J. Math. Soc. Japan
Vol. 58, No. 1, 2006

Classification of totally real and totally geodesic submanifolds of compact 3-symmetric spaces

By Koji Tojo

(Received Jul. 13, 2002)
(Revised Sept. 22, 2004)

Abstract. It is known that each 3-symmetric space admits an invariant almost complex structure J, so-called a canonical almost complex structure. By making use of simple graded Lie algebras and an affine Lie algebra, we classify half dimensional, totally real (with respect to J) and totally geodesic submanifolds of compact 3-symmetric spaces.

1. Introduction.

Let G be a Lie group and K a compact subgroup of G. A homogeneous space $(G/K, \langle , \rangle)$ with a G-invariant Riemannian metric $\langle , \rangle$ is called *a Riemannian* 3-*symmetric space* if it is not isometric to a Riemannian symmetric space and there exists an automorphism σ of order 3 on G such that

(1) ${G^\sigma}_o \subset K \subset G^\sigma$. Here G^σ and ${G^\sigma}_o$ denote the set of fixed points of σ and its identity component, respectively.

(2) The transformation of G/K induced by σ is an isometry.

We denote by $(G/K, \langle , \rangle, \sigma)$ a Riemannian 3-symmetric space with an automorphism σ.

According to Gray [**G**], Wolf and Gray [**WG**], a compact simply connected Riemannian 3-symmetric space (M, g) may be decomposed as a Riemannian product:

$$M = M_1 \times M_2 \times \cdots \times M_r. \tag{1.1}$$

Here M_i $(1 \leq i \leq r)$ is an irreducible Hermitian symmetric space of compact type or a compact irreducible Riemannian 3-symmetric space. Moreover, a compact irreducible Riemannian 3-symmetric space $(G/K, \langle , \rangle, \sigma)$ has one of the following forms:

(i) G is a compact simple Lie group and σ is inner. Furthermore the dimension of the center Z_K of K is equal to 1, 2 or 0.

(ii) $G = L \times L \times L$ where L is a compact simple Lie group, and $\sigma(X, Y, Z) = (Z, X, Y)$ $(X, Y, Z \in L)$.

(iii) G is a compact simple Lie group of type D_4 and σ is outer.

Gray [**G**] also proved that any Riemannian 3-symmetric space admits an almost complex structure J which is called a canonical almost complex structure.

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 17B70, 53C30.

Key Words and Phrases. 3-symmetric space, graded Lie algebra, totally geodesic submanifold, affine Lie algebra.

In Hermitian symmetric spaces, Chen and Nagano [**CN**] proved that any totally geodesic submanifold is either Kähler or totally real, and totally geodesic Kähler submanifolds were classified by Satake [**Sa2**] and Ihara [**I**]. Moreover, half dimensional, totally real and totally geodesic submanifolds of compact Hermitian symmetric spaces were classified by Takeuchi [**T**]. In the case of 3-symmetric spaces, [**To**] proves that if N is a totally complex or half dimensional, totally real and totally geodesic submanifold (with respect to J) of a compact 3-symmetric space $(G/K, \langle,\rangle, \sigma)$, then there exists a Lie subgroup B of G such that N is expressed as an orbit of B. Moreover, we classified half dimensional, totally real and totally geodesic submanifolds of some 3-symmetric spaces of inner type. More precisely, let $\mathfrak{g}^*$ be a noncompact simple Lie algebra over $\boldsymbol{R}$ and let $\mathfrak{g}^* = \mathfrak{b} + \mathfrak{m}^*$ be a Cartan decomposition of $\mathfrak{g}^*$ such that $\mathfrak{b}$ and $\mathfrak{m}^*$ denote a Lie subalgebra and a subspace of $\mathfrak{g}^*$, respectively. Take a gradation

$$\mathfrak{g}^* = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_{-1} + \mathfrak{g}^*_0 + \mathfrak{g}^*_1 + \mathfrak{g}^*_2 \quad (\mathfrak{g}^*_1 \neq \{0\})$$

of the second kind on $\mathfrak{g}^*$ with the characteristic element $Z \in \mathfrak{m}^* \cap \mathfrak{g}^*_0$. Define an inner automorphism σ of order 3 on the compact dual $\mathfrak{g} = \mathfrak{b} + \sqrt{-1}\mathfrak{m}^*$ of $\mathfrak{g}^*$ by

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}Z\right),$$

and put $\mathfrak{k} = \mathfrak{g}^\sigma$, the set of fixed points of σ. Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$ and K the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Then we proved in [**To**] that $\exp \mathfrak{b} \cdot o$ ($o = \{K\} \in G/K$) is a half dimensional, totally real (with respect to J) and totally geodesic submanifold of $(G/K, \langle,\rangle, \sigma)$. Here $\langle,\rangle$ denotes a Riemannian metric on G/K induced by a biinvariant metric on G, and therefore $(G/K, \langle,\rangle)$ is a naturally reductive homogeneous space (see [**KN**] for the definition of naturally reductive homogeneous spaces). By [**G**], a Riemannian 3-symmetric space $(G/K, \langle,\rangle, \sigma)$ is naturally reductive if and only if $(G/K, \langle,\rangle, J)$ is nearly Kählerian, and naturally reductive Riemannian metric is unique up to a scalar multiple. Moreover, we proved in [**To**] that every such submanifold of a compact irreducible Riemannian 3-symmetric space $(G/K, \langle,\rangle, \sigma)$ of inner type with a naturally reductive metric $\langle,\rangle$ is obtained by the above method if $\dim Z_K \neq 0$.

To complete the classification of half dimensional, totally real and totally geodesic submanifolds of compact and naturally reductive Riemannian 3-symmetric spaces, the paper deals with a half dimensional, totally real and totally geodesic submanifolds of a Riemannian 3-symmetric space $(G/K, \langle,\rangle, \sigma)$ which satisfies one of the following conditions:

(T1) G is a compact simple Lie group, σ is inner and $\dim Z_K = 0$.

(T2) $G = L \times L \times L$ where L is a compact simple Lie group, and $\sigma(X, Y, Z) = (Z, X, Y)$ $(X, Y, Z \in L)$.

(T3) G is a compact simple Lie group of type D_4 and σ is outer.

More precisely, let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. According to [**To**, Proposition 3.2], a half dimensional, totally real and totally geodesic submanifold N of $(G/K, \langle,\rangle, \sigma)$ is expressed as an orbit of some Lie subgroup B of G. Let $\mathfrak{b}$ denote the

Lie algebra of B. We call a pair of $((G/K, \langle,\rangle, \sigma), N)$ of a simply connected Riemannian 3-symmetric space and a half dimensional, totally real and totally geodesic submanifold *a TRG-pair*. Then we shall prove the following theorem.

THEOREM 1.1. *Let $(G/K, \langle,\rangle, \sigma)$ be a compact, simply connected and naturally reductive Riemannian* 3*-symmetric space which satisfies one of the conditions* (T1), (T2) *and* (T3). *Let N be a half dimensional, totally real and totally geodesic submanifolds of $(G/K, \langle,\rangle, \sigma)$. Then $((G/K, \langle,\rangle, \sigma), N)$ is equivalent to one of TRG-pairs listed in Table* 1, 2 *and* 3.

Table 1. $(G/K, \langle,\rangle, \sigma)$ is of type (T1) and $N = \exp \mathfrak{b} \cdot o$.

G	K	$\mathfrak{b}$
$E_6/\boldsymbol{Z}_3$	$\{SU(3) \times SU(3) \times SU(3)\}/\{\boldsymbol{Z}_3 \times \boldsymbol{Z}_3\}$	$\mathfrak{sp}(4)$
		$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
$E_7/\boldsymbol{Z}_2$	$\{SU(3) \times (SU(6)/\boldsymbol{Z}_2)\}/\boldsymbol{Z}_3$	$\mathfrak{su}(8)$
		$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$
E_8	$SU(9)/\boldsymbol{Z}_3$	$\mathfrak{so}(16)$
E_8	$\{SU(3) \times E_6\}/\boldsymbol{Z}_3$	$\mathfrak{so}(16)$
		$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$
F_4	$\{SU(3) \times SU(3)\}/\boldsymbol{Z}_3$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$

Table 2. $(G/K, \langle,\rangle, \sigma)$ is of type (T2) and $N = \exp \mathfrak{b} \cdot o$.

G	K	$\mathfrak{b}$
$\{L \times L \times L\}/\Delta Z$	$\Delta L/\Delta Z$	$\mathfrak{l} \oplus \{(X, X); X \in \mathfrak{l}\}$

Z is the center of L and $\mathfrak{l}$ is the Lie algebra of L. $\Delta(x) := (x, x, x)$ $(x \in L)$.

Table 3. $(G/K, \langle,\rangle, \sigma)$ is of type (T3) and $N = \exp \mathfrak{b} \cdot o$.

G	K	$\mathfrak{b}$
Spin(8)	$SU(3)/\boldsymbol{Z}_3$	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$
Spin(8)	G_2	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$
		$\mathfrak{so}(7)$

In §2 we shall recall some notions and facts of graded Lie algebras and affine Kac-Moody Lie algebras.

In §3 we shall give a necessary and sufficient condition of the existence of a half dimensional, totally real and totally geodesic submanifold in a Riemannian 3-symmetric space by using automorphisms of Lie algebras.

In §4 and §5, by making use of graded Lie algebras of finite kind, we shall deal with the classification problem in the case (T1) (Theorem 5.4).

We shall devote §6 to the classification of half dimensional, totally real and totally geodesic submanifolds in the case (T2) (Theorem 6.1).

In §7 and §8, by making use of an affine Lie algebra of type ${D_4}^{(3)}$, we shall classify half dimensional, totally real and totally geodesic submanifolds in the case (T3) (Theorem 8.5).

The author would like to thank the referee for useful advices.

2. Preliminaries.

2.1.

In this subsection we recall notions and some results on root systems of semisimple Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{t}$ be a compact semisimple Lie algebra and a maximal abelian subalgebra of $\mathfrak{g}$, respectively. We denote by $\mathfrak{g}_c$ and $\mathfrak{t}_c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ be the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$ and put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_c; [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{t}_c\}. \tag{2.1}$$

As in Helgason [**H**], we take the Weyl basis $\{E_\alpha \in \mathfrak{g}_\alpha; \alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)\}$ of $\mathfrak{g}_c$ so that

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= \alpha, \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, \quad N_{\alpha,\beta} \in \boldsymbol{R}, \\ N_{\alpha,\beta} &= -N_{-\alpha,-\beta}. \end{aligned} \tag{2.2}$$

Here, using the Killing form of $\mathfrak{g}_c$, we identify ${\mathfrak{t}_c}^*$ with $\mathfrak{t}_c$. We define A_α, $B_\alpha \in \mathfrak{g}$ by

$$A_\alpha := E_\alpha - E_{-\alpha}, \quad B_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha}). \tag{2.3}$$

Lemma 2.1. (1) *For any* $\sqrt{-1}H \in \mathfrak{t}$ *and* $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, *we have*

$$\begin{aligned} \mathrm{Ad}\big(\exp \sqrt{-1}H\big)(A_\alpha) &= \cos\alpha(H) \cdot A_\alpha + \sin\alpha(H) \cdot B_\alpha, \\ \mathrm{Ad}\big(\exp \sqrt{-1}H\big)(B_\alpha) &= \cos\alpha(H) \cdot B_\alpha - \sin\alpha(H) \cdot A_\alpha. \end{aligned}$$

(2) *For any* $\sqrt{-1}H \in \mathfrak{t}$ *and* $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, *we have*

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \big(\mathrm{ad}\sqrt{-1}H\big)^{2m}(A_\alpha) &= \cos\alpha(H) \cdot A_\alpha, \\ \sum_{m=0}^{\infty} \frac{1}{(2m)!} \big(\mathrm{ad}\sqrt{-1}H\big)^{2m}(B_\alpha) &= \cos\alpha(H) \cdot B_\alpha, \end{aligned}$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left(\mathrm{ad}\sqrt{-1}H\right)^{2m+1}(A_\alpha) = \sin\alpha(H) \cdot B_\alpha,$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left(\mathrm{ad}\sqrt{-1}H\right)^{2m+1}(B_\alpha) = -\sin\alpha(H) \cdot A_\alpha.$$

PROOF. (1) For $\sqrt{-1}H \in \mathfrak{t}$, we have

$$\left[\sqrt{-1}H, E_\alpha\right] = \sqrt{-1}\alpha(H)E_\alpha.$$

Therefore we have

$$\mathrm{Ad}\left(\exp\sqrt{-1}H\right)(E_\alpha) = e^{\sqrt{-1}\alpha(H)}E_\alpha = \left\{\cos\alpha(H) + \sqrt{-1}\sin\alpha(H)\right\}E_\alpha. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}\mathrm{Ad}\left(\exp\sqrt{-1}H\right)(A_\alpha) &= \left\{\cos\alpha(H) + \sqrt{-1}\sin\alpha(H)\right\}E_\alpha \\ &\qquad - \left\{\cos\alpha(H) - \sqrt{-1}\sin\alpha(H)\right\}E_{-\alpha} \\ &= \cos\alpha(H)\cdot A_\alpha + \sin\alpha(H)\cdot B_\alpha.\end{aligned}$$

Similarly we obtain

$$\mathrm{Ad}\left(\exp\sqrt{-1}H\right)(B_\alpha) = \cos\alpha(H)\cdot B_\alpha - \sin\alpha(H)\cdot A_\alpha.$$

(2) By (2.3), it follows that $[\sqrt{-1}H, A_\alpha] = \alpha(H)B_\alpha$ and $[\sqrt{-1}H, B_\alpha] = -\alpha(H)A_\alpha$. Hence we have

$$\left(\mathrm{ad}\sqrt{-1}H\right)^{2m}(A_\alpha) = (-1)^m\alpha(H)^{2m}A_\alpha,$$

$$\left(\mathrm{ad}\sqrt{-1}H\right)^{2m}(B_\alpha) = (-1)^m\alpha(H)^{2m}B_\alpha,$$

$$\left(\mathrm{ad}\sqrt{-1}H\right)^{2m+1}(A_\alpha) = (-1)^m\alpha(H)^{2m+1}B_\alpha,$$

$$\left(\mathrm{ad}\sqrt{-1}H\right)^{2m+1}(B_\alpha) = (-1)^{m+1}\alpha(H)^{2m+1}A_\alpha,$$

and (2) of the lemma is easily obtained by these equations. □

2.2.

In this subsection we recall some results of Kaneyuki and Asano [**KA**].

Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\boldsymbol{R}$. Let τ be a Cartan involution of $\mathfrak{g}^*$ and

$$\mathfrak{g}^* = \mathfrak{b} + \mathfrak{m}^*, \quad \tau|_{\mathfrak{b}} = 1, \quad \tau|_{\mathfrak{m}^*} = -1 \tag{2.5}$$

the Cartan decomposition of $\mathfrak{g}^*$ corresponding to τ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}^*$ and let Δ denote the set of restricted roots of $\mathfrak{g}^*$ with respect to $\mathfrak{a}$. We denote by $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ the set of fundamental roots of Δ with respect to a lexicographic ordering of $\mathfrak{a}$. We call subsets $\{\Pi_0, \Pi_1, \cdots, \Pi_n\}$ of Π a *partition* of Π if $\Pi_1 \neq \varnothing$, $\Pi_n \neq \varnothing$ and

$$\Pi = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_n \quad \text{(disjoint union)}.$$

Let Π and $\bar{\Pi}$ be fundamental root systems of noncompact semisimple Lie algebras $\mathfrak{g}^*$ and $\bar{\mathfrak{g}}^*$ respectively. Partitions $\{\Pi_0, \Pi_1, \cdots, \Pi_m\}$ of Π and $\{\bar{\Pi}_0, \bar{\Pi}_1, \cdots, \bar{\Pi}_n\}$ of $\bar{\Pi}$ are said to be *equivalent* if there exists an isomorphism ϕ from the Dynkin diagram of Π to that of $\bar{\Pi}$ such that $m = n$ and $\phi(\Pi_i) = \bar{\Pi}_i$ $(i = 0, 1, \cdots, n)$.

