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Degenerations and fibrations of Riemann surfaces associated with regular polyhedra and soccer ball

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Abstract. To each of regular polyhedra and a soccer ball, we associate degenerating families (*degenerations*) of Riemann surfaces. More specifically: To each orientation-preserving automorphism of a regular polyhedron (and also of a soccer ball), we associate a degenerating family of Riemann surfaces whose topological monodromy is the automorphism. The complete classification of such degenerating families is given. Besides, we determine the Euler numbers of their total spaces. Furthermore, we affirmatively solve the compactification problem raised by Mutsuo Oka — we explicitly construct compact fibrations of Riemann surfaces that compactify the above degenerating families. Their singular fibers and Euler numbers are also determined.

1. Introduction.

Degenerating families (or simply, *degenerations*) of Riemann surfaces of genus 1 are classified by Kodaira [**7**], and those of genus 2 by Namikawa and Ueno [**8**]. These classifications, as for moduli spaces, are considered for *individual* genus. We introduce another class of degenerations from a view point of *symmetry*: To regular polyhedra and the soccer ball, we associate degenerations of Riemann surfaces. They are very symmetric and expected to carry interesting properties. Moreover they are naturally 'compactified' to fibrations over the projective line $\mathbb{P}^1$ (this solves M. Oka's compactification problem).

It is well-known that there are five kinds of regular polyhedra. We embed a regular polyhedron $\mathcal{P}$ in $\mathbb{R}^3$ in such a way that its barycenter is the origin of $\mathbb{R}^3$. The automorphism group $\mathrm{Aut}(\mathcal{P}) := \{f \in O(3) : f(\mathcal{P}) = \mathcal{P}\}$ is a finite subgroup of $O(3)$, and the orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{P}) := \{f \in SO(3) : f(\mathcal{P}) = \mathcal{P}\}$ is a finite subgroup of $SO(3)$. *In what follows, we consider* $\mathrm{Aut}_+(\mathcal{P})$ *rather than* $\mathrm{Aut}(\mathcal{P})$. The following holds (see [**12**, p. 84]) (Table 1):

We remark that the soccer ball is obtained by cutting off vicinities of vertices from the icosahedron, and the orientation-preserving automorphism group of the soccer ball coincides with that of the icosahedron — both are $\mathfrak{A}_5$. The soccer ball is a so-called *semi-regular polyhedron* ([**2**, p. 80]).

Conjugacy classes of $Aut_+(\mathcal{P})$.

The order of elements of a conjugacy class of a group are all equal. This common number is called the *order of the conjugacy class*. Table 2 describes the number m of the

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Key Words and Phrases. degenerating family of Riemann surfaces, fibration, monodromy, regular polyhedron, group action, automorphism group, cyclic quotient singularity.

Table 1. $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote the alternating group and the symmetric group of order n.

$\mathcal{P}$	$\mathrm{Aut}(\mathcal{P})$	$\mathrm{Aut}_+(\mathcal{P})$
Tetrahedron	$\mathfrak{S}_4$	$\mathfrak{A}_4$
Hexahedron	$\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$	$\mathfrak{S}_4$
Octahedron	$\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$	$\mathfrak{S}_4$
Dodecahedron	$\mathbb{Z}/2\mathbb{Z} \times \mathfrak{A}_5$	$\mathfrak{A}_5$
Icosahedron	$\mathbb{Z}/2\mathbb{Z} \times \mathfrak{A}_5$	$\mathfrak{A}_5$

conjugacy classes of order n ($n = 2, 3, \ldots$); the trivial conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{P})$ is omitted. See [**4**, p. 18, p. 29] for details.

Table 2

$\mathcal{P}$	n (m)
Tetrahedron	2 (one), 3 (two)
Hexahedron	2 (two), 3 (one), 4 (one)
Octahedron	2 (two), 3 (one), 4 (one)
Dodecahedron	2 (one), 3 (one), 5 (two)
Icosahedron	2 (one), 3 (one), 5 (two)
Soccer ball	2 (one), 3 (one), 5 (two)

Recall the definition of degeneration. A surjective proper holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to the unit disk $\Delta := \{t \in \mathbb{C} : |t| < 1\}$ is called a *degenerating family* (or simply, a *degeneration*) *of Riemann surfaces* (of genus g) if $\pi^{-1}(0)$ is singular and every $\pi^{-1}(s)$ ($s \neq 0$) is a Riemann surface (of genus g). For example, see Figure 1.

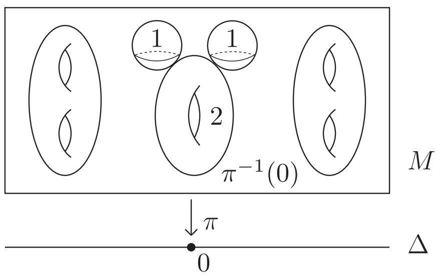

Figure 1. A degenerating family of Riemann surfaces of genus 2: The integer on each irreducible component of $\pi^{-1}(0)$ is the multiplicity of that component. Each intersection is transversal.

Main Result 1 (New Construction).

To each orientation-preserving automorphism of a regular polyhedron, we associate a degeneration of Riemann surfaces.

We introduce some terminology. The *cable surface* Σ of a regular polyhedron (and

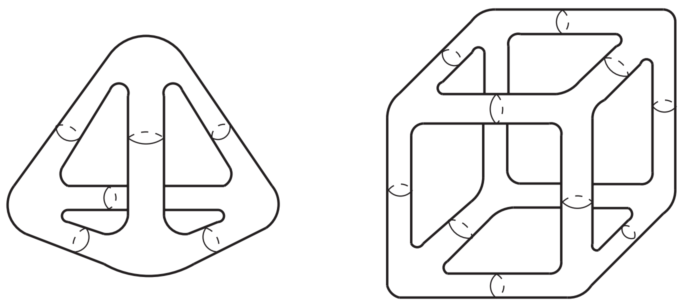

Figure 2

also of a soccer ball) $\mathcal{P}$ is a closed oriented surface obtained from $\mathcal{P}$ by 'thickening' its edges. If the number of faces of $\mathcal{P}$ is n, then the genus of Σ is $n-1$; for instance, it is 3 if $\mathcal{P}$ is the tetrahedron and 5 if $\mathcal{P}$ is the hexahedron:

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{P})$ of a regular polyhedron $\mathcal{P}$ naturally acts on its cable surface Σ. By Kerckhoff's theorem [**6**], there exists a complex structure on Σ such that $\mathrm{Aut}_+(\mathcal{P})$ acts holomorphically. We may thus regard Σ as a Riemann surface on which $\mathrm{Aut}_+(\mathcal{P})$ acts holomorphically. (Alternatively: Let Σ be the boundary of an ε-neighborhood of $\mathcal{P}$ in $\mathbb{R}^3$ with respect to the Euclidean metric of $\mathbb{R}^3$. Then $\mathrm{Aut}_+(\mathcal{P})$ acts on it *isometrically*, and by the uniformization theorem, there exists a complex structure on it, on which $\mathrm{Aut}_+(\mathcal{P})$ acts holomorphically.)

Note that since the order of $\mathrm{Aut}_+(\mathcal{P})$ is finite, the order of any element $f \in \mathrm{Aut}_+(\mathcal{P})$ is necessarily finite. Say that the order of f is m. Then let $g : \Sigma\times\Delta \to \Sigma\times\Delta$ be a periodic automorphism defined by $(x,t) \mapsto (f^{-1}(x), e^{2\pi \mathrm{i}/m}t)$ and G be the cyclic group generated by g. Let $\phi : \Sigma \times \Delta \to \Delta$ be a G-invariant holomorphic function defined by $(x,t) \mapsto t^m$. It determines a holomorphic map $\overline{\phi} : (\Sigma \times \Delta)/G \to \Delta$. Where $\mathfrak{r} : M \to (\Sigma \times \Delta)/G$ is the resolution that minimally resolves each of the (cyclic quotient) singularities of $(\Sigma\times\Delta)/G$, the composition $\pi := \overline{\phi}\circ\mathfrak{r} : M \to \Delta$ is a *degeneration of Riemann surfaces associated with* f. Any smooth fiber $\pi^{-1}(s)$ $(s \neq 0)$ is Σ, and the topological monodromy of $\pi : M \to \Delta$ is $f : \Sigma \to \Sigma$ (see [**10**]). The singular fiber $X := \pi^{-1}(0)$ is *star-shaped*, that is, *branches* emanate from the *core* — the irreducible component of multiplicity m (for example, see Figure 3). Here:

- A branch is a chain of complex projective lines (that is the exceptional set of the minimal resolution of a cyclic quotient singularity).
- Intersections of irreducible components are transversal.

Strong holomorphic-equivalence.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{P})$ of $\mathcal{P}$ naturally acts on the cable surface Σ, so each element $f \in \mathrm{Aut}_+(\mathcal{P})$ may be regarded as the orientation-preserving automorphism of Σ. Let $\pi_f : M_f \to \Delta$ denote the degeneration of Riemann surfaces associated with f. Then the topological monodromy of $\pi_f : M_f \to \Delta$ is $f : \Sigma \to \Sigma$.

