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Analytic continuation of multiple Hurwitz zeta functions

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Abstract. We obtain the analytic continuation of multiple Hurwitz zeta functions by using a simple and elementary translation formula. We also locate the polar hyperplanes for these functions and express the residues, along these hyperplanes, as coefficients of certain infinite matrices.

1. Introduction.

For any integer $r \geq 1$, let U_r denote the open subset of $\mathbb{C}^r$ consisting of all r-tuples $(s_1, \ldots, s_r)$ of complex numbers satisfying the conditions

$$\mathrm{Re}(s_1 + \cdots + s_i) > i \qquad \text{for } 1 \leq i \leq r.$$

For any real numbers $a_1, \ldots, a_r \in [0, 1)$, the multiple Hurwitz zeta function of depth r on U_r is defined by

$$\zeta(s_1, \ldots, s_r; a_1, \ldots, a_r) = \sum_{n_1 > \cdots > n_r > 0} (n_1 + a_1)^{-s_1} \cdots (n_r + a_r)^{-s_r}. \tag{1}$$

It is holomorphic on U_r as the series on the right hand side of (1) is normally convergent on any compact subset of U_r.

When $a_i = 0$ for all $1 \leq i \leq r$, the multiple Hurwitz zeta function of depth r equals to the multiple zeta function of depth r. Now it is natural to ask whether the multiple Hurwitz zeta function of depth r has analytic continuation to the whole $\mathbb{C}^r$. This question was first considered by Akiyama and Ishikawa in [**2**] where they proved, by generalizing an argument of Akiyama, Egami and Tanigawa [**1**], that the multiple Hurwitz zeta functions of depth r can be meromorphically continued to $\mathbb{C}^r$ by applying the Euler–Maclaurin summation formula to n_1, the first index of the summation, in (1). Murty and Sinha [**12**] prove the meromorphic continuation using the binomial theorem and a theorem of Hartogs, while Kelliher and Masri [**8**] obtain it using the Mellin transformation of meromorphic distributions. Analytic continuation of a more general class of multiple zeta functions, using Mellin–Barnes integrals, has been given by Matsumoto [**10**].

The meromorphic continuation of multiple zeta functions using the translation formula

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$$\zeta(s_1+s_2-1,\, s_3,\dots,s_r) = \sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!}\zeta(s_1+k, s_2,\dots,s_r) \tag{2}$$

is recently given by Mehta, Saha and Viswanadham [**11**]. It is inspired by the work of Ecalle [**4**], [**5**] and based on an idea of Ramanujan [**13**] in the case of Riemann zeta function. A similar method has been used by Essouabri [**6**] to give the analytic continuation of a family of zeta functions associated to Pascal's triangle mod p.

The formula (2) can be generalized to the multiple Hurwitz zeta functions as follows:

$$\begin{aligned}\zeta(s_1+s_2-1, s_3,\dots,s_r; a_2,\dots,a_r)\\ = \sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)\zeta(s_1+k, s_2,\dots,s_r; a_1,\dots,a_r),\end{aligned} \tag{3}$$

where $A_k(a_1,a_2)$ denotes $((1-a_2+a_1)^{k+1}-(a_1-a_2)^{k+1})$. We call (3), the translation formula for multiple Hurwitz zeta functions which is proved in Section 2. We obtain, by means of (3), the meromorphic continuation of multiple Hurwitz zeta functions which is the first goal of our paper. This so far seems to be the simplest proof of this fact, to the best of our knowledge.

While considering the meromorphic continuation, the natural and immediate task is to investigate the poles of multiple Hurwitz zeta functions and to compute residues at these poles. A possible list of polar hyperplanes for the multiple Hurwitz zeta functions was given by Akiyama and Ishikawa ([**2**, Theorem 1]), and they were able to give the exact list of these in some special cases without mentioning about the residues along these poles. Kelliher and Masri ([**8**, Theorem 1.1, 1.2 and 1.3]) gave a possible list of polar hyperplanes and gave formulas for the residues along these hyperplanes. We proceed with this paper, in Section 3, by expressing our translation formula as a product of certain infinite matrices. We conclude the paper by giving the complete list, in some special cases, of the location of polar hyperplanes for multiple Hurwitz zeta functions and express their residues (see Section 4) along them as coefficients of these matrices. The method we use here to obtain the meromorphic continuation of multiple Hurwitz zeta funtions, to locate the singularities and to compute residues at these singularities is same as in [**11**].

