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Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$

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Abstract. For any Wulff shape, its dual Wulff shape is naturally defined. A self-dual Wulff shape is a Wulff shape equaling its dual Wulff shape exactly. In this paper, it is shown that a Wulff shape is self-dual if and only if the spherical convex body induced by it is of constant width $\pi/2$.

1. Introduction.

For a positive integer n, let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Let $\mathbb{R}_+$ be the set consisting of positive real numbers. For any continuous function $\gamma : S^n \to \mathbb{R}_+$ and any $\theta \in S^n$, let $\Gamma_{\gamma,\theta}$ be the set consisting of $x \in \mathbb{R}^{n+1}$ such that $x \cdot \theta \le \gamma(\theta)$, where the dot in the center stands for the scalar product of two vectors $x, \theta \in \mathbb{R}^{n+1}$. Then, the *Wulff shape* associated with the support function γ is the following set $\mathcal{W}_\gamma$:

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$

A Wulff shape $\mathcal{W}_\gamma$ was firstly introduced by Wulff in [**7**] as a geometric model of a crystal at equilibrium. By definition, any Wulff shape is a convex body in $\mathbb{R}^{n+1}$ containing the origin as an interior point. Conversely, it has been known that for any convex body W in $\mathbb{R}^{n+1}$ such that $\mathrm{int}(W)$ contains the origin where $\mathrm{int}(W)$ stands for the set of interior points of W, there exists a continuous function $\gamma : S^n \to \mathbb{R}_+$ such that $W = \mathcal{W}_\gamma$ ([**6**]). By using the polar plot expression of elements of $\mathbb{R}^{n+1} - \{0\}$, $S^n \times \mathbb{R}_+$ may be naturally identified with $\mathbb{R}^{n+1} - \{0\}$. Under this identification, for any Wulff shape $\mathcal{W}_\gamma$ and any $\theta \in S^n$, the intersection $\partial\mathcal{W}_\gamma \cap L_\theta$ is exactly one point (denoted by $(\theta, w(\theta))$), where $\partial\mathcal{W}_\gamma$ is the boundary of $\mathcal{W}_\gamma$ and L_θ is the half line $L_\theta = \{(\theta, r) \mid r \in \mathbb{R}_+\}$. For a Wulff shape $\mathcal{W}_\gamma$, let $\overline{\gamma} : S^n \to \mathbb{R}_+$ be the continuous function defined by $\overline{\gamma}(\theta) = 1/(w(-\theta))$. Then, the Wulff shape $\mathcal{W}_{\overline{\gamma}}$ is called the *dual Wulff shape* of $\mathcal{W}_\gamma$ and is denoted by $\mathcal{DW}_\gamma$. For any Wulff shape $\mathcal{W}_\gamma$, there is a characterization of the dual Wulff shape of $\mathcal{W}_\gamma$. The graph of a continuous function $\gamma : S^n \to \mathbb{R}_+$ is denoted by $\mathrm{graph}(\gamma)$. Let inv: $\mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1}$ be the inversion of $\mathbb{R}^{n+1} - \{0\}$ defined by $\mathrm{inv}(\theta, r) = (-\theta, 1/r)$. Then, for any continuous function $\gamma : S^n \to \mathbb{R}_+$, $\mathcal{DW}_\gamma$ is exactly the convex hull of $\mathrm{inv}(\mathrm{graph}(\gamma))$. By this characterization, it is clear that $\mathcal{DDW}_\gamma$ is $\mathcal{W}_\gamma$ for any $\mathcal{W}_\gamma$ when

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inv(graph(γ)) is the boundary of the convex hull of inv(graph(γ)). A Wulff shape $\mathcal{W}_\gamma$ is said to be *self-dual* if the equality $\mathcal{W}_\gamma = \mathcal{D}\mathcal{W}_\gamma$ holds.

