{N}_0$.

PROPOSITION 2. *Let $1\le p<\infty$, $\gamma\in\boldsymbol{N}_0^n$ be a multi-index, $\ell,\ m\in\boldsymbol{N}_0$, and $\lambda\in\boldsymbol{R}$. Suppose that $c>0$ and $j\in\boldsymbol{N}$ satisfy $\frac{|\gamma|}{2\alpha}+\ell-m+\frac{c}{p}>0$ and $\frac{c}{p}-m-j-1<0$. If there exists a constant $M>0$ such that*

$$t^{\frac{(p-1)c}{p}+m+j-\lambda}\int_{\boldsymbol{R}_+^{n+1}} s^{-(\frac{|\gamma|}{2\alpha}+\ell-m+\frac{c}{p})(p-1)}\left|\partial_x^\gamma\partial_t^{\ell+j+1}W^{(\alpha)}(x-y,s+t)\right| d\mu(y,s)\le M \tag{3.2}$$

for all $(y,s)\in\boldsymbol{R}_+^{n+1}$, then there exists a constant $C>0$ such that

$$\int_{\boldsymbol{R}_+^{n+1}}\left|\partial_x^\gamma\partial_t^\ell u\right|^p d\mu\le C\int_{\boldsymbol{R}_+^{n+1}} t^\lambda\left|\partial_t^m u\right|^p dV$$

for all $u\in b_\alpha^p$.

Proof. By Lemma 1, we have

$$\partial_y^\gamma \partial_s^\ell u(y,s) = -2c_{m+j} \int_{\boldsymbol{R}_+^{n+1}} \partial_t^m u(x,t)\ t^{m+j} (-1)^{|\gamma|} \partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s) dV(x,t).$$

Let $1 < p < \infty$ and q be the exponent conjugate to p. The Hölder inequality implies that

$$\begin{aligned}
\left|\partial_y^\gamma \partial_s^\ell u(y,s)\right| &\le C \int_{\boldsymbol{R}_+^{n+1}} \left|\partial_t^m u(x,t)\right|\ t^{m+j} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| dV(x,t) \\
&= C \int_{\boldsymbol{R}_+^{n+1}} \left|\partial_t^m u(x,t)\right| t^{\frac{c}{pq}} \cdot t^{-\frac{c}{pq}}\ t^{m+j} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| dV(x,t) \\
&\le C \Big(\int_{\boldsymbol{R}_+^{n+1}} \left|\partial_t^m u(x,t)\right|^p t^{\frac{c}{q}}\ t^{m+j} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| dV(x,t) \Big)^{\frac{1}{p}} \\
&\quad \times \Big(\int_{\boldsymbol{R}_+^{n+1}} t^{-\frac{c}{p}}\ t^{m+j} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| dV(x,t) \Big)^{\frac{1}{q}}.
\end{aligned}$$

Put $\delta = -\frac{c}{p}+m+j$, $k = \ell+j+1$, and $\varepsilon = \frac{n}{2\alpha}+1$, then $\left(\frac{n+|\gamma|}{2\alpha}+k\right)-\varepsilon-\delta = \frac{|\gamma|}{2\alpha}+\ell-m+\frac{c}{p} > 0$ and $\frac{n}{2\alpha} - \varepsilon - \delta = \frac{c}{p} - m - j - 1 < 0$. Thus, Theorem 1 implies that

$$\Big(\int_{\boldsymbol{R}_+^{n+1}} t^{-\frac{c}{p}}\ t^{m+j} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| dV(x,t) \Big)^{\frac{1}{q}} \le C s^{-(\frac{|\gamma|}{2\alpha}+\ell-m+\frac{c}{p})\frac{1}{q}}.$$

Therefore, the Fubini theorem implies that

$$\int_{\boldsymbol{R}_+^{n+1}} \left|\partial_y^\gamma \partial_s^\ell u(y,s)\right|^p\ d\mu(y,s) \le C \int_{\boldsymbol{R}_+^{n+1}} t^\lambda \left|\partial_t^m u(x,t)\right|^p I(x,t) dV(x,t),$$

where

$$I(x,t) = t^{\frac{c}{q}+m+j-\lambda} \int_{\boldsymbol{R}_+^{n+1}} s^{-(\frac{|\gamma|}{2\alpha}+\ell-m+\frac{c}{p})\frac{p}{q}} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| d\mu(y,s).$$