Remark 1. For $r \geq 1$ and $0 < a_i \leq 1$, $1 \leq i \leq r$, one can also define the multiple Hurwitz zeta function of depth r on U_r in the following way:

$$\zeta^{\mathrm{H}}(s_1,\dots,s_r; a_1,\dots,a_r) = \sum_{n_1>\cdots>n_r\geq 0} (n_1+a_1)^{-s_1}\cdots(n_r+a_r)^{-s_r}. \tag{4}$$

However, one has

$$\zeta^{\mathrm{H}}(s_1,\dots,s_r; a_1,\dots,a_r) = \zeta(s_1,\dots,s_r; a_1,\dots,a_r) + \frac{1}{{a_r}^{s_r}}\zeta(s_1,\dots,s_{r-1}; a_1,\dots,a_{r-1}).$$

Hence, the analytic continuation of $\zeta^{\mathrm{H}}(s_1,\dots,s_r; a_1,\dots,a_r)$ can be obtained from the analytic continuation of $\zeta(s_1,\dots,s_r; a_1,\dots,a_r)$.

2. Translation formula and meromorphic continuation.

In this section, we obtain the meromorphic continuation of the multiple Hurwitz zeta function. First we show that it satisfies the translation formula (3).

THEOREM 1. *For any $r \geq 2$, $a_1, \dots, a_r \in [0,1)$, and $s_1, \dots, s_r \in U_r$, the following formula holds for multiple Hurwitz zeta functions*:

$$\begin{aligned} &\zeta(s_1+s_2-1, s_3, \dots, s_r; a_2, \dots, a_r) \\ &\quad = \sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)\zeta(s_1+k, s_2, \dots, s_r; a_1, \dots, a_r), \end{aligned} \tag{3}$$

where $A_k(a_1,a_2)$ denotes $((1-a_2+a_1)^{k+1} - (a_1-a_2)^{k+1})$.

PROOF. The meromorphic continuation of multiple Hurwitz zeta functions can be obtained by using the following formula remaked by Mehta, Saha and Viswanadham ([**11**, Remark 4]). For any $n_1 \geq 2$, we have

$$\begin{aligned} &(n_1+a_2-1)^{1-s_1} - (n_1+a_2)^{1-s_1} \\ &\quad = \sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} (n_1+a_1)^{-s_1-k} A_k(a_1,a_2). \end{aligned}$$

Taking summation on both sides over n_1 from n_2+1 to ∞, we get

$$(n_2+a_2)^{1-s_1} = \sum_{n_1=n_2+1}^{\infty} \left(\sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} (n_1+a_1)^{-s_1-k} A_k(a_1,a_2) \right).$$

Multiplying both sides by $\prod_{i=2}^{r}(n_i+a_i)^{-s_i}$ and taking summation over n_r from 1 to ∞, over n_{r-1} from n_r+1 to ∞ and so on up to over n_2 from n_3+1 to ∞, we get

$$\begin{aligned} &\zeta(s_1+s_2-1, s_3, \dots, s_r; a_2, a_3, \dots, a_r) \\ &\quad = \sum_{n_1>\cdots>n_r>0} \sum_{k\geq 0} \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)(n_1+a_1)^{-s_1-k} \cdots (n_r+a_r)^{-s_r}. \end{aligned}$$

Hence it is enough to prove that the family of functions

$$\left(\frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)(n_1+a_1)^{-s_1-k} \cdots (n_r+a_r)^{-s_r} \right)_{n_1>\cdots>n_r>0,\ k\geq 0}$$

is normally summable on any compact subset of U_r.

Let C be any compact subset of U_r, and c be the maximum of $|s_1-1|$ in C. If we denote the supremum of a complex valued function f on C by $\|f\|_C$, then we have

$$\left\| \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)(n_1+a_1)^{-s_1-k} \cdots (n_r+a_r)^{-s_r} \right\|_C$$

is bounded above by

$$\frac{c(c+1)\cdots(c+k)}{2^k(k+1)!}|A_k(a_1,a_2)|\,\left\|(n_1+a_1)^{-s_1}\cdots(n_r+a_r)^{-s_r}\right\|_C.$$

We know that the family $(\|(n_1+a_1)^{-s_1}\cdots(n_r+a_r)^{-s_r}\|_C)$ is summable. Now the claim follows since $|A_k(a_1,a_2)|<b^{k+1}$ for some $b<2$ and the series

$$\sum_{k\geq 0}\frac{c(c+1)\cdots(c+k)}{(k+1)!}\left(\frac{b}{2}\right)^k$$

converges. This completes the proof the theorem. □

In the theorem below, we show that the multiple Hurwitz zeta function of depth r extends meromorphically to the whole $\mathbb{C}^r$.