In this paper, a simple and useful characterization for a self-dual Wulff shape in $\mathbb{R}^{n+1}$ is given. In order to state our characterization, several notions in S^{n+1} are defined. For any point P of S^{n+1}, let $H(P)$ be the hemisphere centered at P, namely $H(P)$ is the subset of S^{n+1} consisting of $Q \in S^{n+1}$ satisfying $P \cdot Q \geq 0$, where the dot in the center stands for the scalar product of two vectors $P, Q \in \mathbb{R}^{n+2}$. A subset $\widetilde{W}$ of S^{n+1} is said to be *hemispherical* if there exists a point $P \in S^{n+1}$ such that $\widetilde{W} \cap H(P) = \emptyset$. A hemispherical subset $\widetilde{W} \subset S^{n+1}$ is said to be *spherical convex* if for any $P, Q \in \widetilde{W}$ the following arc PQ is contained in $\widetilde{W}$:

$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \;\middle|\; t \in [0,1] \right\}.$$

A hemispherical subset $\widetilde{W}$ is called a *spherical convex body* if it is closed, spherical convex and has an interior point. A hemisphere $H(P)$ is said to *support a spherical convex body* $\widetilde{W}$ if both $\widetilde{W} \subset H(P)$ and $\partial\widetilde{W} \cap \partial H(P) \neq \emptyset$ hold. For a spherical convex body $\widetilde{W}$ and a hemisphere $H(P)$ supporting $\widetilde{W}$, following [**2**], [**3**], the width of $\widetilde{W}$ determined by $H(P)$ is defined as follows. For any two $P, Q \in S^{n+1}$ $(P \neq \pm Q)$, the intersection $H(P) \cap H(Q)$ is called a *lune* of S^{n+1}. The *thickness of the lune* $H(P) \cap H(Q)$, denoted by $\triangle(H(P) \cap H(Q))$, is the real number $\pi - |PQ|$, where $|PQ|$ stands for the length of the arc PQ. For a spherical convex body $\widetilde{W}$ and a hemisphere $H(P)$ supporting $\widetilde{W}$, the *width of* $\widetilde{W}$ *determined by* $H(P)$, denoted by $\mathrm{width}_{H(P)}\widetilde{W}$, is the minimum of the following set:

$$\left\{ \triangle(H(P) \cap H(Q)) \;\middle|\; \widetilde{W} \subset H(P) \cap H(Q), H(Q) \text{ supports } \widetilde{W} \right\}.$$

For any $\rho \in \mathbb{R}_+$ less than π, a spherical convex body $\widetilde{W} \subset S^{n+1}$ is said to be *of constant width* ρ if $\mathrm{width}_{H(P)}\widetilde{W} = \rho$ for any $H(P)$ supporting $\widetilde{W}$.

Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$, $N \in S^{n+1}$ and $\alpha_N : S^{n+1} - H(-N) \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by $Id(x) = (x, 1)$, the point $(0, \dots, 0, 1) \in S^{n+1}$ and the central projection defined as follows respectively.

$$\alpha_N (P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}}, 1 \right)$$
$$(\forall\, (P_1, \dots, P_{n+1}, P_{n+2}) \in S^{n+1} - H(-N)).$$

Then, for any Wulff shape $\mathcal{W}_\gamma$, it is clear that $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is a spherical convex body. The set $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is called the *spherical convex body induced by* $\mathcal{W}_\gamma$.

THEOREM 1. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then, the Wulff shape W_γ is self-dual if and only if the spherical convex body induced by W_γ is of constant width $\pi/2$.*

The unit disc $D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ of $\mathbb{R}^{n+1}$ is clearly self-dual. Let R be

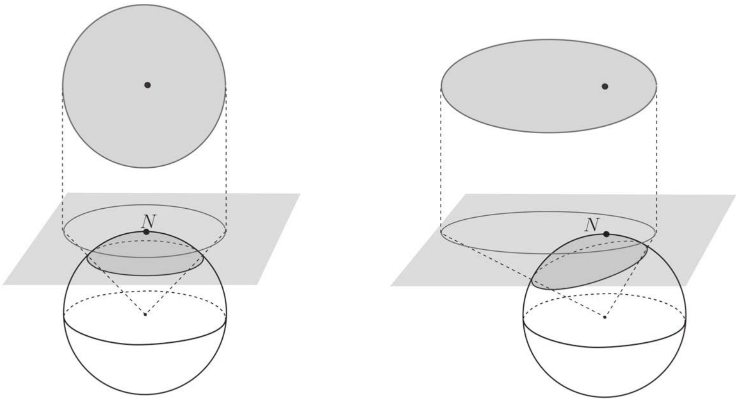

Figure 1. Self-dual Wulff shapes include central projections of spherical caps of width $\pi/2$.

