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A class of minimal submanifolds in spheres

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Abstract. We introduce a class of minimal submanifolds M^n, $n \geq 3$, in spheres $\mathbb{S}^{n+2}$ that are ruled by totally geodesic spheres of dimension $n-2$. If simply-connected, such a submanifold admits a one-parameter associated family of equally ruled minimal isometric deformations that are genuine. As for compact examples, there are plenty of them but only for dimensions $n = 3$ and $n = 4$. In the first case, we have that M^3 must be a $\mathbb{S}^1$-bundle over a minimal torus T^2 in $\mathbb{S}^5$ and in the second case M^4 has to be a $\mathbb{S}^2$-bundle over a minimal sphere $\mathbb{S}^2$ in $\mathbb{S}^6$. In addition, we provide new examples in relation to the well-known Chern-do Carmo–Kobayashi problem since taking the torus T^2 to be flat yields minimal submanifolds M^3 in $\mathbb{S}^5$ with constant scalar curvature.

Introduction.

In several directions, this paper should be considered as a continuation of our work in [**8**] where a new class of minimal ruled submanifolds M^n of Euclidean space $\mathbb{R}^{n+2}$, $n \geq 3$, were studied. These submanifolds lie in codimension two and may be metrically complete regardless the dimension. The rulings are of codimension two in the manifold whereas the rank, that is, the complement of the index of relative nullity, is $\rho = 4$ (unless $n = 3 = \rho$) along an open dense subset. If simply-connected, the submanifolds admit a $\mathbb{S}^1$-parameter family of genuine isometric deformations. Hence, this class of examples should be seen as a new addition to the possible, local or global, classification of Euclidean submanifolds in codimension two that admit genuine isometric deformations; see [**8**] for a discussion of that open problem.

In this paper, we consider a similar construction but for the round sphere as ambient space. We obtain minimal submanifolds M^n in $\mathbb{S}^{n+2}$, $n \geq 3$, with similar properties as the ones in the Euclidean space. Notice that being ruled now means that the submanifold carries a foliation by (open subsets of) totally geodesic spheres in $\mathbb{S}^{n+2}$ of dimension $n-2$. If the manifold is simply-connected, by taking the cones in $\mathbb{R}^{n+3}$ of the components in the associated family in $\mathbb{S}^{n+2}$ we obtain a new class of genuinely deformable Euclidean submanifolds in codimension two but, of course, these are not complete.

New examples of minimal submanifolds in spheres are certainly welcome since the explicitly known ones are usually quite elaborate and certainly less abundant than in the Euclidean space. Frequently, they are spheres of constant sectional curvature or products of them. On the other hand, the submanifolds here introduced can only be complete, or even compact, for dimensions $n = 3$ or 4. If compact and according to the dimension, the

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submanifold must be topologically either a $\mathbb{S}^1$-bundle over $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{S}^5$ or a $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ in $\mathbb{S}^6$.

The compact examples in the case of the torus are of particular interest by two quite different reasons. First, if we replace the torus by its universal cover we obtain a three-dimensional manifold that is not longer compact but has an $\mathbb{S}^1$-parameter family of isometric minimal deformations. But the compact submanifold itself only admits, at most, a finite set of isometric minimal deformations. The second reason, has to do with the well-known Chern-do Carmo–Kobayashi problem [**5**] that concerns compact minimal submanifolds of the sphere with constant scalar curvature. We show that if the torus considered is flat, and these were all parametrically described by Miyaoka [**12**], then M^3 has constant scalar curvature.

1. The results.

This section is devoted to state the results of the paper while proofs are left for the following one. Up to the last two results, the other theorems in this paper can be seen as the "spherical version" of the results obtained in [**8**] for submanifolds in Euclidean space.

Let $g\colon L^2 \to \mathbb{S}^{n+2}$ denote a substantial oriented minimal surface. As already recalled in [**8**] the normal bundle N_gL of g splits along an open dense subset of L^2 as

$$N_gL = N_1^g \oplus N_2^g \oplus \cdots \oplus N_m^g, \quad m = [(n+1)/2],$$

where each subbundle N_s^g, $1 \leq s \leq m$, is spanned by the corresponding $(s+1)^{th}$-fundamental form $\alpha_g^{s+1}\colon TL \times \cdots \times TL \to N_gL$ and has rank two except possible the last one that has rank one if n is odd.