THEOREM 2. *Let $r\geq 2$ be an integer. Then for any $a_1,\ldots,a_r\in[0,1)$, the multiple Hurwitz zeta function of depth r, $\zeta(s_1,\ldots,s_r;a_1,\ldots,a_r)$, is a meromorphic function on $\mathbb{C}^r$ and satisfies the translation formula*:

$$\begin{aligned}&\zeta(s_1+s_2-1,s_3,\ldots,s_r;a_2,\ldots,a_r)\\&\quad=\sum_{k\geq 0}\frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!}A_k(a_1,a_2)\zeta(s_1+k,s_2,\ldots,s_r;a_1,\ldots,a_r),\end{aligned}\tag{5}$$

where the series of meromorphic functions on the right hand side converges normally, except for finite terms, on any compact subset of $\mathbb{C}^r$.

PROOF. We prove this theorem by induction on the depth r. When $r=2$ the left hand side of (5) is a meromorphic function since it is the Hurwitz zeta function and we know that it is a meromorphic function on $\mathbb{C}$. For $r\geq 3$ it is a meromorphic function by the induction hypothesis since it has depth $r-1$.

For any $N\geq 0$, let $U_r(N)$ denote the open subset of $\mathbb{C}^r$ defined by

$$\mathrm{Re}(s_1+\cdots+s_i)>i-N \text{ for } 1\leq i\leq r.$$

We prove, by induction on N, that $\zeta(s_1,\ldots,s_r;a_1,\ldots,a_r)$ is meromorphic in $U_r(N)$ for each $N\geq 0$, which concludes the theorem.

When $N=0$ this is nothing but Theorem 1. Now assume that the multiple Hurwitz zeta function of depth r can be extended to $U_r(N-1)$ meromorphically. Hence all the terms on the right hand side of (5), except possibly the first one, are meromorphic in $U_r(N)$ and the terms with $k\geq N$ are holomorphic in $U_r(N)$. In order to show that the series

$$\sum_{k\geq N}\frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!}A_k(a_1,a_2)\zeta(s_1+k,s_2,\ldots,s_r;a_1,\ldots,a_r)\tag{6}$$

is a holomorphic function on $U_r(N)$ it suffices to show that it is normally convergent on any compact subset of $U_r(N)$. As a consequence $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ is a finite sum of meromorphic functions and hence it is a meromorphic function on $U_r(N)$.

As in the previous theorem, let C be a compact subset of $U_r(N)$ and c be the maximum of $|s_1-1|$ in $U_r(N)$. Then we have for any $k \geq N$

$$\left\| \frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!} A_k(a_1,a_2)(n_1+a_1)^{-s_1-k}\cdots(n_r+a_r)^{-s_r} \right\|_C$$

is bounded above by

$$\frac{c(c+1)\cdots(c+k)}{2^{k-N}(k+1)!}|A_k(a_1,a_2)|\|(n_1+a_1)^{-s_1-N}\cdots(n_r+a_r)^{-s_r}\|_C.$$

We have the family $(\|(n_1+a_1)^{-s_1-N}\cdots(n_r+a_r)^{-s_r}\|_C)_{n_1>\cdots>n_r}$ is summable, since $(s_1+N,s_2,\dots,s_r)\in U_r$, and the series $\sum_{k\geq N}\frac{c(c+1)\cdots(c+k)}{2^{k-N}(k+1)!}|A_k(a_1,a_2)|$ is convergent as $|A_k(a_1,a_2)|<b^{k+1}$ for some $b<2$. □