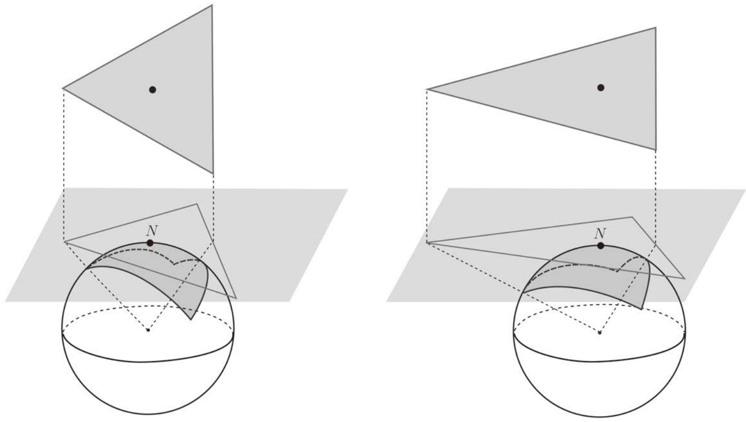

Figure 2. Self-dual Wulff shapes include triangles which are central projections of constant-width spherical triangles of width $\pi/2$.

a rotation of $\mathbb{R}^{n+2}$ about an n dimensional linear subspace with a small angle. Then, since the property of constant width is an invariant property by R, by Theorem 1, $Id^{-1} \circ \alpha_N \left(R\left(\alpha_N^{-1} \circ Id(D^{n+1})\right)\right)$ is self-dual as well (see Figure 1). Moreover, let $\widetilde{\triangle}$ be a spherical triangle of constant width $\pi/2$ in S^2 containing N as an interior point. Then, by Theorem 1, not only $Id^{-1} \circ \alpha_N(\widetilde{\triangle})$ itself, but also any $Id^{-1} \circ \alpha_N\left(R(\widetilde{\triangle})\right)$ is self-dual (see Figure 2). For more consideration on simple, explicit examples, see Section 4.

On the other hand, any Reuleaux triangle in $\mathbb{R}^2$ containing the origin as an interior point (see Figure 3) is not a self-dual Wulff shape, although it is a Wulff shape of constant width in $\mathbb{R}^2$. This is because any Reuleaux triangle is strictly convex, and thus the boundary of it must be smooth by [**1**] if it is self-dual. However, there are three non-smooth points for any Reuleaux triangle in $\mathbb{R}^2$. By Theorem 1, its spherical convex body is not of constant width $\pi/2$.

In Section 2, preliminaries for the proof of Theorem 1 are given. The proof of

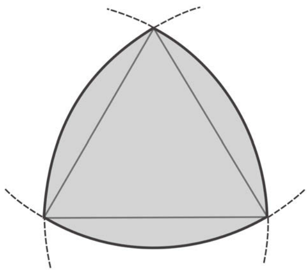

Figure 3. Reuleaux triangle.

Theorem 1 is given in Section 3. Finally, more consideration on simple, explicit examples are given.

2. Preliminaries.

The following two theorems given in [**2**] are keys for the proof of Theorem 1.

THEOREM 2 ([**2**]). *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body and let $H(P)$ be a hemisphere which supports $\widetilde{W}$.*

1. *If $P \notin \widetilde{W}$, then there exists a unique hemisphere $H(Q)$ supporting $\widetilde{W}$ such that the lune $H(P) \cap H(Q)$ contains $\widetilde{W}$ and has thickness* $\mathrm{width}_{H(P)}(\widetilde{W})$. *This hemisphere supports $\widetilde{W}$ at the point R at which the largest ball $B(P,r)$ touches $\widetilde{W}$ from outside. We have $\Delta(H(P) \cap H(Q)) = (\pi/2) - r$.*
2. *If $P \in \partial\widetilde{W}$, then there exists at least one hemisphere $H(Q)$ supporting $\widetilde{W}$ such that $H(P) \cap H(Q)$ is a lune containing $\widetilde{W}$ which has thickness* $\mathrm{width}_{H(P)}(\widetilde{W})$. *This hemisphere supports $\widetilde{W}$ at $R = P$. We have $\Delta(H(P) \cap H(Q)) = \pi/2$.*
3. *If $P \in \mathrm{int}(\widetilde{W})$, then there exists at least one hemisphere $H(Q)$ supporting $\widetilde{W}$ such that $H(P) \cap H(Q)$ is a lune containing $\widetilde{W}$ which has thickness* $\mathrm{width}_{H(P)}(\widetilde{W})$. *Every such $H(Q)$ supports $\widetilde{W}$ at exactly one point $R \in \partial\widetilde{W} \cap B(P,r)$, where $B(P,r)$ denotes the largest ball with center P contained in $\widetilde{W}$, and for every such R this hemisphere $H(Q)$, denoted by $H_R(Q)$, is unique. For every R we have $\Delta(H(P) \cap H_R(Q)) = (\pi/2) + r$.*

