On subadditivity of the logarithmic Kodaira dimension

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Abstract. We reduce Iitaka's subadditivity conjecture for the logarithmic Kodaira dimension to a special case of the generalized abundance conjecture by establishing an Iitaka type inequality for Nakayama's numerical Kodaira dimension. Our proof heavily depends on Nakayama's theory of ω-sheaves and $\widehat{\omega}$-sheaves. As an application, we prove the subadditivity of the logarithmic Kodaira dimension for affine varieties by using the minimal model program for projective klt pairs with big boundary divisor.

1. Introduction.

In this paper, we discuss Iitaka's subadditivity conjecture on the logarithmic Kodaira dimension $\overline{\kappa}$.

CONJECTURE 1.1 (Subadditivity of logarithmic Kodaira dimension). *Let $g : V \to W$ be a dominant morphism between algebraic varieties. Then we have the following inequality*

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

Conjecture 1.1 is usually called Conjecture $\overline{C}_{n,m}$ when $\dim V = n$ and $\dim W = m$. If V is complete in Conjecture 1.1, then it is nothing but the famous Iitaka subadditivity conjecture for the Kodaira dimension κ. We see that Conjecture 1.1 is equivalent to:

CONJECTURE 1.2. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that $\mathrm{Supp} f^* D_Y \subset \mathrm{Supp} D_X$. Then we have*

$$\kappa(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)$$

where F is a sufficiently general fiber of $f : X \to Y$.

One of the main purposes of this paper is to prove:

THEOREM 1.3 (Main theorem).   *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that $\operatorname{Supp} f^* D_Y \subset \operatorname{Supp} D_X$. Then we have*

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)$$

where F is a sufficiently general fiber of $f : X \to Y$.

Note that κ_σ denotes Nakayama's numerical Kodaira dimension and that the inequality $\kappa_\sigma \geq \kappa$ always holds, where κ is Iitaka's D-dimension. Theorem 1.3 is a variant of Nakayama's theorem (see [**N**, V.4.1. Theorem] and Remark 3.8). By Theorem 1.3, Conjecture 1.2 is reduced to:

CONJECTURE 1.4.   *Let X be a smooth projective variety and let D_X be a simple normal crossing divisor on X. Then the equality*

$$\kappa_\sigma(X, K_X + D_X) = \kappa(X, K_X + D_X)$$

holds.

Conjecture 1.4 is known as a special case of the generalized abundance conjecture (see Conjecture 2.10), which is one of the most important conjectures for higher-dimensional algebraic varieties. As an easy corollary of Theorem 1.3, we have:

COROLLARY 1.5.   *In Theorem 1.3, we further assume that $\dim X \leq 3$. Then we have*

$$
\begin{aligned}
\kappa(X, K_X + D_X) &= \kappa_\sigma(X, K_X + D_X) \\
&\geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y) \\
&\geq \kappa(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y).
\end{aligned}
$$

In particular, if $g : V \to W$ is a dominant morphism between algebraic varieties with $\dim V \leq 3$, then we have the inequality

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

Note that the equality $\kappa_\sigma(X, K_X + D_X) = \kappa(X, K_X + D_X)$ in Corollary 1.5 follows from the minimal model program and the abundance theorem for (X, D_X) (see Proposition 4.3). We also note that Corollary 1.5 is new when $\dim V = 3$ and $\dim W = 1$. Anyway, Conjecture 1.1 now becomes a consequence of the minimal model program and the abundance conjecture by Theorem 1.3 (see Remark 4.5). This fact strongly supports Conjecture 1.1.

As an application of Theorem 1.3, we obtain:

COROLLARY 1.6 (Subadditivity of the logarithmic Kodaira dimension for affine varieties).   *Let $g : V \to W$ be a dominant morphism from an affine variety V. Then we*

have

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

Note that W is not necessarily assumed to be affine in Corollary 1.6. In order to prove Corollary 1.6, we construct (X, D_X) with $\overline{\kappa}(V) = \kappa(X, K_X + D_X)$ such that (X, D_X) has a good minimal model or a Mori fiber space structure by using the minimal model program for projective klt pairs with big boundary divisor. Note that $\kappa_\sigma(X, K_X + D_X) = \kappa(X, K_X + D_X)$ holds for such (X, D_X).

REMARK 1.7. By the proof of Corollary 1.6, we see that the inequality

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

holds for every *strictly* rational dominant map $g : V \dashrightarrow W$ from an affine variety V.

In this paper, we use Nakayama's theory of ω-sheaves and $\widehat{\omega}$-sheaves in order to prove an Iitaka type inequality for Nakayama's numerical Kodaira dimension (see Theorem 1.3). It is closely related to Viehweg's clever covering trick and weak positivity. We also use the minimal model program for projective klt pairs with big boundary divisor for the study of affine varieties (see Section 4).

REMARK 1.8. Let $f : X \to Y$ be a projective surjective morphism between smooth projective varieties with connected fibers. In [**K**], Kawamata proved that the inequality

$$\kappa(X) \geq \kappa(F) + \kappa(Y),$$

where F is a sufficiently general fiber of $f : X \to Y$, holds under the assumption that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ has a good minimal model. His approach is completely different from ours. For the details, see [**K**].

Finally the following theorem, which is a slight generalization of Theorem 1.3, was suggested by the referee.