Take a gradation of the ν-th kind on $\mathfrak{g}^*$:

$$\begin{aligned} &\mathfrak{g}^* = \mathfrak{g}^*_{-\nu} + \cdots + \mathfrak{g}^*_0 + \cdots + \mathfrak{g}^*_\nu, \quad \mathfrak{g}^*_1 \neq \{0\}, \\ &[\mathfrak{g}^*_p, \mathfrak{g}^*_q] \subset \mathfrak{g}^*_{p+q}, \quad \tau(\mathfrak{g}^*_p) = \mathfrak{g}^*_{-p}, \quad -\nu \leq p,\ q \leq \nu. \end{aligned} \tag{2.6}$$

We denote by Z the characteristic element of the gradation, i.e. Z is a unique element in $\mathfrak{m}^* \cap \mathfrak{g}^*_0$ such that

$$\mathfrak{g}^*_p = \{X \in \mathfrak{g}^*; [Z, X] = pX\}, \quad -\nu \leq p \leq \nu. \tag{2.7}$$

Let

$$\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}^*_i, \quad \bar{\mathfrak{g}^*} = \sum_{i=-\bar{\nu}}^{\bar{\nu}} \bar{\mathfrak{g}}^*_i$$

be two graded Lie algebras. These gradations are said to be *isomorphic* if $\nu = \bar{\nu}$ and there exists an isomorphism $\phi : \mathfrak{g}^* \longrightarrow \bar{\mathfrak{g}}^*$ such that $\phi(\mathfrak{g}^*_i) = \bar{\mathfrak{g}}^*_i$ $(-\nu \leq i \leq \nu)$. Then the following holds.

Theorem 2.2 (Kaneyuki and Asano [**KA**]). *Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\boldsymbol{R}$ and Π a fundamental root system of $\mathfrak{g}^*$. Then there exists a bijection between the set of equivalent classes of partitions of Π and the set of isomorphic classes of gradations of $\mathfrak{g}^*$.*

The bijection in Theorem 2.2 is constructed as follows: Let $\{\Pi_0, \Pi_1, \cdots, \Pi_n\}$ be a partition of Π. Define $h_\Pi : \Delta \longrightarrow \boldsymbol{Z}$ by

$$h_\Pi(\alpha) := \sum_{\alpha_i \in \Pi_1} m_i + 2 \sum_{\alpha_j \in \Pi_2} m_j + \cdots + n \sum_{\alpha_k \in \Pi_n} m_k, \quad \alpha = \sum_{i=1}^{l} m_i \alpha_i \in \Delta. \tag{2.8}$$

Then there is a unique Z in $\mathfrak{a}$ such that $\alpha(Z) = h_\Pi(\alpha)$ for all $\alpha \in \Delta$. For a partition $\{\Pi_0, \Pi_1, \cdots, \Pi_n\}$ we obtain a gradation $\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}^*_i$ whose characteristic element equals Z. This correspondence induces a bijection mentioned in the theorem.

2.3.

Next, we recall the notion of an affine Lie algebra of type ${D_4}^{(3)}$. We denote by A the generalized Cartan matrix of type ${D_4}^{(3)}$:

$$A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

Let $\mathfrak{h}$ be a 4-dimensional complex vector space. We choose linear independent systems

$$\Pi^\vee(A) = \{{\gamma_1}^\vee, {\gamma_2}^\vee, {\gamma_3}^\vee\} \subset \mathfrak{h}, \quad \Pi(A) = \{\gamma_1, \gamma_2, \gamma_3\} \subset \mathfrak{h}^*$$

so that

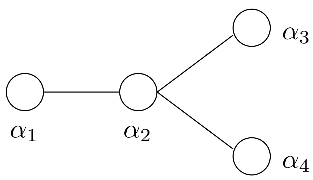
$$\gamma_j({\gamma_i}^\vee) = a_{ij} \quad (a_{ij} : (i,j)\text{-component of } A). \tag{2.9}$$

Let $\mathfrak{g}(A)$ be the Lie algebra with the generators e_i, f_i $(i = 1, 2, 3)$ and $\mathfrak{h}$, and with the following defining relations:

$$\begin{aligned} [h, h'] &= 0 \quad (h, h' \in \mathfrak{h}), \\ [e_i, f_j] &= \delta_{i,j}{\gamma_i}^\vee \quad (i, j = 1, 2, 3), \\ [h, e_i] &= \gamma_i(h) \cdot e_i \quad (h \in \mathfrak{h}, \quad i = 1, 2, 3), \\ [h, f_i] &= -\gamma_i(h) \cdot f_i \quad (h \in \mathfrak{h}, \quad i = 1, 2, 3), \\ (\mathrm{ad} e_i)^{1-a_{ij}}(e_j) &= 0, \\ (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) &= 0. \end{aligned} \tag{2.10}$$

We call $\mathfrak{g}(A)$ an affine Lie algebra of type ${D_4}^{(3)}$, $\{e_i, f_i\}$ Chevalley generators of $\mathfrak{g}(A)$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}(A)$.

Next, we describe another construction of $\mathfrak{g}(A)$. Let $\mathfrak{g}_c$ be a complex simple Lie algebra of type D_4 and $\mathfrak{t}_c$ a Cartan subalgebra of $\mathfrak{g}_c$. Let $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ be the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$ as in subsection 2.1 and let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a set of fundamental roots whose Dynkin diagram is as follows:



Let ν be an automorphism of order 3 on $\mathfrak{g}_c$ induced from the following automorphism $\bar{\nu}$ of the Dynkin diagram of $\mathfrak{g}_c$:

$$\bar{\nu}(\alpha_1) = \alpha_4, \quad \bar{\nu}(\alpha_2) = \alpha_2, \quad \bar{\nu}(\alpha_3) = \alpha_1, \quad \bar{\nu}(\alpha_4) = \alpha_3. \tag{2.11}$$

We denote by $\mathfrak{g}_{\bar{i}}$ ($\bar{i} \in \boldsymbol{Z}_3$) the eigenspace of ν with the eigenvalue ξ^i ($\xi^3 = 1$, $\xi \neq 1$). Since ν is an automorphism of $\mathfrak{g}_c$ we can get a gradation of $\mathfrak{g}_c$:

$$\mathfrak{g}_c = \mathfrak{g}_{-\bar{1}} + \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}.$$

We define an infinite dimensional graded Lie algebra $\mathscr{L}(\mathfrak{g}_c)$ by

$$\mathscr{L}(\mathfrak{g}_c) := \bigoplus_{i \in \boldsymbol{Z}} \mathscr{L}(\mathfrak{g}_c, i), \quad \mathscr{L}(\mathfrak{g}_c, i) := t^i \otimes \mathfrak{g}_{\bar{i}} \subset \boldsymbol{C}[t, t^{-1}] \otimes \mathfrak{g}_c, \tag{2.12}$$

where $\boldsymbol{C}[t, t^{-1}]$ denotes the Laurent polynomial ring with a variable t, and the bracket operation is determined by

$$[P \otimes X, Q \otimes Y] = PQ \otimes [X, Y] \quad (P, Q \in \boldsymbol{C}[t, t^{-1}],\ X, Y \in \mathfrak{g}_c).$$

Then we have a Lie algebra

$$\hat{\mathscr{L}}(\mathfrak{g}_c) := \mathscr{L}(\mathfrak{g}_c) \oplus \boldsymbol{C}K \oplus \boldsymbol{C}d, \tag{2.13}$$

with the bracket $[,]$ defined by

$$\begin{aligned}
[t^i \otimes X, t^j \otimes Y] &= t^{i+j} \otimes [X, Y] + \frac{i}{3}\delta_{i+j,0}(X, Y)K, \\
[K, \hat{\mathscr{L}}(\mathfrak{g}_c)] &= \{0\}, \\
[d, t^i \otimes X] &= it^i \otimes X,
\end{aligned}$$

where $(,)$ denotes the normalized invariant form of $\mathfrak{g}_c$. We choose root vectors e_α ($\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$) of $\mathfrak{g}_c$ so that $\{e_{\alpha_i}, e_{-\alpha_i}; 1 \leq i \leq 4\}$ constitute Chevalley generators of $\mathfrak{g}_c$ and

$$[e_\alpha, e_{-\alpha}] = \alpha, \quad [e_\alpha, e_\beta] = \pm e_{\alpha+\beta}, \tag{2.14}$$

if α, β, $\alpha + \beta \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$. Here we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ by $(,)$. In order to construct Chevalley generators of the Lie algebra $\hat{\mathscr{L}}(\mathfrak{g}_c)$, we set

$$\begin{aligned}
&\theta := \alpha_1 + \alpha_2 + \alpha_3, \\
&E_2 := e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4}, \quad E_3 := e_{\alpha_2}, \\
&F_2 := e_{-\alpha_1} + e_{-\alpha_3} + e_{-\alpha_4}, \quad F_3 := e_{-\alpha_2}, \\
&E_1 := e_{-\theta} + \xi^2 e_{-\bar{\nu}(\theta)} + \xi e_{-\bar{\nu}^2(\theta)}, \quad F_1 := -e_\theta - \xi e_{\bar{\nu}(\theta)} - \xi^2 e_{\bar{\nu}^2(\theta)},
\end{aligned}$$

$$\begin{aligned}
&\hat{E}_1 := t \otimes E_1, \quad \hat{E}_i := t^0 \otimes E_i \qquad (i = 2, 3), \\
&\hat{F}_1 := t^{-1} \otimes F_1, \quad \hat{F}_i := t^0 \otimes F_i \qquad (i = 2, 3), \\
&\hat{\mathfrak{t}} := \mathfrak{t}_c \cap \mathfrak{g}_{\bar{0}}. \qquad\qquad (2.15)
\end{aligned}$$

Then it is known (see Kac [**K**]) that there exists an isomorphism from $\hat{\mathscr{L}}(\mathfrak{g}_c)$ to $\mathfrak{g}(A)$ such that

$$\hat{E}_i \longrightarrow e_i, \quad \hat{F}_i \longrightarrow f_i \qquad (i = 1, 2, 3).$$

Note that

$$\mathfrak{h} \cong \hat{\mathfrak{t}} + \boldsymbol{C}K + \boldsymbol{C}d.$$

Define a homomorphism $\phi_a : \mathscr{L}(\mathfrak{g}_c) \longrightarrow \mathfrak{g}_c$ ($a \in \boldsymbol{C}^{\times}$) by

$$\phi_a(t^i \otimes X) := a^i X. \tag{2.16}$$

Lemma 2.3 ([**K**]). (1) *Every proper maximal ideal of* $\mathscr{L}(\mathfrak{g}_c)$ *is of the form* $(1 - (at)^3)\mathscr{L}(\mathfrak{g}_c)$ $(a \in \boldsymbol{C}^{\times})$.

(2) $\mathrm{Ker}\phi_a = (1 - (a^{-1}t)^3)\mathscr{L}(\mathfrak{g}_c)$.

3. Existence of totally real totally geodesic submanifolds.

Let G be a compact semisimple Lie group and $(G/K, \langle , \rangle, \sigma)$ a simply connected irreducible Riemannian 3-symmetric space with a naturally reductive metric $\langle , \rangle$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, and let $h(,)$ be the Killing form of $\mathfrak{g}$. Since G is compact, the G-invariant metric of G/K induced by $-h(,)$ is positive definite and naturally reductive. Therefore, throughout this paper we assume that

$$\langle , \rangle = -h(,). \tag{3.1}$$

Moreover, in the remaining part of this paper, we use the following notation:

$$\begin{aligned}
\mathfrak{g}^{\mu} &: \text{the set of fixed points of an automorphism } \mu \text{ of a Lie algebra } \mathfrak{g}, \\
G^{\mu} &: \text{the set of fixed points of an automorphism } \mu \text{ of a Lie group } G, \\
[,]_U &: \text{the } U\text{-component of } [,], \text{ where } U \text{ is a vector subspace of } \mathfrak{g}.
\end{aligned}$$

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ with respect to $\langle , \rangle$. Then by (3.1) we have an $\mathrm{Ad}(K)$- and σ-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$. Define a linear automorphism J_o of $\mathfrak{p}$ by

$$\sigma|_{\mathfrak{p}} = -\frac{1}{2}\mathrm{Id}_{\mathfrak{p}} + \frac{\sqrt{3}}{2}J_o. \tag{3.2}$$

Since ${J_o}^2 = -\mathrm{Id}_{\mathfrak{p}}$ and $\mathrm{Ad}(k)J_o = J_o\mathrm{Ad}(k)$ $(k \in K)$, the linear automorphism J_o induces a G-invariant almost complex structure J of G/K. We call J the canonical almost complex structure of $(G/K, \langle , \rangle, \sigma)$. Note that $(G/K, \langle , \rangle, J)$ is an almost Hermitian manifold.

Assume that there exists a connected totally real (with respect to J) and totally geodesic submanifold N of $(G/K, \langle , \rangle)$ such that $2 \dim N = \dim G/K$. Since J is G-invariant and there is $g \in G$ such that the origin o belongs to $g \cdot N$, we may assume that $o \in N$. We denote by V $(V \subset \mathfrak{p} \cong T_oG/K)$ the tangent space of N at o and set $\mathfrak{b} = V + [V, V]_{\mathfrak{k}}$. Let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{b}$ in $\mathfrak{g}$ with respect to $\langle , \rangle$.

Proposition 3.1. *Let μ be a linear automorphism of $\mathfrak{g}$ defined by $\mu|_{\mathfrak{b}} = 1$, $\mu|_{\mathfrak{m}} = -1$. Then μ is involutive and*

$$\mu\sigma = \sigma^2\mu. \tag{3.3}$$

Conversely, suppose that there exists an involutive automorphism μ satisfying (3.3)*. Then $N := G^\mu \cdot o$ (G^μ: the set of fixed points of μ) is a half dimensional, totally real and totally geodesic submanifold of $(G/K, \langle , \rangle, \sigma)$.*

Proof. Since $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{b}$ (see [**To**]), a mapping μ is an involutive automorphism of $\mathfrak{g}$. It follows from (3.2) that

$$\sigma|_{\mathfrak{p}} = -\frac{1}{2}\mathrm{Id}_{\mathfrak{p}} + \frac{\sqrt{3}}{2}J_o, \quad \sigma^2|_{\mathfrak{p}} = -\frac{1}{2}\mathrm{Id}_{\mathfrak{p}} - \frac{\sqrt{3}}{2}J_o.$$

Since

$$\mu(X) = X, \quad \mu(J_oX) = -J_oX \qquad (X \in V),$$

we have for $X \in V$

$$\mu\sigma(X) = \mu\Big(-\frac{1}{2}X + \frac{\sqrt{3}}{2}J_oX\Big) = -\frac{1}{2}X - \frac{\sqrt{3}}{2}J_oX = \sigma^2(X) = \sigma^2\mu(X),$$

and

$$\begin{aligned}\mu\sigma(J_oX) &= \mu\Big(-\frac{1}{2}J_oX - \frac{\sqrt{3}}{2}X\Big) = \frac{1}{2}J_oX - \frac{\sqrt{3}}{2}X \\ &= -\frac{1}{2}(-J_oX) - \frac{\sqrt{3}}{2}J_o(-J_oX) = \sigma^2\mu(J_oX).\end{aligned}$$

For $X \in \mathfrak{k}$, it follows that $\sigma(X) = X$ and $\mu(\mathfrak{k}) = \mathfrak{k}$. So, we have

$$\sigma^2\mu(X) = \mu(X) = \mu\sigma(X).$$

Thus (3.3) holds for any $X \in \mathfrak{g}$.