Remark 2. Let $e(z)$ denote $e^{2\pi i z}$. For any $a_1,\dots,a_r\in[0,1)$ and $\lambda_1,\dots,\lambda_r\in\mathbb{R}$, the multiple Lerch zeta function of depth r on U_r is defined by

$$\zeta(s_1,\dots,s_r;a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)=\sum_{n_1>\cdots>n_r>0}\frac{e(\lambda_1 n_1+\cdots+\lambda_r n_r)}{(n_1+a_1)^{s_1}\cdots(n_r+a_r)^{s_r}}$$

and it is a holomorphic function on U_r as the series on the right hand side above is normally convergent on any compact subset of U_r. The analytic continuation of these functions can be obtained by the following version of the translation formula.

$$\begin{aligned}&\zeta(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r;\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_r)\\&\quad=\sum_{k\geq 0}\frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!}A_k(\lambda_1,a_1,a_2)\zeta(s_1+k,s_2,\dots,s_r;a_1,\dots,a_r,\lambda_1,\dots,\lambda_r),\end{aligned} \tag{7}$$

where $A_k(\lambda_1,a_1,a_2)=e(-\lambda_1)(1-a_2+a_1)^{k+1}-(a_1-a_2)^{k+1}$. The formula (7) can be deduced from the following formula.

$$\begin{aligned}&e(\lambda_1(n_1-1))(n_1+a_2-1)^{1-s_1}-e(\lambda_1 n_1)(n_1+a_2)^{1-s_1}\\&\quad=\sum_{k\geq 0}\frac{(s_1-1)\cdots(s_1+k-1)}{(k+1)!}(n_1+a_1)^{-s_1-k}A_k(\lambda_1,a_1,a_2).\end{aligned} \tag{8}$$

Analytic continuation of a more general class of multiple Hurwitz–Lerch zeta functions using certain integrals has been given by Komori ([**9**, Theorem 3.14]) along with a description of the set of its possible singularities. The analytic coninuation and singularities of multiple Lerch zeta functions, using a variant of translation formula, is studied by Gun and Saha [**3**] and the results are included in the thesis work of Saha [**14**].

3. Matrix representation of the translation formula.

We can view the translation formula as the first one in an infinite family of relations, each obtained successively by applying the translation $s_1 \mapsto s_1+1$ to both sides of (5). By viewing the translation formula in this way, we can express this infinite family of relations in terms of infinite matrices. More precisely, we can write this family of relations as

$$Z(s_1+s_2-1, s_3, \ldots, s_r; a_2, \ldots, a_r) \; = \; M(s_1-1) Z(s_1, \ldots, s_r; a_1, \ldots, a_r), \tag{9}$$

where $Z(s_1, \ldots, s_r; a_1, \ldots, a_r)$ is the infinite column vector given by

$$Z(s_1, \ldots, s_r; a_1, \ldots, a_r) = \begin{pmatrix} \zeta(s_1, s_2 \ldots, s_r; a_1, a_2, \ldots, a_r) \\ \zeta(s_1+1, s_2 \ldots, s_r; a_1, a_2, \ldots, a_r) \\ \zeta(s_1+2, s_2 \ldots, s_r; a_1, a_2, \ldots, a_r) \\ \vdots \end{pmatrix}, \tag{10}$$

and

$$M(t) = \begin{pmatrix} tA_0 & \frac{t(t+1)}{2!} A_1 & \frac{t(t+1)(t+2)}{3!} A_2 & \cdots \\ 0 & (t+1)A_0 & \frac{(t+1)(t+2)}{2!} A_1 & \cdots \\ 0 & 0 & (t+2)A_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{11}$$

In the definition of matrix M we denoted $A_k(a_1, a_2)$ by A_k for simplicity. Observe that $M(t)$ can be written as a product of two matrices $D(t)$ and $G(t)$ as

$$M(t) = G(t)D(t) = D(t)G(t+1), \tag{12}$$

where

$$D(t) = \begin{pmatrix} t & 0 & 0 & \cdots \\ 0 & t+1 & 0 & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \; G(t) = \begin{pmatrix} A_0 & \frac{t}{2!} A_1 & \frac{t(t+1)}{3!} A_2 & \cdots \\ 0 & A_0 & \frac{(t+1)}{2!} A_1 & \cdots \\ 0 & 0 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

Let $B(t)$ denote the following matrix:

$$B(t) = \begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t+1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

Then we can write $G(t) = g(B(t))$, where $g(t)$ is the formal power series $(e^{(a_1-a_2)t}(e^t-1))/t$. Let $h(t)$ be the formal power series which is inverse of $g(t)$, i.e.

$$h(t) = \frac{te^{(a_2-a_1)t}}{e^t - 1} = \sum_{n\geq 0} B_n(a_2 - a_1)\frac{t^n}{n!}\ ,$$

where $B_n(x)$ denote the nth Bernoulli polynomial.