THEOREM 1.9. *Let $f : X \to Y$ be a proper surjective morphism from a normal variety X onto a smooth complete variety Y with connected fibers. Let D_X be an effective $\mathbb{Q}$-divisor on X such that (X, D_X) is lc and let D_Y be a simple normal crossing divisor on Y. Assume that $\mathrm{Supp} f^* D_Y \subset \lfloor D_X \rfloor$, where $\lfloor D_X \rfloor$ is the round-down of D_X. Then we have*

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)$$

where F is a sufficiently general fiber of $f : X \to Y$.

The formulation of Theorem 1.9 seems to be natural and useful from the minimal model theoretic viewpoint.

We summarize the contents of this paper. In Section 2, we briefly recall Iitaka's logarithmic Kodaira dimension, Nakayama's numerical Kodaira dimension, Nakayama's ω-sheaves and $\widehat{\omega}$-sheaves, and some related topics. In Section 3, we prove Theorem 1.3, which is the main theorem of this paper. Our proof heavily depends on Nakayama's argument in his book [**N**], which is closely related to Viehweg's covering trick and weak positivity. In Section 4, we discuss the minimal model program for affine varieties. For any affine variety, we see that there is a smooth compactification which has a good minimal model or a Mori fiber space structure. As an application, we obtain the subadditivity of the logarithmic Kodaira dimension for affine varieties by Theorem 1.3 (see Corollary 1.6).

We will work over $\mathbb{C}$, the complex number field, throughout this paper. For the standard notation of the minimal model program, see [**F1**] and [**F3**].

2. Preliminaries.

In this section, we quickly explain the logarithmic Kodaira dimension introduced by Iitaka, Nakayama's numerical Kodaira dimension, ω-sheaves, and $\widehat{\omega}$-sheaves.

2.1 (SUFFICIENTLY GENERAL FIBERS). Let us recall the definition of sufficiently general fibers for the reader's convenience.

DEFINITION 2.2 (Sufficiently general fibers). Let $f : X \to Y$ be a morphism between algebraic varieties. Then a *sufficiently general fiber* F of $f : X \to Y$ means that $F = f^{-1}(y)$ where y is any point contained in a countable intersection of nonempty Zariski open subsets of Y.

A sufficiently general fiber is sometimes called a *very general fiber* in the literature.

2.3 (LOGARITHMIC KODAIRA DIMENSION). The notion of the logarithmic Kodaira dimension was introduced by Shigeru Iitaka (see [**I1**]).

DEFINITION 2.4 (Logarithmic Kodaira dimension). Let V be an irreducible algebraic variety. By Nagata's theorem, we have a complete algebraic variety $\overline{V}$ which contains V as a dense Zariski open subset. By Hironaka's theorem, we have a smooth projective variety $\overline{W}$ and a projective birational morphism $\mu : \overline{W} \to \overline{V}$ such that if $W = \mu^{-1}(V)$, then $\overline{D} = \overline{W} - W = \mu^{-1}(\overline{V} - V)$ is a simple normal crossing divisor on $\overline{W}$. The *logarithmic Kodaira dimension* $\overline{\kappa}(V)$ of V is defined as

$$\overline{\kappa}(V) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$$

where κ denotes Iitaka's D-dimension.

It is well-known and is easy to see that $\overline{\kappa}(V)$ is well-defined, that is, it is independent of the choice of the pair $(\overline{W}, \overline{D})$.

As we have already explained, the following conjecture (see Conjecture 1.1) is usually called Conjecture $\overline{C}_{n,m}$ when $\dim V = n$ and $\dim W = m$.

CONJECTURE 2.5 (Subadditivity of logarithmic Kodaira dimension). *Let $g : V \to W$ be a dominant morphism between algebraic varieties. Then we have the following inequality*

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

Note that Conjecture 1.2 is a special case of Conjecture 2.5 by putting $V = X \setminus D_X$ and $W = Y \setminus D_Y$. On the other hand, we can easily check that Conjecture 2.5 follows from Conjecture 1.2. For the details, see the proof of Corollary 1.6. Anyway, Conjecture 2.5 (see Conjecture 1.1) is equivalent to Conjecture 1.2. We note that Conjecture 1.2 is easier to handle than Conjecture 2.5 from the minimal model theoretic viewpoint.

2.6 (NAKAYAMA'S NUMERICAL KODAIRA DIMENSION). Let us recall the definition of Nakayama's numerical Kodaira dimension.

DEFINITION 2.7 (Nakayama's numerical Kodaira dimension). Let X be a smooth projective variety and let D be a Cartier divisor on X. We put

$$\sigma(D; A) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \;\middle|\; \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(A + mD))}{m^k} > 0 \right\}$$

and

$$\kappa_\sigma(X, D) = \max\{\sigma(D; A) \,|\, A \text{ is a divisor}\}.$$

Note that if $H^0(X, \mathcal{O}_X(A + mD)) \neq 0$ only for finitely many $m \in \mathbb{Z}_{\geq 0}$ then we define $\sigma(D; A) = -\infty$. It is obvious that $\kappa_\sigma(X, D) \geq \kappa(X, D)$, where $\kappa(X, D)$ denotes Iitaka's D-dimension of D. We also note that $\kappa_\sigma(X, D) \geq 0$ if and only if D is pseudo-effective (see [**N**, V.1.4. Corollary]).