Proposition 2.2 states that if two elements $f, f' \in \mathrm{Aut}_+(\mathcal{P})$ are conjugate, then two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are holomorphically equivalent (see

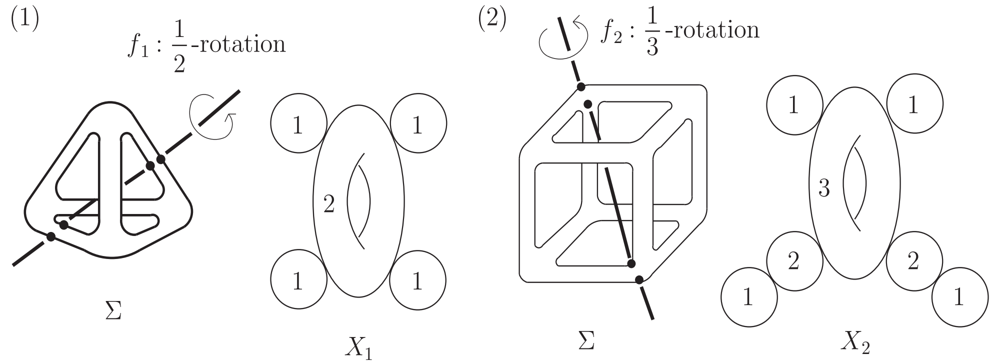

Figure 3. (1) $\mathcal{P}$ is the tetrahedron, $f_1 : \Sigma \to \Sigma$ is a 1/2-rotation and X_1 is the singular fiber. (2) $\mathcal{P}$ is the hexahedron, $f_2 : \Sigma \to \Sigma$ is a 1/3-rotation and X_2 is the singular fiber. (Each circle denotes a projective line and the integer on each irreducible component is the multiplicity of that component.)

the paragraph containing (1)). In this case, they are said to be *strongly* holomorphically equivalent. Note that

$$\begin{array}{ccc} \text{conjugacy class} & \overset{\text{1-1}}{\longleftrightarrow} & \text{strong holomorphic equivalence class} \\ [f] & & [\pi_f : M_f \to \Delta]. \end{array}$$

Clearly if two degenerations are strongly holomorphically equivalent, they are topologically equivalent. The converse is however not true, in fact:

Main Result 2.

In Section 3.4*, we give an example of two degenerations that are topologically equivalent but not strongly holomorphically equivalent — they are associated with automorphisms of order* 2 *of the octahedron.*

The results in Section 3 are summarized as follows (below, the five kinds of regular polyhedra are denoted by their initials $\mathcal{T}$, $\mathcal{H}$, $\mathcal{O}$, $\mathcal{D}$, $\mathcal{I}$ and the soccer ball by $\mathcal{S}$):

Main Result 3 (Classification Theorem).

(A) *The number of strong holomorphic equivalence classes of degenerations associated with the regular polyhedra and the soccer ball is twenty three.* (Table 3 describes their singular fibers and the Euler numbers of their total spaces.)

(B) *The two degenerations associated with each of the following are topologically equivalent* (see $*_1, *_2, *_3, *_4$ in Table 3 for their singular fibers for each case):

(1) *the two conjugacy classes of order* 3 *in* $\mathrm{Aut}_+(\mathcal{T})$*,*

(2) *the two conjugacy classes of order* 2 *in* $\mathrm{Aut}_+(\mathcal{O})$*,*

(3) *the two conjugacy classes of order* 5 *in* $\mathrm{Aut}_+(\mathcal{D})$*,*

(4) *the two conjugacy classes of order* 5 *in* $\mathrm{Aut}_+(\mathcal{S})$*.*

In Table 3, the singular fibers [T.1], [T.2], [T.3] appear in Section 3.1, [H.1], [H.2], [H.3], [H.4] in Section 3.2, [O.1], [O.2], [O.3], [O.4] in Section 3.3, [D.1], [D.2], [D.3], [D.4] in Section 3.4, [I.1], [I.2], [I.3], [I.4] in Section 3.5, and [S.1], [S.2], [S.3], [S.4] in Section 3.6.

Table 3. For a representative f of a conjugacy class, $X_f := \pi_f^{-1}(0)$ denotes the singular fiber of $\pi_f : M_f \to \Delta$.

$\mathcal{P}$	ord(f)	X_f	$\chi(M_f)$
Tetrahedron	2	[T.1]	4
	3	$*_1$ [T.2], [T.3]	3
Hexahedron	2	[H.1]	2
	2	[H.2]	-4
	3	[H.3]	6
	4	[H.4]	-2
Octahedron	2	$*_2$ [O.1], [O.2]	0
	3	[O.3]	-4
	4	[O.4]	8
Dodecahedron	2	[D.1]	-4
	3	[D.2]	2
	5	$*_3$ [D.3], [D.4]	-4
Icosahedron	2	[I.1]	-12
	3	[I.2]	-12
	5	[I.3]	6
	5	[I.4]	4
Soccer ball	2	[S.1]	-24
	3	[S.2]	-20
	5	$*_4$ [S.3], [S.4]	-12

Concerning the above result, Mutsuo Oka raised a question: *Is there natural way to compactify the above degenerations?* Or: *Are there fibrations over $\mathbb{P}^1$ whose singular fibers are appearing above?* We construct such fibrations. Each of them is associated with a regular polyhedral group $\mathrm{Aut}_+(\mathcal{P})$ *itself* rather than its conjugacy classes, and is referred to as the *cable fibration* associated with $\mathcal{P}$. We determine the singular fibers of *all* cable fibrations as well as the Euler numbers of their total spaces. In fact, in Section 4 the following is shown:

Main Result 4 (Classification of cable fibrations).

Let $\pi : W_{\mathcal{P}} \to \mathbb{P}^1$ be the cable fibration associated with $\mathcal{P}$. The singular fibers of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1$ and the local monodromies around them and the Euler number $\chi(W_{\mathcal{P}})$ are as follows:

On further development of our work, we give some comments:

(i) The total space of any cable fibration actually admits another fibering. We show that after blowing-down, it becomes a ruled surface.

(ii) Degenerations and cable fibrations of Riemann surfaces associated with *all* semi-regular polyhedra will be described.

(iii) The second author [**11**] developed the theory of *splitting deformation*. It is expected that splitting singular fibers of a cable fibration yields a Lefshetz fibration, which

Table 4

$\mathcal{P}$	Singular fibers	Local monodromies	$\chi(W_{\mathcal{P}})$
Tetrahedron	[T.1], [T.2], [T.3]	f_1, f_2, $f_3 = (f_2)^2$ in Section 3.1	14
Hexahedron	[H.1], [H.3], [H.4]	f_1, f_3, f_4 in Section 3.2	14
Octahedron	[O.1], [O.3], [O.4]	f_1, f_3, f_4 in Section 3.3	16
Dodecahedron	[D.1], [D.2], [D.3]	f_1, f_2, f_3 in Section 3.4	14
Icosahedron	[I.1], [I.2], [I.3]	f_1, f_2, f_3 in Section 3.5	18
Soccer ball	[S.1], [S.2], [S.3]	f_1, f_2, f_3 in Section 3.6	4

may have interesting properties (inherited from regular polyhedra). This will be investigated.

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2. Preparation.

2.1. Equivalences of degenerations of Riemann surfaces.

A surjective proper holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to $\Delta := \{t \in \mathbb{C} : |t| < 1\}$ is a *degenerating family* (or simply, a *degeneration*) *of Riemann surfaces* (of genus g) if $\pi^{-1}(0)$ is singular and every $\pi^{-1}(s)$ ($s \neq 0$) is a Riemann surface (of genus g). Two degenerations of Riemann surfaces $\pi : M \to \Delta$ and $\pi' : M' \to \Delta$ are *topologically equivalent* if there exist homeomorphisms $\Psi : M \to M'$ and $\psi : \Delta \to \Delta$ that make the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\ \Psi\ } & M' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \Delta & \xrightarrow{\ \psi\ } & \Delta. \end{array} \tag{1}$$

If Ψ and ψ are biholomorphic maps, $\pi : M \to \Delta$ and $\pi' : M' \to \Delta$ are said to be *holomorphically equivalent*.

Now given a periodic automorphism of a Riemann surface, we may construct a degeneration of Riemann surfaces whose topological monodromy is the automorphism (here an automorphism means a *biholomorphic* map).

Let Σ be a Riemann surface and $f : \Sigma \to \Sigma$ be a periodic automorphism of order m. Consider then a periodic automorphism $g : \Sigma \times \Delta \to \Sigma \times \Delta$ (of order m) defined by $g(x, t) := (f^{-1}(x), e^{2\pi \mathrm{i}/m} t)$, and let G denote the cyclic group generated by g. Define then a holomorphic function $\phi : \Sigma \times \Delta \to \Delta$ by $(x, t) \mapsto t^m$. Then ϕ is G-invariant, so determines a holomorphic map $\overline{\phi} : (\Sigma \times \Delta)/G \to \Delta$. Here the quotient space $(\Sigma \times \Delta)/G$ is a complex surface with cyclic quotient singularities. Let $\mathfrak{r} : M \to (\Sigma \times \Delta)/G$ be the resolution that minimally resolves each of the (cyclic quotient) singularities of $(\Sigma \times \Delta)/G$. The composite map $\pi := \overline{\phi} \circ \mathfrak{r} : M \to \Delta$ is then a degeneration of Riemann surfaces *associated with* f (the topological monodromy of $\pi : M \to \Delta$ is f). This construction of a degeneration is called the *cyclic quotient construction*.