When $p=1$, the Fubini theorem implies that

$$\int_{\boldsymbol{R}_+^{n+1}} \left|\partial_y^\gamma \partial_s^\ell u(y,s)\right|\ d\mu(y,s) \le C \int_{\boldsymbol{R}_+^{n+1}} t^\lambda \left|\partial_t^m u(x,t)\right| J(x,t) dV(x,t),$$

where

$$J(x,t) = t^{m+j-\lambda} \int_{\boldsymbol{R}_+^{n+1}} \left|\partial_x^\gamma \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)\right| d\mu(y,s).$$

Therefore, we have the proposition. □

When $p=1$, the assumptions $\frac{|\gamma|}{2\alpha}+\ell+\frac{c}{p}-m>0$ and $\frac{c}{p}-m-j-1<0$ are not needed in the proof of Proposition 2. We give a necessary condition for a measure μ to satisfy the estimate (1.3).

PROPOSITION 3. *Let $1\le p<\infty$, $\gamma\in \boldsymbol{N}_0^n$ be a multi-index, $\ell,m\in\boldsymbol{N}_0$, and $\lambda>-1$. If there exists a constant $C>0$ such that*

$$\int_{\boldsymbol{R}_+^{n+1}} \left|\partial_x^\gamma\partial_t^\ell u\right|^p\,d\mu \le C\int_{\boldsymbol{R}_+^{n+1}} t^\lambda\left|\partial_t^m u\right|^p\,dV$$

for all $u\in b_\alpha^p$, then there exists a constant $K>0$ such that

$$\mu(Q^{(\alpha)}(y,s))\le Ks^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p}$$

for all $(y,s)\in\boldsymbol{R}_+^{n+1}$.

PROOF. Let $(y,s)\in\boldsymbol{R}_+^{n+1}$ and $j\ge 2$. Then, Theorem 1 implies that a function $u(x,t)=\partial_x^\gamma\partial_t^j W^{(\alpha)}(x-y,t+s)$ is in b_α^p for $1\le p<\infty$. Therefore, Corollary 1 implies that

$$\begin{aligned} C\int_{\boldsymbol{R}_+^{n+1}} t^\lambda\left|\partial_x^\gamma\partial_t^{m+j}W^{(\alpha)}(x-y,t+s)\right|^p\,dV &\ge \int_{\boldsymbol{R}_+^{n+1}}\left|\partial_x^{2\gamma}\partial_t^{\ell+j}W^{(\alpha)}(x-y,t+s)\right|^p\,d\mu\\ \ge \int_{Q^{(\alpha)}(y,\rho s)}\left|\partial_x^{2\gamma}\partial_t^{\ell+j}W^{(\alpha)}(x-y,t+s)\right|^p\,d\mu &\ge C's^{-(\frac{n+2|\gamma|}{2\alpha}+\ell+j)p}\int_{Q^{(\alpha)}(y,\rho s)}d\mu. \end{aligned}$$

Since we can choose an integer j such that $(\frac{n+|\gamma|}{2\alpha}+m+j)p-(\frac{n}{2\alpha}+1)>\lambda$, Theorem 1 implies that

$$\int_{\boldsymbol{R}_+^{n+1}} t^\lambda\left|\partial_x^\gamma\partial_t^{m+j}W^{(\alpha)}(x-y,t+s)\right|^p\,dV\le Cs^{\lambda-(\frac{n+|\gamma|}{2\alpha}+m+j)p+\frac{n}{2\alpha}+1}.$$

Thus, we have $\mu(Q^{(\alpha)}(y,\rho s))\le Cs^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p}$. Since s is arbitrary, we obtain

$$\mu(Q^{(\alpha)}(y,s))\le C(s/\rho)^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p}=Ks^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p}. \qquad \square$$

4. Carleson type measures on b_α^p.

We give a characterization of Carleson type measures on b_α^p.

