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Reducing subspaces of multiplication operators on weighted Hardy spaces over bidisk

By Shuhei Kuwahara

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Abstract. We consider weighted Hardy spaces over bidisk $\mathbb{D}^2$ which generalize the weighted Bergman spaces $A^2_\alpha(\mathbb{D}^2)$. Let z, w be coordinate functions and $M_{z^N w^N}$ the multiplication by $z^N w^N$ for a natural number N. In this paper, we study the reducing subspaces of $M_{z^N w^N}$. In particular, we obtain the minimal reducing subspaces of M_{zw}.

1. Introduction.

Let X be a closed subspace in a Hilbert space. Then, X is said to be invariant under an operator A if $AX \subset X$. Moreover, X is called a reducing subspace of an operator A if X is invariant under both A and its adjoint A^*. We consider the problem of determining the reducing subspaces of multiplication operators on weighted Hardy spaces $H^2_\omega(\mathbb{D}^2)$. Let (n_1, n_2) be a multi-index of non-negative integers. The weighted Hardy space $H^2_\omega(\mathbb{D}^2)$ over bidisk $\mathbb{D}^2$ with the weight ω consists of analytic functions

$$f(z,w) = \sum_{(n_1,n_2)} a(n_1,n_2) z^{n_1} w^{n_2}$$

such that

$$\|f\|^2 = \sum_{(n_1,n_2)} \omega(n_1,n_2)|a(n_1,n_2)|^2 < \infty,$$

where $\omega = \{\omega(n_1,n_2)\}$ is a set of positive numbers.

One of the examples of weighted Hardy spaces is the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$, where the weight is

$$\omega(n_1,n_2) = \frac{n_1!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n_1)} \cdot \frac{n_2!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n_2)}$$

for $\alpha > -1$.

For the definition of *transparent function*, see section 2. Let M_φ denote the multiplication operator defined by φ. Now we state our main result.

Theorem 1.1. *We fix a natural number N.*
(1) *The reducing subspaces of $M_{z^N w^N}$ on $H^2_\omega(\mathbb{D}^2)$ contain the minimal reducing subspace*

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X_p where p is a transparent function.
(2) X is a minimal reducing subspace of $M_{z^N w^N}$ if and only if there exists a transparent function p such that $X = X_p$.

In this paper, we will determine the reducing subspaces of the multiplication operator by $z^N w^N$ $(N \geq 1)$ on weighted Hardy spaces over bidisk by using the technique in [**9**] and [**12**]. We note that we solve the problem of determining the reducing subspaces of $M_{z^N w^N}$ when $N = 1$ as well as $N > 1$.

Zhu [**13**] determined the reducing subspaces of the multiplication by Blaschke products with two zeros on the Bergman space over the unit disk. Stessin and Zhu [**12**] determined the reducing subspaces of the multiplication by z^N in weighted Hardy spaces on the unit disk. Motivated by these results, many mathematicians have studied the reducing subspaces of the multiplication operators on weighted Hardy spaces over the unit disk and bidisk. We can find the results of Blaschke product in [**2**], [**3**] and [**6**]. In particular, the general case of finite Blaschke products has been done on the Bergman space in [**2**]. Guo and Huang [**4**], [**5**], [**6**], [**7**] studied a broad range of topics of the reducing subspace from various points of view. Dan and Huang [**1**] determined the reducing subspaces of the multiplication operators on the Bergman space $A^2(\mathbb{D}^2)$ defined by a class of polynomials which are in the form of $z^k + w^l$.

Let N_1, N_2 be natural numbers. In association with this paper, the author [**9**] studied the reducing subspaces of $M_{z^{N_1}}$ and $M_{w^{N_2}}$ on weighted Hardy spaces over bidisk. Shi and Lu [**11**] determined the reducing subspaces of $T_{z^{N_1} w^{N_2}} = PM_{z^{N_1} w^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$, where $N_1 \neq N_2$ and P is the projection onto $A^2_\alpha(\mathbb{D}^2)$. Lu and Zhou [**10**] determined the reducing subspaces of $T_{z^N w^N} = PM_{z^N w^N}$ on $A^2_\alpha(\mathbb{D}^2)$.

2. Preliminaries.

Throughout this paper, we fix a natural number N and the weight satisfying

$$\sup_{(n_1,n_2)} \frac{\omega(n_1+N, n_2+N)}{\omega(n_1,n_2)} < \infty$$

so that the multiplication operator $M_{z^N w^N}$ and its adjoint $M^*_{z^N w^N}$ are bounded.