We show that if two periodic automorphisms f, f' of Σ are conjugate in the group of automorphisms of Σ, then the degenerations obtained from them by the cyclic quotient construction are holomorphically equivalent. To that end, we need some preparation. As above, let g and g' be the periodic automorphisms of $\Sigma \times \Delta$ given by $g(x,t) := (f^{-1}(x), e^{2\pi \mathrm{i}/m}t)$ and $g'(x,t) := (f'^{-1}(x), e^{2\pi \mathrm{i}/m}t)$, and G and G' denote the cyclic groups generated by g and g'. Let $\mathfrak{r} : M \to (\Sigma \times \Delta)/G$ and $\mathfrak{r}' : M' \to (\Sigma \times \Delta)/G'$ be the resolutions that minimally resolve each singularity of $(\Sigma \times \Delta)/G$ and $(\Sigma \times \Delta)/G'$. Then the following holds:

LEMMA 2.1. *If there exists a biholomorphic map* $\psi : (\Sigma \times \Delta)/G \to (\Sigma \times \Delta)/G'$, *then there exists a biholomorphic map* $\Psi : M \to M'$ *that makes the following diagram commute*:

$$\begin{array}{ccc} M & \overset{\Psi}{\dashrightarrow} & M' \\ {\scriptstyle \mathfrak{r}}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}'} \\ (\Sigma \times \Delta)/G & \xrightarrow{\ \psi\ } & (\Sigma \times \Delta)/G'. \end{array}$$

PROOF. $\psi \circ \mathfrak{r} : M \to (\Sigma \times \Delta)/G'$ and $\mathfrak{r}' : M' \to (\Sigma \times \Delta)/G'$ are resolutions of $(\Sigma \times \Delta)/G'$ that minimally resolve each singularity of $(\Sigma \times \Delta)/G'$. Such resolutions are unique up to isomorphism, so there exists a biholomorphic map $\Psi : M \to M'$ such that $\psi \circ \mathfrak{r} = \mathfrak{r}' \circ \Psi$. □

Now $\mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ denotes the group of automorphisms of Σ. If $\mathrm{genus}(\Sigma) \geq 2$, this group is *finite*. In fact, $\#\mathrm{Aut}_{\mathrm{hol}}(\Sigma) \leq 84(\mathrm{genus}(\Sigma) - 1)$ by Hurwitz's theorem. In particular, any automorphism of Σ is of *finite* order, so periodic. If $f, f' \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ are conjugate in $\mathrm{Aut}_{\mathrm{hol}}(\Sigma)$, then their orders are equal. Moreover:

PROPOSITION 2.2. *Two degenerations* $\pi : M \to \Delta$ *and* $\pi' : M' \to \Delta$ (*constructed from* f *and* f' *by the cyclic quotient construction*) *are holomorphically equivalent, in fact there exists a biholomorphic map* $\Psi : M \to M'$ *that makes the following diagram commute*:

$$\begin{array}{ccc} M & \xrightarrow{\ \Psi\ } & M' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \Delta & \xrightarrow{\ \mathrm{id}\ } & \Delta. \end{array} \tag{2}$$

PROOF. Say $f' = hfh^{-1}$ ($h \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$). Next let g and g' be the periodic automorphisms of $\Sigma \times \Delta$ (of order m) given by $g(x,t) := (f^{-1}(x), e^{2\pi \mathrm{i}/m}t)$ and $g'(x,t) := (f'^{-1}(x), e^{2\pi \mathrm{i}/m}t)$, then the following diagram commutes:

$$\begin{array}{ccc} \Sigma \times \Delta & \xrightarrow{h \times \mathrm{id}} & \Sigma \times \Delta \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \Sigma \times \Delta & \xrightarrow{h \times \mathrm{id}} & \Sigma \times \Delta. \end{array}$$

Hence the biholomorphic map $h \times \mathrm{id} : \Sigma \times \Delta \to \Sigma \times \Delta$ induces a biholomorphic map $\overline{h \times \mathrm{id}} : (\Sigma \times \Delta)/G \to (\Sigma \times \Delta)/G'$.

Now the holomorphic function $\phi : \Sigma \times \Delta \to \Delta$ determines holomorphic maps $\overline{\phi} : (\Sigma \times \Delta)/G \to \Delta$ and $\overline{\phi'} : (\Sigma \times \Delta)/G' \to \Delta$. Next noting that $\phi = \phi \circ (h \times \mathrm{id})$, $\overline{\phi} = \overline{\phi'} \circ \overline{h \times \mathrm{id}}$, that is, the following diagram commutes:

$$\begin{array}{ccc} (\Sigma \times \Delta)/G & \xrightarrow{\overline{h \times \mathrm{id}}} & (\Sigma \times \Delta)/G' \\ {\scriptstyle \overline{\phi}}\downarrow & & \downarrow{\scriptstyle \overline{\phi'}} \\ \Delta & \xrightarrow{\mathrm{id}} & \Delta. \end{array} \tag{3}$$

On the other hand, by Lemma 2.1 (applied to the case $\psi = \overline{h \times \mathrm{id}}$) there exists a biholomorphic map $\Psi : M \to M'$ that makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\Psi} & M' \\ {\scriptstyle \mathfrak{r}}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}'} \\ (\Sigma \times \Delta)/G & \xrightarrow{\overline{h \times \mathrm{id}}} & (\Sigma \times \Delta)/G'. \end{array} \tag{4}$$

Combining the diagrams (3) and (4) yields the desired diagram (2). □

2.2. Automorphism groups and mapping class groups.

The *mapping class group* $\mathrm{MCG}(\Sigma)$ of Σ is the group of isotopy classes of orientation-preserving self-homeomorphisms of Σ. The isotopy class of $f \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ is denoted by $[f]$. Note that *if* $\mathrm{genus}(\Sigma) \geq 2$, *then* $\mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ *may be regarded as a finite subgroup of* $\mathrm{MCG}(\Sigma)$. In fact, the following holds:

Lemma 2.3. *The map* $\rho : f \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma) \mapsto [f] \in \mathrm{MCG}(\Sigma)$ *gives a group homomorphism. Moreover if* $\mathrm{genus}(\Sigma) \geq 2$, ρ *is injective.* (Note: If $\mathrm{genus}(\Sigma) \geq 2$, then $\mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ is finite, while $\mathrm{MCG}(\Sigma)$ is infinite — a Dehn twist is an element of infinite order. If $\mathrm{genus}(\Sigma) = 0, 1$, then ρ is *not* injective.)

Proof. The first assertion is obvious. We show the second assertion. If $\mathrm{genus}(\Sigma) \geq 2$, then the order of any $f \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ is finite (that is, f is periodic), accordingly the quotient map $\Sigma \to \Sigma/\langle f \rangle$ is a *finite* covering. We now show that ρ is injective, or equivalently that $[f] = \mathrm{id}$ implies that $f = \mathrm{id}$. This may be confirmed by contradiction. If $f \neq \mathrm{id}$, then $\mathrm{genus}(\Sigma/\langle f \rangle) < \mathrm{genus}(\Sigma)$ (from the Riemann–Hurwitz formula). However $\mathrm{genus}(\Sigma/\langle f \rangle) = \mathrm{genus}(\Sigma)$ (because $[f] = \mathrm{id}$). This is a contradiction. □

Now let $\mathcal{P}$ be a regular polyhedron (or a soccer ball) and Σ be the cable surface of $\mathcal{P}$, which is obtained by thickening the edges of $\mathcal{P}$. Note that if $\mathcal{P}$ is a regular n-hedron where $n = 4, 6, 8, 12, 20$, then genus$(\Sigma) = n - 1$.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{P})$ of $\mathcal{P}$ naturally acts on Σ. Noting that genus$(\Sigma) \geq 2$, the same argument as that used in the proof of Lemma 2.3 shows the following:

LEMMA 2.4. *The group homomorphism* $f \in \mathrm{Aut}_+(\mathcal{P}) \mapsto [f] \in \mathrm{MCG}(\Sigma)$ *is injective.*

Kerckhoff's theorem [**6**] then ensures the existence of a complex structure on Σ such that the action of the finite group $\mathrm{Aut}_+(\mathcal{P})$ on Σ is holomorphic, so $\mathrm{Aut}_+(\mathcal{P}) \subset \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$. This with Lemma 2.4 yields

$$\mathrm{Aut}_+(\mathcal{P}) \subset \mathrm{Aut}_{\mathrm{hol}}(\Sigma) \subset \mathrm{MCG}(\Sigma). \tag{5}$$

EXAMPLE 2.5. Let Σ be the cable surface of the octahedron $\mathcal{P}$. It appears that there is an isotopy between 1/2-rotations f and f' as illustrated in Figure 4. This is actually not true: since $f \neq f'$ in $\mathrm{Aut}_+(\mathcal{P})$, necessarily $[f] \neq [f']$ in $\mathrm{MCG}(\Sigma)$, so f and f' are *not* isotopic. Moreover they are *not* conjugate in $\mathrm{Aut}_+(\mathcal{P})$ (Example 2.8 below). In contrast, $[f]$ and $[f']$ are conjugate in $\mathrm{MCG}(\Sigma)$, which follows from Nielsen's theorem (see Section 2.4), because (i) the valency data of f and f' are equal (both are $(1/2, 1/2, 1/2, 1/2)$) and (ii) the ramification data of the cyclic coverings $\Sigma \to \Sigma/\langle f \rangle$ and $\Sigma \to \Sigma/\langle f' \rangle$ are equal.

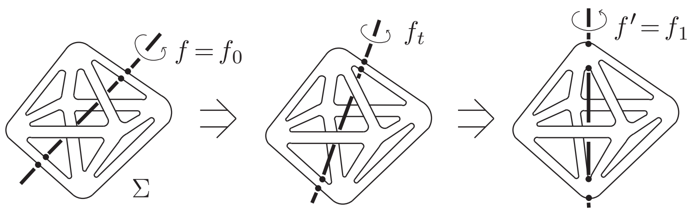

Figure 4. Each f_t $(0 \leq t \leq 1)$ is a 1/2-rotation.