DEFINITION 1 ([**2**]). Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Then, the following number is called the *diameter* of $\widetilde{W}$ and is denoted by $\mathrm{diam}(\widetilde{W})$:

$$\max\left\{ |PQ| \;\middle|\; P, Q \in \widetilde{W} \right\}.$$

THEOREM 3 ([**2**]). *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body such that* $\mathrm{diam}\left(\widetilde{W}\right) \leq \pi/2$. *Then, the following holds*:

$$\mathrm{diam}(\widetilde{W}) = \max\left\{ \mathrm{width}_{H(P)}(\widetilde{W}) \;\middle|\; H(P) \text{ is a supporting hemisphere of } \widetilde{W} \right\}.$$

DEFINITION 2 ([**5**]). For any hemispherical subset $\widetilde{W}$ of S^{n+1}, the following set (denoted by s-conv$(\widetilde{W})$) is called the *spherical convex hull of* $\widetilde{W}$:

$$\text{s-conv}(\widetilde{W}) = \left\{ \frac{\sum_{i=1}^{k} t_i P_i}{\| \sum_{i=1}^{k} t_i P_i \|} \;\middle|\; P_i \in \widetilde{W},\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0, k \in \mathbb{N} \right\}.$$

It is clear that s-conv$(\widetilde{W}) = \widetilde{W}$ if $\widetilde{W}$ is spherical convex. More generally, we have the following:

LEMMA 2.1 ([**5**]). *Let $\widetilde{W}$ be a hemispherical subset of S^{n+1}. Then, the spherical convex hull of $\widetilde{W}$ is the smallest spherical convex set containing $\widetilde{W}$.*

DEFINITION 3 ([**5**]). For any subset $\widetilde{W}$ of S^{n+1}, the set

$$\bigcap_{P \in \widetilde{W}} H(P)$$

is called the *spherical polar set* of $\widetilde{W}$ and is denoted by $\widetilde{W}^{\circ}$.

For the spherical polar sets, the following lemma is fundamental.

LEMMA 2.2 ([**5**]). *For any non-empty closed hemispherical subset $\widetilde{W} \subset S^{n+1}$, the equality* $\text{s-conv}(\widetilde{W}) = \left(\text{s-conv}(\widetilde{W}) \right)^{\circ\circ}$ *holds.*

3. Proof of Theorem 1.

By the definition of the dual Wulff shape $\mathcal{DW}_{\gamma}$ for a given Wulff shape $\mathcal{W}_{\gamma}$, it is sufficient to show the following:

PROPOSITION 1. *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Then, $\widetilde{W} = \widetilde{W}^{\circ}$ if and only if $\widetilde{W}$ is of constant width $\pi/2$.*

3.1. Proof of the "if" part of Proposition1.

In this subsection, we show that $\widetilde{W} = \widetilde{W}^{\circ}$ under the assumption that $\widetilde{W}$ is of constant width $\pi/2$. We first show the inclusion $\widetilde{W} \subset \widetilde{W}^{\circ}$. Let P_1, Q_1 be two points of $\partial\widetilde{W}$ such that $|P_1Q_1| = \text{diam}(\widetilde{W})$. Set $P_1 = (r\theta, x_{n+2})$ $(0 < r, x_{n+2} < 1, \theta \in S^n)$. Since $\widetilde{W}$ is a spherical convex body, for the $\theta \in S^n$, there exists the unique real number t $(0 < t < 1)$ such that $H\left((t\theta + (1-t)N)/\|t\theta + (1-t)N\|\right)$ supports $\widetilde{W}$. For the t, set $P = (t\theta + (1-t)N)/\|t\theta + (1-t)N\|$. Then, since we have assumed that $\widetilde{W}$ is of constant width $\pi/2$, by Theorem 2, we have that $P \in \partial\widetilde{W}$. This implies $P_1 = P$ and hemisphere $H(P_1)$ supports $\widetilde{W}$. Since $Q_1 \in \widetilde{W} \subset H(P_1)$, we have the following,