Now we introduce some notions of multi-index in this paper. First we set the order on a set of multi-indices as follows; (1)$(k,m) > (l,n)$ (2)$(m,k) > (n,k)$ for all non-negative integers k,l,m,n with $m > n \geq 0$. Let I be a subset of multi-indices such that

$$I = \{(m_1,m_2); 0 \leq m_1 \leq N-1 \text{ or } 0 \leq m_2 \leq N-1\}.$$

If

$$\frac{\omega(m_1+kN, m_2+kN)}{\omega(m_1,m_2)} = \frac{\omega(n_1+kN, n_2+kN)}{\omega(n_1,n_2)}$$

for all positive integer k, then we say that (m_1,m_2) and (n_1,n_2) are equivalent. In this case, we write $(m_1,m_2) \sim (n_1,n_2)$. Using this relation, we define a class of functions. We say that a function in the form of

$$p(z,w)=\sum_{(n_1,n_2)\in I} a(n_1,n_2)z^{n_1}w^{n_2}$$

is a transparent function if we have $(n_1,n_2)\sim(n_1',n_2')$ for any two nonzero coefficient $a(n_1,n_2)$ and $a(n_1',n_2')$ of p. If we partition the set I into equivalence classes $\Omega_1,\Omega_2,\dots$, then $g_j(z,w)=\sum_{(n_1,n_2)\in\Omega_j} a(n_1,n_2)z^{n_1}w^{n_2}$ is transparent for all j.

Next we will show an example of the reducing subspaces which plays an important role. Let $\mathbf{S}$ be the algebra over $\mathbb{C}$ generated by the multiplication operator $M_{z^Nw^N}$ and its adjoint $M^*_{z^Nw^N}$. For any nonzero function $f\in H^2_\omega(\mathbb{D}^2)$, we put $\mathbf{S}f=\{Tf;T\in\mathbf{S}\}$. We set X_f the closure of $\mathbf{S}f$ in $H^2_\omega(\mathbb{D}^2)$.

Lemma 2.1. (1) *X_f is the smallest reducing subspace containing f.*
(2) *If $f(z,w)=\sum_{(n_1,n_2)\in I} a(n_1,n_2)z^{n_1}w^{n_2}$ is a transparent function, then*

$$X_f=\mathrm{Span}\{f(z,w)z^{kN}w^{kN};k=0,1,2,\dots\},$$

where we denote by Span Y *the closed linear span of a subset Y in $H^2_\omega(\mathbb{D}^2)$.*

Proof. (1) Obvious.
(2) Let $X=\mathrm{Span}\{f(z,w)z^{kN}w^{kN}:k=0,1,2,\dots\}$. Then $f\in X\subset X_f$. From (1), it is enough to show that X is a reducing subspace of $M_{z^Nw^N}$. It is easy to see that $M_{z^Nw^N}X\subset X$. We will compute $M^*_{z^Nw^N}\{f(z,w)z^{kN}w^{kN}\}$ to show that $M^*_{z^Nw^N}X\subset X$. Without loss of generality, we may assume $k>0$. Let $k=l+1$. We note that both $M_{z^Nw^N}$ and $M^*_{z^Nw^N}$ are bounded. Since f is a transparent function, by calculation we have

$$\begin{aligned}
M^*_{z^Nw^N}\{f(z,w)z^{kN}w^{kN}\} &= M^*_{z^Nw^N}M_{z^Nw^N}\{f(z,w)z^{lN}w^{lN}\}\\
&= M^*_{z^Nw^N}M_{z^Nw^N}\left\{\sum_{(n_1,n_2)\in I} a(n_1,n_2)z^{n_1+lN}w^{n_2+lN}\right\}\\
&= \sum_{(n_1,n_2)\in I} a(n_1,n_2)\frac{\omega(n_1+kN,n_2+kN)}{\omega(n_1+lN,n_2+lN)}z^{n_1+lN}w^{n_2+lN}\\
&= \frac{\omega(m_1+kN,m_2+kN)}{\omega(m_1+lN,m_2+lN)}\sum_{(n_1,n_2)\in I} a(n_1,n_2)z^{n_1+lN}w^{n_2+lN}\\
&= \frac{\omega(m_1+kN,m_2+kN)}{\omega(m_1+lN,m_2+lN)}f(z,w)z^{lN}w^{lN}\in X,
\end{aligned}$$