If L^2 is simply-connected, there exists a one-parameter associated family of minimal isometric immersions. In fact, for each constant $\theta \in \mathbb{S}^1 = [0, \pi)$ consider the parallel orthogonal tensor field

$$J_\theta = \cos\theta I + \sin\theta J$$

where I is the identity map and J the complex structure determined by the metric and orientation. Then, the symmetric section $\alpha_g(J_\theta\cdot, \cdot)$ of the bundle $\mathrm{Hom}(TL \times TL, N_gL)$ satisfies the Gauss, Codazzi and Ricci equations with respect to the same induced normal connection; see [**6**] for details. Therefore, there exists an isometric minimal immersion $g_\theta\colon L^2 \to \mathbb{S}^{n+2}$ whose second fundamental form is

$$\alpha_{g_\theta}(X, Y) = \phi_\theta \alpha_g(J_\theta X, Y)$$

where $\phi_\theta\colon N_gL \to N_{g_\theta}L$ is the parallel vector bundle isometry that identifies the normal bundles as well as each normal subbundles N_s^g with $N_s^{g_\theta}$ for any $1 \leq s \leq m$.

In the sequel, let $g\colon L^2 \to \mathbb{S}^{n+2}$, $n \geq 2$ be a substantial 1-*isotropic* surface. This means that g is minimal and that the ellipse of curvature (of first order) at any point is a circle. Let L_0 be the open subset of L^2 where $\dim N_1^g(p) = 2$. It was shown in [**7**] that

$L^2 \smallsetminus L_0$ consists of isolated points and that the vector bundle $N_1^g|_{L_0}$ smoothly extends to a plane bundle over L^2, that we still denote by N_1^g.

Let $\pi\colon \Lambda_g \to L^2$ denote the vector bundle of rank $n-2$ whose fibers are the orthogonal complement in the normal bundle N_gL of g of its extended first normal bundle N_1^g. Then $F_g\colon M^n \to \mathbb{S}^{n+2}$ is the submanifold of $\mathbb{S}^{n+2}$ associated to g constructed by attaching at each point of the surface g the totally geodesic sphere $\mathbb{S}^{n-2}$ whose tangent space at that point is the fiber of Λ_g, that is,

$$(p,v) \in \Lambda_g \mapsto F_g(p,v) = \exp_{g(p)} v, \tag{1}$$

while dropping the singular points whenever they exist, i.e., points where the induced metric is singular. By definition F_g is an $(n-2)$-ruled submanifold, that is, there is an integrable tangent distribution of dimension $n-2$ whose leaves are mapped diffeomorphically by F_g onto open subsets of totally geodesic $(n-2)$-spheres of $\mathbb{S}^{n+2}$.

For simplicity, it is very convenient to do computations in terms of the cone of M^n in $\mathbb{S}^{n+2} \subset \mathbb{R}^{n+3}$, and then view M^n as the intersection of that cone with $\mathbb{S}^{n+2}$. More precisely, we consider the map $G_g\colon \mathbb{R} \times \Lambda_g \to \mathbb{R}^{n+3}$ given by

$$G_g(s,p,v) = sg(p) + v \tag{2}$$

and set $S_G = \{0\} \times \big(\Lambda_g^*(p) \smallsetminus \{0\}\big)$ where

$$\Lambda_g^*(p) = \{(p,v) \in \Lambda_g : v \perp N_2^g(p)\}.$$

In the next section, we show that the set of singular points of the metric induced by G_g consists of the vertex $V = (0,p,0)$ and the set S_G. Set

$$N^{n+1} = \mathbb{R} \times \Lambda_g \smallsetminus (V \cup S_G)$$

and denote $G = G_g|_{N^{n+1}}$. Thus, we have

$$M^n = \{(s,p,v) \in \mathbb{R} \times \Lambda_g \smallsetminus S_G : s^2 + \|v\|^2 = 1\}$$

and $F_g = G|_{M^n}$ where M^n is endowed with the induced metric. Observe that M^n is complete (respectively, compact) if and only if g is complete (respectively, compact) and S_G is empty. Notice also that S_G can only be empty for $n = 3, 4$.

In the sequel, we denote by $\mathcal{H}$ the tangent distribution orthogonal to the rulings. An embedded surface $j\colon L^2 \to M^n$ is called an *integral surface* of $\mathcal{H}$ if $j_*T_pL = \mathcal{H}(j(p))$ at every point $p \in L^2$.