Since $G(t)$ and $D(t)$ are upper triangular matrices whose diagonal elements are invertible, we have $G(t)$ and $D(t)$ are invertible. If we denote the inverse of $M(t)$ by $N(t)$, then we have

$$N(t) = D(t)^{-1}G(t)^{-1} = G(t+1)^{-1}D(t)^{-1}. \tag{15}$$

Since $G(t) = g(B(t))$ and $h(t)$ is the inverse of $g(t)$, formula (15) can be equivalently written as

$$N(t) = D(t)^{-1}h(B(t)) = h(B(t+1))D(t)^{-1}. \tag{16}$$

Hence we have

$$N(t) \;=\; N(t; a_1, a_2) \;=\; \begin{pmatrix} \frac{1}{t} & \frac{B_1(a_2-a_1)}{1!} & \frac{(t+1)B_2(a_2-a_1)}{2!} & \frac{(t+1)(t+2)B_3(a_2-a_1)}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1(a_2-a_1)}{1!} & \frac{(t+2)B_2(a_2-a_1)}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1(a_2-a_1)}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{17}$$

From the above description one may be tempted to express the column vector $Z(s_1, \ldots, s_r; a_1, \ldots, a_r)$ as a product of $N(s_1 - 1; a_1, a_2)$ with the column vector $Z(s_1 + s_2 - 1, s_3 \ldots, s_r; a_2, \ldots, a_r)$. However, this does not seem to be possible as the entries of the formal product of the matrices $N(s_1-1; a_1, a_2)$ and $Z(s_1+s_2-1, s_3 \ldots, s_r; a_2, \ldots, a_r)$ are not convergent series. To overcome this hurdle, in view of (15), we express formula (9) as

$$D^{-1}(s_1 - 1)Z(s_1 + s_2 - 1, \ldots, s_r; a_2, \ldots, a_r) = G(s_1)Z(s_1, \ldots, s_r; a_1, \ldots, a_r). \tag{18}$$

We now choose an integer $K \geq 1$ and let $I = \{n \in \mathbb{N} : 0 \leq n \leq K-1\}$, and $J = \{n \in \mathbb{N} : n \geq K\}$, where $\mathbb{N}$ denotes the set of integers $n \geq 0$. With these notations we can write infinite matrices as block matrices, for example

$$G(s_1) = \begin{pmatrix} G_{II}(s_1) & G_{IJ}(s_1) \\ 0_{JI} & G_{JJ}(s_1) \end{pmatrix}.$$

From (18) we have

$$\begin{aligned} &D_{II}^{-1}(s_1 - 1)Z_I(s_1 + s_2 - 1, s_3, \ldots, s_r; a_2, a_3 \ldots, a_r) \\ &\quad = G_{II}(s_1)Z_I(s_1, \ldots, s_r; a_1, \ldots, a_r) + G_{IJ}(s_1)Z_J(s_1, \ldots, s_r; a_1, \ldots, a_r). \end{aligned} \tag{19}$$

Hence we deduce that

$$Z_I(s_1,\dots,s_r;a_1,\dots,a_r) = G_{II}^{-1}(s_1)D_{II}^{-1}(s_1-1)Z_I(s_1+s_2-1,s_3\dots,s_r;a_2,\dots,a_r) \\ - G_{II}^{-1}(s_1)G_{IJ}(s_1)Z_J(s_1,\dots,s_r;a_1,\dots,a_r). \quad (20)$$

The entries in the product of the matrices in the first term of the right hand side of (20) are meromorphic in $\mathbb{C}^r$ and the entries of the product

$$G_{II}^{-1}G_{IJ}(s_1)Z_J(s_1,\dots,s_r;a_1,\dots,a_r) \quad (21)$$

are holomorphic in $U_r(K)$. Comparing the first entry of the column vectors of both sides of (20) we get

$$\zeta(s_1,\dots,s_r;a_1,\dots,a_r) = \frac{1}{s_1-1}\zeta(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r) \\ + \sum_{k=0}^{K-2}\frac{s_1(s_1+1)\cdots(s_1+k-1)}{(k+1)!}B_{k+1}(a_2-a_1)\zeta(s_1+s_2+k,s_3,\dots,s_r;a_2,\dots,a_r) \\ + \xi_K(s_1,\dots,s_r;a_1,\dots,a_r), \quad (22)$$

where $\xi_K(s_1,\dots,s_r;a_1,\dots,a_r)$ is a holomorphic function in $U_r(K)$. This is what we get by applying the Euler–Maclaurin summation formula to n_1, the first index of summation, in (1) as obtained by Akiyama and Ishikawa (see [**2**, Section 3]), except here we have a more precise error term.