where (m_1,m_2) is the minimal multi-index of non-zero coefficient of f. □

For $f\in\mathrm{Hol}(\mathbb{D}^2)$, we denote $f^{(k_1,k_2)}(0,0)=(\partial^{k_1+k_2})/\partial z^{k_1}\partial w^{k_2}f(0,0)$. For any subspace X in $H^2_\omega(\mathbb{D}^2)$ with $X\neq\{0\}$, let (m_1,m_2) be the minimal multi-index such that there exists some $f\in X$ with $f^{(m_1,m_2)}(0,0)\neq0$ but $g^{(k_1,k_2)}(0,0)=0$ for all $g\in X$ and $(k_1,k_2)<(m_1,m_2)$. We will call (m_1,m_2) the order of X at the origin.

Proposition 2.2. *Let X be a nonzero reducing subspace of $M_{z^Nw^N}$ and (m_1,m_2)*

the order of X at the origin. Then the extremal problem

$$\sup\{\mathrm{Re} f^{(m_1,m_2)}(0,0); f \in X, \|f\| \le 1\}$$

has a unique solution G with $\|G\| = 1$ and $G^{(m_1,m_2)}(0,0) > 0$. Furthermore, G is in the form of

$$G(z,w) = \sum_{(n_1,n_2)\in I} b(n_1,n_2) z^{n_1} w^{n_2}.$$

Proof. Since the mapping $f \mapsto f^{(m_1,m_2)}(0,0)$ is a bounded linear functional on $H^2_\omega(\mathbb{D}^2)$, by Riesz representation theorem the extremal problem has a unique solution G with $\|G\| = 1$ and $G^{(m_1,m_2)}(0,0) > 0$.

Next we prove $M^*_{z^N w^N} G = 0$. We put $g_f = (G + M_{z^N w^N} f) \cdot \|G + M_{z^N w^N} f\|^{-1}$ for all $f \in X$. Since $\mathrm{Re} g_f^{(m_1,m_2)}(0,0) \le G^{(m_1,m_2)}(0,0)$ and

$$\mathrm{Re} g_f^{(m_1,m_2)}(0,0) = \mathrm{Re} G^{(m_1,m_2)}(0,0) \cdot \|G + M_{z^N w^N} f\|^{-1},$$

we obtain $\|G + M_{z^N w^N} f\| \ge 1$ for all $f \in X$. From this inequality, we see that $G \perp M_{z^N w^N} X$. Therefore $M^*_{z^N w^N} G = 0$ because $M^*_{z^N w^N} G \in X$. $\square$

We will call the function G in Proposition 2.2 the extremal function of X.

Proposition 2.3. *The extremal function of any reducing subspace of $M_{z^N w^N}$ in $H^2_\omega(\mathbb{D}^2)$ is a transparent function.*

Proof. Let X be a reducing subspace of $M_{z^N w^N}$ and (m_1, m_2) be the order of X at the origin. From Proposition 2.2, there exists a unique extremal function G. From the definition of (m_1, m_2), the function G contains the term $(G^{(m_1,m_2)}(0,0)/m_1!m_2!) z^{m_1} w^{m_2}$.

We recall that for $n = 1, 2, \ldots,$

$$g_n(z,w) = \sum_{(k_1,k_2)\in\Omega_n} \frac{G^{(k_1,k_2)}(0,0)}{k_1!k_2!} z^{k_1} w^{k_2}$$

is a transparent function and that $G(z,w) = \sum_n g_n(z,w)$, where Ω_n is the equivalence class as above. Without loss of generality, we may assume $g_1(z,w)$ contains the term $(G^{(m_1,m_2)}(0,0)/m_1!m_2!) z^{m_1} w^{m_2}$.