When X is a normal projective variety, we take a resolution $\varphi : X' \to X$, where X' is a smooth projective variety, and put

$$\kappa_\sigma(X, D) = \kappa_\sigma(X', \varphi^* D).$$

It is not difficult to see that $\kappa_\sigma(X, D)$ is well-defined and has various good properties. For the details, see [**N**, V. Section 2], [**L**] and [**E**].

The following lemma, which is lacking in [**N**], will play a crucial role in the proof of Theorem 1.3.

Lemma 2.8 ([**L**, Theorem 6.7 (7)]). *Let D be a pseudo-effective Cartier divisor on a smooth projective variety X. We fix some sufficiently ample Cartier divisor A on X. Then there exist positive constants C_1 and C_2 such that*

$$C_1 m^{\kappa_\sigma(X,D)} \le \dim H^0(X, \mathcal{O}_X(mD + A)) \le C_2 m^{\kappa_\sigma(X,D)}$$

for every sufficiently large m.

For the details, see [**L**, Theorem 6.7 (7)] and [**E**, 2.8, 2.10, and Theorem 0.2].

Remark 2.9.   Nakayama's numerical Kodaira dimension can be defined for $\mathbb{R}$-Cartier $\mathbb{R}$-divisors and has many equivalent definitions and several nontrivial characterizations. For the details, see [**N**, V. Section 2], [**L**, Theorem 1.1], and [**E**, Theorem 0.2]. Note that [**E**, 2.9] describes a gap in Lehmann's paper [**L**].

The following conjecture is one of the most important conjectures for higher-dimensional algebraic varieties. Conjecture 1.4 is a special case of Conjecture 2.10.

Conjecture 2.10 (Generalized abundance conjecture).   *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair. Then $\kappa_\sigma(X, K_X + \Delta) = \kappa_\iota(X, K_X + \Delta)$.*

It is obvious that if Conjecture 2.10 holds for (X, D_X) in Theorem 1.3 then Theorem 1.3 implies Conjecture 1.2 in full generality.

Remark 2.11 (On the definition of $\kappa_\iota(X, K_X + \Delta)$).   We have to be careful when Δ is an $\mathbb{R}$-divisor in Conjecture 2.10. If there exists an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_\mathbb{R} D$, then we put

$$\kappa_\iota(X, K_X + \Delta) = \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{\log m}.$$

Otherwise, we put $\kappa_\iota(X, K_X + \Delta) = -\infty$. The above definition of $\kappa_\iota(X, K_X + \Delta)$ is well-defined, that is, $\kappa_\iota(X, K_X + \Delta)$ is independent of the choice of D (see, for example, [**C**, Definition 2.2.1] and [**F3**]). Note that if $K_X + \Delta$ is a $\mathbb{Q}$-divisor then $\kappa_\iota(X, K_X + \Delta)$ coincides with $\kappa(X, K_X + \Delta)$, that is,

$$\kappa_\iota(X, K_X + \Delta) = \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))}{\log m}.$$

Example 2.12.   We put $X = \mathbb{P}^1$. Let Δ be an effective $\mathbb{R}$-divisor on X. We assume that $\deg \Delta = 2$ and that Δ is not a $\mathbb{Q}$-divisor. Then we can easily see that $K_X + \Delta \sim_\mathbb{R} 0$, $\kappa_\sigma(X, K_X + \Delta) = \kappa_\iota(X, K_X + \Delta) = 0$, and $\kappa(X, K_X + \Delta) = -\infty$.

Anyway, we do not use $\mathbb{R}$-divisors in this paper. So, we do not discuss subtle problems on $\mathbb{R}$-divisors here. However, we note that it is indispensable to treat $\mathbb{R}$-divisors when we discuss Conjecture 2.10 and Conjecture 2.13 below.

Note that Conjecture 2.10 holds in dimension $\le n$ if and only if Conjecture 2.13 holds in dimension $\le n$.

CONJECTURE 2.13 (Good minimal model conjecture).   *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair. Assume that $K_X + \Delta$ is pseudo-effective. Then (X, Δ) has a good minimal model.*

For the relationships among various conjectures on the minimal model program, see [**FG**].

We will use the following easy well-known lemma in the proof of Corollary 1.6.

LEMMA 2.14.   *Let $f : X \to Y$ be a generically finite surjective morphism between smooth projective varieties. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that $\mathrm{Supp} f^* D_Y \subset \mathrm{Supp} D_X$. Then we have*

$$\kappa(X, K_X + D_X) \geq \kappa(Y, K_Y + D_Y)$$

and

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(Y, K_Y + D_Y).$$

PROOF.   We put $n = \dim X = \dim Y$. Then we have

$$f^* \Omega_Y^n(\log D_Y) \subset \Omega_X^n(\log D_X).$$

Therefore, we can write

$$K_X + D_X = f^*(K_Y + D_Y) + R$$

for some effective Cartier divisor R. Thus, we have the desired inequalities.   □

2.15 (NAKAYAMA'S ω-SHEAVES AND $\widehat{\omega}$-SHEAVES).   Let us briefly recall the theory of Nakayama's ω-sheaves and $\widehat{\omega}$-sheaves.

The following definition of ω-sheaf is equivalent to Nakayama's original definition of ω-sheaf in the category of projective varieties (see [**N**, V.3.8. Definition]).