2.3. Examples of equivalent degenerations.

Let $\mathcal{P}$ be a regular polyhedron (or a soccer ball) and Σ be its cable surface. The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{P})$ of $\mathcal{P}$ naturally acts on Σ. Kerckhoff's theorem [**6**] then ensures the existence of a complex structure on Σ such that the action of $\mathrm{Aut}_+(\mathcal{P})$ on Σ is holomorphic. We regard Σ as a Riemann surface equipped with this complex structure. Any $f \in \mathrm{Aut}_+(\mathcal{P})$ then acts on Σ biholomorphically and is periodic. Let $\pi_f : M_f \to \Delta$ denote the degeneration of Riemann surfaces associated with f (see Section 2.1) and $X_f := \pi_f^{-1}(0)$ denote its singular fiber.

If two elements $f, f' \in \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ are conjugate in $\mathrm{Aut}_{\mathrm{hol}}(\Sigma)$, then two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are holomorphically equivalent (Proposition 2.2). Since $\mathrm{Aut}_+(\mathcal{P}) \subset \mathrm{Aut}_{\mathrm{hol}}(\Sigma)$ (see (5)), the following holds:

LEMMA 2.6. *If two elements $f, f' \in \mathrm{Aut}_+(\mathcal{P})$ are conjugate in $\mathrm{Aut}_+(\mathcal{P})$, then two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are holomorphically equivalent.* (In this case, they are said to be *strongly* holomorphically equivalent.)

EXAMPLE 2.7. The 1/4-rotation f and the $-(1/4)$-rotation f^{-1} of the hexahedron $\mathcal{P}$ are *conjugate* in $\mathrm{Aut}_+(\mathcal{P})$. (This follows from the facts that (i) the orders of both f and f^{-1} are 4 and (ii) $\mathrm{Aut}_+(\mathcal{P}) \cong \mathfrak{S}_4$ and any two elements of order 4 in $\mathfrak{S}_4$ are conjugate.) Hence two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f^{-1}} : M_{f^{-1}} \to \Delta$ are strongly holomorphically equivalent.

Even if f and f' are not conjugate in $\mathrm{Aut}_+(\mathcal{P})$, two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are possibly topologically equivalent:

EXAMPLE 2.8. The automorphism group $\mathrm{Aut}_+(\mathcal{P})$ of the octahedron $\mathcal{P}$ may be identified $\mathfrak{S}_4$. The 1/2-rotation f of $\mathcal{P}$ about the axis through the midpoints of two opposite edges is an odd permutation, and the 1/2-rotation f' of $\mathcal{P}$ about the axis through two opposite vertices is an even permutation. In particular, they are *not* conjugate in $\mathrm{Aut}_+(\mathcal{P})$, so Lemma 2.6 is not applicable, and it is not clear whether two degenerations $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are holomorphically equivalent. However they are indeed topologically equivalent, because $[f]$ and $[f']$ are conjugate in $\mathrm{MCG}(\Sigma)$ (see Example 2.5).

2.4. Singular fibers.

Let Σ be a Riemann surface and $f : \Sigma \to \Sigma$ be a periodic automorphism of order m. For a point $p \in \Sigma$, the least positive integer c $(1 \le c \le m)$ satisfying $f^c(p) = p$ is the *recurrence number* of p. Here:

- If $c < m$, then p is a *ramification point* of f (or, a ramification point of a cyclic covering $\psi : \Sigma \to \Sigma/\langle f \rangle$). In the special case $c = 1$, p is a fixed point of f.
- If $c = m$, then p is a *generic point* with respect to the action of f.

REMARK 2.9. For the action of $\mathrm{Aut}_+(\mathcal{P})$ on the cable surface Σ of a regular polyhedron $\mathcal{P}$, each element of $\mathrm{Aut}_+(\mathcal{P})$ is a rotation about some axis and for any $f \in \mathrm{Aut}_+(\mathcal{P})$, any ramification point of Σ is a fixed point of f.

Given a periodic automorphism $f : \Sigma \to \Sigma$ of order m, the m-fold cyclic covering $\psi : \Sigma \to \Sigma/\langle f \rangle$ maps the ramification points to the branch points. Say $v_1, v_2, \ldots, v_l$ are the branch points of ψ. The recurrence number of a ramification point $\widetilde{v_j}$ over v_j (that is, $\widetilde{v_j} \in \psi^{-1}(v_j)$) is independent of the choice of a ramification point. Denote it by c_j. Below, the genus of $\Sigma/\langle f \rangle$ is denoted by h and the *ramification index* of v_j by $n_j := m/c_j$. The set $(h; n_1, n_2, \ldots, n_l)$ is called the *ramification data* of f.

For any ramification point $\widetilde{v_j}$ over v_j, the automorphism $f^{-c_j} : \Sigma \to \Sigma$ is locally a rotation about $\widetilde{v_j}$, say, a $2\pi a_j/n_j$-rotation where a_j $(0 < a_j < n_j)$ is an integer relatively prime to n_j. Take the integer q_j $(0 < q_j < n_j)$ satisfying $a_j q_j \equiv 1 \bmod n_j$.

DEFINITION 2.10. The fraction q_j/n_j is the *valency* of v_j and the set $(q_1/n_1, q_2/n_2, \ldots, q_l/n_l)$ is the *valency data* of f.

Nielsen's theorem [**9**] states that if two periodic automorphisms f and f' of a Riemann surface Σ have the same ramification data and the same valency data, then f and f' are conjugate: $f' = \Phi f \Phi^{-1}$ for some orientation-preserving homeomorphism $\Phi : \Sigma \to \Sigma$. In this case, Φ descends to an orientation-preserving homeomorphism $\varphi : \Sigma/\langle f \rangle \to \Sigma/\langle f' \rangle$, and two cyclic coverings $\Sigma \to \Sigma/\langle f \rangle$ and $\Sigma \to \Sigma/\langle f' \rangle$ are *topologically equivalent*, that is, the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Phi} & \Sigma \\ \downarrow & & \downarrow \\ \Sigma/\langle f \rangle & \xrightarrow{\varphi} & \Sigma/\langle f' \rangle. \end{array} \tag{6}$$

We now return to degenerations of Riemann surfaces. Let $f : \Sigma \to \Sigma$ be a periodic automorphism of order m. Where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$, let $g : \Sigma \times \Delta \to \Sigma \times \Delta$ be a periodic automorphism (of order m) defined by $(x, t) \mapsto (f^{-1}(x), e^{2\pi \mathrm{i}/m} t)$. Denote by G the cyclic group generated by g.

In what follows, suppose that *any* of the ramification points of f is a fixed point — this is, for instance, the case when $f \in \mathrm{Aut}_+(\mathcal{P})$ and Σ is the cable surface of $\mathcal{P}$. Regard a fixed point of f as a fixed point of g under the identification of Σ with $\Sigma \times \{0\}$.

The image $\overline{p} \in (\Sigma \times \Delta)/G$ of a fixed point $p \in \Sigma \times \{0\}$ under the quotient map $\Sigma \times \Delta \to (\Sigma \times \Delta)/G$ is a cyclic quotient singularity. Let $\mathfrak{r} : M \to (\Sigma \times \Delta)/G$ be the resolution that minimally resolves each of the (cyclic quotient) singularities of $(\Sigma \times \Delta)/G$. Now let $\phi : \Sigma \times \Delta \to \Delta$ be a G-invariant holomorphic function given by $(x, t) \mapsto t^m$. It determines a holomorphic map $\overline{\phi} : (\Sigma \times \Delta)/G \to \Delta$. The composition $\pi := \overline{\phi} \circ \mathfrak{r} : M \to \Delta$ is a *degeneration associated with* f.

Resolving a cyclic quotient singularity of $(\Sigma \times \Delta)/G$ yields a chain of (complex) projective lines. We explain how the multiplicity of each component of this chain is determined. First let $\overline{p} \in (\Sigma \times \Delta)/G$ be the cyclic quotient singularity that is the image of a fixed point p of the automorphism g. Let q/m be the valency of $\overline{p}$ (with respect to the automorphism f) and $m_0, m_1, m_2, \ldots, m_\lambda$ be the sequence of integers determined from $m_0 := m$, $m_1 := q$ by the *negative* division algorithm:

$$\begin{cases} m_{i-1} = r_i m_i - m_{i+1} \quad (0 \leq m_{i+1} < m_i), \qquad i = 1, 2, \ldots, \lambda - 1, \\ m_{\lambda-1} = r_\lambda m_\lambda, \end{cases} \tag{7}$$

where r_i is an integer greater than 1. Then the minimal resolution of $\overline{p}$ yields the following chain of projective lines with multiplicities:

$$m_0 D_0 + m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_\lambda \Theta_\lambda,$$

where D_0 is a disk and $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ are projective lines such that D_0 and Θ_1 intersect transversally and Θ_i and Θ_{i+1} ($i = 1, 2, \ldots, \lambda - 1$) intersect transversally. We say that $m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_\lambda \Theta_\lambda$ is a *branch*.

Let $p_1, p_2, \ldots, p_l \in \Sigma$ be the fixed points of f. The degeneration $\pi : M \to \Delta$ is then obtained by resolving the (cyclic quotient) singularities $\overline{p_1}, \overline{p_2}, \ldots, \overline{p_l}$ of $(\Sigma \times \Delta)/G$. Let

$E_j := m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ be the branch emanating from $\overline{p_j}$ (λ_j is called the *length* of E_j). Then the singular fiber $X := \pi^{-1}(0)$ of $\pi\colon M \to \Delta$ is *star-shaped*, that is, from the *core* $C := \Sigma/\langle f\rangle$ the branches $E_1, E_2, \ldots, E_l$ emanate:

$$X = mC + \sum_{j=1}^{l} E_j \qquad \text{(for example, see Figure 3).}$$

Convention 2.11. Write C of genus g as $C_{\text{genus}\, g}$ and $E_j = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ as $E_j = (m_1^{(j)}, m_2^{(j)}, \ldots, m_{\lambda_j}^{(j)})$. For example, $X = 3C + \Theta_1^{(1)} + 2\Theta_1^{(2)} + \Theta_2^{(2)}$ is expressed as $X = 3C_{\text{genus}\, g} + (1) + (2,1)$.