Next, we suppose that there is an involutive automorphism μ satisfying (3.3). Since $\sigma\mu(X) = \mu\sigma^2(X) = \mu(X)$ $(X \in \mathfrak{k})$, we have $\mu(\mathfrak{k}) = \mathfrak{k}$, $\mu(\mathfrak{p}) = \mathfrak{p}$ and

$$\mathfrak{g}^\mu = \mathfrak{g}^\mu \cap \mathfrak{k} + \mathfrak{g}^\mu \cap \mathfrak{p}.$$

According to Sagle [**S**], if a subspace U of a tangent space at o of a naturally reductive homogeneous space G/K satisfies

$$[U, U]_\mathfrak{p} \subset U, \quad [U, [U, U]_\mathfrak{k}] \subset U$$

then $\exp(U + [U, U]) \cdot o$ is a totally geodesic submanifold of G/K. Hence it follows that $G^\mu \cdot o$ is totally geodesic. Furthermore, for $X \in \mathfrak{g}^\mu \cap \mathfrak{p}$, we have

$$\begin{aligned}\mu(J_o X) &= \frac{2}{\sqrt{3}}\mu\left(\sigma(X) + \frac{1}{2}X\right) = \frac{2}{\sqrt{3}}\left(\sigma^2\mu(X) + \frac{1}{2}\mu(X)\right)\\ &= \frac{2}{\sqrt{3}}\left(\sigma^2(X) + \frac{1}{2}X\right) = -J_o X.\end{aligned}$$

Consequently, we obtain that

$$\mathfrak{p} = \mathfrak{g}^\mu \cap \mathfrak{p} \oplus J_o(\mathfrak{g}^\mu \cap \mathfrak{p}), \quad \text{(orthogonal decomposition)}$$

which implies that $G^\mu \cdot o$ is a half dimensional, totally real and totally geodesic submanifold of G/K. □

4. The case (T1).

In this section we use the same notation as in Section 2. Let $(G/K, \langle,\rangle, \sigma)$ be a simply connected, compact irreducible Riemannian 3-symmetric space such that σ is inner and $\dim Z_K = 0$. Since σ is inner, there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. According to [**G**], 3-symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ satisfying the condition (T1) are given by

$$\begin{aligned}&(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3)), \quad (\mathfrak{e}_7, \mathfrak{su}(3) \oplus \mathfrak{su}(6)), \quad (\mathfrak{e}_8, \mathfrak{su}(9)),\\ &(\mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6), \quad (\mathfrak{f}_4, \mathfrak{su}(3) \oplus \mathfrak{su}(3)), \quad (\mathfrak{g}_2, \mathfrak{su}(3)).\end{aligned} \tag{4.1}$$

Note that if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{su}(3))$, then $(G/K, \langle,\rangle, \sigma)$ is isometric to the standard 6-sphere S^6. Accordingly we deal with 3-symmetric pairs except for $(\mathfrak{g}_2, \mathfrak{su}(3))$.

First, we construct a half dimensional, totally real and totally geodesic submanifold by using graded Lie algebras. Let $\mathfrak{g}^*$ be a noncompact simple Lie algebra over $\boldsymbol{R}$ such that the complexification $\mathfrak{g}_c$ of $\mathfrak{g}^*$ is simple. Consider a Cartan decomposition $\mathfrak{g}^* = \mathfrak{b} + \mathfrak{m}^*$ and a Cartan involution τ given by (2.5). We take a gradation $\mathfrak{g}^* = \sum_{p=-\nu}^{\nu} \mathfrak{g}_p^*$ of the ν-th kind on $\mathfrak{g}^*$ as in (2.6) and we define an inner automorphism σ of order 3 on the compact dual $\mathfrak{g} = \mathfrak{b} + \mathfrak{m}$ $(\mathfrak{m} := \sqrt{-1}\mathfrak{m}^*)$ as follows:

$$\sigma := \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}Z\right). \tag{4.2}$$

Here Z denotes the characteristic element defined by (2.7). Since $Z \in \mathfrak{m}^*$, we have

$$\begin{aligned} &\mathfrak{g}_0^* = \mathfrak{b} \cap \mathfrak{g}_0^* + \mathfrak{m}^* \cap \mathfrak{g}_0^*, \\ &\mathfrak{g}_p^* + \mathfrak{g}_{-p}^* = \mathfrak{b} \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*) + \mathfrak{m}^* \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*), \quad p = 1, \cdots, \nu. \end{aligned} \tag{4.3}$$

Then it follows from (2.7) and (4.2) that $\mathfrak{k} := \mathfrak{g}^\sigma$ coincides with

$$\mathfrak{k} = \mathfrak{b} \cap \mathfrak{g}_0^* + \sqrt{-1}(\mathfrak{m}^* \cap \mathfrak{g}_0^*) + \sum_{p \equiv 0 \mod 3} \left\{\mathfrak{b} \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*) + \sqrt{-1}\big(\mathfrak{m}^* \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*)\big)\right\}. \tag{4.4}$$

We put

$$\begin{aligned} \mathfrak{p} &:= \sum_{p \not\equiv 0 \mod 3} \left\{\mathfrak{b} \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*) + \sqrt{-1}\big(\mathfrak{m}^* \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*)\big)\right\}, \\ V &:= \sum_{p \not\equiv 0 \mod 3} \mathfrak{b} \cap \big(\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*\big), \\ W &:= \sum_{p \not\equiv 0 \mod 3} \sqrt{-1}\big(\mathfrak{m}^* \cap (\mathfrak{g}_p^* + \mathfrak{g}_{-p}^*)\big). \end{aligned} \tag{4.5}$$

Then it is obvious that

$$\mathfrak{b} \cap \mathfrak{p} = V, \quad \mathfrak{p} = V \oplus W \quad \text{(orthogonal direct sum)}. \tag{4.6}$$

By (2.6) and (2.7) we can see that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $\sigma(\mathfrak{p}) = \mathfrak{p}$. Therefore the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is $\mathrm{ad}(\mathfrak{k})$- and σ-invariant. Moreover the following lemma holds.

Lemma 4.1. *Let μ be the involutive automorphism of $\mathfrak{g}$ defined by $\mu|_{\mathfrak{b}} = \mathrm{Id}_{\mathfrak{b}}$, $\mu|_{\mathfrak{m}} = -\mathrm{Id}_{\mathfrak{m}}$. Then*

$$\mu\sigma = \sigma^2\mu.$$

Proof. Because $\sqrt{-1}Z \in \mathfrak{m}$, we have $\mu(\sqrt{-1}Z) = -\sqrt{-1}Z$. Therefore, by (4.2) we obtain

$$\begin{aligned} \mu\sigma\mu^{-1} &= \mu\mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}Z\right)\mu^{-1} = \mathrm{Ad}\left(\exp\frac{2\pi}{3}\mu(\sqrt{-1}Z)\right) \\ &= \mathrm{Ad}\left(\exp -\frac{2\pi}{3}\sqrt{-1}Z\right) = \sigma^{-1} = \sigma^2. \end{aligned}$$

□

In the remaining part of this section, we shall prove the following theorem.

THEOREM 4.2. (1) *Let $\mathfrak{g}^* = \mathfrak{b} + \mathfrak{m}^*$ be a Cartan decomposition of a noncompact simple Lie algebra $\mathfrak{g}^*$ as in (2.5) and let $\mathfrak{g} = \mathfrak{b} + \sqrt{-1}\mathfrak{m}^*$ be the compact dual of $\mathfrak{g}^*$. Take a gradation $\mathfrak{g}^* = \sum_{p=-\nu}^{\nu} \mathfrak{g}_p^*$ on $\mathfrak{g}^*$ with the characteristic element $Z \in \mathfrak{m}^* \cap \mathfrak{g}_0^*$ and set $\sigma = \mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}Z)$. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$ and let K be a connected Lie subgroup of G corresponding to $\mathfrak{k}$ given by (4.4). Then $\exp \mathfrak{b} \cdot o$ is a half dimensional, totally real and totally geodesic submanifold of a compact Riemannian 3-symmetric space $(G/K, \langle , \rangle, \sigma)$.*

(2) *Let $(G/K, \langle , \rangle, \sigma)$ be a compact Riemannian 3-symmetric space such that σ is inner and $\dim Z_K = 0$. Then every half dimensional, totally real and totally geodesic submanifold is obtained from a graded Lie algebra by the method described in the above (1).*

We note that (1) of the theorem is immediate from Proposition 3.1 and Lemma 4.1, and so, we shall prove (2) of the theorem in the following.

Since G/K is simply connected, we may assume that G is centerless, K is connected and the center Z_K of K is isomorphic to $\boldsymbol{Z}_3$ (see [**G**]). Then since $\mathfrak{k} = \mathfrak{g}^\sigma$, there is $g \in Z_K$ such that $\sigma = \mathrm{Ad}(g)$. Assume that there exists a half dimensional, totally real and totally geodesic submanifold of G/K. Then by Proposition 3.1, there exists an involutive automorphism μ of $\mathfrak{g}$ satisfying (3.3). As before, put

$$\mathfrak{b} := \mathfrak{g}^\mu, \quad \mathfrak{m} := \mathfrak{b}^\perp, \quad V := \mathfrak{b} \cap \mathfrak{p} \tag{4.7}$$

where $\perp$ means the orthogonal complement with respect to $\langle , \rangle$. Since the pair $(\mathfrak{g}, \mathfrak{b})$ is locally symmetric and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, we have

$$\begin{aligned} &[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{b}, \quad [\mathfrak{b}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{b}_\mathfrak{k}, V] \subset \mathfrak{b} \cap \mathfrak{p} = V, \\ &[\mathfrak{m} \cap \mathfrak{k}, V] \subset \mathfrak{m} \cap \mathfrak{p} = J_o V, \quad [\mathfrak{m} \cap \mathfrak{k}, J_o V] \subset V. \end{aligned} \tag{4.8}$$

Since σ is inner, there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. Let $\sqrt{-1}t_0$ be an element of $\mathfrak{t}$ such that $g_0 := \exp \pi\sqrt{-1}t_0 \in Z_K$, $g_0 \neq e$. We decompose $\sqrt{-1}t_0$ to

$$\sqrt{-1}t_0 = \sqrt{-1}t_1 + \sqrt{-1}t_2, \quad \sqrt{-1}t_1 \in \mathfrak{m} \cap \mathfrak{k}, \quad \sqrt{-1}t_2 \in \mathfrak{b}_\mathfrak{k}. \tag{4.9}$$

Let A_α, B_α ($\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$) be as in (2.3). Since $\mathfrak{t} \subset \mathfrak{k}$, we have for any $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$

$$A_\alpha, \ B_\alpha \in \mathfrak{k}, \quad \text{or} \quad A_\alpha, \ B_\alpha \in \mathfrak{p},$$

So, we put

$$\begin{aligned} \Delta_\mathfrak{k} &:= \{\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c); A_\alpha, B_\alpha \in \mathfrak{k}\}, \\ \Delta_\mathfrak{p} &:= \{\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c); A_\alpha, B_\alpha \in \mathfrak{p}\}. \end{aligned}$$

Then

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha), \quad \mathfrak{k} = \mathfrak{t} + \sum_{\alpha \in \Delta_{\mathfrak{k}}} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha). \tag{4.10}$$

Replacing H in Lemma 2.1 with t_0, we can see that for any $\alpha \in \Delta_{\mathfrak{k}}$

$$\alpha(t_0) = \alpha(t_1) + \alpha(t_2) \in 2\boldsymbol{Z} \tag{4.11}$$

because $g_0 \in Z_K$.

To prove Theorem 4.2 we prepare some lemmas.

Lemma 4.3. *For $\alpha \in \Delta_{\mathfrak{k}}$, we have*

$$\alpha(t_1),\ \alpha(t_2) \in \boldsymbol{Z}, \quad \alpha(t_1) \equiv \alpha(t_2) \pmod 2.$$

Proof. Let

$$X = \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha X_\alpha$$

be an element of $\mathfrak{k}$. Here $X_\alpha = a_\alpha A_\alpha + b_\alpha B_\alpha$, ${a_\alpha}^2 + {b_\alpha}^2 = 1$. We define a subset $\mathrm{supp}(X)$ of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ by

$$\mathrm{supp}(X) := \{\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c); c_\alpha \neq 0\}. \tag{4.12}$$

First, we assume that $X \in \mathfrak{b}_{\mathfrak{k}}$. By Lemma 2.1 and (4.8), we obtain

$$\begin{aligned} \sum_{m=0} \frac{1}{(2m+1)!} \left(\mathrm{ad}\pi\sqrt{-1}t_1\right)^{2m+1}(X) &= \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_1) \cdot Y_\alpha \in \mathfrak{m}, \\ \sum_{m=0} \frac{1}{(2m+1)!} \left(\mathrm{ad}\pi\sqrt{-1}t_2\right)^{2m+1}(X) &= \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_2) \cdot Y_\alpha \in \mathfrak{b}_{\mathfrak{k}}, \end{aligned} \tag{4.13}$$

where $Y_\alpha = a_\alpha B_\alpha - b_\alpha A_\alpha$. By (4.11), we get

$$\sin \pi\alpha(t_1) = -\sin \pi\alpha(t_2), \quad \alpha \in \Delta_{\mathfrak{k}},$$

and it follows from (4.13) that

$$\sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_1) \cdot Y_\alpha = \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_2) \cdot Y_\alpha = 0.$$

Consequently, we obtain

$$\alpha(t_1),\ \alpha(t_2) \in \boldsymbol{Z}, \quad \alpha \in \mathrm{supp}(X), \quad X \in \mathfrak{b}_{\mathfrak{k}}. \tag{4.14}$$

Next, we assume that $X \in \mathfrak{m} \cap \mathfrak{k}$. By the similar computation as above, we can see that

$$\sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_1) \cdot Y_\alpha \in \mathfrak{b}_{\mathfrak{k}}, \qquad \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha \sin \pi\alpha(t_2) \cdot Y_\alpha \in \mathfrak{m} \cap \mathfrak{k},$$

and

$$\alpha(t_1),\ \alpha(t_2) \in \boldsymbol{Z}, \quad \alpha \in \operatorname{supp}(X), \quad X \in \mathfrak{m} \cap \mathfrak{k}. \tag{4.15}$$

By using (4.14) and (4.15) together with (4.11), the lemma is proved. □

LEMMA 4.4. *$\exp 2\pi\sqrt{-1}t_1 \in Z_K$, $\exp 2\pi\sqrt{-1}t_1 \neq e$.*

PROOF. Note that

$$g_0 = \exp \pi\sqrt{-1}t_0 = \exp \pi\sqrt{-1}t_1 \cdot \exp \pi\sqrt{-1}t_2,$$
$${g_0}^2 = \exp 2\pi\sqrt{-1}t_0 = \exp 2\pi\sqrt{-1}t_1 \cdot \exp 2\pi\sqrt{-1}t_2.$$

By substituting $2t_1$ or $2t_2$ for H in Lemma 2.1, it follows from Lemma 4.3 that

$$\exp 2\pi\sqrt{-1}t_1, \quad \exp 2\pi\sqrt{-1}t_2 \in Z_K.$$

Suppose that $\exp 2\pi\sqrt{-1}t_2 \neq e$, then $\operatorname{Ad}(\exp 2\pi\sqrt{-1}t_2)$ coincides with σ or σ^2 since $Z_K \cong \boldsymbol{Z}_3$. Since $\mathfrak{b}$ is a Lie algebra and $\sqrt{-1}t_2 \in \mathfrak{b}_{\mathfrak{k}}$, we have $\sigma(V) = V$. However, this contradicts the fact that N is totally real with respect to J. So, we obtain $\exp 2\pi\sqrt{-1}t_2 = e$ and

$$\exp 2\pi\sqrt{-1}t_1 = \exp 2\pi\sqrt{-1}t_0 = {g_0}^2 \in Z_K, \quad {g_0}^2 \neq e. \qquad \square$$

By Lemma 4.4, it follows that $\operatorname{Ad}(\exp 2\pi\sqrt{-1}t_1)$ coincides with σ or σ^2. Therefore, by putting $h = t_1$ or $-t_1$, we see that there is a vector $\sqrt{-1}h \in \mathfrak{g}$ such that

$$\sqrt{-1}h \in \mathfrak{m} \cap \mathfrak{t}, \quad \alpha(h) \in 2\boldsymbol{Z}, \quad \alpha \in \Delta_{\mathfrak{k}} \tag{4.16}$$

and

$$\operatorname{Ad}\big(\exp \pi\sqrt{-1}h\big) = \sigma. \tag{4.17}$$

For any $\alpha \in \Delta_{\mathfrak{p}}$, we have by Lemma 2.1 and (4.17) that

$$3\alpha(h) \in 2\boldsymbol{Z} \quad (\alpha \in \Delta_{\mathfrak{p}}).$$

For $k \in \boldsymbol{Z}$, we put

$$
\begin{aligned}
\Delta_1(k) &:= \left\{\alpha \in \Delta_{\mathfrak{p}}; \alpha(h) = \frac{2}{3} + 2k\right\}, \quad \Delta_2(k) := \left\{\alpha \in \Delta_{\mathfrak{p}}; \alpha(h) = -\frac{2}{3} + 2k\right\}, \\
\mathfrak{p}_i(k) &:= \sum_{\alpha \in \Delta_i(k)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha), \quad i = 1, 2, \\
\Delta_{\mathfrak{k}}(k) &:= \{\alpha \in \Delta_{\mathfrak{k}}; \alpha(h) = 2k\}, \quad \mathfrak{k}_k := \sum_{\alpha \in \Delta_{\mathfrak{k}}(k)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha). \qquad (4.18)
\end{aligned}
$$