THEOREM 1. *Let $g\colon L^2 \to \mathbb{S}^{n+2}$, $n \geq 3$, be a 1-isotropic substantial surface. Then the associated immersion $F_g\colon M^n \to \mathbb{S}^{n+2}$ is an $(n-2)$-ruled minimal submanifold with rank $\rho = 4$ (unless $n = 3 = \rho$) on an open dense subset of M^n. Moreover, the integral surface L^2 of $\mathcal{H}$ is totally geodesic and unique up to the one obtained by composing with the antipodal map.*

Conversely, let $F\colon M^n \to \mathbb{S}^{n+2}$ be an $(n-2)$-ruled minimal immersion with $n \geq 4$ and $\rho = 4$ (unless $n = 3 = \rho$) on an open dense subset of M^n. Assume that $\mathcal{H}$ admits a *totally geodesic integral surface $j\colon L^2 \to M^n$ which is a global cross section to the rulings. Then the surface $g = F \circ j\colon L^2 \to \mathbb{S}^{n+2}$ is 1-isotropic and F can be parametrized as F_g.*

The existence of genuine deformations is considered in the following result.

THEOREM 2. *Let $g\colon L^2 \to \mathbb{S}^{n+2}, n \geq 3$, be a simply-connected 1-isotropic substantial surface. Then F_g allows a smooth one-parameter family of minimal genuine isometric deformations $F_\theta\colon M^n \to \mathbb{S}^{n+2}$, $\theta \in \mathbb{S}^1$, such that $F_0 = F_g$ and each F_θ carries the same rulings and relative nullity leaves as F_g.*

The relation between the second fundamental forms of members of the associated family is given next, for simplicity, in terms of their cones.

THEOREM 3. *Let $g\colon L^2 \to \mathbb{S}^{n+2}, n \geq 3$, be a simply-connected 1-isotropic substantial surface. Then G allows an associated smooth one-parameter family of minimal genuine isometric immersions $G_\theta\colon N^{n+1} \to \mathbb{R}^{n+3}$, $\theta \in \mathbb{S}^1$, such that $G_0 = G$ and each G_θ carries the same rulings and relative nullity leaves as G.*

Moreover, there is a parallel vector bundle isometry $\Psi_\theta\colon N_G N \to N_{G_\theta} N$ such that the relation between the second fundamental forms is given by

$$\alpha_{G_\theta}(X,Y) = \Psi_\theta\big(R_{-\theta}\alpha_G(X,Y) + 2\kappa\sin(\theta/2)\beta(\mathcal{J}_{-\theta/2}X,Y)\big) \tag{3}$$

where R_θ is the rotation of angle θ on $N_G N$ that preserves orientation, κ is the radius of the ellipse of curvature of g and β is the traceless bilinear form defined by (17).

A substantial surface in even codimension $g\colon L^2 \to \mathbb{S}^{n+2}$ is called *pseudoholomorphic* when the ellipses of curvature of any order are circles at any point. In odd codimension, the surface is called *isotropic* when the ellipses of curvature of any order but for the last one-dimensional normal subbundle are circles at any point.

If $g\colon L^2 \to \mathbb{S}^{n+2}$ is pseudoholomorphic, then taking a rotation of angle $\theta \in \mathbb{S}^1$ that preserves orientation in each N_s^g, $s \geq 2$, induces an intrinsic isometry S_θ on M^n. The next result says that F_g is equivariant with respect to the one-parameter family of intrinsic isometries S_θ.

THEOREM 4. *If $g\colon L^2 \to \mathbb{S}^{n+2}$ is pseudoholomorphic, then $F_g \circ S_{-\theta}$ is congruent to F_θ for any $\theta \in \mathbb{S}^1$.*

We have that $F_g\colon M^3 \to \mathbb{S}^5$ or $F_g\colon M^4 \to \mathbb{S}^6$ is compact if and only if L^2 is compact and g is regular. The latter condition means that L_0 is empty and that N_2^g has constant dimension. According to a result of Asperti [**1**] any compact regular substantial minimal surface in $\mathbb{S}^5$ is a topological torus and in $\mathbb{S}^6$ is a topological sphere. For both cases, there are plenty of 1-isotropic examples. In fact, the tori in $\mathbb{S}^5$ include the flat ones described parametrically by Miyaoka [**12**] and those that are holomorphic with respect to the nearly Kaehler structure of $\mathbb{S}^6$ considered in [**3**], [**10**] and [**11**]. Other examples of 1-isotropic surfaces in $\mathbb{S}^5$ are the Legendrian surfaces given in [**13**].