Let $(M_1^{(j)}, M_2^{(j)})$ be the minimal multi-index of p_j. It is clear that

$$(M^*_{z^N w^N})^k (M_{z^N w^N})^k g_n = \frac{\omega(M_1^{(n)} + kN, M_2^{(n)} + kN)}{\omega(M_1^{(n)}, M_2^{(n)})} g_n.$$

We note that $(M_1^{(j)}, M_2^{(j)})$ and (m_1, m_2) are not equivalent for $j = 2, 3, \ldots$. Therefore there exists an integer k depending on j such that

$$\frac{\omega(M_1^{(j)}+kN, M_2^{(j)}+kN)}{\omega(M_1^{(j)}, M_2^{(j)})} \neq \frac{\omega(m_1+kN, m_2+kN)}{\omega(m_1, m_2)}.$$

We compute

$$\begin{aligned}&\left(\frac{\omega(M_1^{(j)}+kN, M_2^{(j)}+kN)}{\omega(M_1^{(j)}, M_2^{(j)})} - (M^*_{z^Nw^N})^k(M_{z^Nw^N})^k\right)G(z,w)\\ &\quad = \sum_{n\neq j}\left(\frac{\omega(M_1^{(j)}+kN, M_2^{(j)}+kN)}{\omega(M_1^{(j)}, M_2^{(j)})} - \frac{\omega(M_1^{(n)}+kN, M_2^{(n)}+kN)}{\omega(M_1^{(n)}, M_2^{(n)})}\right)g_n \in X,\end{aligned}$$

where we note that the function g_1 does not vanish. By this calculation we can remove transparent functions other than g_1 one by one from G. Repeating this process and taking the limit in the norm of $H^2_\omega(\mathbb{D}^2)$, we can force a transparent function g_1 to be in X. The transparent function g_1 satisfies the conditions of the extremal problem in Proposition 2.2; $\|g_1\| \leq \|G\| = 1, g_1^{(m_1,m_2)}(0,0) = G^{(m_1,m_2)}(0,0)$ and $g_1 \in X$. The fact that G is extremal implies that G is equal to g_1 and is a transparent function. □

The reducing subspace X is called minimal if $\{0\}$ and X are the only reducing subspaces contained in X.

PROPOSITION 2.4. *If p is a transparent function, then the reducing subspace X_p is minimal.*

PROOF. Let X be a reducing subspace contained in X_p. Assume that $X \neq \{0\}$. From Proposition 2.2, we have the extremal function G_X on X which is a transparent function in the form of

$$G_X(z,w) = \sum_{(n_1,n_2)\in I} b(n_1,n_2)z^{n_1}w^{n_2}.$$

Since the transparent function G_X is in X_p, there is some function $f(z,w) \in \mathrm{Hol}(\mathbb{D}^2)$ such that $f(z,w) = \sum_{n\geq 0} c_n(zw)^{nN}$ and $pf = G_X$. We see that f is constant, forcing $p \in X$. This implies that $X_p = X$. Thus X_p is minimal, finishing the proof of Proposition 2.4. □

3. Main results.

THEOREM 3.1. *The reducing subspaces of $M_{z^Nw^N}$ on $H^2_\omega(\mathbb{D}^2)$ contain the minimal reducing subspace X_p where p is a transparent function in the form of*

$$p(z,w) = \sum_{(n_1,n_2)\in I} a(n_1,n_2)z^{n_1}w^{n_2}.$$

Moreover, X is a minimal reducing subspace of $M_{z^Nw^N}$ if and only if there exists a transparent function p such that $X = X_p$.

PROOF. Without loss of generality, we may assume X is a nonzero reducing subspace of $M_{z^N w^N}$. From Proposition 2.2, there exists a unique extremal function $p(z,w)$ in X. From Proposition 2.3, this extremal function is a transparent function. By Lemma 2.1, X_p is the smallest reducing subspace containing p and therefore $X_p \subset X$. In addition, if X is minimal, then it is clear that $X = X_p$. The converse is true from Proposition 2.4. □

Next we consider the case of the weighted Bergman spaces over bidisk. Corollary 3.2 is first obtained in [**10**, Theorem 1.1]. We note that this statement holds when $N = 1$.

COROLLARY 3.2. *The minimal reducing subspaces of $M_{z^N w^N}$ on the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ over bidisk are*

$$\mathrm{Span}\{(az^m w^n + bz^n w^m)(zw)^{kN}; k = 0, 1, 2, \ldots\}$$

for $(m,n), (n,m) \in I$ and complex numbers a, b.