The Euler number $\chi(|X|)$ of the underlying topological space $|X|$ of X is equal to $(2 - 2\,\text{genus}(C)) + \sum_{j=1}^{l} \lambda_j$, where λ_j is the length of E_j. Since $|X|$ is a deformation retract of M (see the proof of [**3**, Corollary 17, p. 178]), $\chi(M) = \chi(|X|)$, so

$$\chi(M) = \chi(|X|) = (2 - 2\,\text{genus}(C)) + \sum_{j=1}^{l} \lambda_j. \tag{8}$$

3. Degenerations associated with regular polyhedra and soccer ball.

Let $\text{Aut}_+(\mathcal{P})$ denote the orientation-preserving automorphism group of a regular polyhedron $\mathcal{P}$. For each $f \in \text{Aut}_+(\mathcal{P})$, we shall describe the degeneration $\pi_f : M_f \to \Delta$ associated with it. Here recall that if $f, f' \in \text{Aut}_+(\mathcal{P})$ are conjugate in $\text{Aut}_+(\mathcal{P})$, then $\pi_f : M_f \to \Delta$ and $\pi_{f'} : M_{f'} \to \Delta$ are holomorphically equivalent — in fact, *strongly* holomorphically equivalent (Lemma 2.6). It thus suffices to describe the degeneration $\pi_f : M_f \to \Delta$ for each representative f of a conjugacy class of $\text{Aut}_+(\mathcal{P})$. Note that distinct strong holomorphic equivalence class may be topologically equivalent — we will describe which strong holomorphic classes are topologically equivalent.

3.1. Tetrahedron case.

The orientation-preserving automorphism group $\text{Aut}_+(\mathcal{T})$ of the tetrahedron $\mathcal{T}$ has three conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \text{Aut}_+(\mathcal{T})$): one conjugacy class of order 2 and two conjugacy classes of order 3 (Table 2). The 1/2-rotation f_1 about the axis through the midpoints of two opposite edges is a representative of the conjugacy class of order 2, and the 1/3-rotation f_2 about the axis through a vertex and the barycenter of the opposite face is a representative of a conjugacy class of order 3, and $f_3 := (f_2)^2$ is a representative of the other conjugacy class of order 3.

The automorphisms f_1 and f_2 of the tetrahedron $\mathcal{T}$ naturally act on the cable surface Σ of $\mathcal{T}$. The orientation-preserving homeomorphism $f_1 : \Sigma \to \Sigma$ has four fixed points and the orientation-preserving homeomorphism $f_2 : \Sigma \to \Sigma$ has two fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ ($i = 1, 2, 3$) be the degeneration associated with f_i. The singular fibers X_{f_1}, X_{f_2} and X_{f_3} are as follows (see Convention 2.11):

$$\begin{aligned}
&[\text{T.1}] \quad X_{f_1} = 2C_{\text{genus }1} + (1) + (1) + (1) + (1),\\
&[\text{T.2}] \quad X_{f_2} = 3C_{\text{genus }1} + (1) + (2,1),\\
&[\text{T.3}] \quad X_{f_3} = 3C_{\text{genus }1} + (1) + (2,1).
\end{aligned}$$

Note that although f_2 and f_3 are *not* conjugate in $\mathrm{Aut}_+(\mathcal{T})$, their isotopy classes $[f_2]$ and $[f_3]$ are conjugate in the mapping class group $\mathrm{MCG}(\Sigma)$ (Remark 3.1). Consequently, the degenerations $\pi_{f_2} : M_{f_2} \to \Delta$ and $\pi_{f_3} : M_{f_3} \to \Delta$ are topologically equivalent.

REMARK 3.1. The valency data of f_2 and f_3 are equal (both are $(1/3, 2/3)$) and the ramification data of the cyclic coverings $\Sigma \to \Sigma/\langle f_2\rangle$ and $\Sigma \to \Sigma/\langle f_3\rangle$ are equal. By Nielsen's theorem [**9**], $[f_2]$ and $[f_3]$ are thus conjugate in $\mathrm{MCG}(\Sigma)$.

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = 4, \chi(M_{f_2}) = \chi(M_{f_3}) = 3$.

We summarize the above result as follows:

PROPOSITION 3.2. (1) *The singular fibers of the degenerations associated with* f_1, f_2, f_3 *are* [T.1], [T.2], [T.3], *and their Euler numbers* $\chi(X_{f_i})\,(= \chi(M_{f_i}))$ *are as follows*: $\chi(X_{f_1}) = 4, \chi(X_{f_2}) = \chi(X_{f_3}) = 3$.

(2) *The two degenerations* $\pi_{f_2} : M_{f_2} \to \Delta$ *and* $\pi_{f_3} : M_{f_3} \to \Delta$ *are topologically equivalent.*

3.2. Hexahedron case.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{H})$ of the hexahedron $\mathcal{H}$ has four conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{H})$): two conjugacy classes of order 2, one conjugacy class of order 3 and one conjugacy class of order 4 (Table 2). Representatives of the two conjugacy classes of order 2 are given by 1/2-rotations f_1 and f_2, respectively, about the axis through the midpoints of two opposite edges and about the axis through the barycenters of two opposite faces. A representative of the conjugacy class of order 3 is a 1/3-rotation f_3 about the axis through two opposite vertices, and a representative of the conjugacy class of order 4 is a 1/4-rotation f_4 about the axis through the barycenters of two opposite faces. (We may take $(f_4)^2$ as f_2.)

The automorphisms f_1, f_2, f_3 and f_4 of the hexahedron $\mathcal{H}$ naturally act on the cable surface Σ of $\mathcal{H}$. The orientation-preserving homeomorphisms $f_1 : \Sigma \to \Sigma$ and $f_3 : \Sigma \to \Sigma$ have four fixed points and the orientation-preserving homeomorphisms $f_2 : \Sigma \to \Sigma$ and $f_4 : \Sigma \to \Sigma$ have no fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ $(i = 1, 2, 3, 4)$ be the degeneration associated with f_i. The singular fibers $X_{f_1}, X_{f_2}, X_{f_3}$ and X_{f_4} are as follows (see Convention 2.11):

$$\begin{aligned}
&[\text{H.1}] \quad X_{f_1} = 2C_{\text{genus }2} + (1) + (1) + (1) + (1),\\
&[\text{H.2}] \quad X_{f_2} = 2C_{\text{genus }3},\\
&[\text{H.3}] \quad X_{f_3} = 3C_{\text{genus }1} + (1) + (1) + (2,1) + (2,1),\\
&[\text{H.4}] \quad X_{f_4} = 4C_{\text{genus }2}.
\end{aligned}$$

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3, 4)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = 2$, $\chi(M_{f_2}) = -4$, $\chi(M_{f_3}) = 6$, $\chi(M_{f_4}) = -2$.

We summarize the above result as follows:

PROPOSITION 3.3. *The singular fibers of the degenerations associated with* f_1, f_2, f_3, f_4 *are* [H.1], [H.2], [H.3], [H.4], *and their Euler numbers* $\chi(X_{f_i}) (= \chi(M_{f_i}))$ *are as follows*: $\chi(X_{f_1}) = 2$, $\chi(X_{f_2}) = -4$, $\chi(X_{f_3}) = 6$, $\chi(X_{f_4}) = -2$.

3.3. Octahedron case.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{O})$ of the octahedron $\mathcal{O}$ has four conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{O})$): two conjugacy classes of order 2, one conjugacy class of order 3 and one conjugacy class of order 4 (Table 2). Representatives of the two conjugacy classes of order 2 are given by 1/2-rotations f_1 and f_2, respectively, about the axis through the midpoints of two opposite edges and about the axis through two opposite vertices. A representative of the conjugacy class of order 3 is a 1/3-rotation f_3 about the axis through the barycenters of two opposite faces, and a representative of the conjugacy class of order 4 is a 1/4-rotation f_4 about the axis through two opposite vertices. (We may take $(f_4)^2$ as f_2.)

The automorphisms f_1, f_2, f_3 and f_4 of the octahedron $\mathcal{O}$ naturally act on the cable surface Σ of $\mathcal{O}$. The orientation-preserving homeomorphisms $f_1 : \Sigma \to \Sigma$, $f_2 : \Sigma \to \Sigma$ and $f_4 : \Sigma \to \Sigma$ have four fixed points and the orientation-preserving homeomorphism $f_3 : \Sigma \to \Sigma$ has no fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ $(i = 1, 2, 3, 4)$ be the degeneration associated with f_i. The singular fibers $X_{f_1}, X_{f_2}, X_{f_3}$ and X_{f_4} are as follows (see Convention 2.11):

$$
\begin{aligned}
&[\text{O.1}] \quad X_{f_1} = 2C_{\text{genus}\,3} + (1) + (1) + (1) + (1),\\
&[\text{O.2}] \quad X_{f_2} = 2C_{\text{genus}\,3} + (1) + (1) + (1) + (1),\\
&[\text{O.3}] \quad X_{f_3} = 3C_{\text{genus}\,3},\\
&[\text{O.4}] \quad X_{f_4} = 4C_{\text{genus}\,1} + (1) + (1) + (3,2,1) + (3,2,1).
\end{aligned}
$$

Note that although f_1 and f_2 are *not* conjugate in $\mathrm{Aut}_+(\mathcal{O})$, their isotopy classes $[f_1]$ and $[f_2]$ are conjugate in the mapping class group $\mathrm{MCG}(\Sigma)$ (Remark 3.4). Consequently, the degenerations $\pi_{f_1} : M_{f_1} \to \Delta$ and $\pi_{f_2} : M_{f_2} \to \Delta$ are topologically equivalent.