DEFINITION 2.16 (ω-sheaf).   A coherent sheaf $\mathcal{F}$ on a projective variety Y is called an ω-*sheaf* if there exists a projective morphism $f : X \to Y$ from a smooth projective variety X such that $\mathcal{F}$ is a direct summand of $f_* \omega_X$.

We also need the notion of $\widehat{\omega}$-sheaf (see [**N**, V.3.16. Definition]).

DEFINITION 2.17 ($\widehat{\omega}$-sheaf).   A coherent torsion-free sheaf $\mathcal{F}$ on a normal projective variety Y is called an $\widehat{\omega}$-*sheaf* if there exist an ω-sheaf $\mathcal{G}$ and a generically isomorphic inclusion $\mathcal{G} \hookrightarrow \mathcal{F}^{**}$ into the double dual $\mathcal{F}^{**}$ of $\mathcal{F}$.

Although the following lemma is easy to prove, it plays a crucial role in the proof of Theorem 1.3.

LEMMA 2.18.   *Let Y be a projective variety. Then there exists an ample Cartier divisor A on Y such that $\mathcal{F} \otimes \mathcal{O}_Y(A)$ is generated by global sections for every ω-sheaf $\mathcal{F}$ on Y.*

PROOF.   We may assume that $\mathcal{F} = f_*\omega_X$ for a projective morphism $f : X \to Y$ from a smooth projective variety X. Let H be an ample Cartier divisor on Y such that $|H|$ is free. We put $A = (\dim Y + 1)H$. Then we have

$$H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y(A) \otimes \mathcal{O}_Y(-iH)) = 0$$

for every $i > 0$ by Kollár's vanishing theorem. Therefore, by using the Castelnuovo–Mumford regularity, we see that $\mathcal{F} \otimes \mathcal{O}_Y(A)$ is generated by global sections. $\square$

As an obvious corollary of Lemma 2.18, we have:

COROLLARY 2.19.   *Let Y be a normal projective variety. Then there exists an ample Cartier divisor A on Y such that $\mathcal{F} \otimes \mathcal{O}_Y(A)$ is generically generated by global sections for every reflexive $\widehat{\omega}$-sheaf $\mathcal{F}$ on Y.*

2.20 (STRICTLY RATIONAL MAP).   We close this section with the notion of strictly rational maps. For the details, see [**I2**, Lecture 2] and [**I3**, Section 2.12 Strictly Rational Maps].

DEFINITION 2.21 (Strictly rational map).   Let $f : X \dashrightarrow Y$ be a rational map between irreducible varieties. If there is a proper birational morphism $\mu : Z \to X$ from an irreducible variety Z such that $f \circ \mu$ is a morphism, then $f : X \dashrightarrow Y$ is called a *strictly rational map*.

$$
\begin{array}{ccc}
Z & & \\
\mu \downarrow & \searrow^{f \circ \mu} & \\
X & \dashrightarrow & Y \\
& f &
\end{array}
$$

Note that a rational map $f : X \dashrightarrow Y$ from X to a complete variety Y is always strictly rational.

EXAMPLE 2.22.   Let X be a smooth projective variety and let U be a dense open subset of X such that $U \subsetneq X$. Then the natural open immersion $\iota : U \hookrightarrow X$ is strictly rational. On the other hand, $f = \iota^{-1} : X \dashrightarrow U$ is not strictly rational.

3.   Subadditivity of Nakayama's numerical Kodaira dimension.

In this section, we prove Theorem 1.3 by using Nakayama's theory of ω-sheaves and $\widehat{\omega}$-sheaves. The following lemma is a special case of [**N**, V.3.34. Lemma]. It is a reformulation and a generalization of Viehweg's deep result (see [**V**, Corollary 5.2]).

LEMMA 3.1 (cf. [**N**, V.3.34. Lemma]).   *Let $f : X \to Y$ be a projective surjective morphism from a normal projective variety X onto a smooth projective variety Y with connected fibers. Let L be a Cartier divisor on X, let Δ be an effective $\mathbb{Q}$-divisor on X, and let k be an integer greater than one satisfying the following conditions:*

(i) *(X, Δ) is klt.*

(ii) $L - k(K_{X/Y} + \Delta)$ *is ample.*

Then we obtain that

$$\omega_Y((k-1)H) \otimes f_*\mathcal{O}_X(L)$$

is an $\widehat{\omega}$-sheaf for any ample Cartier divisor H on Y.

REMARK 3.2. In Lemma 3.1, it is sufficient to assume that (X, Δ) is lc and that there is a positive rational number δ such that $(X, (1 - \delta)\Delta)$ is klt. This is because $L - k(K_{X/Y} + (1 - \varepsilon)\Delta)$ is ample and $(X, (1 - \varepsilon)\Delta)$ is klt for $0 < \varepsilon \ll \delta$. Therefore, we can replace (X, Δ) with $(X, (1 - \varepsilon)\Delta)$ and may assume that (X, Δ) is klt.

We do not repeat the proof of [**N**, V.3.34. Lemma] here. For the details, see [**N**]. Note that the essence of Viehweg's theory of weakly positive sheaves is contained in the proof of Lemma 3.1. Therefore, Lemma 3.1 is highly nontrivial.

We make a small remark on the proof of [**N**, V.3.34. Lemma] for the reader's convenience.