PROOF. Let $\gamma_n = n!\Gamma(2+\alpha)/\Gamma(2+\alpha+n)$ for $\alpha > -1$. Recall that the weight of the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ over bidisk is

$$\omega(n_1, n_2) = \frac{n_1!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n_1)} \cdot \frac{n_2!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n_2)} = \gamma_{n_1}\gamma_{n_2}.$$

By calculations, for all non-negative integers m, n and k,

$$\omega(m+k, n+k) = \gamma_{m+k}\gamma_{n+k} = \omega(n+k, m+k).$$

In particular, we see that

$$\omega(m,n) = \gamma_m\gamma_n = \omega(n,m).$$

Clearly these equalities imply that $(m,n) \sim (n,m)$ for $(m,n), (n,m) \in I$.

On the other hand, we assume that distinct multi-indices (m_1, m_2) and (n_1, n_2) are equivalent; (m_1, m_2) and (n_1, n_2) satisfy

$$\frac{\gamma_{m_1+kN}\gamma_{m_2+kN}}{\gamma_{m_1}\gamma_{m_2}} = \frac{\gamma_{n_1+kN}\gamma_{n_2+kN}}{\gamma_{n_1}\gamma_{n_2}}$$

for all integers $k \geq 0$. Then we see that

$$\gamma_{m_1}\gamma_{m_2} = \gamma_{n_1}\gamma_{n_2}, \tag{1}$$

considering that the sequence $\{\gamma_n\}$ is strictly decreasing and

$$\lim_{k\to\infty} \frac{\gamma_{m_1+kN}\gamma_{m_2+kN}}{\gamma_{n_1+kN}\gamma_{n_2+kN}} = 1.$$

The same argument to prove the equality (1) can be found in the proof of [**11**, Theorem 3.2]. The equality (1) implies

$$\gamma_{m_1+kN}\gamma_{m_2+kN} = \gamma_{n_1+kN}\gamma_{n_2+kN}$$

for all integers $k \geq 0$. Moreover, tracing the proofs of [**11**, Lemma 2.3] and [**11**, Lemma 3.1] in similar ways, we will show that $m_1 = n_2$ and that $m_2 = n_1$ as below.

First we assume that $\alpha = 0$. Here we consider the case of the Bergman space. The equality $\gamma_{m_1+kN}\gamma_{m_2+kN} = \gamma_{n_1+kN}\gamma_{n_2+kN}$ for all $k \geq 0$ implies that

$$h(\lambda) = (m_1 + \lambda N + 1)(m_2 + \lambda N + 1) - (n_1 + \lambda N + 1)(n_2 + \lambda N + 1)$$

is analytic on $\mathbb{C}$ and hence $h(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. We see that $n_1 - m_1 = m_2 - n_2$. In addition, the equality $\gamma_{m_1}\gamma_{m_2} = \gamma_{n_1}\gamma_{n_2}$ implies that $(m_1+1)(m_2+1) = (n_1+1)(n_2+1)$. Since $(m_1, m_2) \neq (n_1, n_2)$, we obtain $m_1 = n_2$ and $m_2 = n_1$.

Next we consider the case when $\alpha \neq 0$. Without loss of generality, we may assume that $n_1 > m_1$ and $m_2 > n_2$. If $m_1 > n_1, m_2 > n_2$ or $m_1 < n_1, m_2 < n_2$, then $\gamma_{m_1+kN}\gamma_{m_2+kN} \neq \gamma_{n_1+kN}\gamma_{n_2+kN}$ since the sequence $\{\gamma_n\}$ is strictly decreasing. The equality $\gamma_{m_1+kN}\gamma_{m_2+kN} = \gamma_{n_1+kN}\gamma_{n_2+kN}$ for all $k \geq 0$ implies that

$$\begin{aligned} &\prod_{j=1}^{n_1-m_1} (\lambda N + j + m_1) \prod_{j=1}^{m_2-n_2} (\lambda N + 2 + \alpha + m_2 - j) \\ &\quad = \prod_{j=1}^{n_1-m_1} (\lambda N + 2 + \alpha + n_1 - j) \prod_{j=1}^{m_2-n_2} (\lambda N + j + n_2). \end{aligned} \tag{2}$$

Comparing the coefficient of $\lambda^{n_1-m_1+m_2-n_2-1}$, we get

$$\sum_{j=1}^{n_1-m_1} (j + m_1) + \sum_{j=1}^{m_2-n_2} (2 + \alpha + m_2 - j) = \sum_{j=1}^{n_1-m_1} (2 + \alpha + n_1 - j) + \sum_{j=1}^{m_2-n_2} (j + n_2).$$

This equality implies that $n_1 - m_1 = m_2 - n_2$.