REMARK 3.4. The valency data of f_1 and f_2 are equal (both are $(1/2, 1/2, 1/2, 1/2)$) and the ramification data of the cyclic coverings $\Sigma \to \Sigma/\langle f_1 \rangle$ and $\Sigma \to \Sigma/\langle f_2 \rangle$ are equal. By Nielsen's theorem [**9**], $[f_1]$ and $[f_2]$ are thus conjugate in $\mathrm{MCG}(\Sigma)$.

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3, 4)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = \chi(M_{f_2}) = 0$, $\chi(M_{f_3}) = -4$, $\chi(M_{f_4}) = 8$.

We summarize the above result as follows:

PROPOSITION 3.5. (1) *The singular fibers of the degenerations associated with* f_1, f_2, f_3, f_4 *are* [O.1], [O.2], [O.3], [O.4], *and their Euler numbers* $\chi(X_{f_i}) (= \chi(M_{f_i}))$ *are as follows*: $\chi(X_{f_1}) = \chi(X_{f_2}) = 0$, $\chi(X_{f_3}) = -4$, $\chi(X_{f_4}) = 8$.

(2) *The two degenerations $\pi_{f_1} : M_{f_1} \to \Delta$ and $\pi_{f_2} : M_{f_2} \to \Delta$ are topologically equivalent.*

3.4. Dodecahedron case.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{D})$ of the dodecahedron $\mathcal{D}$ has four conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{D})$): one conjugacy class of order 2, one conjugacy class of order 3 and two conjugacy classes of order 5 (Table 2). The 1/2-rotation f_1 about the axis through the midpoints of two opposite edges is a representative of the conjugacy class of order 2, and the 1/3-rotation f_2 about the axis through two opposite vertices is a representative of the conjugacy class of order 3, and the 1/5-rotation f_3 about the axis through the barycenters of two opposite faces is a representative of a conjugacy class of order 5, and $f_4 := (f_3)^2$ is a representative of the other conjugacy class of order 5.

The automorphisms f_1, f_2 and f_3 of the dodecahedron $\mathcal{D}$ naturally act on the cable surface Σ of $\mathcal{D}$. The orientation-preserving homeomorphisms $f_1 : \Sigma \to \Sigma$ and $f_2 : \Sigma \to \Sigma$ have four fixed points and the orientation-preserving homeomorphism $f_3 : \Sigma \to \Sigma$ has no fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ $(i = 1, 2, 3, 4)$ be the degeneration associated with f_i. The singular fibers $X_{f_1}, X_{f_2}, X_{f_3}$ and X_{f_4} are as follows (see Convention 2.11):

$$
\begin{aligned}
&[\mathrm{D}.1] \quad X_{f_1} = 2C_{\mathrm{genus}\,5} + (1) + (1) + (1) + (1),\\
&[\mathrm{D}.2] \quad X_{f_2} = 3C_{\mathrm{genus}\,3} + (1) + (1) + (2,1) + (2,1),\\
&[\mathrm{D}.3] \quad X_{f_3} = 5C_{\mathrm{genus}\,3},\\
&[\mathrm{D}.4] \quad X_{f_4} = 5C_{\mathrm{genus}\,3}.
\end{aligned}
$$

Note that although f_3 and f_4 are *not* conjugate in $\mathrm{Aut}_+(\mathcal{D})$, their isotopy classes $[f_3]$ and $[f_4]$ are conjugate in the mapping class group $\mathrm{MCG}(\Sigma)$ (Remark 3.6). Consequently, the degenerations $\pi_{f_3} : M_{f_3} \to \Delta$ and $\pi_{f_4} : M_{f_4} \to \Delta$ are topologically equivalent.

Remark 3.6. The valency data of f_3 and f_4 are equal (indeed, they have no ramification points, so their valency data are both vacuous) and the ramification data of the cyclic coverings $\Sigma \to \Sigma/\langle f_3 \rangle$ and $\Sigma \to \Sigma/\langle f_4 \rangle$ are equal. By Nielsen's theorem [**9**], $[f_3]$ and $[f_4]$ are thus conjugate in $\mathrm{MCG}(\Sigma)$.

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3, 4)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = \chi(M_{f_3}) = \chi(M_{f_4}) = -4$, $\chi(M_{f_2}) = 2$.

We summarize the above result as follows:

Proposition 3.7. (1) *The singular fibers of the degenerations associated with f_1, f_2, f_3, f_4 are* [D.1], [D.2], [D.3], [D.4], *and their Euler numbers $\chi(X_{f_i})$ $(= \chi(M_{f_i}))$ are as follows*: $\chi(X_{f_1}) = \chi(X_{f_3}) = \chi(X_{f_4}) = -4$, $\chi(X_{f_2}) = 2$.

(2) *The two degenerations $\pi_{f_3} : M_{f_3} \to \Delta$ and $\pi_{f_4} : M_{f_4} \to \Delta$ are topologically equivalent.*

3.5. Icosahedron case.

The orientation-preserving automorphism group $\mathrm{Aut}_+(\mathcal{I})$ of the icosahedron $\mathcal{I}$ has four conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{I})$): one conjugacy class of order 2, one conjugacy class of order 3 and two conjugacy classes of order 5 (Table 2). The 1/2-rotation f_1 about the axis through the midpoints of two opposite edges is a representative of the conjugacy class of order 2, and the 1/3-rotation f_2 about the axis through the barycenters of two opposite faces is a representative of the conjugacy class of order 3, and the 1/5-rotation f_3 about the axis through two opposite vertices is a representative of a conjugacy class of order 5, and $f_4 := (f_3)^2$ is a representative of the other conjugacy class of order 5.

The automorphisms f_1, f_2 and f_3 of the icosahedron $\mathcal{I}$ naturally act on the cable surface Σ of $\mathcal{I}$. The orientation-preserving homeomorphisms $f_1 : \Sigma \to \Sigma$ and $f_3 : \Sigma \to \Sigma$ have four fixed points and the orientation-preserving homeomorphism $f_2 : \Sigma \to \Sigma$ has no fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ $(i = 1, 2, 3, 4)$ be the degeneration associated with f_i. The singular fibers $X_{f_1}, X_{f_2}, X_{f_3}$ and X_{f_4} are as follows (see Convention 2.11):

[I.1] $X_{f_1} = 2C_{\text{genus}\,9} + (1) + (1) + (1) + (1)$,
[I.2] $X_{f_2} = 3C_{\text{genus}\,7}$,
[I.3] $X_{f_3} = 5C_{\text{genus}\,3} + (1) + (1) + (4, 3, 2, 1) + (4, 3, 2, 1)$,
[I.4] $X_{f_4} = 5C_{\text{genus}\,3} + (2, 1) + (2, 1) + (3, 1) + (3, 1)$.

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3, 4)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = \chi(M_{f_2}) = -12$, $\chi(M_{f_3}) = 6$, $\chi(M_{f_4}) = 4$.

We summarize the above result as follows:

Proposition 3.8. *The singular fibers of the degenerations associated with* f_1, f_2, f_3, f_4 *are* [I.1], [I.2], [I.3], [I.4], *and their Euler numbers* $\chi(X_{f_i})\,(= \chi(M_{f_i}))$ *are as follows*: $\chi(X_{f_1}) = \chi(X_{f_2}) = -12$, $\chi(X_{f_3}) = 6$, $\chi(X_{f_4}) = 4$.

3.6. Soccer ball case.

A convex polyhedron is a *semi-regular polyhedron* if its faces are regular polygons and all vertices are congruent. As the cases of the regular polyhedra, we can construct degenerations associated with semi-regular polyhedra. In this paper, we consider only the soccer ball case. (The cases of the other semi-regular polyhedra are considered elsewhere.)

The soccer ball $\mathcal{S}$ is a convex polyhedron whose faces are twelve regular pentagons and twenty regular hexagons. The soccer ball is also called the *truncated icosahedron*, for it is obtained by cutting off parts of the icosahedron about vertices. Therefore the orientation-preserving automorphism group of the soccer ball is equal to that of the icosahedron: $\mathrm{Aut}_+(\mathcal{S}) \cong \mathrm{Aut}_+(\mathcal{I})$.

The automorphism group $\mathrm{Aut}_+(\mathcal{S})$ has four conjugacy classes other than the *trivial* conjugacy class (that is the conjugacy class of $1 \in \mathrm{Aut}_+(\mathcal{S})$): one conjugacy class of order 2, one conjugacy class of order 3 and two conjugacy classes of order 5 (Table 2). The 1/2-rotation f_1 about the axis through the barycenters of two opposite edges adjacent to hexagonal faces is a representative of the conjugacy class of order 2, and the 1/3-rotation f_2 about the axis through the barycenters of two opposite pentagonal faces is a representative of the conjugacy class of order 3, and the 1/5-rotation f_3 about the axis through the barycenters of two opposite hexagonal faces is a representative of a conjugacy class of order 5, and $f_4 := (f_3)^2$ is a representative of the other conjugacy class of order 5.

The automorphisms f_1, f_2 and f_3 of the soccer ball $\mathcal{S}$ naturally act on the cable surface Σ of $\mathcal{S}$. The orientation-preserving homeomorphism $f_1 : \Sigma \to \Sigma$ has four fixed points and the orientation-preserving homeomorphisms $f_2 : \Sigma \to \Sigma$ and $f_3 : \Sigma \to \Sigma$ have no fixed points.