REMARK 3.3. In the proof of [**N**, V.3.34. Lemma], P is nef and big in our setting. By taking more blow-ups and perturbing the coefficients of Δ slightly, we may further assume that P is ample. Therefore, it is easy to see that $f_*\mathcal{O}_X(K_X + \lceil P \rceil)$ is an ω-big ω-sheaf. For the definition of ω-*big ω-sheaves*, see [**N**, V.3.16. Definition (1)].

By the proof of [**N**, V.3.35. Theorem], we can check the following theorem. It is an application of Lemma 3.1.

THEOREM 3.4 (cf. [**N**, V.3.35. Theorem]). *Let $f : X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with the following properties*:

(i) *f has connected fibers.*

(ii) *$f : (U_X \subset X) \to (U_Y \subset Y)$ is toroidal and is equidimensional.*

(iii) *f is smooth over U_Y.*

(iv) *X has only quotient singularities.*

(v) *$\Delta_Y = Y \setminus U_Y$.*

(vi) *Δ_X is a reduced divisor contained in $X \setminus U_X$.*

(vii) $\operatorname{Supp} f^*\Delta_Y \subset \operatorname{Supp} \Delta_X$.

Let L be a Cartier divisor on X and let k be a positive integer with $k \geq 2$ such that $k(K_X + \Delta_X)$ is Cartier. Assume that

$$L - k(K_{X/Y} + \Delta_X - f^*\Delta_Y)$$

is very ample. Then

$$\omega_Y(\Delta_Y) \otimes f_*\mathcal{O}_X(L)$$

is an $\widehat{\omega}$-sheaf.

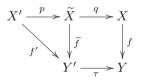REMARK 3.5.   A key point of Theorem 3.4 is that Δ_Y does not depend on L.

REMARK 3.6.   We note that $f_*\mathcal{O}_X(L)$ in Theorem 3.4 is reflexive. This is because $\mathcal{O}_X(L)$ is a locally free sheaf on a normal variety X and f is equidimensional. For the details, see, for example, [**H**, Corollary 1.7]. We also note that f is flat because f is equidimensional, X is Cohen–Macaulay, and Y is smooth.

COROLLARY 3.7.   *In Theorem* 3.4, *there is an ample Cartier divisor A' on Y such that $\mathcal{O}_Y(A') \otimes f_*\mathcal{O}_X(L)$ is generically generated by global sections. Moreover A' is independent of L and depends only on Y and Δ_Y.*

PROOF.   Let A be an ample Cartier divisor on Y as in Corollary 2.19. Then $\mathcal{O}_Y(A) \otimes \omega_Y(\Delta_Y) \otimes f_*\mathcal{O}_X(L)$ is generically generated by global sections. Let A_1 be an ample Cartier divisor on Y such that $A_1 - K_Y - \Delta_Y$ is very ample. Then $A' = A + A_1$ is the desired ample Cartier divisor on Y. Note that $f_*\mathcal{O}_X(L)$ is reflexive. $\square$

Let us prove Theorem 3.4.

PROOF OF THEOREM 3.4.   We take an ample Cartier divisor H on Y such that $H = A_1 - A_2$, where A_1 and A_2 are both smooth general very ample divisors on Y. Let $\tau : Y' \to Y$ be a finite Kawamata cover from a smooth projective variety Y' such that $\tau^*H = mH'$ for some Cartier divisor H' on Y' with $m \gg 0$. We put

$$X' \xrightarrow{p} \widetilde{X} \xrightarrow{q} X$$

with f' (from X'), $\widetilde{f}$ (from $\widetilde{X}$), f (from X) mapping down, and $Y' \xrightarrow{\tau} Y$

and $\lambda = q \circ p$, where $\widetilde{X} = X \times_Y Y'$ and X' is the normalization of $\widetilde{X}$. We may assume that $f' : X' \to Y'$ is a weak semistable reduction by [**AK**, Proposition 5.1 and Proposition 5.10]. We put $L' = \lambda^*L$. Since $L - k(K_{X/Y} + \Delta_X - f^*\Delta_Y)$ is very ample, we may assume that

$$L = k(K_{X/Y} + \Delta_X - f^*\Delta_Y) + B$$

where B is a general smooth very ample divisor on X. Then, by the arguments for the proof of [**F2**, Lemma 10.4 and Lemma 10.5], there exists a generically isomorphic injection

$$f'_*\mathcal{O}_{X'}(L') \hookrightarrow \tau^*(f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(\Delta_Y)). \tag{3.1}$$

We put $\Delta = \Delta_X - f^*\Delta_Y$ and define Δ' by

$$k(K_{X'/Y'} + \Delta') = \lambda^* k(K_{X/Y} + \Delta).$$

Since $\tau : Y' \to Y$ is a finite Kawamata cover, we can write $\Delta = \Sigma_X - f^*\Sigma_Y$ such that Σ_Y is a simple normal crossing divisor on Y, $\Delta_Y \le \Sigma_Y$, and τ is étale over $Y \setminus \Sigma_Y$. Then we have $K_{X'} + \Sigma_{X'} = \lambda^*(K_X + \Sigma_X)$ and $K_{Y'} + \Sigma_{Y'} = \tau^*(K_Y + \Sigma_Y)$ such that $\Sigma_{X'}$ and $\Sigma_{Y'}$ are effective and reduced. Of course, $\Delta' = \Sigma_{X'} - f'^*\Sigma_{Y'}$. By construction, $\mathrm{Supp}\Sigma_{X'} \supset \mathrm{Supp}f'^*\Sigma_{Y'}$. Since f' is weakly semistable, $f'^*\Sigma_{Y'}$ is reduced. Therefore, $\Sigma_{X'} \ge f'^*\Sigma_{Y'}$. This means that $\Delta' = \Sigma_{X'} - f'^*\Sigma_{Y'}$ is effective. We can find a positive rational number α such that