4. Poles and residues.

In this section, we obtain a list of possible polar hyperplanes of the multiple Hurwitz zeta function and express their residues, along these hyperplanes, as matrix coefficients. For any $j \geq 0$, let $\mathbb{Z}_{\leq j}$ denote the set of all integers n such that $n \leq j$.

The following theorem gives a list of possible singularities of multiple Hurwitz zeta function:

Theorem 3. *The multiple Hurwitz zeta function of depth r has at most a simple pole along the following hyperplanes*

$$s_1 = 1, \quad s_1+s_2+\cdots+s_i \in \mathbb{Z}_{\leq i} \quad \textit{for } i = 2,3,\dots,r, \quad (23)$$

while it is a holomorphic function on $\mathbb{C}^r$ outside the union of these hyperplanes.

Proof. We prove the theorem by induction on the depth r. Since the Hurwitz zeta function is a holomorphic function on $\mathbb{C}$ except at $s=1$, where it has a simple pole, the theorem holds when $r=1$.

Let $r \geq 2$ and assume that the theorem is true for multiple Hurwitz zeta functions of depth $\leq r-1$. Choose an integer $K \geq 1$. As earlier, let I be the set of all integers n such that $0 \leq n \leq K-1$, and J be the set of integers n such that $n \geq K$. Then by (20) we have

$$Z_I(s_1,\dots,s_r;a_1,\dots,a_r) = N_{II}(s_1-1;a_1,a_2)Z_I(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r)$$

$$- G_{II}^{-1}(s_1)G_{IJ}(s_1)Z_J(s_1,\dots,s_r;a_1,\dots,a_r).$$

The entries of the column vector $G_{II}^{-1}(s_1)G_{IJ}(s_1)Z_J(s_1,\dots,s_r;a_1,\dots,a_r)$ are holomorphic on $U_r(K)$. Hence the polar hyperplanes in $U_r(K)$ of the first entry of the column vector on the left hand side, which is $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$, are the union of polar hyperplanes of the first row of $N_{II}(s_1-1;a_1,a_2)$ and the entries of the column vector $Z_I(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r)$.

All the entries of the first row of the matrix $N_{II}(s_1-1;a_1,a_2)$ are holomorphic except the first entry where it has a simple pole at $s_1=1$. By the induction hypothesis the polar hyperplanes for the entries of the column vector $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ in $U_r(K)$ are

$$s_1+\cdots+s_i=n, \quad \text{where } -(K-1)+i\le n\le i \quad \text{and } 2\le i\le r. \tag{24}$$

Since $U_r(K)$ is an open covering of $\mathbb{C}^r$, the theorem follows. □

We define the residue along the hyperplane given by the equation $s_1+\cdots+s_i=i-k$ to be the restriction to this hyperplane of the meromorphic function $(s_1+\cdots+s_i-i+k)\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$. To verify if a hyperplane listed in the above theorem is indeed a polar hyperplane, we compute the residue of the multiple Hurwitz zeta function of depth r along that hyperplane.

Theorem 4. *The residue of the multiple Hurwitz zeta function along the hyperplane $s_1=1$ is the restriction of the function $\zeta(s_2,\dots,s_r;a_2,\dots,a_r)$ to the hyperplane $s_1=1$. The residue along the hyperplane $s_1+\cdots+s_i=i-k$, $2\le i\le r$ and $k\ge 0$, is the product of the $(0,k)^{th}$ entry of the matrix $\prod_{d=1}^{i-1}N(s_1+\cdots+s_d-d;a_d,a_{d+1})$ by the restriction of the function $\zeta(s_{i+1},\dots,s_r;a_{i+1},\dots,a_r)$ to the hyperplane $s_1+\cdots+s_i=i-k$.*

Proof. To prove the first assertion, consider the function

$$\zeta(s_1,\dots,s_r;a_1,\dots,a_r)-\frac{\zeta(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r)}{s_1-1}. \tag{25}$$

We can see by (20) or (22) that the function above has no pole along the hyperplane $s_1=1$. Hence the residue of the function $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ is restriction of the meromorphic function $\zeta(s_1+s_2-1,s_3,\dots,s_r;a_2,\dots,a_r)$ to $s_1=1$ which is same as $\zeta(s_2,\dots,s_r;a_2,\dots,a_r)$.