Moreover we obtain $m_1 = n_2$ by substituting $\lambda = -(m_1+1)/N$ and $\lambda = -(n_2+1)/N$ for the equality (2) and using the same argument in the proof of [**11**, Lemma 3.1]. Combining with $n_1 - m_1 = m_2 - n_2$, we see that $m_2 = n_1$.

Therefore we conclude that two multi-indices (m_1, m_2) and (n_1, n_2) in I are equivalent if and only if $m_1 = n_2$ and $n_1 = m_2$. This fact implies that transparent functions are in the form of $az^m w^n + bz^n w^m$. Thus the statement holds from Theorem 3.1. □

Finally we assume $N = 1$. We can put $I = \{(n_1, n_2); n_1 = 0 \text{ or } n_2 = 0\}$. In this case, we express a transparent function $p(z, w)$ as follows;

$$p(z, w) = a_0 + \sum_{j=1}^{\infty} (a_j z^j + b_j w^j).$$

Corollary 3.3. *The minimal reducing subspaces of M_{zw} on the weighted Hardy space $H^2_\omega(\mathbb{D}^2)$ over bidisk are*

$$\mathrm{Span}\{p(z, w)(zw)^k; k = 0, 1, 2, \ldots\},$$

where $p(z, w)$ is a transparent function.

Proof. This is a special case of Theorem 3.1 when $N = 1$. □

Corollary 3.4. *The minimal reducing subspaces of M_{zw} on the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ over bidisk are*

$$\mathrm{Span}\{(az^n + bw^n)(zw)^k; k = 0, 1, 2, \dots\}$$

for a non-negative integer n and complex numbers a, b.

Proof. We can prove this statement by combining Corollary 3.2 and Corollary 3.3. □

For example, let $p(z, w) = z - w$. We easily see that $p(z, w)$ is a transparent function and X_p is a minimal reducing subspace of $M_{z^N w^N}$ on the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ over bidisk. This statement holds when $N = 1$.

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References

[1] H. Dan and H. Huang, Multiplication operators defined by a class of polynomials on $L^2_a(\mathbb{D}^2)$, Integr. Equ. Oper. Theory, **80** (2014), 581–601.

[2] R. Douglas, M. Putinar and K. Wang, Reducing subspaces for analytic multipliers of the Bergman space, J. Funct. Anal., **263** (2012), 1744–1765.

[3] R. Douglas, S. Sun and D. Zheng, Multiplication operators on the Bergman space via analytic continuation, Adv. Math., **226** (2011), 541–583.

[4] K. Guo and H. Huang, Multiplication operators defined by covering maps on the Bergman space: the connection between operator theory and von Neumann algebras, J. Funct. Anal., **260** (2011), 1219–1255.

[5] K. Guo and H. Huang, Reducing subspaces of multiplication operators on function spaces: Dedicated to the Memory of Chen Kien-Kwong on the Occasion of his 120th Birthday, Appl. Math. J. Chinese Univ., **28** (2013), 395–404.

[6] K. Guo and H. Huang, Geometric constructions of thin Blaschke products and reducing subspace problem, Proc. London Math. Soc., **109** (2014), 1050–1091.

[7] K. Guo and H. Huang, Multiplication operators on the Bergman space, Lecture Notes in Math., **2145** (2015).

[8] H. Hedenmalm, A factorization theorem for square area-integrable analytic functions, J. Reine Angew. Math., **422** (1991), 45–68.

[9] S. Kuwahara, Reducing subspaces of weighted Hardy spaces on polydisks, Nihonkai Math. J., **25** (2014), 77–83.

[10] Y. Lu and X. Zhou, Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk, J. Math. Soc. Japan, **62** (2010), 745–765.

[11] Y. Shi and Y. Lu, Reducing subspaces for Toeplitz operators on the polydisk, Bull. Korean Math. Soc., **50** (2013), 687–696.

[12] M. Stessin and K. Zhu, Reducing subspaces of weighted shift operators, Proc. Amer. Math. Soc., **130** (2002), 2631–2639.

[13] K. Zhu, Reducing subspaces for a class of multiplication operators, J. London Math. Soc., **62** (2000), 553–568.

Shuhei Kuwahara
Sapporo Seishu High School
Sapporo 064–0916, Japan
E-mail: s.kuwahara@sapporoseishu.ed.jp