Minimal 2-spheres in spheres have been investigated by Calabi, Barbosa and Chern among others. From their work, we know that these surfaces must be substantial in even codimension and pseudoholomorphic. It was then shown by Calabi [**4**] that any such surface in $\mathbb{S}^6$ is regular if its area is 24π. Then Barbosa [**2**] proved that the space of these surfaces is diffeomorphic to $SO(7,\mathbb{C})/SO(7,\mathbb{R})$, where $SO(7,\mathbb{C})$ denotes the set of 7×7 complex matrices that satisfy $AA^t = I$ and $\det A = 1$.

Concerning the set of genuine minimal isometric deformations of compact submanifolds constructed from tori we have the following result.

THEOREM 5. *Let $g\colon L^2 \to \mathbb{S}^5$ be a regular substantial isotropic surface. Then, the set of all equally ruled minimal isometric immersions of M^3 into $\mathbb{S}^5$ as $F_g\colon M^3 \to \mathbb{S}^5$ is finite or parametrized by a circle $\mathbb{S}^1$. If L^2 is compact then the set is necessarily finite.*

As discussed in the introduction the last result is of independent interest.

THEOREM 6. *Let g be a flat 1-isotropic torus in $\mathbb{S}^5$. Then $F_g\colon M^3 \to \mathbb{S}^5$ is a compact minimal submanifold with constant normalized scalar curvature $s = -1/3$.*

2. The proofs.

In this section, we provide several proofs for $n \geq 4$ but similar arguments take care of the case $n = 3$.

First we have already discussed the set of singular points of F_g.

PROPOSITION 7. *Let $g\colon L^2 \to \mathbb{S}^{n+2}, n \geq 4$, be a substantial oriented minimal surface. Then, the set of singular points of the map $G\colon \mathbb{R}\times\Lambda_g \to \mathbb{R}^{n+3}$ given by* (2) *consists of $V = (0,p,0)$ and S_G.*

PROOF. Fix $(s_0,p_0,v_0) \in \mathbb{R}\times\Lambda_g \smallsetminus \{V\}$. Choose a smooth orthonormal frame $\{e_5,\ldots,e_{n+2}\}$ of Λ_g on a neighborhood U of p_0 and set

$$v_0 = \sum_{i\geq 1} a_i e_{i+4}(p_0).$$

Consider the projection $\Pi\colon \mathbb{R}\times\Lambda_g \to L^2$ and parametrize $\Pi^{-1}(U)$ via the diffeomorphism $h\colon U\times\mathbb{R}^{n-1} \to \Pi^{-1}(U)$ given by

$$h(p,s,t_1,\ldots,t_{n-2}) = \Big(s,p,\sum_{i\geq 1} t_i e_{i+4}\Big).$$

That $(s_0,p_0,v_0) \in S_G$ means that there exists a non-zero vector

$$Z = X + \lambda_0\partial/\partial s + \sum_{i\geq 1}\lambda_i\partial/\partial t_i \in \ker(G\circ h)_*(p_0,s_0,a_1,\ldots,a_{n-2})$$

where $X \in T_{p_0}L$. Thus,

$$\lambda_0 g(p_0) + s_0 g_*(p_0)X + \sum_{i\geq 1} a_i\nabla^{\perp}_X e_{i+4}(p_0) + \sum_{i\geq 1}\lambda_i e_{i+4}(p_0) = 0.$$

Since $Z \neq 0$, we obtain that $\lambda_0 = 0$, $s_0 = 0$, $X \neq 0$ and

$$\sum_{i\geq 1} a_i \nabla^\perp_X e_{i+4}(p_0) + \sum_{i\geq 1} \lambda_i e_{i+4}(p_0) = 0.$$

It follows that

$$\langle v_0, \nabla^\perp_X \xi\rangle(p_0) = 0$$

for any $\xi \in N^g_1$. We easily conclude that $v_0 \perp N^g_2(p_0)$. The converse is immediate. $\square$