THEOREM 2. *Let $1\le p<\infty$, $\gamma\in\boldsymbol{N}_0^n$ be a multi-index, and $\ell,m\in\boldsymbol{N}_0$. Suppose that $\lambda>-1$ and $1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p>0$. Then, there exists a constant $C>0$ such that*

$$\int_{\boldsymbol{R}_+^{n+1}} \left|\partial_x^\gamma \partial_t^\ell u\right|^p \, d\mu \le C \int_{\boldsymbol{R}_+^{n+1}} t^\lambda \left|\partial_t^m u\right|^p \, dV$$

for all $u \in b_\alpha^p$ *if and only if there exists a constant* $K > 0$ *such that*

$$\mu(Q^{(\alpha)}(y,s)) \le K s^{\frac{n}{2\alpha}+1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p} \tag{4.1}$$

for all $(y,s) \in \boldsymbol{R}_+^{n+1}$.

Proof. Suppose that there exists a constant $K > 0$ such that $\mu(Q^{(\alpha)}(y,s)) \le Ks^\varepsilon$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$, where $\varepsilon = \frac{n}{2\alpha} + 1 + \lambda + (\frac{|\gamma|}{2\alpha} + \ell - m)p$. Let $p > 1$. Since $1+\lambda+(\frac{|\gamma|}{2\alpha}+\ell-m)p > 0$, there exists a constant $c > 0$ such that $-(\frac{|\gamma|}{2\alpha}+\ell-m)p < c < (1+\lambda+\frac{|\gamma|}{2\alpha}+\ell-m)\frac{p}{p-1}$. Let j be a non-negative integer such that $\frac{c}{p} - m - j - 1 < 0$ and $j - \lambda + m + \frac{(p-1)c}{p} > 0$. Put $\delta = -(\frac{|\gamma|}{2\alpha} + \ell - m + \frac{c}{p})(p-1)$ and $k = \ell + j + 1$. By Proposition 2, we only show that there exists a constant $M > 0$ such that

$$\int_{\boldsymbol{R}_+^{n+1}} s^\delta \left|\partial_x^\gamma \partial_t^k W^{(\alpha)}(x-y,s+t)\right| \, d\mu(y,s) \le M t^{-(\frac{(p-1)c}{p}+m+j-\lambda)},$$

because $c > 0$ and $j \in \boldsymbol{N}$ satisfy $\frac{|\gamma|}{2\alpha} + \ell - m + \frac{c}{p} > 0$ and $\frac{c}{p} - m - j - 1 < 0$. Since $(\frac{n+|\gamma|}{2\alpha}+k)-\varepsilon-\delta = j-\lambda+m+\frac{(p-1)c}{p} > 0$ and $\frac{n}{2\alpha}-\varepsilon-\delta = \frac{(p-1)c}{p}-(1+\lambda+\frac{|\gamma|}{2\alpha}+\ell-m) < 0$, Theorem 1 implies that

$$\int_{\boldsymbol{R}_+^{n+1}} s^\delta \left|\partial_x^\gamma \partial_t^k W^{(\alpha)}(x-y,s+t)\right| \, d\mu(y,s) \le M t^{\delta-(\frac{n+|\gamma|}{2\alpha}+k)+\varepsilon} = M t^{-(\frac{(p-1)c}{p}+m+j-\lambda)}.$$

When $p = 1$, by the remark below Proposition 2 we only consider the conditions $(\frac{n+|\gamma|}{2\alpha} + k) - \varepsilon - \delta > 0$ and $\frac{n}{2\alpha} - \varepsilon - \delta < 0$. It is easier than the above.

The converse of the implication is a consequence of Proposition 3. Thus, we have the theorem. □

In Theorem 2, we can not remove the condition $1 + \lambda + (\frac{|\gamma|}{2\alpha} + \ell - m)p > 0$. In fact, consider Carleson type measures on the unit disk D in the complex plane ($n = 1$), when $\alpha = \frac{1}{2}$, $p = 2$, $\gamma = (0,\dots,0)$, $\ell = 0$, $m = 1$, and $\lambda \le 1$. For $\lambda < 1$, Stegenga [**9**] proved that a measure μ on D satisfies the inequality $\int_D |f|^2 d\mu \le C \int_D (1-|z|)^\lambda |f'|^2 dV$ for all holomorphic functions f on D if and only if $\mu(\cup S(I_j)) \le K\mathrm{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$ ($I_j \subset \partial D$), where Cap is an appropriate Bessel capacity. Moreover, when $\lambda = 1$, Stegenga [**9**] also proved that μ satisfies the inequality $\int_D |f|^2 d\mu \le C \int_D (1-|z|)|f'|^2 dV$ if and only if $\mu(S(I)) \le K|I|$ for all intervals $I \subset \partial D$. It is known that these conditions are stronger than the condition (4.1) in Theorem 2 (see [**9**, p. 122] and [**12**, p. 170]).