Note that

$$
\Delta_{\mathfrak{k}} = \bigcup_{k \in \boldsymbol{Z}} \Delta_{\mathfrak{k}}(k), \quad \Delta_{\mathfrak{p}} = \bigcup_{k \in \boldsymbol{Z}} (\Delta_1(k) \cup \Delta_2(k)).
$$

Lemma 4.5.

$$
V = \sum_{k \in \boldsymbol{Z}} V \cap (\mathfrak{p}_1(k) + \mathfrak{p}_2(-k)), \quad \mathfrak{b}_{\mathfrak{k}} = (\mathfrak{b} \cap \mathfrak{t}) \oplus \sum_{k \in \boldsymbol{Z}} \mathfrak{b} \cap (\mathfrak{k}_k + \mathfrak{k}_{-k}).
$$

Proof. Let

$$
X = \sum_{\alpha \in \Delta_{\mathfrak{p}}} c_\alpha X_\alpha
$$

be an element of V, where $X_\alpha = a_\alpha A_\alpha + b_\alpha B_\alpha$, ${a_\alpha}^2 + {b_\alpha}^2 = 1$. It follows from Lemma 2.1 and (4.18) that for any $s \in \boldsymbol{R}$

$$
\begin{aligned}
\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\mathrm{ad}\ s\sqrt{-1}h\right)^{2m}(X) &= \sum_{\alpha \in \Delta_{\mathfrak{p}}} c_\alpha \cos s\alpha(h) \cdot X_\alpha \\
&= \sum_{k \in \boldsymbol{Z}} \left\{ \sum_{\alpha \in \Delta_1(k)} c_\alpha \cos s\alpha(h) \cdot X_\alpha + \sum_{\alpha \in \Delta_2(k)} c_\alpha \cos s\alpha(h) \cdot X_\alpha \right\} \\
&= \sum_{k \in \boldsymbol{Z}} \left\{ \sum_{\alpha \in \Delta_1(k)} c_\alpha \cos s\left(\frac{2}{3} + 2k\right) \cdot X_\alpha + \sum_{\alpha \in \Delta_2(k)} c_\alpha \cos s\left(\frac{2}{3} - 2k\right) \cdot X_\alpha \right\} \\
&= \sum_{k \in \boldsymbol{Z}} \cos s\left(\frac{2}{3} + 2k\right) \cdot X_k,
\end{aligned}
$$

where

$$
X_k = \sum_{\alpha \in \Delta_1(k)} c_\alpha X_\alpha + \sum_{\alpha \in \Delta_2(-k)} c_\alpha X_\alpha.
$$

By (4.8), each $(\mathrm{ad}\sqrt{-1}h)^{2m}(X)$ is contained in V, therefore we have

$$\sum_{k \in \boldsymbol{Z}} \cos s\left(\frac{2}{3} + 2k\right) \cdot X_k \in V$$

for any $s \in \boldsymbol{R}$. Consequently, we obtain $X_k \in V$ for any $k \in \boldsymbol{Z}$.

Next, let

$$W = \sum_{\alpha \in \Delta_{\mathfrak{k}}} c_\alpha W_\alpha$$

be an element of $\mathfrak{b}_{\mathfrak{k}}$, where $W_\alpha = a_\alpha A_\alpha + b_\alpha B_\alpha$, ${a_\alpha}^2 + {b_\alpha}^2 = 1$. By the same argument as above, we can see that

$$\sum_{k \in \boldsymbol{Z}} \cos 2ks \cdot W_k \in \mathfrak{b}_{\mathfrak{k}},$$

for any $s \in \boldsymbol{R}$. Here

$$W_k := \sum_{\alpha \in \Delta_{\mathfrak{k}}(k)} c_\alpha W_\alpha + \sum_{\alpha \in \Delta_{\mathfrak{k}}(-k)} c_\alpha W_\alpha.$$

Consequently we have $W_k \in \mathfrak{b}_{\mathfrak{k}}$. $\square$

Under the same notation as in the proof of Lemma 4.5, we obtain by (4.8)

$$\begin{aligned}
\left[\sqrt{-1}h, X_k\right] &= \left(\frac{2}{3} + 2k\right)\left(\sum_{\alpha \in \Delta_1(k)} c_\alpha Y_\alpha - \sum_{\alpha \in \Delta_2(-k)} c_\alpha Y_\alpha\right) \in J_o V, \\
\left[\sqrt{-1}h, W_k\right] &= 2k\left(\sum_{\alpha \in \Delta_{\mathfrak{k}}(k)} c_\alpha T_\alpha - \sum_{\alpha \in \Delta_{\mathfrak{k}}(-k)} c_\alpha T_\alpha\right) \in \mathfrak{m} \cap \mathfrak{k},
\end{aligned} \tag{4.19}$$

where $Y_\alpha := a_\alpha B_\alpha - b_\alpha A_\alpha$ ($\alpha \in \Delta_{\mathfrak{p}}$), $T_\alpha := a_\alpha B_\alpha - b_\alpha A_\alpha$ ($\alpha \in \Delta_{\mathfrak{k}}$). We denote by $\mathfrak{g}^* = \mathfrak{b} + \sqrt{-1}\mathfrak{m}$ the noncompact dual of $\mathfrak{g}$ and put

$$\begin{aligned}
\sqrt{-1}Y_k :&= \sqrt{-1}\left(\sum_{\alpha \in \Delta_1(k)} c_\alpha Y_\alpha - \sum_{\alpha \in \Delta_2(-k)} c_\alpha Y_\alpha\right) \in \sqrt{-1}J_o V, \\
\sqrt{-1}T_k :&= \left(\sum_{\alpha \in \Delta_{\mathfrak{k}}(k)} c_\alpha T_\alpha - \sum_{\alpha \in \Delta_{\mathfrak{k}}(-k)} c_\alpha T_\alpha\right) \in \mathfrak{m} \cap \mathfrak{k}.
\end{aligned} \tag{4.20}$$

Then, it is obvious that for any $k \in \boldsymbol{Z}$

$$X_k, \quad \sqrt{-1}Y_k, \quad W_k, \quad \sqrt{-1}T_k \in \mathfrak{g}^*.$$

Lemma 4.6. *For any $k \in \boldsymbol{Z}$, we have*

$$\begin{aligned}
\operatorname{ad} h\ \big(X_k + \sqrt{-1}Y_k\big) &= -\Big(\frac{2}{3} + 2k\Big)\big(X_k + \sqrt{-1}Y_k\big), \\
\operatorname{ad} h\ \big(X_k - \sqrt{-1}Y_k\big) &= \Big(\frac{2}{3} + 2k\Big)\big(X_k - \sqrt{-1}Y_k\big), \\
\operatorname{ad} h\ \big(W_k + \sqrt{-1}T_k\big) &= -2k\big(W_k + \sqrt{-1}T_k\big), \\
\operatorname{ad} h\ \big(W_k - \sqrt{-1}T_k\big) &= 2k\big(W_k - \sqrt{-1}T_k\big).
\end{aligned}$$

Proof. By (4.18) and (4.20) we have

$$\begin{aligned}
\big[\sqrt{-1}h, \sqrt{-1}Y_k\big] &= \sqrt{-1}\Big[\sqrt{-1}h, \sum_{\alpha\in\Delta_1(k)} c_\alpha Y_\alpha - \sum_{\alpha\in\Delta_2(-k)} c_\alpha Y_\alpha\Big] \\
&= -\sqrt{-1}\sum_{\alpha\in\Delta_1(k)} \alpha(h)c_\alpha X_\alpha + \sqrt{-1}\sum_{\alpha\in\Delta_2(-k)} \alpha(h)c_\alpha X_\alpha \\
&= -\sqrt{-1}\Big(\frac{2}{3} + 2k\Big)\Big(\sum_{\alpha\in\Delta_1(k)} c_\alpha X_\alpha - \sum_{\alpha\in\Delta_2(-k)} c_\alpha X_\alpha\Big) \\
&= -\sqrt{-1}\Big(\frac{2}{3} + 2k\Big)X_k.
\end{aligned}$$

Therefore, it follows from (4.19) that

$$\big[h, X_k \pm \sqrt{-1}Y_k\big] = -\sqrt{-1}\Big(\frac{2}{3} + 2k\Big)\big(Y_k \mp \sqrt{-1}X_k\big) = \mp\Big(\frac{2}{3} + 2k\Big)\big(X_k \pm \sqrt{-1}Y_k\big).$$

Similarly, we obtain

$$\big[h, W_k \pm \sqrt{-1}T_k\big] = \mp 2k\big(W_k \pm \sqrt{-1}T_k\big). \qquad \square$$

Now we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. As mentioned below Theorem 4.2, it is sufficient to prove (2) of the theorem. By Lemma 4.5 and Lemma 4.6, all the eigenvalues of $\operatorname{ad} h : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ are

$$0,\ \pm 2k,\ \pm\Big(\frac{2}{3} + 2k\Big) \quad (\text{for some } k \in \boldsymbol{Z}).$$

Put $Z = \frac{3}{2}h \in \sqrt{-1}\mathfrak{m} \subset \mathfrak{g}^*$. Then the eigenvalues of $\operatorname{ad} Z$ are contained in $\boldsymbol{Z}$ and therefore Z is a characteristic element of a graded Lie algebra. Moreover, by (4.7) and (4.17), we have

$$N = \exp \mathfrak{b} \cdot o, \quad \sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3} \sqrt{-1} Z \right),$$

and (2) follows. □

As in [**T**], we call a pair $((G/K, \langle , \rangle, \sigma), N)$ of a simply connected Riemannian 3-symmetric space and a half dimensional, totally real and totally geodesic submanifold a *TRG-pair*. Moreover we call a TRG-pair constructed from a graded Lie algebra $\mathfrak{g}^* = \sum_i \mathfrak{g}_i^*$ *a TRG-pair associated to* $\mathfrak{g}^* = \sum_i \mathfrak{g}_i^*$. Let

$$\mathfrak{g}^* = \mathfrak{g}_{-\nu}^* + \cdots + \mathfrak{g}_0^* + \cdots + \mathfrak{g}_\nu^*,$$
$$\bar{\mathfrak{g}}^* = \bar{\mathfrak{g}}_{-\bar{\nu}}^* + \cdots + \bar{\mathfrak{g}}_0^* + \cdots + \bar{\mathfrak{g}}_{\bar{\nu}}^*$$

be two graded simple Lie algebras which have $Z \in \mathfrak{m}^*$ and $\bar{Z} \in \bar{\mathfrak{m}}^*$ as characteristic elements, respectively. Let $((G/K, \langle , \rangle, \sigma), N)$ and $((\bar{G}/\bar{K}, \langle , \rangle, \bar{\sigma}), \bar{N})$ be two TRG-pairs associated to $\{\mathfrak{g}_i^*\}_{-\nu \le i \le \nu}$ and $\{\bar{\mathfrak{g}}_i^*\}_{-\bar{\nu} \le i \le \bar{\nu}}$, respectively. TRG-pairs $((G/K, \langle , \rangle, \sigma), N)$ and $((\bar{G}/\bar{K}, \langle , \rangle, \bar{\sigma}), \bar{N})$ are said to be *equivalent* if there exists an isometry $\varphi : (G/K, \langle , \rangle) \longrightarrow (\bar{G}/\bar{K}, \langle , \rangle)$ such that $\varphi(N) = \bar{N}$. We note that if $\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}_i^*$ is isomorphic to $\bar{\mathfrak{g}}^* = \sum_{i=-\bar{\nu}}^{\nu} \bar{\mathfrak{g}}_i^*$, then $((G/K, \langle , \rangle, \sigma, N))$ and $((\bar{G}/\bar{K}, \langle , \rangle, \bar{\sigma}, \bar{N}))$ are equivalent (cf. Remark 5.4 of [**To**]).

5. Classification of TRG-pairs satisfying (T1).

In this section we shall classify the equivalent classes of TRG-pairs satisfying the condition (T1). Then, considering Theorem 4.2, it is sufficient to determine the isomorphism classes of simple graded Lie algebras such that $\dim Z_K = 0$.

Let $\mathfrak{g}^*$ be a noncompact simple Lie algebra such that its complexification is simple, and let $\mathfrak{g}^* = \mathfrak{b} + \mathfrak{m}^*$ be a Cartan decomposition of $\mathfrak{g}^*$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}^*$ and $\mathfrak{t}^*$ a Cartan subalgebra of $\mathfrak{g}^*$ such that $\mathfrak{a} \subset \mathfrak{t}^*$. We denote by Δ and $\Delta({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$ the sets of roots of $(\mathfrak{g}^*, \mathfrak{a})$ and of $({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$, respectively. Moreover, we denote by

$$\Pi := \{\alpha_1, \cdots, \alpha_l\}, \quad \Pi({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c) := \{\beta_1, \cdots, \beta_n\}$$

fundamental root systems of Δ and $\Delta({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$, respectively, with respect to a compatible orderings of Δ and $\Delta({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$. Put

$$\mathfrak{t} := \sqrt{-1}\mathfrak{a} + \mathfrak{t}^* \cap \mathfrak{b}.$$

Then $\mathfrak{t}$ is a maximal abelian subalgebra of the compact dual $\mathfrak{g} = \mathfrak{b} + \mathfrak{m}$ ($\mathfrak{m} := \sqrt{-1}\mathfrak{m}^*$) of $\mathfrak{g}^*$. We define $h_i \in \mathfrak{a}$ and $H_i \in \sqrt{-1}\mathfrak{t}$ by

$$\alpha_i(h_j) = \delta_{ij}, \quad \beta_i(H_j) = \delta_{ij}, \tag{5.1}$$

and denote the highest root of Δ by

$$\delta := \sum_{j=1}^{l} m_j \alpha_j, \quad m_j \in \boldsymbol{Z}. \tag{5.2}$$

First, by applying Wolf and Gray [**WG**, Theorem 3.3] to restricted root systems, we shall prove the following proposition.

Proposition 5.1. *Let $W(\mathfrak{g}, \mathfrak{b})$ be the Weyl group of the symmetric pair $(\mathfrak{g}, \mathfrak{b})$ and $\sqrt{-1}h$ a vector in $\sqrt{-1}\mathfrak{a}$. Suppose that an inner automorphism $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h)$ of $\mathfrak{g}$ is of order 3. Then*
(1) If Δ is a reduced root system, then there exist $w \in W(\mathfrak{g}, \mathfrak{b})$ and $\sqrt{-1}t \in \sqrt{-1}\mathfrak{a}$ such that $\frac{1}{3}\sqrt{-1}h = \frac{1}{3}w(\sqrt{-1}\bar{h}) + \sqrt{-1}t$, $\mathrm{Ad}(\exp 2\pi\sqrt{-1}t) = \mathrm{Id}$, where $\bar{h}$ is either

$$\bar{h} = h_i \quad (m_i = 1, 2, \text{ or } 3) \quad \text{or} \quad (h_i + h_j) \quad (m_i = m_j = 1).$$

(2) If Δ is a nonreduced root system, then there exist $w \in W(\mathfrak{g}, \mathfrak{b})$ and $\sqrt{-1}t \in \sqrt{-1}\mathfrak{a}$ such that $\frac{1}{3}\sqrt{-1}h = \frac{1}{3}w(\sqrt{-1}\bar{h}) + \sqrt{-1}t$, $\mathrm{Ad}(\exp 2\pi\sqrt{-1}t) = \mathrm{Id}$, where

$$\bar{h} = h_k \quad (m_k = 2).$$

Proof. By Lemma 2.1, we can see that for $\sqrt{-1}H \in \mathfrak{t}$ an automorphism $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}H)$ is of order 3 if and only if

$$\beta_i(H) \in \boldsymbol{Z}, \quad 1 \leq i \leq n. \tag{5.3}$$

Similarly, for $\sqrt{-1}h \in \sqrt{-1}\mathfrak{a}$ an automorphism $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h)$ is of order 3 if and only if

$$\alpha_i(h) \in \boldsymbol{Z}, \quad 1 \leq i \leq l. \tag{5.4}$$

(1) Considering (5.3) and (5.4), by applying the argument of the proof of [**WG**, Theorem 3.3] to this case, we can prove that there exist $w \in W(\mathfrak{g}, \mathfrak{b})$ and $\sqrt{-1}t \in \sqrt{-1}\mathfrak{a}$ such that

$$\frac{1}{3}\sqrt{-1}h = \frac{1}{3}w\big(\sqrt{-1}\bar{h}\big) + \sqrt{-1}t, \quad \alpha_i(t) \in \boldsymbol{Z}, \quad 1 \leq i \leq l. \tag{5.5}$$

Here $\bar{h}$ is one of the following form:

$$\bar{h} = h_i \quad (m_i = 1, 2, \text{ or } 3) \quad \text{or} \quad (h_i + h_j) \quad (m_i = m_j = 1).$$

By Lemma 2.1, we have $\mathrm{Ad}(\exp 2\pi\sqrt{-1}t) = \mathrm{Id}$, and (1) is proved.