$$\text{diam}(\widetilde{W}) = \mid P_1Q_1 \mid \leq \frac{\pi}{2}.$$

Let R be an arbitrary point of $\widetilde{W}$. Since $\text{diam}(\widetilde{W}) \leq \pi/2$, the following holds,

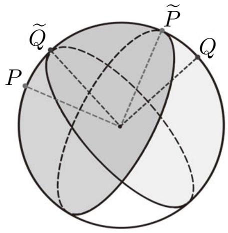

Figure 4. $\mid PQ \mid > \pi/2$.

$$R \in \bigcap_{\widetilde{R} \in \widetilde{W}} H(\widetilde{R}) = \widetilde{W}^\circ.$$

Therefore, we have $\widetilde{W} \subset \widetilde{W}^\circ$.

Next we show the converse inclusion $\widetilde{W}^\circ \subset \widetilde{W}$. Suppose that there exists a point $P \in \widetilde{W}^\circ$ such that $P \notin \widetilde{W}$. By Lemma 2.2, it follows that $P \notin \widetilde{W} = \bigcap_{Q \in \widetilde{W}^\circ} H(Q)$. This implies that there exist two points P and Q of $\widetilde{W}^\circ$ such that $|PQ| > \pi/2$. For these two points $P, Q \in \widetilde{W}^\circ$, set $\widetilde{P} = PQ \cap \partial H(P), \widetilde{Q} = PQ \cap \partial H(Q)$ (see Figure 4). Then we have the following,

$$\pi = \mid P\widetilde{P} \mid + \mid \widetilde{Q}Q \mid = \mid P\widetilde{Q} \mid + \mid \widetilde{Q}\widetilde{P} \mid + \mid \widetilde{Q}\widetilde{P} \mid + \mid \widetilde{P}Q \mid = \mid PQ \mid + \mid \widetilde{P}\widetilde{Q} \mid .$$

By the assumption, it follows that $\mid \widetilde{P}\widetilde{Q} \mid < \pi/2$. Let $H(\widetilde{R})$ be a supporting hemisphere of $\widetilde{W}$ whose boundary is perpendicular to the arc PQ at the intersecting point. Then, the following holds:

$$\mathrm{width}_{H(\widetilde{R})}(\widetilde{W}) \leq \mid \widetilde{P}\widetilde{Q} \mid < \frac{\pi}{2}.$$

This contradicts the assumption that $\widetilde{W}$ is of constant width $\pi/2$. Therefore, it follows that $\widetilde{W}^\circ \subset \widetilde{W}$. □

3.2. Proof of the "only if" part of Proposition 1.

In this subsection, we show that $\widetilde{W}$ is of constant width $\pi/2$ under the assumption that $\widetilde{W} = \widetilde{W}^\circ$. Suppose that there exists a hemisphere $H(P)$ supporting $\widetilde{W}$ such that $\mathrm{width}_{H(P)}(\widetilde{W}) > \pi/2$. By Theorem 3, it follows that $\mathrm{diam}(\widetilde{W}) \geq \mathrm{width}_{H(P)}(\widetilde{W}) > \pi/2$. This implies that there exist two points $P, Q \in \widetilde{W}$ such that $P \notin H(Q)$. Then, we have the following:

$$P \notin \bigcap_{Q \in \widetilde{W}} H(Q) = \widetilde{W}^\circ.$$

This contradicts the assumption $\widetilde{W} = \widetilde{W}^\circ$.