Now let $\pi_{f_i} : M_{f_i} \to \Delta$ $(i = 1, 2, 3, 4)$ be the degeneration associated with f_i. The singular fibers $X_{f_1}, X_{f_2}, X_{f_3}$ and X_{f_4} are as follows (see Convention 2.11):

[S.1] $X_{f_1} = 2C_{\text{genus } 15} + (1) + (1) + (1) + (1)$,
[S.2] $X_{f_2} = 3C_{\text{genus } 11}$,
[S.3] $X_{f_3} = 5C_{\text{genus } 7}$,
[S.4] $X_{f_4} = 5C_{\text{genus } 7}$.

Note that although f_3 and f_4 are *not* conjugate in $\mathrm{Aut}_+(\mathcal{S})$, their isotopy classes $[f_3]$ and $[f_4]$ are conjugate in the mapping class group $\mathrm{MCG}(\Sigma)$ (Remark 3.9). Consequently, the degenerations $\pi_{f_3} : M_{f_3} \to \Delta$ and $\pi_{f_4} : M_{f_4} \to \Delta$ are topologically equivalent.

Remark 3.9. The valency data of f_3 and f_4 are equal (indeed, they have no ramification points, so their valency data are both vacuous) and the ramification data of the cyclic coverings $\Sigma \to \Sigma/\langle f_3 \rangle$ and $\Sigma \to \Sigma/\langle f_4 \rangle$ are equal. By Nielsen's theorem [**9**], $[f_3]$ and $[f_4]$ are thus conjugate in $\mathrm{MCG}(\Sigma)$.

Next the Euler numbers of M_{f_i} $(i = 1, 2, 3, 4)$ may be determined by the formula (8) as follows: $\chi(M_{f_1}) = -24$, $\chi(M_{f_2}) = -20$, $\chi(M_{f_3}) = \chi(M_{f_4}) = -12$.

We summarize the above result as follows:

Proposition 3.10. (1) *The singular fibers of the degenerations associated with* f_1, f_2, f_3, f_4 *are* [S.1], [S.2], [S.3], [S.4], *and their Euler numbers* $\chi(X_{f_i})\,(= \chi(M_{f_i}))$ *are as follows*: $\chi(X_{f_1}) = -24$, $\chi(X_{f_2}) = -20$, $\chi(X_{f_3}) = \chi(X_{f_4}) = -12$.

(2) *The two degenerations* $\pi_{f_3} : M_{f_3} \to \Delta$ *and* $\pi_{f_4} : M_{f_4} \to \Delta$ *are topologically equivalent.*

4. Compactification problem.

Let $\mathcal{P}$ be a regular polyhedron or the soccer ball. To conjugacy classes of $\mathrm{Aut}_+(\mathcal{P})$, we associated degenerations of Riemann surfaces. Mutsuo Oka raised a question: *Is there a natural way to compactify these degenerations?* Or: *Is there a compact fibration of Riemann surfaces over* $\mathbb{P}^1$ *whose singular fibers are those of the above degenerations?* From $\mathrm{Aut}_+(\mathcal{P})$ itself (not from each of conjugacy classes of $\mathrm{Aut}_+(\mathcal{P})$), we explicitly construct such a fibration.

Note the fact that the automorphism group $\Gamma := \mathrm{Aut}_+(\mathcal{P})$, being a subgroup of $SO(3)$, acts on the sphere S^2 (as a spherical triangle group) — observe that S^2 is regarded as the 'sphering' of $\mathcal{P}$, so the action of Γ on $\mathcal{P}$ naturally defines its action on S^2.

In the sequel, we regard S^2 as $\mathbb{P}^1$ on which Γ acts *holomorphically.* Γ also acts on the cable surface Σ of $\mathcal{P}$ holomorphically. Next let Γ act holomorphically on the product space $\Sigma \times \mathbb{P}^1$ by $\gamma(x, y) \mapsto (\gamma x, \gamma y)$.

Remark 4.1. This action is slightly different from the action $(x, t) \mapsto (f^{-1}(x), e^{2\pi \mathrm{i}/m})$ in Section 2.1. Notice that $(x, y) \mapsto (\gamma^{-1}x, \gamma y)$ does *not* define a group action, because Γ is *not* abelian, so in general $\gamma_1^{-1}\gamma_2^{-1} \neq (\gamma_1\gamma_2)^{-1}$. (On the other hand, the group generated by f is cyclic.)

Now the projection $\mathrm{pr} : \Sigma \times \mathbb{P}^1 \to \mathbb{P}^1$ is Γ-equivariant, so determines a holomorphic map $\overline{\mathrm{pr}} : (\Sigma \times \mathbb{P}^1)/\Gamma \to \mathbb{P}^1/\Gamma \cong \mathbb{P}^1$. Here $(\Sigma \times \mathbb{P}^1)/\Gamma$ is a *compact* complex surface with singularities — all are cyclic quotient singularities. Take the resolution map $\mathfrak{r} : W_{\mathcal{P}} \to (\Sigma \times \mathbb{P}^1)/\Gamma$ that minimally resolves every singularity. We say that the composition $\pi := \overline{\mathrm{pr}} \circ \mathfrak{r} : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma\, (\cong \mathbb{P}^1)$ is a *cable fibration of Riemann surfaces associated with* $\mathcal{P}$. By construction, every smooth fiber of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ is Σ. The singular fibers lie over the branch points of the quotient map $\psi : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma$. In fact the following holds:

Lemma 4.2. *Take a sufficiently small disk D centered at a branch point $p \in \mathbb{P}^1/\Gamma$ of $\psi : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma$. Then the restriction $\pi : M \to D$ of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ to $M := \pi^{-1}(D)$ is a degeneration associated with $\mathcal{P}$.*

Proof. Take a point $q \in \mathbb{P}^1$ such that $\psi(q) = p$ and the disk $\widetilde{D} \subset \mathbb{P}^1$ centered at q such that $\psi(\widetilde{D}) = D$. Then q is a ramification point of ψ over p and $\widetilde{D}$ is invariant under the action of the *stabilizer* $G := \{\gamma \in \Gamma : \gamma(q) = q\}$. Say that the ramification index of q is m, then G is a cyclic group of order m and a generator $\gamma \in G$ acts on the disk $\widetilde{D}$ by a $1/m$-rotation about q, and the action of G on $\Sigma \times \widetilde{D}$ is generated by the automorphism

$$(x, y) \in \Sigma \times \widetilde{D} \longmapsto (\gamma x, e^{2\pi \mathrm{i}/m} y) \in \Sigma \times \widetilde{D}. \tag{9}$$

Now $\pi : M \to D$ is reconstructed by the cyclic quotient construction as follows: First note that the quotient of $\Sigma \times \widetilde{D} \to \widetilde{D}$ under the G-action is naturally embedded in $\overline{\mathrm{pr}} : (\Sigma \times \mathbb{P}^1)/\Gamma \to \mathbb{P}^1/\Gamma$:

$$\begin{array}{ccccccc} \Sigma \times \widetilde{D} & \subset & \Sigma \times \mathbb{P}^1 & & (\Sigma \times \widetilde{D})/G & \subset & (\Sigma \times \mathbb{P}^1)/\Gamma \\ \downarrow & & \downarrow \mathrm{pr} & \xrightarrow{\text{quotient}} & \downarrow & & \downarrow \overline{\mathrm{pr}} \\ \widetilde{D} & \subset & \mathbb{P}^1 & & \widetilde{D}/G & \subset & \mathbb{P}^1/\Gamma\,. \end{array} \tag{10}$$

Accordingly $\pi : M \to D := \widetilde{D}/G$ is embedded in $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ and is a degeneration of Riemann surfaces associated with γ^{-1} (which is the *local monodromy* of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ around the singular fiber $\pi^{-1}(p)$):

$$\begin{array}{ccc} M & \subset & W_{\mathcal{P}} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ D & \subset & \mathbb{P}^1/\Gamma. \end{array} \tag{11}$$

□

REMARK 4.3. Precisely speaking, "local monodromy" γ^{-1} should be the conjugacy class of γ^{-1} in the mapping class group of Σ. However by convention, any representative of the conjugacy class is also called "local monodromy".

For any $\mathcal{P}$, the number of singular fibers of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ is three, because so is the number of branch points of $\psi : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma$; their ramification indices appear in Table 5 (see [**12**, p. 86]).

Table 5

	$\mathcal{T}$	$\mathcal{H}$	$\mathcal{O}$	$\mathcal{D}$	$\mathcal{I}$	$\mathcal{S}$
Ramification indices	$2,3,3$	$2,3,4$	$2,3,4$	$2,3,5$	$2,3,5$	$2,3,5$

For the three branch points of $\psi : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma$, the list of triples (m, f, X_f), where m is the ramification index of a branch point and f is the local monodromy of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ around the singular fiber X_f of π over it, is as follows:

Table 6

	(m, f, X_f)
$\mathcal{T}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_2^{-1}, X_{f_2^{-1}})$, $(3, f_2^{-2}, X_{f_2^{-2}})$; f_1, f_2 in Section 3.1
$\mathcal{H}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_3^{-1}, X_{f_3^{-1}})$, $(4, f_4^{-1}, X_{f_4^{-1}})$; f_1, f_3, f_4 in Section 3.2
$\mathcal{O}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_3^{-1}, X_{f_3^{-1}})$, $(4, f_4^{-1}, X_{f_4^{-1}})$; f_1, f_3, f_4 in Section 3.3
$\mathcal{D}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_2^{-1}, X_{f_2^{-1}})$, $(5, f_3^{-1}, X_{f_3^{-1}})$; f_1, f_2, f_3 in Section 3.4
$\mathcal{I}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_2^{-1}, X_{f_2^{-1}})$, $(5, f_3^{-1}, X_{f_3^{-1}})$; f_1, f_2, f_3 in Section 3.5
$\mathcal{S}$	$(2, f_1^{-1}, X_{f_1^{-1}})$, $(3, f_2^{-1}, X_{f_2^{-1}})$, $(5, f_3^{-1}, X_{f_3^{-1}})$; f_1, f_2, f_3 in Section 3.6

Here note that if $\mathcal{P} \neq \mathcal{T}$, then any $f \in \Gamma \setminus \{1\}$ is conjugate to $f^{-1} \in \Gamma \setminus \{1\}$. In fact $f^{-1} = h^{-1} f h$, where $h \in \Gamma$ is the 1/2-rotation that reverses the rotation axis of f as illustrated in Figure 5. In contrast if $\mathcal{P} = \mathcal{T}$, then for a 1/3-rotation f there exists no such h and in fact f is *not* conjugate to f^{-1} in Γ (Figure 5).