$$L - k(K_{X/Y} + \Delta) - \alpha f^* H$$

is ample. Let $\tau : Y' \to Y$ be the finite Kawamata cover as above for $m > (k-1)/\alpha$ and let H' be the same ample divisor as above. Then

$$L' - k(K_{X'/Y'} + \Delta') - (k-1)f'^*H' = \lambda^*\left(L - k(K_{X/Y} + \Delta) - \frac{k-1}{m}f^*H\right)$$

is ample. We apply Lemma 3.1 to $L' - (k-1)f'^*H'$ (see also Remark 3.2). Thus $\omega_{Y'} \otimes f'_*\mathcal{O}_{X'}(L')$ is an $\widehat{\omega}$-sheaf. Let G be the Galois group of $\tau : Y' \to Y$. By the proof of Lemma 3.1 (see the proof of [**N**, V.3.34. Lemma]), we can make everything G-equivariant and have an ω-sheaf $\mathcal{F}'$ and a generically isomorphic G-equivariant injection

$$\mathcal{F}' \hookrightarrow \omega_{Y'} \otimes f'_*\mathcal{O}_{X'}(L').$$

Hence there is a generically isomorphic injection

$$\mathcal{F} \hookrightarrow \omega_Y \otimes f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(\Delta_Y)$$

from a direct summand $\mathcal{F}$ of $\tau_*\mathcal{F}'$. Therefore, $\omega_Y(\Delta_Y) \otimes f_*\mathcal{O}_X(L)$ is an $\widehat{\omega}$-sheaf. $\qquad\square$

Let us prove Theorem 1.3.

PROOF OF THEOREM 1.3. Without loss of generality, we may assume that $\kappa_\sigma(F, K_F + D_X|_F) \ne -\infty$. By [**AK**, Theorem 2.1, Proposition 4.4, and Remark 4.5], we may assume that $f : X \to Y$ satisfies the conditions (i)–(v) in Theorem 3.4. We may also assume that $D_X \subset X \setminus U_X$ and $D_Y \subset Y \setminus U_Y$. We take $\Delta_X = \mathrm{Supp}(D_X + f^*\Delta_Y)$. Then Δ_X satisfies the conditions (vi) and (vii) in Theorem 3.4. We put

$$P = k(K_{X/Y} + \Delta_X - f^*\Delta_Y)$$

and

$$D = k(K_{X/Y} + D_X - f^*D_Y)$$

where k is a positive integer ≥ 2 such that D and P are both Cartier. We take a very ample Cartier divisor A on X. We put

$$r(mD; A) = \mathrm{rank} f_*\mathcal{O}_X(mD + A).$$

Since $D = P$ over the generic point of Y,

$$r(mD; A) = \mathrm{rank} f_* \mathcal{O}_X(mP + A).$$

Note that

$$\sigma(D|_F; A|_F) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \, \middle| \, \limsup_{m \to \infty} \frac{r(mD; A)}{m^k} > 0 \right\}$$

for a sufficiently general fiber F of $f : X \to Y$. We also note that

$$\kappa_\sigma(F, K_F + D_X|_F) = \kappa_\sigma(F, D|_F)$$
$$= \max\{\sigma(D|_F; A|_F) \, | \, A \text{ is very ample}\}.$$

Since $f_* \mathcal{O}_X(mP + A) \otimes \omega_Y(\Delta_Y)$ is a reflexive $\widehat{\omega}$-sheaf for every positive integer m by Theorem 3.4, there is an ample Cartier divisor H on Y such that we have a generically isomorphic injection

$$\mathcal{O}_Y^{\oplus r(mD;A)} \hookrightarrow \mathcal{O}_Y(H) \otimes f_* \mathcal{O}_X(mP + A)$$

for every $m \geq 1$ (see Corollary 3.7). Therefore, we have generically isomorphic injections

$$\mathcal{O}_Y(mk(K_Y + D_Y) + H)^{\oplus r(mD;A)}$$
$$\hookrightarrow \mathcal{O}_Y(mk(K_Y + D_Y) + 2H) \otimes f_* \mathcal{O}_X(mP + A)$$
$$\hookrightarrow \mathcal{O}_Y(mk(K_Y + D_Y) + 2H) \otimes f_* \mathcal{O}_X(mD + A).$$

This implies that

$$\dim H^0(X, \mathcal{O}_X(mk(K_X + D_X) + A + 2f^*H))$$
$$\geq r(mD; A) \cdot \dim H^0(Y, \mathcal{O}_Y(mk(K_Y + D_Y) + H)).$$

We assume that H is sufficiently ample and that A is also sufficiently ample. Then, by Lemma 2.8, we can find a constant C such that

$$r(mD; A) \cdot \dim H^0(Y, \mathcal{O}_Y(mk(K_Y + D_Y) + H))$$
$$\geq Cm^{\kappa_\sigma(F, D|_F) + \kappa_\sigma(Y, K_Y + D_Y)}$$

for every sufficiently large m. Hence we have

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y).$$

This is the desired inequality. $\qquad \square$

We give a remark on Nakayama's proof of [**N**, V.4.1. Theorem] for the reader's convenience.