Suppose that there exists a hemisphere $H(P)$ supporting $\widetilde{W}$ such that the following holds:

$$\mathrm{width}_{H(P)}(\widetilde{W}) < \frac{\pi}{2}.$$

Then, there exists a hemisphere $H(Q)$ supporting $\widetilde{W}$ such that the following holds:

$$\Delta(H(P) \cap H(Q)) = \mathrm{width}_{H(P)}(\widetilde{W}) < \frac{\pi}{2}.$$

Since $\Delta(H(P) \cap H(Q)) = \pi- \mid PQ \mid$, we have the following:

$$\mid PQ \mid > \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

On the other hand, since $\widetilde{W}$ is a subset of $H(P)$ (resp. $H(Q)$), it follows that $P \in \widetilde{W}^\circ = \widetilde{W}$ (resp. $Q \in \widetilde{W}^\circ = \widetilde{W}$). This implies $\mathrm{diam}(\widetilde{W}) \geq \mid PQ \mid > \pi/2$. Thus, we have a contradiction. □

4. More on simple, explicit examples.

4.1. Centrally symmetric self-dual Wulff shapes.

In this subsection, we determine centrally symmetric Wulff shapes. Here, a convex body $W \subset \mathbb{R}^{n+1}$ is said to be *centrally symmetric* if $x \in W$ implies $-x \in W$.

PROPOSITION 2. *Let $W \subset \mathbb{R}^{n+1}$ be a self-dual Wulff shape. Then, W is centrally symmetric if and only if W is the unit disc D^{n+1}.*

PROOF. The "if"part is clear. We show the "only if"part. Suppose that there exists a centrally symmetric self-dual Wulff shape W which is not the unit disc D^{n+1}. Then one of the following holds.

(1) There exists a point $p \in W$ such that $\|p\| > 1$.

(2) The inequality $\|p\| \leq 1$ holds for any point p of W and there exists a point $q \in \partial W$ such that $\|q\| < 1$.

Here, $\|x\|$ is the distance from the origin to the point $x \in \mathbb{R}^{n+1}$.

Suppose that (1) holds. Then, since W is centrally symmetric, it follows that $-p \in W$. Set $\widetilde{p} = p/\|p\| \in S^n$. For any point $x \in \mathbb{R}^{n+1}$, set $X_+ = \alpha_N^{-1} \circ Id(x)$ and $X_- = \alpha_N^{-1} \circ Id(-x)$. Notice that $P_- \in \widetilde{W} = \alpha_N^{-1} \circ Id(W)$. Since the distance $|\widetilde{P}_+\widetilde{P}_-|$ is equal to $\pi/2$, we have the following:

$$\frac{\pi}{2} = |\widetilde{P}_+\widetilde{P}_-| < |P_+P_-|.$$

This implies $P_+ \notin H(P_-)$. Thus, it follows that

$$P_+ \notin \bigcap_{Q \in \widetilde{W}} H(Q) = \widetilde{W}^\circ.$$

On the other hands, since W is a self-dual Wulff shape and $p \in W$, we have that $P_+ \in \widetilde{W} = \widetilde{W}^\circ$. Therefore, we have a contradiction.

Next, suppose that (2) holds. Since there exists a point $q \in \partial W$ such that $\|q\| < 1$, it follows that the point $q/\|q\| \in S^n$ does not belong to W. Set $\widetilde{q} = q/\|q\|$. Then, since W is a self-dual Wulff shape, it follows that $\widetilde{Q}_+ \notin \widetilde{W} = \widetilde{W}^\circ$. On the other hands, by the assumption (2), the following holds.

$$\widetilde{W} \subset \alpha_N^{-1} \circ Id(D^{n+1}) \subset H(\widetilde{Q}_+).$$

Thus, $\widetilde{Q}_+$ is a point of $\widetilde{W}^\circ$ and we have a contradiction. □

4.2. Self-dual Wulff shapes of polytope type.

A Wulff shape is said to be *of polytope type* if there exist finitely many points $P_1, \ldots, P_k \in S^{n+1}$ such that $\widetilde{W} = \bigcap_{i=1}^k H(P_i)$, where $\widetilde{W}$ is the spherical convex body induced by W and $k \geq n+2 \in \mathbb{N}$. For crystallines, we have the following proposition:

LEMMA 4.1 (Maehara's Lemma [**4**], [**5**]). *For any hemispherical finite subset $X = \{P_1, \ldots, P_k\}$, the following holds*:

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \;\middle|\; P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ = \bigcap_{i=1}^k H(P_i).$$

PROPOSITION 3. *Let $W \subset \mathbb{R}^{n+1}$ be a Wulff shape of polytope type and let $\widetilde{W}$ be a spherical convex body induced by W. Set $\widetilde{W} = \bigcap_{i=1}^k H(P_i) \subset S^{n+1}$. Then, W is a self-dual Wulff shape if and only if P_i is a vertex of $\widetilde{W}$ for any i $(1 \leq i \leq k)$.*

PROOF.