(2) Note that Δ is of type $\mathfrak{bc}$ and $m_i = 2$ for all i in this case. Furthermore the Dynkin diagram of type $\mathfrak{bc}$ is isomorphic to that of type $\mathfrak{b}$ and there are not α_i and α_j $(i \neq j)$ with $m_i = m_j = 1$ in a fundamental root system of type $\mathfrak{b}$. Then by the same

argument as in the proof of (1), it follows that there exist $w \in W(\mathfrak{g}, \mathfrak{b})$ and $\sqrt{-1}t \in \sqrt{-1}\mathfrak{a}$ such that

$$\frac{1}{3}\sqrt{-1}h = \frac{1}{3}w\big(\sqrt{-1}h_k\big) + \sqrt{-1}t, \quad \alpha_i(t) \in \boldsymbol{Z},$$

for some k and for all i. This proves (2). $\square$

LEMMA 5.2. *Let $h \in \mathfrak{a}$ be one of h_i $(m_i = 1, 2)$ and $h_i + h_j$ $(m_i = m_j = 1)$. Then the center of the fixed point set of* $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h)$ *in $\mathfrak{g}$ is non-trivial.*

PROOF. It follows from Theorem 2.2 and (2.8) that there exists a graded Lie algebra of the first or the second kind:

$$\mathfrak{g}^* = \mathfrak{g}^*_{-\nu} + \cdots + \mathfrak{g}^*_0 + \cdots + \mathfrak{g}^*_\nu, \quad \nu = 1 \text{ or } 2$$

whose characteristic element coincides with h. Then the fixed point set of $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h)$ equals

$$\mathfrak{g}^*_0 \cap \mathfrak{b} + \sqrt{-1}(\mathfrak{g}^*_0 \cap \mathfrak{m}^*),$$

which contains $\sqrt{-1}h$ as central element. $\square$

Next, we consider the case where $h = h_k$, $m_k = 3$. In this case the following lemma holds.

LEMMA 5.3. *Suppose that $\alpha_k \in \Pi$ with $m_k = 3$. Then the center of the fixed point set of* $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h_k)$ *in $\mathfrak{g}$ is trivial.*

PROOF. It is known that there is α_k such that $m_k = 3$ if and only if Π is of exceptional type. Therefore a pair $(\mathfrak{g}^*, \mathfrak{b})$ is one of the following (cf. Araki [**A**], [**H**] and [**Sa1**]):

$$\begin{aligned}&(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4)), \quad (\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2)), \quad (\mathfrak{e}_{7(7)}, \mathfrak{su}(8)), \quad (\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2)),\\ &(\mathfrak{e}_{8(8)}, \mathfrak{so}(16)), \quad (\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2)), \quad (\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) \oplus \mathfrak{su}(2)).\end{aligned} \tag{5.6}$$

Moreover, in the Satake diagram of $(\mathfrak{g}^*, \mathfrak{b})$, there exists a unique $\beta_{i_k} \in \Pi({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$ such that

$$\beta_{i_k}|_{\mathfrak{a}} = \alpha_k, \quad n_{i_k} = 3.$$

Here we denote the highest root of $\Delta({\mathfrak{g}^*}_c, {\mathfrak{t}^*}_c)$ by $\tilde{\delta} := \sum_{i=1}^n n_i\beta_i$. Since for any β_i, $i \neq i_k$,

$$\beta_i|_{\mathfrak{a}} = 0 \text{ or } \beta_i|_{\mathfrak{a}} = \alpha_j \text{ for some } j \neq k,$$

it follows that

$$\beta_i(h_k) = 0, \quad \beta_{i_k}(h_k) = \alpha_k(h_k) = 1.$$

Hence, by (5.1), we have $h_k = H_{i_k}$. Finally, by the Table in [**WG**, Theorem 3.3], we can check that the center of the fixed point set of $\mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}H_{i_k})$ is trivial. □

Now, we prove the following theorem which classifies the TRG-pairs satisfying (T1).

Theorem 5.4. *Let $((G/K, \langle,\rangle, \sigma), N)$ be a TRG-pair such that σ is inner and the center of $\mathfrak{k}$ is trivial. Then the equivalent class of $((G/K, \langle,\rangle, \sigma), N)$ is associated to one of isomorphism classes of simple graded Lie algebras listed in Table* 4.

Proof. By Theorem 4.2, we may assume that $((G/K, \langle,\rangle, \sigma), N)$ is a TRG-pair associated to

$$\mathfrak{g}^* = \mathfrak{g}^*_{-\nu} + \cdots + \mathfrak{g}^*_0 + \cdots + \mathfrak{g}^*_\nu,$$

whose characteristic element is $Z \in \mathfrak{a} \subset \mathfrak{m}^* \cap \mathfrak{g}^*_0$. Then we have $\sigma = \mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}Z)$. By (4.4) and the fact that $\mathfrak{t}^* \subset \mathfrak{g}^*_0$, we obtain $\mathfrak{t} \subset \mathfrak{k}$ and $\sqrt{-1}Z \in \mathfrak{t}$. Since the center of $\mathfrak{k}$ is trivial, we may assume that G is a centerless exceptional Lie group and not of type G_2 (cf. (4.1)). It follows from Proposition 5.1, Lemma 5.2 and Lemma 5.3 that there exist $h_k \in \mathfrak{a}$, $w \in W(\mathfrak{g}^*, \mathfrak{b})$ and $t \in \mathfrak{a}$ such that $m_k = 3$, $\mathrm{Ad}(\exp 2\pi\sqrt{-1}t) = \mathrm{Id}$ and

$$\frac{1}{3}\sqrt{-1}Z = \frac{1}{3}w(\sqrt{-1}h_k) + \sqrt{-1}t.$$

Therefore we have

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3} w\big(\sqrt{-1}h_k\big)\right) = \tilde{w} \circ \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}h_k\right) \circ \tilde{w}^{-1},$$

where $\tilde{w}$ denotes an element of $\mathrm{Int}(\mathfrak{b})$ such that $\tilde{w}|_{\mathfrak{a}} = w$. Put $\nu := \mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h_k)$. Then it is easy to see that

$$\tilde{w}(\mathfrak{b}) = \mathfrak{b}, \quad \mathfrak{k} = \tilde{w}(\mathfrak{g}^\nu), \tag{5.7}$$

and h_k is a characteristic element of a graded Lie algebra of the third kind (see Theorem 2.2). More precisely, the gradation is corresponding to a partition

$$\Pi = \Pi_0 \cup \Pi_1, \quad \Pi_1 = \{\alpha_k\}.$$

Let $\tilde{K}$ be the connected Lie subgroup of G with Lie algebra $\mathfrak{g}^\nu$. Then, it follows from (5.7) that a TRG-pair $((G/K, \langle,\rangle, \sigma), N)$ is equivalent to $((G/\tilde{K}, \langle,\rangle, \nu), \exp \mathfrak{b} \cdot o)$. As stated in the proof of Lemma 5.3, $(\mathfrak{g}^*, \mathfrak{b})$ is one of pairs in (5.6). Moreover, using the classification of the Satake diagrams in [**A**] or [**H**], we can easily get α_k with $m_k = 3$.

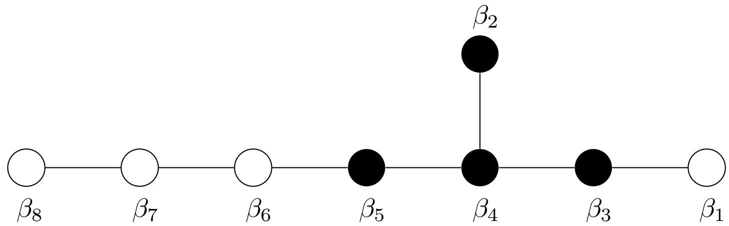
Now, suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_8$. Then, by (4.2), a pair $(\mathfrak{g}, \mathfrak{k})$ is one of

$$(\mathfrak{e}_8, \mathfrak{su}(9)), \quad (\mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6).$$

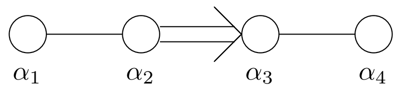
Moreover, it follows from (5.6) that $(\mathfrak{g}^*, \mathfrak{b})$ is one of

$$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2)), \quad (\mathfrak{e}_{8(8)}, \mathfrak{so}(16)).$$

The Satake diagram of $\mathfrak{e}_{8(-24)}$ is as follows:



In this case, the Dynkin diagram of Π is of type $\mathfrak{f}_4$:



Here

$$\alpha_1 = \beta_8|_{\mathfrak{a}}, \quad \alpha_2 = \beta_7|_{\mathfrak{a}}, \quad \alpha_3 = \beta_6|_{\mathfrak{a}}, \quad \alpha_4 = \beta_1|_{\mathfrak{a}}.$$

It is well-known that

$$\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,$$

and so σ is conjugate to $\nu = \mathrm{Ad}(\exp \frac{2\pi}{3}\sqrt{-1}h_2)$. Noting that $h_2 = H_7$, the Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{su}(3) \oplus \mathfrak{e}_6$ (cf. the Table in [**WG**, Theorem 3.3] and Theorem 5.15 of Chapter X of [**H**]). Hence, in this case, we have

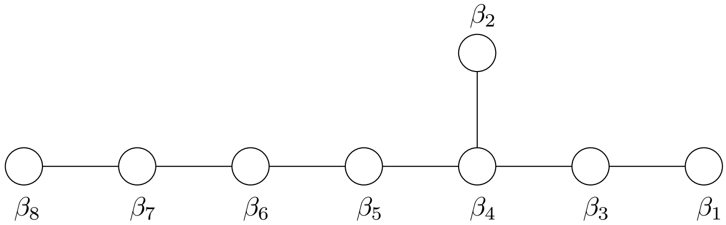
$$(\mathfrak{g}^*, \mathfrak{b}, \mathfrak{g}, \mathfrak{k}) = \big(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2), \mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6\big),$$

and $\Pi_1 = \{\alpha_2\}$.

Next, we consider the case $\mathfrak{g}^* = \mathfrak{e}_{8(8)}$. The Satake diagram of $\mathfrak{e}_{8(8)}$ is as follows: In this case we have $\alpha_i = \beta_i$, $1 \leq i \leq 8$, and

$$\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_2.$$

Therefore, σ is conjugate to one of



$$\nu_1 := \mathrm{Ad}\Big(\exp \frac{2\pi}{3}\sqrt{-1}h_2\Big), \quad \nu_2 := \mathrm{Ad}\Big(\exp \frac{2\pi}{3}\sqrt{-1}h_7\Big).$$

Noting that $h_2 = H_2$ and $h_7 = H_7$, the Lie algebra $\mathfrak{k}$ is isomorphic to

$$\mathfrak{g}^{\nu_1} = \mathfrak{su}(9) \quad \text{or} \quad \mathfrak{g}^{\nu_2} = \mathfrak{su}(3) \oplus \mathfrak{e}_6.$$

Hence $(\mathfrak{g}^*, \mathfrak{b}, \mathfrak{g}, \mathfrak{k})$ is one of

$$\big(\mathfrak{e}_{8(8)}, \mathfrak{so}(16), \mathfrak{e}_8, \mathfrak{su}(9)\big), \quad \big(\mathfrak{e}_{8(8)}, \mathfrak{so}(16), \mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6\big).$$

In the case where $\mathfrak{g}$ is one of type $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{f}_4$, by the similar argument as above, we can get all $(\mathfrak{g}^*, \mathfrak{b}, \mathfrak{g}, \mathfrak{k})$. We list $(\mathfrak{g}^*, \mathfrak{b})$, Π, Π_1 and $(\mathfrak{g}, \mathfrak{k})$ in Table 4. $\square$

Table 4. The case where σ is inner.

$(\mathfrak{g}^*, \mathfrak{b})$	$\Pi = \Pi_0 \cup \Pi_1$	Π_1	$(\mathfrak{g}, \mathfrak{k})$
EI $(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	E_6	$\{\alpha_4\}$	$(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3))$
EII $(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$	F_4	$\{\alpha_2\}$	$(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3))$
EV $(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	E_7	$\{\alpha_3\}$	$(\mathfrak{e}_7, \mathfrak{su}(3) \oplus \mathfrak{su}(6))$
EVI $(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$	F_4	$\{\alpha_2\}$	$(\mathfrak{e}_7, \mathfrak{su}(3) \oplus \mathfrak{su}(6))$
EVIII $(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	E_8	$\{\alpha_2\}$	$(\mathfrak{e}_8, \mathfrak{su}(9))$
EVIII $(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	E_8	$\{\alpha_7\}$	$(\mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6)$
EIX $(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$	F_4	$\{\alpha_2\}$	$(\mathfrak{e}_8, \mathfrak{su}(3) \oplus \mathfrak{e}_6)$
FI $(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) \oplus \mathfrak{su}(2))$	F_4	$\{\alpha_2\}$	$(\mathfrak{f}_4, \mathfrak{su}(3) \oplus \mathfrak{su}(3))$

Remark 5.5. In the above table, we use the numbering of simple roots given in [**H**, p. 477].

6. The case where $G = L \times L \times L$.

In this section we classify a TRG-pair $((G/K, \langle , \rangle, \sigma), N)$ such that

$$G = L \times L \times L \ \ (L : \text{a compact simple Lie group}), \quad \sigma(x, y, z) = (z, x, y) \ \ (x, y, z \in L).$$

Then we have

$$\mathfrak{k} = \mathfrak{g}^\sigma = \{(X, X, X); X \in \mathfrak{l}\},$$

where $\mathfrak{l}$ denotes the Lie algebra of L. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ with respect to $\langle , \rangle$. We denote by $V \subset \mathfrak{p}$ the tangent space of N at $o = \{K\}$ and put $\mathfrak{b} = V + [V, V]_{\mathfrak{k}}$. Then as stated in Section 3, the orthogonal complement $\mathfrak{m}$ of $\mathfrak{b}$ satisfies $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{b}$. So, $\mathfrak{b}$ is conjugate under Aut(G) to one of the followings:

(1) $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$. $(\mathfrak{l}, \mathfrak{b}_i)$ is a locally symmetric pair.
(2) $\mathfrak{b} = \mathfrak{b}_1 \oplus \{(X, X); X \in \mathfrak{l}\}$. $(\mathfrak{l}, \mathfrak{b}_1)$ is a locally symmetric pair.
(3) $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{l}$. $(\mathfrak{l}, \mathfrak{b}_i)$ $(i = 1, 2)$ is a locally symmetric pair.
(4) $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{l} \oplus \mathfrak{l}$. $(\mathfrak{l}, \mathfrak{b}_1)$ is a locally symmetric pair.
(5) $\mathfrak{b} = \mathfrak{l} \oplus \{(X, X); X \in \mathfrak{l}\}$.