REMARK 3.8. The proof of [**N**, V.4.1. Theorem] is insufficient. We think that we need the inequality as in [**L**, Theorem 6.7 (7)] (see [**E**, 2.8, 2.10, and Theorem 0.2] and

Lemma 2.8).

Let D be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X. Then the inequality in [**L**, Theorem 6.7 (7)] says that

$$\kappa_\sigma(X, D) = \lim_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))}{\log m}$$

where A is a sufficiently ample Cartier divisor on X. This useful characterization is not in [**N**].

From now on, we freely use the notation in the proof of [**N**, V.4.1. Theorem]. Nakayama proved the following inequality

$$h^0(X, \lceil m(D + f^*Q) \rceil + A + 2f^*H) \geq r(mD; A) \cdot h^0(Y, \lfloor mQ \rfloor + H) \qquad (3.2)$$

in the proof of [**N**, V.4.1. Theorem (1)]. We think that we need [**L**, Theorem 6.7 (7)] (see also [**E**, 2.8, 2.10, and Theorem 0.2]), which can not directly follow from the results in [**N**], to obtain

$$\kappa_\sigma(D + f^*Q) \geq \kappa_\sigma(D; X/Y) + \kappa_\sigma(Q)$$

from the inequality (3.2). The same trouble is in the proof of [**N**, V.4.1. Theorem (2)].

We close this section with a sketch of the proof of Theorem 1.9. We leave the details as an exercise for the reader.

SKETCH OF THE PROOF OF THEOREM 1.9. Here, we will only explain how to modify the proof of Theorem 1.3 for Theorem 1.9. First, we note that we can easily check that Theorem 3.4 holds true even when the coefficients of the horizontal part of Δ_X are in $[0, 1] \cap \mathbb{Q}$. All we have to do is to check the generically isomorphic injection (3.1)

$$f'_*\mathcal{O}_{X'}(L') \hookrightarrow \tau^*(f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(\Delta_Y)).$$

exists when the horizontal part of Δ_X is not necessarily reduced in the proof of Theorem 3.4 (see the arguments for the proof of [**F2**, Lemma 10.4 and Lemma 10.5]). Next, by [**AK**, Theorem 2.1, Proposition 4.4, and Remark 4.5], we may assume that $f : X \to Y$ satisfies the conditions (i)–(v) in Theorem 3.4. For the proof of Theorem 1.9, we may further assume that the coefficients of the vertical part of D_X are one by replacing D_X with $D_X^{\mathrm{h}} + \lfloor D_X^{\mathrm{v}} \rfloor$, where D_X^{h} (resp. D_X^{v}) is the horizontal (resp. vertical) part of D_X. Then we put $\Delta_X = D_X^{\mathrm{h}} + \mathrm{Supp} f^*\Delta_Y$. Finally, the proof of Theorem 1.3 works for Theorem 1.9 by the generalization of Theorem 3.4 discussed above. $\square$

We strongly recommend the interested reader to see [**N**, V. Section 4] for various related results.

4. Minimal model program for affine varieties.

In this section, we discuss the minimal model program for affine varieties and prove Corollary 1.6 as an application.

Let us start with Yoshinori Gongyo's observation. Proposition 4.1 says that the minimal model program works well for affine varieties.

PROPOSITION 4.1 (Yoshinori Gongyo).   *Let V be an affine variety. We can take a pair $(\overline{W}, \overline{D})$ as in Definition 2.4 such that*

$$\kappa_\sigma(\overline{W}, K_{\overline{W}} + \overline{D}) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D}) = \overline{\kappa}(V).$$

PROOF.   We take an embedding $V \subset \mathbb{A}^N$. Let $\overline{V}$ be the closure of V in $\mathbb{P}^N$. Then there is an effective ample Cartier divisor H on $\overline{V}$ such that $\operatorname{Supp} H = \overline{V} \setminus V$. We take a resolution $\mu : \overline{W} \to \overline{V}$ as in Definition 2.4. Then $\mu^* H$ is an effective Cartier divisor such that $\operatorname{Supp} \mu^* H = \operatorname{Supp} \overline{D}$. Let ε be a small positive rational number such that $\overline{D} - \varepsilon \mu^* H$ is effective. Since $\mu^* H$ is semi-ample, we can take an effective $\mathbb{Q}$-divisor B on $\overline{W}$ such that $B \sim_{\mathbb{Q}} \varepsilon \mu^* H$ and that $(\overline{W}, (\overline{D} - \varepsilon \mu^* H) + B)$ is klt. Note that

$$K_{\overline{W}} + \overline{D} \sim_{\mathbb{Q}} K_{\overline{W}} + (\overline{D} - \varepsilon \mu^* H) + B$$

and that $(\overline{D} - \varepsilon \mu^* H) + B$ is big. By [**BCHM**, Theorem 1.1, Corollary 1.3.3, Corollary 3.9.2], $(\overline{W}, \overline{D})$ has a good minimal model or a Mori fiber space structure. Hence, we obtain $\kappa_\sigma(\overline{W}, K_{\overline{W}} + \overline{D}) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D}) = \overline{\kappa}(V)$. More precisely, by running a minimal model program with ample scaling, we have a finite sequence of flips and divisorial contractions

$$(\overline{W}, \overline{D}) = (\overline{W}_0, \overline{D}_0) \dashrightarrow (\overline{W}_1, \overline{D}_1) \dashrightarrow \cdots \dashrightarrow (\overline{W}_k, \overline{D}_k)$$

such that $(\overline{W}_k, \overline{D}_k)$ is a good minimal model or has a Mori fiber space structure. Therefore, $\kappa(\overline{W}_k, K_{\overline{W}_k} + \overline{D}_k) = \kappa_\sigma(\overline{W}_k, K_{\overline{W}_k} + \overline{D}_k)$ holds. Note that in each step of the minimal model program κ and κ_σ are preserved. Thus, we obtain $\kappa_\sigma(\overline{W}, K_{\overline{W}} + \overline{D}) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$. □