In the condition (4.1) of Theorem 2, we can not replace $Q^{(\alpha)}(y,s)$ by $Q^{(\beta)}(y,s)$ when $\alpha \ne \beta$. In fact, suppose that $\alpha > \beta$, $n = 1, \gamma = (0,\dots,0)$ and $\ell = m = \lambda = 0$. Since $\frac{1}{2\alpha} < \frac{1}{2\beta}$, we can choose a constant ε such that $0 < \varepsilon < \frac{1}{2\beta} - \frac{1}{2\alpha}$. Let

$$\varphi_1(x,t) = \begin{cases} t^{\frac{1}{2\alpha}-\frac{1}{2\beta}+\varepsilon} & (|2^{\frac{1}{2\beta}+1}x|^{2\beta} \le t) \\ 0 & (|2x|^{2\beta} \le t < |2^{\frac{1}{2\beta}+1}x|^{2\beta}) \\ 1 & (t < |2x|^{2\beta}), \end{cases}$$

$$\varphi_2(x,t) = \begin{cases} t^{\varepsilon} & (|2^{\frac{1}{2\alpha}+1}x|^{2\alpha} \le t) \\ 0 & (|2x|^{2\alpha} \le t < |2^{\frac{1}{2\alpha}+1}x|^{2\alpha}) \\ 1 & (t < |2x|^{2\alpha}), \end{cases}$$

$d\mu_1 = \varphi_1(x,t)\chi_{\{t\le 1\}}(x,t)dV$, and $d\mu_2 = \varphi_2(x,t)\chi_{\{t\ge 1\}}(x,t)dV$, where χ_E denotes the characteristic function of a set E. Then, it is easy to see that $\mu_1(Q^{(\alpha)}(y,s)) \le Ks^{\frac{1}{2\alpha}+1}$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$. However, $\mu_1(Q^{(\beta)}(0,s)) \sim s^{\frac{1}{2\alpha}+1+\varepsilon}$ $(s \to 0)$. Therefore, μ_1 can not satisfy that $\mu_1(Q^{(\beta)}(y,s)) \le Ks^{\frac{1}{2\beta}+1}$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$. Conversely, it is also easy to see that $\mu_2(Q^{(\beta)}(y,s)) \le Ks^{\frac{1}{2\beta}+1}$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$ and $\mu_2(Q^{(\alpha)}(0,s)) \sim s^{\frac{1}{2\alpha}+1+\varepsilon}$ $(s \to \infty)$.

The following corollary is Propositions 5.5 and 6.8 of [**7**] (see also Theorem 4.4 of [**8**]).

Corollary 2. *Let $1 \le p < \infty$, $\gamma \in \boldsymbol{N}_0^n$ be a multi-index, and $\ell, m \in \boldsymbol{N}_0$.*
(1) *There exists a constant $C > 0$ such that*

$$C^{-1}\int_{\boldsymbol{R}_+^{n+1}} \left|t^{\frac{|\gamma|}{2\alpha}+\ell}\partial_x^{\gamma}\partial_t^{\ell}u\right|^p\, dV \le \int_{\boldsymbol{R}_+^{n+1}} |u|^p\, dV \le C\int_{\boldsymbol{R}_+^{n+1}} \left|t^m\partial_t^m u\right|^p\, dV$$

for all $u \in b_\alpha^p$.
(2)

$$\sum_{|\gamma|+\ell=m}\int_{\boldsymbol{R}_+^{n+1}} \left|t^{\frac{|\gamma|}{2\alpha}+\ell}\partial_x^{\gamma}\partial_t^{\ell}u\right|^p\, dV \approx \int_{\boldsymbol{R}_+^{n+1}} |u|^p\, dV \approx \int_{\boldsymbol{R}_+^{n+1}} \left|t^m\partial_t^m u\right|^p\, dV$$

for all $u \in b_\alpha^p$.