To prove the second assertion, let $s_1+\cdots+s_i=i-k$, $2\le i\le r$ and $k\ge 0$ be the given hyperplane. Let K be an integer such that $K>k$. Since the entries of the second term on the right hand side of (20) are holomorphic along the hyperplane $s_1+\cdots+s_i=i-k$, this term itself can be neglected. Hence by iterating the formula (20) $i-1$ times we get

$$Z_I(s_1,\dots,s_r;a_1,\dots,a_r)$$
$$=\prod_{d=1}^{i-1} N_{II}(s_1+\cdots+s_d-d;a_d,a_{d+1})Z_I(s_1+\cdots+s_i-i+1,s_{i+1},\dots,s_r;a_i,\dots,a_r)\,. \tag{26}$$

The entries of the matrix $\prod_{d=1}^{i-1} N_{II}(s_1+\cdots+s_d-d;a_d,a_{d+1})$ have no pole along the hyperplane $s_1+\cdots+s_i=i-k$ since they are rational functions in $s_1,\dots,s_{i-1}$. The first $k-1$ entries of the column vector are $\zeta(s_1+\cdots+s_i-i+1+j,s_{i+1},\dots,s_r;a_i,a_{i+1},\dots,a_r)$, $(0\le j\le k-1)$ and the entries with index $>k$ are $\zeta(s_1+\cdots+s_i-i+1+j,s_{i+1},\dots,s_r;a_i,a_{i+1},\dots,a_r)$, $(j>k)$. Hence these entries have no pole along the hyperplane $s_1+\cdots+s_i=i-k$ (by Theorem 3). Hence only the kth entry of $Z_I(s_1+\cdots+s_i-i+1)$ possibly has a pole along the hyperplane $s_1+\cdots+s_i=i-k$. Hence the residue of $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ along the hyperplane is the product of the $(0,k)^{th}$ entry of the matrix $\prod_{d=1}^{i-1} N_{II}(s_1+\cdots+s_d-d;a_d,a_{d+1})$ by the residue of the function $\zeta(s_1+\cdots+s_i-i+1+k,s_{i+1},\dots,s_r;a_i,\dots,a_r)$. By considering the function

$$\zeta(s_1+\cdots+s_i-i+1+k,s_{i+1},\dots,s_r;a_i,\dots,a_r)$$
$$-\frac{\zeta(s_1+\cdots+s_{i+1}-i+k,s_{i+2},\dots,s_r;a_{i+1},\dots,a_r)}{s_1+\cdots+s_i-i+k}$$

we can see that the residue of $\zeta(s_1+\cdots+s_i-i+1+k,s_{i+1},\dots,s_r;a_i,a_{i+1},\dots,a_r)$ along the hyperplane $s_1+\cdots+s_i=i-k$ is the restriction of the function $\zeta(s_{i+1},\dots,s_r;a_{i+1},\dots,a_r)$ to $s_1+\cdots+s_i=i-k$, from which the theorem follows. □

Now, using Theorem 4, we deduce the exact list of polar hyperplanes for multiple Hurwitz zeta functions of depth r in the case when $a_{i+1}-a_i$ $(1\le i\le r-1)$ are rationals. The lack of appropriate information about the real zeros of Bernoulli polynomials restricts us to this particular case. To obtain the exact list of polar hyperplanes, we rule out the hyperplanes listed in Theorem 3 for which the computed residue is 0.

Theorem 5. *Let $r\ge 2$, and assume that $a_{i+1}-a_i\in\mathbb{Q}\cap(-1,1)$ for $1\le i\le r-1$. The precise list of the polar hyperplanes for the multiple Hurwitz zeta function of depth r is the following*:

$$s_1=1;$$
$$s_1+s_2=2-k\ \ (k\ge 0)\quad \text{if and only if } B_k(a_2-a_1)\neq 0;$$
$$s_1+\cdots+s_i=i-k\ \ (i\ge 3 \text{ and } k\ge 0).$$

Proof. Since the residue of $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ along the hyperplane $s_1=1$ is $\zeta(s_2,\dots,s_r;a_2,\dots,a_r)$ which is a non-zero meromorphic function, the first assertion follows.