The Case (1): In this case, an involutive automorphism μ of $\mathfrak{g}$ such that $\mathfrak{g}^\mu = \mathfrak{b}$ is given by

$$\mu(X, Y, Z) = (\mu_1(X), \mu_2(Y), \mu_3(Z)) \quad (X, Y, Z \in \mathfrak{l}),$$

where $\mu_i : \mathfrak{l} \longrightarrow \mathfrak{l}$ denotes an involutive automorphism of $\mathfrak{l}$ satisfying $\mathfrak{l}^{\mu_i} = \mathfrak{b}_i$ $(i = 1, 2, 3)$. Assume that μ satisfies (3.3). Then for $(0, 0, X) \in \mathfrak{g}$ $(X \neq 0)$ it follows that

$$\sigma^2\mu(0, 0, X) = (0, \mu_3(X), 0) = (\mu_1(X), 0, 0) = \mu\sigma(0, 0, X).$$

This means that $X = 0$. Therefore there is no involutive automorphism satisfying (3.3) in this case.

The case (2): Let μ_1 be an involutive automorphism of $\mathfrak{l}$ such that $\mathfrak{l}^{\mu_1} = \mathfrak{b}_1$. It is easy to see that an involutive automorphism

$$\mu(X, Y, Z) = (\mu_1(X), Z, Y)$$

of $\mathfrak{g}$ satisfies (3.3) if and only if

$$(\mu_1(Z), Y, X) = (Z, Y, \mu_1(X)),$$

i.e. $\mu_1(X) = X$ for any $X \in \mathfrak{l}$. So, in the case (2) there is no TRG-pair.

By a similar computation as above, we can see that there is no TRG-pair in the case (3) and (4).

The case (5): In this case an involutive automorphism μ of $\mathfrak{g}$ such that $\mathfrak{g}^\mu = \mathfrak{b}$ is given by

$$\mu(X, Y, Z) = (X, Z, Y).$$

It is easy to show that μ satisfies (3.3).

Consequently we obtain the following theorem.

THEOREM 6.1. *Let $((G/K,\langle,\rangle,\sigma),N)$ be a TRG-pair such that $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{l}\oplus\mathfrak{l}$. Then it is equivalent to a TRG-pair*

$$((G/K,\langle,\rangle,\sigma),\exp\mathfrak{b}\cdot o),$$

where $\mathfrak{b}=\mathfrak{l}\oplus\{(X,X);X\in\mathfrak{l}\}$.

7. Constructions of some TRG-pairs for the case (T3).

In the remaining part of this paper we use the same notation as in Section 2. By making use of an affine Lie algebra of type ${D_4}^{(3)}$, we shall give examples of TRG-pairs $((G/K,\langle,\rangle,\sigma),N)$ such that G is of type D_4 and σ is an outer automorphism of G. Let $\mathfrak{g}$ be a compact simple Lie algebra of type $\mathfrak{d}_4$ and $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$. Let

$$\mathscr{L}(\mathfrak{g}_c)=\bigoplus_{i\in\boldsymbol{Z}}\mathscr{L}(\mathfrak{g}_c,i),\quad \hat{\mathscr{L}}(\mathfrak{g}_c)=\mathscr{L}(\mathfrak{g}_c)+\boldsymbol{C}K+\boldsymbol{C}d$$

and ϕ_a be as in Section 2. We define mappings $\tilde{\sigma}_i:\hat{\mathscr{L}}(\mathfrak{g}_c)\longrightarrow\hat{\mathscr{L}}(\mathfrak{g}_c)$ $(i=1,2)$ by the following relations:

$$\begin{aligned}
&\tilde{\sigma}_1(\hat{E}_1)=\xi\hat{E}_1,\quad \tilde{\sigma}_1(\hat{F}_1)=\xi^2\hat{F}_1,\quad \tilde{\sigma}_1(\hat{E}_j)=\hat{E}_j,\quad \tilde{\sigma}_1(\hat{F}_j)=\hat{F}_j\quad (j=2,3),\\
&\tilde{\sigma}_2(\hat{E}_3)=\xi\hat{E}_3,\quad \tilde{\sigma}_2(\hat{F}_3)=\xi^2\hat{F}_3,\quad \tilde{\sigma}_2(\hat{E}_k)=\hat{E}_k,\quad \tilde{\sigma}_2(\hat{F}_k)=\hat{F}_k\quad (k=1,2),\\
&\tilde{\sigma}_i(H)=H\quad (H\in\mathfrak{h}=\hat{\mathfrak{t}}+\boldsymbol{C}K+\boldsymbol{C}d,\ \ i=1,2),
\end{aligned}\tag{7.1}$$

where $\xi^3=1$ $(\xi\neq 1)$. It is easy to check that $\tilde{\sigma}_i$ $(i=1,2)$ preserves a defining relations (2.10). Therefore we can extend $\tilde{\sigma}$ to an automorphism of $\hat{\mathscr{L}}(\mathfrak{g}_c)$ (and to $\mathscr{L}(\mathfrak{g}_c)$). Then the following holds (see Theorem 8.6 of [**K**] and Theorem 5.15 of [**H**], Chapter X. Also, see [**G**], [**WG**]).

PROPOSITION 7.1. *Define a mapping $\sigma_i:\mathfrak{g}_c\longrightarrow\mathfrak{g}_c$ $(i=1,2)$ by*

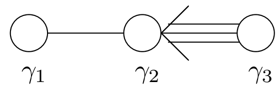
$$\phi_1\circ\tilde{\sigma}_i=\sigma_i\circ\phi_1.$$

Then σ_i is a well-defined outer automorphism of order 3 on $\mathfrak{g}_c$. Conversely, any outer automorphism of order 3 on $\mathfrak{g}_c$ is conjugate within $\mathrm{Aut}(\mathfrak{g}_c)$ to one of σ_i $(i=1,2)$.

Furthermore, $\mathfrak{g}^{\sigma_1}$ and $\mathfrak{g}^{\sigma_2}$ are generated by $\{E_i,\ F_i;\ i=2,3\}$ and $\{E_i,\ F_i;\ i=1,2\}$ respectively, and

$$\mathfrak{g}^{\sigma_1}=\mathfrak{g}_2,\quad \mathfrak{g}^{\sigma_2}=\mathfrak{su}(3).$$

FACT. (1) The Dynkin Diagram of a generalized Cartan matrix A of type ${D_4}^{(3)}$ is as follows ([**K**]):



(2) Put $\theta_0 := 2\gamma_2 + \gamma_3$. Then we have $\theta_0|_{\hat{\mathfrak{t}}} = \theta = -\gamma_1|_{\hat{\mathfrak{t}}}$ ([**K**]). Furthermore, we have $\gamma_2|_{\hat{\mathfrak{t}}} = \alpha_1$, $\gamma_3|_{\hat{\mathfrak{t}}} = \alpha_2$.

(3) By the definition of ν, we have $\nu = \sigma_1$ and $\mathfrak{g}_{\bar{0}} = \mathfrak{g}^{\sigma_1} \cong \mathscr{L}(\mathfrak{g}_c, 0)$.

For any root γ of $\hat{\mathscr{L}}(\mathfrak{g}_c)$, we denote the root space of γ by $\mathscr{L}_\gamma$ and define a root δ of $\hat{\mathscr{L}}(\mathfrak{g}_c)$ by

$$\delta := \theta_0 + \gamma_1.$$

By the above Fact, we get $\delta|_{\hat{\mathfrak{t}}} = 0$.

Proposition 7.2. *$\mathscr{L}(\mathfrak{g}_c, 1)$ (resp. $\phi_1(\mathscr{L}(\mathfrak{g}_c, 1))$) is an irreducible representation space of $\mathfrak{g}_{\bar{0}}$ with the lowest weight $-\theta$ and with the lowest weight vector $\hat{E}_1$ (resp. E_1). Similarly, $\mathscr{L}(\mathfrak{g}_c, -1)$ (resp. $\phi_1(\mathscr{L}(\mathfrak{g}_c, -1))$) is an irreducible representation space of $\mathfrak{g}_{\bar{0}}$ with the highest weight θ and with the highest weight vector $\hat{F}_1$ (resp. F_1).*

Moreover we have the following root space decompositions of $\mathscr{L}(\mathfrak{g}_c, \pm 1)$:

$$\begin{aligned} \mathscr{L}(\mathfrak{g}_c, 1) &= \mathscr{L}_{\delta-\theta_0} + \mathscr{L}_{\gamma_1+\gamma_2} + \mathscr{L}_{\delta-\gamma_2} + \mathscr{L}_{\delta} + \mathscr{L}_{\delta+\gamma_2} + \mathscr{L}_{\delta+\gamma_2+\gamma_3} + \mathscr{L}_{\delta+\theta_0}, \\ \mathscr{L}(\mathfrak{g}_c, -1) &= \mathscr{L}_{-\delta+\theta_0} + \mathscr{L}_{-(\gamma_1+\gamma_2)} + \mathscr{L}_{-\delta+\gamma_2} + \mathscr{L}_{-\delta} + \mathscr{L}_{-(\delta+\gamma_2)} \\ &\quad + \mathscr{L}_{-(\delta+\gamma_2+\gamma_3)} + \mathscr{L}_{-(\delta+\theta_0)}. \end{aligned}$$

In particular, the highest weight of $\mathscr{L}(\mathfrak{g}_c, 1)$ is $\theta = (\delta + \theta_0)|_{\hat{\mathfrak{t}}}$, and the lowest weight of $\mathscr{L}(\mathfrak{g}_c, -1)$ is $-\theta = -(\delta + \theta_0)|_{\hat{\mathfrak{t}}}$.

Proof. The early part of the proposition is immediately from Proposition 8.3 of [**K**].

Let

$$\{\beta + j\gamma_i; p \leq j \leq q\}$$

be the γ_i-series through β. Then it is known that

$$p + q = -\beta({\gamma_i}^\vee). \tag{7.2}$$

Using (7.2) and the fact that $\mathfrak{g}_{\bar{0}}$ is generated by $\{E_i,\ F_i;\ i = 2, 3\}$ and $\mathscr{L}(\mathfrak{g}_c, \pm 1)$ are irreducible representation space of $\mathfrak{g}_{\bar{0}}$, we can show the latter part of the proposition. □

Now, we give some examples of TRG-pairs. Let μ_1 be an involutive automorphism of $\mathfrak{g}_c$ (and $\mathfrak{g}$) satisfying the following relations:

$$\mu_1(e_{\alpha_1}) = e_{\alpha_1}, \quad \mu_1(e_{\alpha_2}) = e_{\alpha_2}, \quad \mu_1(e_{\alpha_3}) = e_{\alpha_4},$$
$$\mu_1(e_{-\alpha_1}) = e_{-\alpha_1}, \quad \mu_1(e_{-\alpha_2}) = e_{-\alpha_2}, \quad \mu_1(e_{-\alpha_3}) = e_{-\alpha_4}. \tag{7.3}$$

Proposition 7.3. *$((G/K, \langle,\rangle, \sigma_1), \exp \mathfrak{g}^{\mu_1} \cdot o)$ is a TRG-pair. In particular, we have $\mathfrak{g}^{\mu_1} \cong \mathfrak{so}(7)$.*

Proof. In this case we may assume that

$$\mu_1(e_{\alpha_1+\alpha_2+\alpha_3}) = e_{\alpha_1+\alpha_2+\alpha_4}, \quad \mu_1(e_{\alpha_2+\alpha_3+\alpha_4}) = e_{\alpha_2+\alpha_3+\alpha_4},$$
$$\mu_1(e_{-(\alpha_1+\alpha_2+\alpha_3)}) = e_{-(\alpha_1+\alpha_2+\alpha_4)}, \quad \mu_1(e_{-(\alpha_2+\alpha_3+\alpha_4)}) = e_{-(\alpha_2+\alpha_3+\alpha_4)}. \tag{7.4}$$

Then by (2.15) and (7.3) it is easy to see that

$$\mu_1(E_i) = E_i, \quad \mu_1(F_i) = F_i \quad (i = 2, 3). \tag{7.5}$$

Put

$$X := \mu_1(E_1) = e_{-(\alpha_1+\alpha_2+\alpha_4)} + \xi e_{-(\alpha_2+\alpha_3+\alpha_4)} + \xi^2 e_{-(\alpha_1+\alpha_2+\alpha_3)},$$
$$Y := \mu_1(F_1) = -e_{\alpha_1+\alpha_2+\alpha_4} - \xi^2 e_{\alpha_2+\alpha_3+\alpha_4} - \xi e_{\alpha_1+\alpha_2+\alpha_3}.$$

Since $\nu(X) = \xi^2 X$, $\nu(Y) = \xi Y$ and $\sigma_1 = \nu$, we have $X \in \mathfrak{g}_{-\bar{1}} = \phi_1(\mathscr{L}(\mathfrak{g}_c, -1))$ and $Y \in \mathfrak{g}_{\bar{1}} = \phi_1(\mathscr{L}(\mathfrak{g}_c, 1))$. More precisely, since

$$\theta(= \alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_4 = \alpha_2 + \alpha_3 + \alpha_4 \quad \text{on } \hat{\mathfrak{t}},$$

it follows that

$$X \in \phi_1(\mathscr{L}_{-(\delta+\theta_0)}) \subset \phi_1(\mathscr{L}(\mathfrak{g}_c, -1)), \quad Y \in \phi_1(\mathscr{L}_{\delta+\theta_0}) \subset \phi_1(\mathscr{L}(\mathfrak{g}_c, 1)). \tag{7.6}$$

By (7.5), (7.6) and Proposition 7.2, we obtain

$$\mu_1(\phi_1(\mathscr{L}(\mathfrak{g}_c, 1))) = \phi_1(\mathscr{L}(\mathfrak{g}_c, -1)),$$

and so $\sigma_1 \circ \mu_1 = \mu_1 \circ {\sigma_1}^2$, which concludes that $((G/K, \langle,\rangle, \sigma_1), \exp \mathfrak{g}^{\mu_1} \cdot o)$ $(o = \{K\})$ is a TRG-pair. We note that $\mathfrak{g}^{\mu_1} \cong \mathfrak{so}(7)$ (cf. [**H**], [**G**]). □

Put

$$X_1 := e_{\alpha_1}, \quad X_2 := e_{\alpha_2}, \quad X_3 := e_{\alpha_3} + e_{\alpha_4},$$
$$Y_1 := e_{-\alpha_1}, \quad Y_2 := e_{-\alpha_2}, \quad Y_3 := e_{-\alpha_3} + e_{-\alpha_4},$$
$$X_0 := e_{-(\alpha_1+\alpha_2+\alpha_3)} - e_{-(\alpha_1+\alpha_2+\alpha_4)}, \quad Y_0 := e_{\alpha_1+\alpha_2+\alpha_3} - e_{\alpha_1+\alpha_2+\alpha_4}.$$

According to Theorem 5.15 of Chapter X of [**H**], there exists an (outer) involutive automorphism μ_2 of $\mathfrak{g}$ satisfying the following relations:

$$\begin{aligned}&\mu_2(X_0) = X_0, \quad \mu_2(X_1) = X_1, \quad \mu_2(X_2) = -X_2, \quad \mu_2(X_3) = X_3,\\&\mu_2(Y_0) = Y_0, \quad \mu_2(Y_1) = Y_1, \quad \mu_2(Y_2) = -Y_2, \quad \mu_2(Y_3) = Y_3.\end{aligned} \tag{7.7}$$

Proposition 7.4. *$((G/K, \langle,\rangle, \sigma), \exp \mathfrak{g}^{\mu_2} \cdot 0)$ is a TRG-pair. In particular, we have $\mathfrak{g}^{\mu_2} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(5)$.*

Proof. By (7.7), it is easy to see that

$$\mu_2(\alpha_1) = \alpha_1, \quad \mu_2(\alpha_2) = \alpha_2, \quad \mu_2(\alpha_3 + \alpha_4) = \alpha_3 + \alpha_4.$$

Since ${\mu_2}^2 = 1$ and $\alpha_3 - \alpha_4$ is perpendicular to α_1, α_2 and $\alpha_3 + \alpha_4$, we obtain

$$\mu_2(\alpha_1) = \alpha_1, \quad \mu_2(\alpha_2) = \alpha_2, \quad \mu_2(\alpha_3) = \alpha_4. \tag{7.8}$$