5. Toeplitz operators on the parabolic Bergman spaces.

For $0 < \alpha \le 1$, we define Toeplitz operators on the parabolic Bergman spaces b_α^2. Since the Huygens property implies that each point evaluation is a bounded linear functional on the parabolic Bergman spaces, the parabolic Bergman spaces b_α^p are closed linear subspaces of $L^p(\boldsymbol{R}_+^{n+1}, dV)$. Therefore, for $0 < \alpha \le 1$ there exists an orthogonal projection R_α from $L^2(\boldsymbol{R}_+^{n+1}, dV)$ onto b_α^2. Given a function $\varphi \in L^1(\boldsymbol{R}_+^{n+1}, dV)$, we define an operator T_φ on b_α^2 by

$$T_\varphi u = R_\alpha(\varphi u), \quad u \in b_\alpha^2. \tag{5.1}$$

We call T_φ the Toeplitz operator on the parabolic Bergman space with symbol φ. In general, the operator T_φ is unbounded. It is well known that the Toeplitz operator T_φ is bounded on the classical Hardy space H^2 (the definition of T_φ is similar) if and only if φ is a essentially bounded function on the unit circle ∂D and $\|T_\varphi\| = \|\varphi\|_\infty$. Similarly, if φ is a bounded function in $L^1(\boldsymbol{R}_+^{n+1}, dV)$, then we clearly have T_φ is bounded on b_α^2 and $\|T_\varphi\| \leq \|\varphi\|_\infty$. However, a complete characterization of the boundedness of T_φ is not known even if $\alpha = \frac{1}{2}$. If $\alpha = \frac{1}{2}$ and φ is a nonnegative function, then a characterization of the boundedness of T_φ is known (see Theorem 6.2.4 in [**12**]). We give a generalization of Theorem 6.2.4 in [**12**].

For $(y,s) \in \boldsymbol{R}_+^{n+1}$, the reproducing property of $-2\partial_t W^{(\alpha)}(x-y,t+s)$ implies that

$$\begin{aligned}\int_{\boldsymbol{R}_+^{n+1}} |-2\partial_t W^{(\alpha)}(x-y,t+s)|^2 dV(x,t) &= -2\partial_t W^{(\alpha)}(y-y,s+s)\\ = -2\partial_t W^{(\alpha)}(0,2s) &= \frac{2}{(2\pi)^n}\int_{\boldsymbol{R}^n} |\xi|^{2\alpha}\exp(-2s|\xi|^{2\alpha})\, d\xi.\end{aligned}$$

Let $w_{(y,s)}^{(\alpha)}(x,t) = -2\partial_t W^{(\alpha)}(x-y,t+s)\{-2\partial_t W^{(\alpha)}(0,2s)\}^{-\frac{1}{2}}$, then we have

$$\int_{\boldsymbol{R}_+^{n+1}} \left|w_{(y,s)}^{(\alpha)}(x,t)\right|^2 dV(x,t) = 1. \tag{5.2}$$

For a function $\varphi \in L^1(\boldsymbol{R}_+^{n+1}, dV)$, we define functions $\widetilde{\varphi}_\alpha$ and $\widehat{\varphi}_\alpha$ on $\boldsymbol{R}_+^{n+1}$ by

$$\widetilde{\varphi}_\alpha(y,s) = \int_{\boldsymbol{R}_+^{n+1}} \left|w_{(y,s)}^{(\alpha)}(x,t)\right|^2 \varphi(x,t)\, dV(x,t) \quad (y,s) \in \boldsymbol{R}_+^{n+1}, \tag{5.3}$$

and

$$\widehat{\varphi}_\alpha(y,s) = \frac{1}{V(Q^{(\alpha)}(y,s))}\int_{Q^{(\alpha)}(y,s)} \varphi(x,t)\, dV(x,t) \quad (y,s) \in \boldsymbol{R}_+^{n+1}, \tag{5.4}$$

respectively.