REMARK 4.2 (Logarithmic canonical ring).   Let V be an affine variety and let $(\overline{W}, \overline{D})$ be a pair as in Definition 2.4. We put

$$\overline{R}(V) = \bigoplus_{m \geq 0} H^0(\overline{W}, \mathcal{O}_{\overline{W}}(m(K_{\overline{W}} + \overline{D})))$$

and call it the *logarithmic canonical ring* of V. It is well-known and is easy to see that $\overline{R}(V)$ is independent of the pair $(\overline{W}, \overline{D})$ and is well-defined. Then $\overline{R}(V)$ is a finitely generated $\mathbb{C}$-algebra. This is because we can choose $(\overline{W}, \overline{D})$ such that it has a good minimal model or a Mori fiber space structure as we saw in the proof of Proposition 4.1.

Note that Conjecture 1.4 follows from the minimal model program and the abundance conjecture.

PROPOSITION 4.3. *Let X be a smooth projective variety and let D_X be a simple normal crossing divisor on X. Assume that the minimal model program and the abundance conjecture hold for (X, D_X). Then we have*

$$\kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X).$$

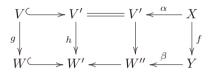*In particular, if* $\dim X \leq 3$, *then we have*

$$\kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X).$$

PROOF. We run the minimal model program. If $K_X + D_X$ is pseudo-effective, then (X, D_X) has a good minimal model. If $K_X + D_X$ is not pseudo-effective, then (X, D_X) has a Mori fiber space structure. Anyway, we obtain $\kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X)$ (see also the proof of Proposition 4.1). Note that in each step of the minimal model program κ and κ_σ are preserved. □

PROOF OF COROLLARY 1.5. This is obvious by Theorem 1.3 and Proposition 4.3. □

Let us prove Corollary 1.6.

PROOF OF COROLLARY 1.6. We take the following commutative diagram:

$$
\begin{array}{ccccccc}
V & \hookrightarrow & V' & =\!=\!= & V' & \xleftarrow{\alpha} & X \\
\downarrow{g} & & \downarrow{h} & & \downarrow & & \downarrow{f} \\
W & \hookrightarrow & W' & \longleftarrow & W'' & \xleftarrow{\beta} & Y
\end{array}
$$

such that $h : V' \to W'$ is a compactification of $g : V \to W$, $V' \to W'' \to W'$ is the Stein factorization of $V' \to W'$, α and β are suitable resolutions. We can take a simple normal crossing divisor D_X on X such that

$$\overline{\kappa}(V) = \kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X)$$

by Proposition 4.1. We have a simple normal crossing divisor D_Y on Y such that $\mathrm{Supp} f^* D_Y \subset \mathrm{Supp} D_X$ and

$$\overline{\kappa}(W) \leq \kappa(Y, K_Y + D_Y) \leq \kappa_\sigma(Y, K_Y + D_Y)$$

by Lemma 2.14. Then, by Theorem 1.3, we obtain

$$
\begin{aligned}
\overline{\kappa}(V) = \kappa(X, K_X + D_X) &= \kappa_\sigma(X, K_X + D_X) \\
&\geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y) \\
&\geq \kappa(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y) \\
&\geq \overline{\kappa}(F') + \overline{\kappa}(W)
\end{aligned}
$$

where F is a sufficiently general fiber of $f : X \to Y$. Note that

$$\overline{\kappa}(F') = \kappa(F, K_F + D_X|_F).$$

Therefore, we obtain the desired inequality of the logarithmic Kodaira dimension. $\square$

REMARK 4.4. If $g : V \dashrightarrow W$ is a strictly rational dominant map, then we can take a proper birational morphism $\mu : \widetilde{V} \to V$ such that $g \circ \mu : \widetilde{V} \to W$ is a morphism. By applying the proof of Corollary 1.6 to $g \circ \mu : \widetilde{V} \to W$, we have $\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$ as pointed out in Remark 1.7.

We close this paper with a remark on Conjecture 2.5 (see Conjecture 1.1).

REMARK 4.5. By the proof of Corollary 1.6, we see that Conjecture 2.5 (see Conjecture 1.1) follows from $\kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X)$. Moreover, the equality $\kappa(X, K_X + D_X) = \kappa_\sigma(X, K_X + D_X)$ follows from the minimal model program and the abundance conjecture for (X, D_X) by Proposition 4.3. Therefore, Conjecture 2.5 (see Conjecture 1.1) now becomes a consequence of the minimal model program and the abundance conjecture by Theorem 1.3. This fact strongly supports Conjecture 2.5 (see Conjecture 1.1 and Conjecture 1.2).

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