From (7.7) and (7.8), we may assume that

$$\begin{aligned}&\mu_2(e_{\pm\alpha_1}) = e_{\pm\alpha_1}, \quad \mu_2(e_{\pm\alpha_2}) = -e_{\pm\alpha_2}, \quad \mu_2(e_{\pm\alpha_3}) = e_{\pm\alpha_4},\\&\mu_2(e_{\pm(\alpha_1+\alpha_2+\alpha_3)}) = -e_{\pm(\alpha_1+\alpha_2+\alpha_4)}, \quad \mu_2(e_{\pm(\alpha_2+\alpha_3+\alpha_4)}) = -e_{\pm(\alpha_2+\alpha_3+\alpha_4)}.\end{aligned}$$

Then we can easily check that

$$\begin{aligned}&\mu_2(E_2) = E_2, \quad \mu_2(E_3) = -E_3, \quad \mu_2(E_1) = -X,\\&\mu_2(F_2) = F_2, \quad \mu_2(F_3) = -F_3, \quad \mu_2(F_1) = -Y,\end{aligned} \tag{7.9}$$

(see the proof of Proposition 7.3 for the definition of X and Y). Then Proposition 7.2 shows that $\mu_2(\phi_1(\mathscr{L}(\mathfrak{g}_c, 1))) = \phi_1(\mathscr{L}(\mathfrak{g}_c, -1))$ and $\sigma_1 \circ \mu_2 = \mu_2 \circ {\sigma_1}^2$. Consequently, we obtain a TRG-pair $((G/K, \langle,\rangle, \sigma_1), \exp \mathfrak{g}^{\mu_2} \cdot o)$. Note that $\mathfrak{g}^{\mu_2} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(5)$. □

Next, suppose that $\sigma = \sigma_2$. Since e_{α_i}, $e_{-\alpha_i}$ $(1 \leq i \leq 4)$ constitute Chevalley generators of $\mathfrak{g}_c$, there exist involutive automorphisms ν_0 and ω_0 of $\mathfrak{g}_c$ satisfying the following relations:

$$\begin{aligned}&\nu_0(e_{\alpha_2}) = e_{\alpha_2}, \quad \nu_0(e_{\alpha_4}) = e_{\alpha_4}, \quad \nu_0(e_{\alpha_1}) = e_{\alpha_3},\\&\nu_0(e_{-\alpha_2}) = e_{-\alpha_2}, \quad \nu_0(e_{-\alpha_4}) = e_{-\alpha_4}, \quad \nu_0(e_{-\alpha_1}) = e_{-\alpha_3},\\&\omega_0(e_{\alpha_i}) = -e_{-\alpha_i} \qquad (1 \leq i \leq 4).\end{aligned} \tag{7.10}$$

Proposition 7.5. *Put $\mu_3 := \nu_0 \circ \omega_0$. Then $((G/K, \langle,\rangle, \sigma_2), \exp \mathfrak{g}^{\mu_3} \cdot o)$ is a TRG-pair. In particular, we have $\mathfrak{g}^{\mu_3} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(5)$.*

Proof. By (7.10), we get

$$\omega_0(h) = -h, \qquad h \in \mathfrak{t}_c, \tag{7.11}$$

and by using the Jacobi identities, we can check that

$$\begin{aligned}
&[[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]], [e_{-\alpha_1}, [e_{-\alpha_2}, e_{-\alpha_3}]]] = \alpha_1 + \alpha_2 + \alpha_3, \\
&[[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_4}]], [e_{-\alpha_1}, [e_{-\alpha_2}, e_{-\alpha_4}]]] = \alpha_1 + \alpha_2 + \alpha_4, \\
&[[e_{\alpha_3}, [e_{\alpha_2}, e_{\alpha_4}]], [e_{-\alpha_3}, [e_{-\alpha_2}, e_{-\alpha_4}]]] = \alpha_2 + \alpha_3 + \alpha_4.
\end{aligned} \tag{7.12}$$

Considering (7.10) and (7.11) together with (7.12) we obtain

$$\omega_0(e_\theta) = -e_{-\theta}, \quad \omega_0(e_{\bar{\nu}(\theta)}) = -e_{-\bar{\nu}(\theta)}, \quad \omega_0(e_{\bar{\nu}^2(\theta)}) = -e_{-\bar{\nu}^2(\theta)}.$$

Then we can show that μ_3 satisfies

$$\mu_3(E_1) = F_1, \quad \mu_3(E_i) = -F_i, \quad i = 2, 3. \tag{7.13}$$

Because $\hat{E}_i$, $\hat{F}_i$ $(i = 1, 2, 3)$ constitute Chevalley generators of $\hat{\mathscr{L}}(\mathfrak{g}_c)$, the vectors E_i, F_i $(i = 1, 2, 3)$ generate $\mathfrak{g}_c$. Consequently, it follows from (7.13) that μ_3 is involutive and

$$\sigma_2 \circ \mu_3 = \mu_3 \circ {\sigma_2}^2,$$

and hence $((G/K, \langle, \rangle, \sigma_2), \exp \mathfrak{g}^{\mu_3} \cdot o)$ is a TRG-pair. By a dimension argument, it is easy to see that $\mathfrak{g}^{\mu_3} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(5)$. $\square$

8. TRG-pairs satisfying (T3).

Let G be a compact simple Lie group of type D_4 and σ an outer automorphism of order 3 on G. In this section we shall prove that any TRG-pair $((G/K, \langle, \rangle, \sigma), N)$ is equivalent to one of TRG-pairs described in Propositions 7.3, 7.4 and 7.5.

First, we consider a Riemannian 3-symmetric space $(G/K, \langle, \rangle, \sigma_1)$. In this case $\mathfrak{k} = \mathfrak{g}^{\sigma_1} (= \mathfrak{g}_{\bar{0}})$ is generated by

$$E_2, \quad E_3, \quad F_2, \quad F_3,$$

and isomorphic to $\mathfrak{g}_2$ (see Proposition 7.1). Suppose that there exists a half dimensional, totally real and totally geodesic submanifold of $(G/K, \langle, \rangle, \sigma_1)$. Then, by Lemma 3.1, there exists an involutive automorphism μ of $\mathfrak{g}$ such that

$$\mu\sigma_1 = {\sigma_1}^2 \mu.$$

Let X be in $\mathscr{L}(\mathfrak{g}_c, 1)$. Then

$${\sigma_1}^2 \mu\phi_1(X) = \mu\sigma_1\phi_1(X) = \mu\phi_1\tilde{\sigma}_1(X) = \xi\mu\phi_1(X),$$

that is

$$\sigma_1\mu\phi_1(X) = \xi^2\mu\phi_1(X).$$

Then since $\nu = \sigma_1$ and $\mathfrak{g}_{\pm 1} = \phi_1(\mathscr{L}(\mathfrak{g}_c, \pm 1))$, it follows that

$$\mu\phi_1(\mathscr{L}(\mathfrak{g}_c, 1)) = \phi_1(\mathscr{L}(\mathfrak{g}_c, -1)). \tag{8.1}$$

By (8.1), we can see that the mapping $\tilde{\mu} : \mathscr{L}(\mathfrak{g}_c) \longrightarrow \mathscr{L}(\mathfrak{g}_c)$ defined by

$$\tilde{\mu}(t^i \otimes X) = t^{-i} \otimes \mu(X), \quad t^i \otimes X \in \mathscr{L}(\mathfrak{g}_c, i) \tag{8.2}$$

induces an automorphism of $\mathscr{L}(\mathfrak{g}_c)$. Since $\phi_1 \circ \tilde{\mu} = \mu \circ \phi_1$, Lemma 2.2 shows that

$$\tilde{\mu}((1 - t^3)\mathscr{L}(\mathfrak{g}_c)) = (1 - t^3)\mathscr{L}(\mathfrak{g}_c). \tag{8.3}$$

Since $\mathfrak{k} \cong \mathfrak{g}_2$ and $\mu^2 = 1$ (on $\mathfrak{k}$), it is known that $\mu|_{\mathfrak{k}}$ is conjugate within $\mathrm{Int}(\mathfrak{k})$ to one of the following:

$$\begin{aligned}
&\tau_1 : \mathfrak{k} \longrightarrow \mathfrak{k}, \quad \tau_1(E_2) = E_2, \quad \tau_1(F_2) = F_2, \quad \tau_1(E_3) = -E_3, \quad \tau_1(F_3) = -F_3,\\
&\tau_2 = \mathrm{Id}_{\mathfrak{k}} : \mathfrak{k} \longrightarrow \mathfrak{k}.
\end{aligned}$$

Hence we may assume that $\mu|_{\mathfrak{k}} = \tau_i$ ($i = 1$ or 2).

Lemma 8.1. $\tilde{\mu}(\hat{E}_1) \in \mathscr{L}_{-(\delta+\theta_0)}$, $\tilde{\mu}(\hat{F}_1) \in \mathscr{L}_{\delta+\theta_0}$.

Proof. By Proposition 7.2, the vector $\hat{E}_1$ is the lowest weight vector of $\mathscr{L}(\mathfrak{g}_c, 1)$. Since $\mathfrak{k} = \mathfrak{g}_{\bar{0}}$ and $\mu|_{\mathfrak{k}} = \tau_1$ or τ_2, it is easy to check that $\tilde{\mu}(\hat{E}_1)$ is the lowest weight vector of $\mathscr{L}(\mathfrak{g}_c, -1)$. Therefore it follows from Proposition 7.2 that $\tilde{\mu}(\hat{E}_1) \in \mathscr{L}_{-(\delta+\theta_0)}$. Similarly we have $\tilde{\mu}(\hat{F}_1) \in \mathscr{L}_{\delta+\theta_0}$. □

We put

$$t \otimes X_{\theta_0} = \tilde{\mu}(\hat{F}_1), \qquad t^{-1} \otimes X_{-\theta_0} = \tilde{\mu}(\hat{E}_1),$$

and choose a, $b \in \boldsymbol{C}$, $t^{\pm 1} \otimes H_{\pm} \in \mathscr{L}_{\pm\delta}$ so that

$$\begin{aligned}
&(\mathrm{ad}\, t \otimes H_+)^3(\hat{E}_2) = t^3\hat{E}_2, \quad (\mathrm{ad}\, t^{-1} \otimes H_-)^3(\hat{F}_2) = t^{-3}\hat{F}_2,\\
&t \otimes H_+ = a[\hat{E}_1, [\hat{E}_2, [\hat{E}_2, \hat{E}_3]]], \quad t^{-1} \otimes H_- = b[\hat{F}_1, [\hat{F}_2, [\hat{F}_2, \hat{F}_3]]].
\end{aligned} \tag{8.4}$$

It follows from (8.2) and (8.4) that

$$\tilde{\mu}((1 - t^3)\hat{E}_2) = \tilde{\mu}(\hat{E}_2) - \tilde{\mu}(t^3\hat{E}_2) = \hat{E}_2 - \tilde{\mu}((\mathrm{ad}\, t \otimes H_+)^3(\hat{E}_2)), \tag{8.5}$$

and

$$\begin{aligned}\tilde{\mu}(t\otimes H_+) &= a[\tilde{\mu}(\hat{E}_1),[\tilde{\mu}(\hat{E}_2),[\tilde{\mu}(\hat{E}_2),\tilde{\mu}(\hat{E}_3)]]]\\ &= a[t^{-1}\otimes X_{-\theta_0},[\hat{E}_2,[\hat{E}_2,\pm\hat{E}_3]]].\end{aligned}$$

Since $t^{-1}\otimes X_{-\theta_0}\in\mathscr{L}_{-(\delta+\theta_0)}$, we can see that $\tilde{\mu}(t\otimes H_+)\in\mathscr{L}_{-\delta}$ and there is $c\in\boldsymbol{C}$ such that

$$\tilde{\mu}(t\otimes H_+) = c\cdot t^{-1}\otimes H_-. \tag{8.6}$$

Considering (8.5) and (8.6) together with (8.3), we obtain

$$\tilde{\mu}((1-t^3)\hat{E}_2) = \hat{E}_2 - c^3(\mathrm{ad}t^{-1}\otimes H_-)^3(\hat{E}_2)\in(1-t^3)\mathscr{L}(\mathfrak{g}_c),$$

and thus

$$c^3(\mathrm{ad}H_-)^3(E_2) = E_2,\quad c^3(\mathrm{ad}t^{-1}\otimes H_-)^3(\hat{E}_2) = t^{-3}\hat{E}_2. \tag{8.7}$$

Now we are in a position to prove the following lemma which is related to the uniqueness of TRG-pairs.

Lemma 8.2. *Let* $\mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$ *be the set of automorphisms of* $\mathfrak{g}_c$ *which preserve* $\mathfrak{k}_c$. *If there exists an involutive automorphism* $\mu':\mathfrak{g}_c\longrightarrow\mathfrak{g}_c$ *such that* $\mu|_{\mathfrak{k}_c}=\mu'|_{\mathfrak{k}_c}$ *and* $\mu'\circ\sigma_1={\sigma_1}^2\circ\mu'$, *then* μ' *is conjugate to* μ *under* $\mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$.

Proof. By the same argument as above, there are an automorphism $\tilde{\mu}'$ of $\mathscr{L}(\mathfrak{g}_c)$ and $s\in\boldsymbol{C}^\times$ satisfying the following:

$$\begin{aligned}&\phi_1\circ\tilde{\mu}' = \mu'\circ\phi_1,\quad \tilde{\mu}'((1-t^3)\mathscr{L}(\mathfrak{g}_c)) = (1-t^3)\mathscr{L}(\mathfrak{g}_c),\\ &\mu'(E_i)=\mu(E_i),\quad \mu'(F_i)=\mu(F_i)\qquad (i=2,3),\\ &\mu'(E_1) = s\mu(E_1) = s\cdot X_{-\theta_0},\quad \mu'(F_1) = s^{-1}\mu(F_1) = s^{-1}X_{\theta_0}.\end{aligned} \tag{8.8}$$

Moreover, by (8.7) and (8.8), we have

$$\begin{aligned}\tilde{\mu}'((1-t^3)\hat{E}_2) &= \hat{E}_2 - s^3c^3(\mathrm{ad}t^{-1}\otimes H_-)^3(\hat{E}_2)\\ &= \hat{E}_2 - s^3t^{-3}\hat{E}_2\in(1-t^3)\mathscr{L}(\mathfrak{g}_c),\end{aligned}$$

and so $s^3=1$.