Theorem 3. *Suppose that $0 < \alpha \leq 1$ and φ is a nonnegative function in $L^1(\boldsymbol{R}_+^{n+1}, dV)$. Then, the following are equivalent*:

(1) *T_φ is a bounded operator on b_α^2;*
(2) *$\widetilde{\varphi}_\alpha$ is a bounded function on $\boldsymbol{R}_+^{n+1}$;*
(3) *$\widehat{\varphi}_\alpha$ is a bounded function on $\boldsymbol{R}_+^{n+1}$.*

Proof. (1)$\Longrightarrow$(2). Let $\langle\cdot\ ,\ \cdot\rangle$ be the usual inner product of $L^2(\boldsymbol{R}_+^{n+1}, dV)$. Since each $w_{(y,s)}^{(\alpha)}$ is a unit vector in b_α^2 and R_α is an orthogonal projection from $L^2(\boldsymbol{R}_+^{n+1}, dV)$ onto b_α^2, we have

$$\begin{aligned}0 \leq \widetilde{\varphi}_\alpha(y,s) &= \left\langle \varphi w_{(y,s)}^{(\alpha)}, w_{(y,s)}^{(\alpha)}\right\rangle = \left\langle \varphi w_{(y,s)}^{(\alpha)}, R_\alpha w_{(y,s)}^{(\alpha)}\right\rangle\\ &= \left\langle R_\alpha\big(\varphi w_{(y,s)}^{(\alpha)}\big), w_{(y,s)}^{(\alpha)}\right\rangle = \left\langle T_\varphi w_{(y,s)}^{(\alpha)}, w_{(y,s)}^{(\alpha)}\right\rangle \leq \|T_\varphi\|.\end{aligned}$$

Thus, $\widetilde{\varphi}_\alpha$ is a bounded function on $\boldsymbol{R}_+^{n+1}$.

(2)$\Longrightarrow$(3). Let $(y,s) \in \boldsymbol{R}_+^{n+1}$. By (1) of Proposition 1, we have $|\partial_t W^{(\alpha)}(0,2s)| \leq Cs^{-(\frac{n}{2\alpha}+1)}$. Moreover, Corollary 1 implies that there are constants ρ, $C>0$ such that $Cs^{-(\frac{n}{2\alpha}+1)} \leq |\partial_t W^{(\alpha)}(x-y,t+s)|$ for all $(x,t) \in Q^{(\alpha)}(y,\rho s)$. Thus, we have

$$\begin{aligned}
\widetilde{\varphi}_\alpha(y,s) &= \int_{\boldsymbol{R}_+^{n+1}} \left|w_{(y,s)}^{(\alpha)}(x,t)\right|^2 \varphi(x,t)\ dV(x,t) \\
&\geq \int_{Q^{(\alpha)}(y,\rho s)} \left|w_{(y,s)}^{(\alpha)}(x,t)\right|^2 \varphi(x,t)\ dV(x,t) \\
&\geq Cs^{-(\frac{n}{2\alpha}+1)} \int_{Q^{(\alpha)}(y,\rho s)} \varphi(x,t)\ dV(x,t) \\
&= C'(\rho s)^{-(\frac{n}{2\alpha}+1)} \int_{Q^{(\alpha)}(y,\rho s)} \varphi(x,t)\ dV(x,t).
\end{aligned}$$

Since $V(Q^{(\alpha)}(y,s)) = s^{\frac{n}{2\alpha}+1}$, the boundedness of $\widetilde{\varphi}_\alpha$ implies that there exists a constant $C>0$ such that $\widehat{\varphi}_\alpha(y,\rho s) \leq C$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$. Therefore, $\widehat{\varphi}_\alpha$ is a bounded function on $\boldsymbol{R}_+^{n+1}$.

(3)$\Longrightarrow$(1). Let $d\mu = \varphi dV$, then the boundedness of $\widehat{\varphi}_\alpha$ implies that there exists a constant $K>0$ such that $\mu(Q^{(\alpha)}(y,s)) \leq Ks^{\frac{n}{2\alpha}+1}$ for all $(y,s) \in \boldsymbol{R}_+^{n+1}$. Therefore, Theorem 2 implies that there exists a constant $C>0$ such that

$$\int_{\boldsymbol{R}_+^{n+1}} |u|^2\ d\mu \leq C \int_{\boldsymbol{R}_+^{n+1}} |u|^2\ dV$$

for all $u \in b_\alpha^2$. It follows that

$$\langle T_\varphi u, u\rangle = \langle \varphi u, R_\alpha u\rangle = \langle \varphi u, u\rangle = \int_{\boldsymbol{R}_+^{n+1}} |u|^2\ d\mu \leq C \int_{\boldsymbol{R}_+^{n+1}} |u|^2\ dV$$

for all $u \in b_\alpha^2$. Since T_φ is positive-definite, T_φ is a bounded operator on b_α^2. □

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