The following thus holds ($f \sim f'$ means that f and f' are conjugate in Γ) (Table 7):

We may accordingly change representatives of local monodromies in their conjugacy classes in Γ. Here note that if f and f^{-1} are conjugate in Γ, then $\pi_f : M_f \to \Delta$ and $\pi_{f^{-1}} : M_{f^{-1}} \to \Delta$ are holomorphically equivalent (Lemma 2.6), consequently the following holds (Table 8):

Noting that $\mathbb{P}^1 \cong \mathbb{P}^1/\Gamma$, in what follows we write $\pi : W_{\mathcal{P}} \to \mathbb{P}^1/\Gamma$ as $\pi : W_{\mathcal{P}} \to \mathbb{P}^1$.

Euler numbers of total spaces.

For arbitrary fibration $\pi : W \to \mathbb{P}^1$ of Riemann surfaces of genus g with singular fibers $X_1, X_2, \ldots, X_l$, the following holds ([**1**, Proposition 11.4, p. 118]):

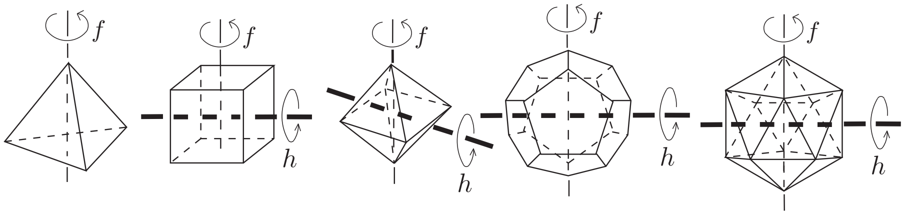

Figure 5. $\mathcal{T}$ is *not* symmetric in the vertical direction.

Table 7. In all cases, $f_1^2 = 1$, so $f_1^{-1} = f_1$.

$\mathcal{T}$	$f_1^{-1} = f_1$,	$f_2^{-1} = f_2^2$,	$f_2^{-2} = f_2$
$\mathcal{H}$	$f_1^{-1} = f_1$,	$f_3^{-1} \sim f_3$,	$f_4^{-1} \sim f_4$
$\mathcal{O}$	$f_1^{-1} = f_1$,	$f_3^{-1} \sim f_3$,	$f_4^{-1} \sim f_4$
$\mathcal{D}$	$f_1^{-1} = f_1$,	$f_2^{-1} \sim f_2$,	$f_3^{-1} \sim f_3$
$\mathcal{I}$	$f_1^{-1} = f_1$,	$f_2^{-1} \sim f_2$,	$f_3^{-1} \sim f_3$
$\mathcal{S}$	$f_1^{-1} = f_1$,	$f_2^{-1} \sim f_2$,	$f_3^{-1} \sim f_3$

Table 8

$\mathcal{T}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{T.1}]$,	$X_{f_2^{-1}} = X_{f_2^2} = [\mathrm{T.3}]$,	$X_{f_2^{-2}} = X_{f_2} = [\mathrm{T.2}]$
$\mathcal{H}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{H.1}]$,	$X_{f_3^{-1}} = X_{f_3} = [\mathrm{H.3}]$,	$X_{f_4^{-1}} = X_{f_4} = [\mathrm{H.4}]$
$\mathcal{O}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{O.1}]$,	$X_{f_3^{-1}} = X_{f_3} = [\mathrm{O.3}]$,	$X_{f_4^{-1}} = X_{f_4} = [\mathrm{O.4}]$
$\mathcal{D}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{D.1}]$,	$X_{f_2^{-1}} = X_{f_2} = [\mathrm{D.2}]$,	$X_{f_3^{-1}} = X_{f_3} = [\mathrm{D.3}]$
$\mathcal{I}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{I.1}]$,	$X_{f_2^{-1}} = X_{f_2} = [\mathrm{I.2}]$,	$X_{f_3^{-1}} = X_{f_3} = [\mathrm{I.3}]$
$\mathcal{S}$	$X_{f_1^{-1}} = X_{f_1} = [\mathrm{S.1}]$,	$X_{f_2^{-1}} = X_{f_2} = [\mathrm{S.2}]$,	$X_{f_3^{-1}} = X_{f_3} = [\mathrm{S.3}]$

$$\chi(W) = (2 - l)(2 - 2g) + \sum_{i=1}^{l} \chi(|X_i|), \tag{12}$$

where $\chi(|X_i|)$ may be computed from (8). For $W = W_{\mathcal{P}}$, noting that $l = 3$, the Euler numbers of the singular fibers of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1$ and the Euler number $\chi(W_{\mathcal{P}})$ of $W_{\mathcal{P}}$ are as follows:

Table 9

$\mathcal{P}$	g	$\chi(X)$: X is a singular fiber	$\chi(W_{\mathcal{P}})$
$\mathcal{T}$	3	$\chi(\lvert\mathrm{T.1}\rvert) = 4$, $\chi(\lvert\mathrm{T.2}\rvert) = 3$, $\chi(\lvert\mathrm{T.3}\rvert) = 3$ (Proposition 3.2)	14
$\mathcal{H}$	5	$\chi(\lvert\mathrm{H.1}\rvert) = 2$, $\chi(\lvert\mathrm{H.3}\rvert) = 6$, $\chi(\lvert\mathrm{H.4}\rvert) = -2$ (Proposition 3.3)	14
$\mathcal{O}$	7	$\chi(\lvert\mathrm{O.1}\rvert) = 0$, $\chi(\lvert\mathrm{O.3}\rvert) = -4$, $\chi(\lvert\mathrm{O.4}\rvert) = 8$ (Proposition 3.5)	16
$\mathcal{D}$	11	$\chi(\lvert\mathrm{D.1}\rvert) = -4$, $\chi(\lvert\mathrm{D.2}\rvert) = 2$, $\chi(\lvert\mathrm{D.3}\rvert) = -4$ (Proposition 3.7)	14
$\mathcal{I}$	19	$\chi(\lvert\mathrm{I.1}\rvert) = -12$, $\chi(\lvert\mathrm{I.2}\rvert) = -12$, $\chi(\lvert\mathrm{I.3}\rvert) = 6$ (Proposition 3.8)	18
$\mathcal{S}$	31	$\chi(\lvert\mathrm{S.1}\rvert) = -24$, $\chi(\lvert\mathrm{S.2}\rvert) = -20$, $\chi(\lvert\mathrm{S.3}\rvert) = -12$ (Proposition 3.10)	4

Generalization.

We may generalize the construction of $\pi : W_{\mathcal{P}} \to \mathbb{P}^1$ by replacing $\mathbb{P}^1$ with a Riemann surface on which $\mathrm{Aut}_+(\mathcal{P})$ acts holomorphically. Any singular fiber of the resulting fibration coincides with the singular fiber of some degeneration associated with $\mathcal{P}$. We will subsequently study these fibrations — among them the fibration constructed in the present paper is most *canonical*.

References

[1] W. Barth, K. Hulek, C. Peters and A. Van de Ven, Compact Complex Surfaces, Springer, 2000.
[2] P. Cromwell, Polyhedra, Cambridge University Press, 1997.
[3] R. Friedman, Algebraic surfaces and holomorphic vector bundles, Springer, 1998.
[4] W. Fulton and J. Harris, Representation Theory, Springer, 2004.
[5] R. Hirakawa and S. Takamura, "Version with More Figures" of this paper, https://www.math.kyoto-u.ac.jp/~takamura/1HiTa_figure.pdf
[6] S. Kerckhoff, The Nielsen realization problem, Ann. of Math., **117** (1983), 235–265.
[7] K. Kodaira, On compact complex analytic surfaces II, Ann. of Math., **77** (1963), 563–626.
[8] Y. Namikawa and K. Ueno, The complete classification of fibers in pencils of curves of genus two, Manuscripta Math., **9** (1973), 143–186.
[9] J. Nielsen, The structure of periodic surface transformations, Mat.-Fys. Medd. Danske Vid. Selsk., **15**, (1937), in Collected Papers, **2** (1986), 65–102, Birkhäuser.
[10] S. Takamura, Towards the classification of atoms of degenerations, II, (Cyclic quotient construction of degenerations of complex curves), RIMS preprint, **1344**, 2001.
[11] S. Takamura, Towards the classification of atoms of degenerations, III, (Splitting Deformations of Degenerations of Complex Curves), Springer Lecture Notes in Math., **1886**, 2006.
[12] J. A. Wolf, Spaces of constant curvature, AMS Chelsea Publishing, 2011.

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