Since $\hat{E}_i$, $\hat{F}_i$ $(i=1,2,3)$ constitute Chevalley generators of $\mathfrak{g}(A)=\hat{\mathscr{L}}(\mathfrak{g}_c)$, we can get an automorphism $\tilde{\varphi}$ of $\hat{\mathscr{L}}(\mathfrak{g}_c)$ and of $\mathscr{L}(\mathfrak{g}_c)$ satisfying the following:

$$\begin{aligned}&\tilde{\varphi}(\hat{E}_1) = s\hat{E}_1,\quad \tilde{\varphi}(\hat{F}_1) = s^{-1}\hat{F}_1,\\ &\tilde{\varphi}(\hat{E}_i) = \hat{E}_i,\quad \tilde{\varphi}(\hat{F}_i) = \hat{F}_i,\quad (i=2,3).\end{aligned} \tag{8.9}$$

Indeed,

$$\tilde{\varphi} = (\tilde{\sigma}_2)^k, \qquad (k = 1, 2 \text{ or } 3), \tag{8.10}$$

because $s^3 = 1$. It follows from (8.4) and (8.9) that

$$\tilde{\varphi}(t^{-1} \otimes H_-) = s^{-1}t^{-1} \otimes H_-,$$

so we obtain

$$\tilde{\varphi}(t^{-1} \otimes X_{-\theta_0}) = s^{-1}t^{-1} \otimes X_{-\theta_0}. \tag{8.11}$$

From (8.10) and Proposition 7.1 we have

$$\tilde{\varphi}((1 - t^3)\mathscr{L}(\mathfrak{g}_c)) = (1 - t^3)\mathscr{L}(\mathfrak{g}_c),$$

hence there exists an automorphism φ of $\mathfrak{g}_c$ such that $\varphi \circ \phi_1 = \phi_1 \circ \tilde{\varphi}$. Using (8.8), (8.9) and (8.11), it is easy to see that $\varphi \in \mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$ and

$$\mu' \circ \varphi = \varphi \circ \mu. \qquad \square$$

Next, we suppose that $\sigma = \sigma_2$. In this case, $\mathfrak{k} = \mathfrak{g}^{\sigma_2}$ is generated by $\{E_i, F_i; i = 1, 2\}$ and

$$\mathfrak{k} \cong \mathfrak{su}(3). \tag{8.12}$$

Suppose that there exists an involutive automorphism μ of $\mathfrak{g}$ such that $\mu \circ \sigma_2 = {\sigma_2}^2 \circ \mu$. By using the classification of symmetric spaces, we know that $\mathfrak{g}^\mu$ is isomorphic to one of $\mathfrak{so}(7)$, $\mathfrak{u}(4)$, $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ and $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$. Since $\dim G/K = \dim \mathfrak{g} - \dim \mathfrak{k} = 20$ and the possibilities of the dimension of $\mathfrak{g}^\mu$ are 21, 16, 13 and 12, the dimension of $\mathfrak{k}^\mu$ must be 11, 6, 3 or 2. Therefore it follows from (8.12) that $\dim \mathfrak{k}^\mu = 3$ and $\mathfrak{k}^\mu \cong \mathfrak{so}(3)$. Moreover, since $(\mathfrak{su}(3), \mathfrak{so}(3))$ is a symmetric pair associated to a normal real form of $\mathfrak{sl}(3, \boldsymbol{C})$, it follows that $\mathfrak{k}^\mu$ is conjugate within $\mathrm{Int}(\mathfrak{k})$ to

$$\boldsymbol{R}(E_1 + F_1) + \boldsymbol{R}(E_2 - F_2) + \boldsymbol{R}[E_1 + F_1, E_2 - F_2].$$

So, we may assume that

$$\mu(E_1) = F_1, \quad \mu(F_1) = E_1, \quad \mu(E_2) = -F_2, \quad \mu(F_2) = -E_2. \tag{8.13}$$

Since $\sigma_2(E_3) = \xi E_3$, $\sigma_2(F_3) = \xi^2 F_3$ ($\xi^3 = 1$) and $\mu \circ \sigma_2 = {\sigma_2}^2 \circ \mu$, we get

$$\sigma_2(\mu(E_3)) = \xi^2 \mu(E_3), \qquad \sigma_2(\mu(F_3)) = \xi \mu(F_3). \tag{8.14}$$

Lemma 8.3. *There is $c \in \boldsymbol{C}^\times$ such that $\mu(E_3) = cF_3$.*

Proof. By a dimension argument it is easy to see that the dimension of the eigenspaces $\mathfrak{g}(\sigma_2, \xi^{\pm 1})$ of σ_2 with the eigenvalues $\xi^{\pm}$ are 10. Moreover, by using (7.2) we can check that

$$\begin{aligned}
\mathrm{ad}U(\mathfrak{k})(\hat{E}_3) = {} & \mathscr{L}_{\gamma_3} + \mathscr{L}_{\gamma_2+\gamma_3} + \mathscr{L}_{2\gamma_2+\gamma_3} + \mathscr{L}_{3\gamma_2+\gamma_3} + \mathscr{L}_{\gamma_1+\gamma_2+\gamma_3} + \mathscr{L}_{\gamma_1+2\gamma_2+\gamma_3} \\
& + \mathscr{L}_{\gamma_1+3\gamma_2+\gamma_2} + \mathscr{L}_{2\gamma_1+2\gamma_2+\gamma_3} + \mathscr{L}_{2\gamma_1+3\gamma_2+\gamma_3} + \mathscr{L}_{3\gamma_1+3\gamma_2+\gamma_3}, \\
\mathrm{ad}U(\mathfrak{k})(\hat{F}_3) = {} & \mathscr{L}_{-\gamma_3} + \mathscr{L}_{-(\gamma_2+\gamma_3)} + \mathscr{L}_{-(2\gamma_2+\gamma_3)} + \mathscr{L}_{-(3\gamma_2+\gamma_3)} + \mathscr{L}_{-(\gamma_1+\gamma_2+\gamma_3)} \\
& + \mathscr{L}_{-(\gamma_1+2\gamma_2+\gamma_3)} + \mathscr{L}_{-(\gamma_1+3\gamma_2+\gamma_2)} + \mathscr{L}_{-(2\gamma_1+2\gamma_2+\gamma_3)} + \mathscr{L}_{-(2\gamma_1+3\gamma_2+\gamma_3)} \\
& + \mathscr{L}_{-(3\gamma_1+3\gamma_2+\gamma_3)},
\end{aligned}$$

where we denote by $U(\mathfrak{k})$ the universal enveloping algebra of $\mathfrak{k}$. Since $\phi_1(\mathrm{ad}U(\mathfrak{k})(\hat{E}_3)) \subset \mathfrak{g}(\sigma_2, \xi)$, $\phi_1(\mathrm{ad}U(\mathfrak{k})(\hat{F}_3)) \subset \mathfrak{g}(\sigma_2, \xi^2)$ and $\dim \mathfrak{g}(\sigma_2, \xi^{\pm 1}) = 10$, we get

$$\phi_1(\mathrm{ad}U(\mathfrak{k})(\hat{E}_3)) = \mathfrak{g}(\sigma_2, \xi), \quad \phi_1(\mathrm{ad}U(\mathfrak{k})(\hat{F}_3)) = \mathfrak{g}(\sigma_2, \xi^2).$$

Considering $\mathfrak{g}(\sigma_2, \xi^{\pm 1})$ to be representation spaces of $\mathfrak{k}$, it is clear that E_3 is the lowest weight vector of $\mathfrak{g}(\sigma_2, \xi)$, and it follows from (8.13) and (8.14) that $\mu(E_3)$ is the highest weight vector of $\mathfrak{g}(\sigma_2, \xi^2)$. Consequently, we obtain

$$\mu(E_3) \in \phi_1(\mathscr{L}_{-\gamma_3}) = \boldsymbol{C}F_3. \qquad \square$$

Define a mapping $\tilde{\mu} : \mathscr{L}(\mathfrak{g}_c) \longrightarrow \mathscr{L}(\mathfrak{g}_c)$ as follows:

$$\tilde{\mu}(t^i \otimes X) := t^{-i} \otimes \mu(X).$$

Then it follows from (8.13) and Lemma 8.3 that $\tilde{\mu}$ is an involutive automorphism satisfying

$$\phi_1 \circ \tilde{\mu} = \mu \circ \phi_1, \qquad \tilde{\mu}((1-t^3)\mathscr{L}(\mathfrak{g}_c)) = (1-t^3)\mathscr{L}(\mathfrak{g}_c). \tag{8.15}$$

Now, as in the case where $\sigma = \sigma_1$, we calculate $\tilde{\mu}((1-t^3)\hat{E}_2)$ by using (8.4). Since

$$\begin{aligned}
\tilde{\mu}(t \otimes H_+) &= a\tilde{\mu}([\hat{E}_1, [\hat{E}_2, [\hat{E}_2, \hat{E}_3]]]) \\
&= a[\hat{F}_1, [-\hat{F}_2, [-\hat{F}_2, c\hat{F}_3]]] = \frac{ac}{b} t^{-1} \otimes H_-,
\end{aligned}$$

we have

$$\begin{aligned}
\tilde{\mu}((1-t^3)\hat{E}_2) &= \tilde{\mu}(\hat{E}_2) - \tilde{\mu}((\mathrm{ad}t \otimes H_+)^3(\hat{E}_2)) \\
&= -\hat{F}_2 - \left(\frac{ac}{b}\right)^3 (\mathrm{ad}t^{-1} \otimes H_-)^3(-\hat{F}_2) = -\hat{F}_2 + \left(\frac{ac}{b}\right)^3 t^{-3}\hat{F}_2.
\end{aligned}$$

Therefore it follows from (8.15) that

$$\left(\frac{ac}{b}\right)^3 = 1. \tag{8.16}$$

Now, we shall prove the following lemma which claims the uniqueness of the TRG-pair in this case.

LEMMA 8.4. *Let* $\mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$ *be as in Lemma* 8.2. *If there exists an involutive automorphism* $\mu' : \mathfrak{g}_c \longrightarrow \mathfrak{g}_c$ *such that* $\mu|_{\mathfrak{k}_c} = \mu'|_{\mathfrak{k}_c}$ *and* $\mu' \circ \sigma_2 = {\sigma_2}^2 \circ \mu'$, *then* μ' *is conjugate within* $\mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$ *to* μ.

PROOF. By the assumption of the lemma, we can see that there is $p \in \boldsymbol{C}^\times$ such that

$$\begin{aligned} &\mu'(E_1) = F_1, \quad \mu'(F_1) = E_1, \quad \mu'(E_2) = -F_2, \\ &\mu'(F_2) = -E_2, \quad \mu'(E_3) = pF_3, \quad \mu'(F_3) = p^{-1}E_3. \end{aligned} \tag{8.17}$$

Moreover there exists an involutive automorphism $\tilde{\mu}'$ of $\mathscr{L}(\mathfrak{g}_c)$ satisfying $\phi_1 \circ \tilde{\mu}' = \mu' \circ \phi_1$ (see (8.13), (8.15), (8.16) and Lemma 8.3). As stated in the case where $\sigma = \sigma_1$, for any $q \in \boldsymbol{C}^\times$ there exists an automorphism $\tilde{\nu}$ of $\mathscr{L}(\mathfrak{g}_c)$ satisfying

$$\begin{aligned} &\tilde{\nu}(\hat{E}_1) = \hat{E}_1, \quad \tilde{\nu}(\hat{E}_2) = \hat{E}_2, \quad \tilde{\nu}(\hat{E}_3) = q\hat{E}_3, \\ &\tilde{\nu}(\hat{F}_1) = \hat{F}_1, \quad \tilde{\nu}(\hat{F}_2) = \hat{F}_2, \quad \tilde{\nu}(\hat{F}_3) = q^{-1}\hat{F}_3, \\ &\left(\frac{ap}{b}\right)^3 = 1, \end{aligned} \tag{8.18}$$

since $\hat{E}_i$, $\hat{F}_i$ $(i = 1, 2, 3)$ are Chevalley generators of $\hat{\mathscr{L}}(\mathfrak{g}_c)$. Then by using (8.4) we obtain $\tilde{\nu}(t \otimes H_+) = q \cdot t \otimes H_+$ and

$$\tilde{\nu}((1 - t^3)\hat{E}_2) = \hat{E}_2 - \tilde{\nu}\big((\mathrm{ad}t \otimes H_+)^3(\hat{E}_2)\big) = \hat{E}_2 - q^3 t^3 \hat{E}_2,$$

and the equations (8.16) and (8.18) imply that

$$\tilde{\nu}((1 - t^3)\mathscr{L}(\mathfrak{g}_c) = (1 - q^3 t^3)\mathscr{L}(\mathfrak{g}_c) = \left(1 - \left(\frac{cq}{p}\right)^3 t^3\right)\mathscr{L}(\mathfrak{g}_c).$$

Therefore there exists an automorphism ν of $\mathfrak{g}_c$ satisfying

$$\nu \circ \phi_1 = \phi_{(\frac{cq}{p})^{-1}} \circ \tilde{\nu}. \tag{8.19}$$

Note that $\nu \in \mathrm{Aut}_{\mathfrak{k}_c}(\mathfrak{g}_c)$. By making use of (8.13), (8.17), (8.18), (8.19) and Lemma 8.3, we can calculate $\mu' \circ \nu(E_i)$ and $\nu \circ \mu(E_i)$ $(i = 1, 2, 3)$:

$$\mu' \circ \nu(E_1) = \left(\frac{cq}{p}\right)^{-1} F_1, \qquad \nu \circ \mu(E_1) = \left(\frac{cq}{p}\right) F_1,$$
$$\mu' \circ \nu(E_2) = -F_2, \qquad \nu \circ \mu(E_2) = -F_2,$$
$$\mu' \circ \nu(E_3) = pqF_3, \quad \nu \circ \mu(E_3) = cq^{-1}F_3.$$

The proof of the lemma is completed by choosing q so that $q^2 = \frac{c}{p}$. $\square$

Finally, we shall prove the following theorem.

THEOREM 8.5. *Let G be a compact simple Lie group of type D_4 and σ an outer automorphism of order 3 on G. Then a TRG-pair $((G/K, \langle,\rangle, \sigma), N)$ is equivalent to one of TRG-pairs*

$$((G/K, \langle,\rangle, \sigma_1), \exp \mathfrak{g}^{\mu_i} \cdot o) \quad (i = 1, 2) \quad \textit{and} \quad ((G/K, \langle,\rangle, \sigma_2), \exp \mathfrak{g}^{\mu_3} \cdot o)$$

described in Propositions 7.3, 7.4 and 7.5.

PROOF. Considering (7.5) and (7.9) combined with Lemma 8.2, it is easy to see that a TRG-pair $((G/K, \langle,\rangle, \sigma_1), N)$ is equivalent to one of TRG-pairs described in Propositions 7.3 and 7.4. Also by (7.13), (8.13) and Lemma 8.4, a TRG-pair $((G/K, \langle,\rangle, \sigma_2), N)$ is equivalent to that in Proposition 7.5. $\square$

REMARK 8.6. Combining Theorem 8.5 with Theorem 6.1, all the possibilities of TRG-pairs $((G/K, \langle,\rangle, \sigma), \exp \mathfrak{b} \cdot o)$ is obtained in the case where σ is outer. We list all $(\mathfrak{g}, \mathfrak{k} = \mathfrak{g}^\sigma, \mathfrak{b} = \mathfrak{g}^\mu, \mu)$ in Table 5 below.

Table 5. The case where σ is outer.

$\mathfrak{g}$	$\mathfrak{k} = \mathfrak{g}^\sigma$	$\mathfrak{g}^\mu$	μ
$\mathfrak{so}(8)$	$\mathfrak{g}_2 (= \mathfrak{g}^{\sigma_1})$	$\mathfrak{so}(7)$	μ_1 in Proposition 7.3.
$\mathfrak{so}(8)$	$\mathfrak{g}_2 (= \mathfrak{g}^{\sigma_1})$	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$	μ_2 in Proposition 7.4.
$\mathfrak{so}(8)$	$\mathfrak{su}(3) (= \mathfrak{g}^{\sigma_2})$	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$	μ_3 in Proposition 7.5.
$\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$	$\{(X, X, X); X \in \mathfrak{l}\}$	$\mathfrak{l} \oplus \{(X, X); X \in \mathfrak{l}\}$	$\mu(X, Y, Z) = (X, Z, Y)$

References

[A] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ., **13** (1962), 1–34.

[CN] Bang-Yen Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces, II, Duke Math. J., **45** (1978), 405–425.

[G] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geom., **7** (1972), 343–369.

[H] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York-London, 1978.

[I] S. Ihara, Holomorphic imbeddings of symmetric domains, J. Math. Soc. Japan, **19** (1967), 261–302.

[K] V. G. Kac, Infinite dimensional Lie algebras, Third edition, Cambridge Univ. Press, 1990.

[Ka] S. Kaneyuki, On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{ev}$ of semisimple graded Lie algebras, J. Math. Soc. Japan, **45** (1993), 1–19 .

[KA] S. Kaneyuki and H. Asano, Graded Lie algebras and generalized Jordan triple systems, Nagoya Math. J., **112** (1988), 81–115.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry Vol. II, Interscience Tracts in Pure and Applied Mathematics, No. 15, Interscience Publishers John Wiley & Sons, Inc., New York, 1969.

[S] A. Sagle, A note on triple systems and totally geodesic submanifolds in a homogeneous space, Nagoya Math. J., **91** (1968), 5–20.

[Sa1] I. Satake, On representations and compactifications of symmetric Riemann spaces, Ann. of Math., **71** (1960), 77–110.

[Sa2] I. Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, Amer. J. Math., **87** (1965), 425–461.

[T] M. Takeuchi, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, Tohoku Math. J., **36** (1984), 293–314.

[To] K. Tojo, Totally real totally geodesic submanifolds of compact 3-symmetric spaces, Tohoku Math. J., **53** (2001), 131–143.

[WG] A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms I, II, J. Differential Geom., **2** (1968), 77–114, 115–159.

Koji Tojo

Department of Mathematics
Chiba Institute of Technology
Shibazono, Narashino
Chiba 275-0023
Japan