The residue of $\zeta(s_1,\dots,s_r;a_1,\dots,a_r)$ along the hyperplane $s_1+s_2=2-k$ is the product of $(0,k)^{th}$ entry of the matrix $N(s_1-1;a_1,a_2)$ by $\zeta(s_3,\dots,s_r;a_3,\dots,a_r)$. The matrix $N(s_1-1;a_1,a_2)$ has $(0,0)^{th}$ entry as $1/(s_1-1)$ and the $(0,k)^{th}$ entry as $(s_1\cdots(s_1+k-2)B_k(a_2-a_1))/(k)!$. By using this and the fact that

$\zeta(s_3, \ldots, s_r; a_3, \ldots, a_r)$ is a non-zero meromorphic function, the second assertion also holds.

Let $i = 3$. Since $\zeta(s_4, \ldots, s_r; a_4, \ldots, a_r)$ is a non-zero meromorphic function to prove the theorem in this case it suffices to prove that the $(0,k)^{th}$ entry of the matrix $N(s_1-1; a_1, a_2)N(s_1+s_2-2; a_2, a_3)$ is non-zero. Observe that, when k is even, $B_k(a_3-a_2)$ is non-zero, and when k is odd, either $B_1(a_2-a_1)B_{k-1}(a_3-a_2)$ or $B_{k-1}(a_2-a_1)B_1(a_3-a_2)$ is non-zero. This follows from the fact that even Bernoulli polynomials does not vanish at rational numbers, and both $a_2 - a_1$, $a_3 - a_2$ can not be equal to $1/2$. Thus in any case, one of the summand of the $(0,k)^{th}$ entry of $N(s_1 - 1; a_1, a_2)N(s_1 + s_2 - 2; a_2, a_3)$ is non-zero. Hence we are done in this case.

Let $i \geq 4$. Since $\zeta(s_{i+1}, \ldots, s_r; a_{i+1}, \ldots, a_r)$ is a non-zero meromorphic function to prove the last assertion it suffices to prove that the $(0,k)^{th}$ entry of the matrix $\prod_{d=1}^{i-1} N(s_1 + \cdots + s_d - d; a_d, a_{d+1})$ is non-zero. The entries of the first row of the matrix $\prod_{d=1}^{i-2} N(s_1+\cdots+s_d-d; a_d, a_{d+1})$ belong to $\mathbb{Q}(s_1-1, s_1+s_2-2, \ldots, s_1+\cdots+s_{i-2}-(i-2))$. We have that the first two entries of the matrix $\prod_{d=1}^{i-2} N(s_1 + \cdots + s_d - d; a_d, a_{d+1})$ is non-zero which can be observed by induction. The entries of the kth column of $N(s_1+\cdots+s_i-i; a_i, a_{i+1})$ belong to $\mathbb{Q}(s_1+\cdots+s_i-i)$. Those which are not equal to zero are linearly independent over $\mathbb{Q}(s_1-1, s_1+s_2-2, \ldots, s_1+\cdots+s_{i-2}-(i-2))$. Since one of $B_k(x)$ and $B_{k-1}(x)$ is non-zero we have at least one of the first two entries in this column is not zero. This implies that the $(0,k)^{th}$ entry of $\prod_{d=1}^{i-1} N(s_1 + \cdots + s_d - d; a_d, a_{d+1})$ is non-zero, which concludes the proof. $\square$

Remark 3. Since the residues of $\zeta(s_1, \ldots, s_r; a_1, \ldots, a_r)$ depend on the values of Bernoulli polynomials at $a_{i+1} - a_i$, $(1 \leq i \leq r-1)$, in general, the exact list of polar hyperplanes of $\zeta(s_1, \ldots, s_r; a_1, \ldots, a_r)$ cannot be given. However, once $a_1, \ldots, a_r$ are given, this seems easily possible by the method of Theorem 5.

Remark 4. Whether a given hyperplane is indeed a polar hyperplane or not depends on the zeros of Bernoulli polynomials. The later is studied by many authors, for example see Inkeri [**7**].

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