PROOF OF THE "ONLY IF" PART. Let W be a self-dual Wulff shape of polytope type. Then, by Maehara's Lemma, we have the following equality:

$$\widetilde{W} = \bigcap_{i=1}^k H(P_i) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \;\middle|\; P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ.$$

Then, by Lemma 2.2, the following holds:

$$\widetilde{W}^\circ = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \;\middle|\; P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}.$$

Since W is a self-dual Wulff shape, it follows that $\widetilde{W} = \widetilde{W}^\circ$. Hence, P_i is a vertex of $\widetilde{W}$ for any i $(1 \leq i \leq 2m+1)$. □

PROOF OF THE "IF" PART. Since P_i is a vertex of $\widetilde{W}$, we have the following:

$$\widetilde{W} = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \;\middle|\; P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}.$$

Thus, by Maehara's Lemma, we have the following:

$$\widetilde{W}^{\circ} = \left\{ \frac{\sum_{i=1}^{k} t_i P_i}{\| \sum_{i=1}^{k} t_i P_i \|} \;\middle|\; P_i \in X, \sum_{i=1}^{k} t_i = 1, t_i \geq 0 \right\}^{\circ} = \bigcap_{i=1}^{k} H(P_i) = \widetilde{W}.$$

Therefore, W is a self-dual Wulff shape. □

4.3. When is the dual Wulff shape congruent to the original Wulff shape?

Finally, as a generalized problem of characterization of self-dual Wulff shapes, we pose the following:

Problem. Under what conditions is the dual Wulff shape merely congruent to the original Wulff shape ?

We have partial results to this problem as follows:

Example. Let X_{2m} be a regular polygon with $2m$ vertices in the plane where $m \geq 2$. Denote the half of the length of its diagonal by a_{2m}. Suppose that the center of X_{2m} is the origin and a_{2m} satisfies the following equation:

$$(*) \qquad \sin\left(\frac{\pi - (2\pi/2m)}{2}\right) = \frac{1/a_{2m}}{a_{2m}}.$$

Then, $X_{2m} \neq \mathcal{D}X_{2m}$ but $\mathcal{D}X_{2m}$ is congruent to X_{2m}.

For instance, consider a square $P_1P_2P_3P_4 \subset \mathbb{R}^2$ such that the origin is its center and the length of its edge is $2/a_4$, where $a_4^2 = \sqrt{2}$. Let $Q_1Q_2Q_3Q_4 \subset \mathbb{R}^2$ be the dual Wulff shape of $P_1P_2P_3P_4$. Then, $P_1P_2P_3P_4 \neq Q_1Q_2Q_3Q_4$ (see Figure 5). And, it is easy to see that $Q_1Q_2Q_3Q_4$ is also a square with properties that the origin is its center and the length of its edge is $2/a_4$. Thus, $Q_1Q_2Q_3Q_4$ is congruent to $P_1P_2P_3P_4$.

It is not difficult to obtain the equation $(*)$ for a_{2m} of general $2m$-gon X_{2m}.

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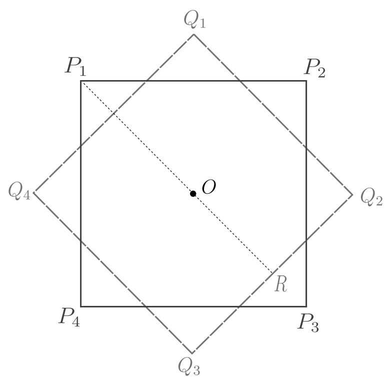

Figure 5. Square $P_1P_2P_3P_4$ and its dual square $Q_1Q_2Q_3Q_4$. $P_1O = a_4$, $RO = 1/a_4$, where $a_4^2 = \sqrt{2}$.

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