In the sequel, we argue for an open set of L^2 where all the normal subspaces N^g_s's of the substantial oriented minimal surface $g\colon L^2 \to \mathbb{S}^{n+2}$ have constant dimension. Choose local positively oriented orthonormal frames $\{e_1, e_2\}$ in TL and $\{e_3, e_4\}$ of N^g_1 such that

$$\alpha_g(e_1, e_1) = \kappa e_3 \quad \text{and} \quad \alpha_g(e_1, e_2) = \mu e_4$$

where κ, μ are the semi-axes of the ellipse of curvature. Take a local orthonormal normal frame $\{e_5, \ldots, e_{n+2}\}$ such that $\{e_{2r+1}, e_{2r+2}\}$ is positively oriented spanning N^g_r for every even r. When $n = 2m + 1$ is odd, then e_{2m+1} spans the last normal bundle. We refer to $\{e_1, \ldots, e_{n+2}\}$ as an *adapted frame* of g and consider the one-forms

$$\omega_{ij} = \langle \tilde{\nabla} e_i, e_j\rangle \ \text{ for } \ 1 \leq i, j \leq n + 2,$$

where $\tilde{\nabla}$ denotes the Riemannian connection in the ambient space. Using that

$$\alpha^3_g(e_1, e_1, e_1) + \alpha^3_g(e_1, e_2, e_2) = 0$$

we easily obtain

$$\omega_{45} = -\frac{1}{\lambda} * \omega_{35} \ \text{ and } \ \omega_{46} = -\frac{1}{\lambda} * \omega_{36} \tag{4}$$

where $\lambda = \mu/\kappa$ and $*$ denotes the Hodge operator, i.e., $*\omega(e) = -\omega(Je)$. Here J is the complex structure of L^2 induced by the orientation. We denote by

$$V = a_1e_1 + a_2e_2, \ \ W = b_1e_1 + b_2e_2, \ \ Y = c_1e_1 + c_2e_2 \ \text{ and } \ Z = d_1e_1 + d_2e_2$$

the dual vector fields of $\omega_{35}, \omega_{36}, \omega_{45}$ and ω_{46}, respectively. Then (4) is equivalent to

$$Y = -\frac{1}{\lambda} JV \ \text{ and } \ Z = -\frac{1}{\lambda} JW,$$

and hence

$$\lambda c_1 = a_2, \ \ \lambda c_2 = -a_1, \ \ \lambda d_1 = b_2 \ \text{ and } \ \lambda d_2 = -b_1. \tag{5}$$

Clearly, we have that $G\colon N^{n+1} \to \mathbb{R}^{n+3}$ is an immersion and

$$T_{(s,p,v)}N = \mathbb{R} \oplus T_{(p,v)}\Lambda_g = \mathbb{R} \oplus \mathcal{H}^G(p, v) \oplus \mathcal{V}(p, v)$$

where $\mathbb{R} = \operatorname{span}\{\partial/\partial s\}$ and $\mathcal{H}^G$ is the orthogonal complement of $\mathcal{V}$ in $T\Lambda_g$. Moreover, $\mathcal{V}$ denotes the vertical bundle of $\pi\colon \Lambda_g \to L^2$ given by $\mathcal{V} = \ker \pi_*$.

Fixed $(p, v) \in \Lambda_g$, let δ_v be the normal vector field defined in a neighborhood of p by

$$\delta_v(q) = \sum_{j\geq 5} \langle v, e_j(p)\rangle e_j(q). \tag{6}$$

Let β_i, $1 \leq i \leq 2$, be the curves in Λ_g satisfying $\beta_i(0) = (p, v)$ given by

$$\beta_i(t) = (c_i(t), \delta_v(c_i(t)))$$

where $c_i(t)$ is a smooth curve in a neighborhood of p satisfying $c_i'(0) = e_i(p)$. Set

$$Y_i = \beta_i'(0) \in T_{(p,v)}\Lambda_g,\ 1 \leq i \leq 2. \tag{7}$$

Let $G_i, H_i \in C^\infty(\Lambda_g),\ 1 \leq i \leq 2$, be the functions

$$G_i = t_2\omega_{56}^i + t_3\omega_{57}^i + t_4\omega_{58}^i,\quad H_i = -t_1\omega_{56}^i + t_3\omega_{67}^i + t_4\omega_{68}^i$$

where $\omega_{ij}^k = \omega_{ij}(e_k)$ and $t_j \in C^\infty(\Lambda_g)$ is defined by

$$t_j(q, w) = \langle w, e_{j+4}(q)\rangle,\ \ 1 \leq j \leq 4.$$

It is clear that $G_*(s, p, v)\mathcal{V} = (N_1^g(p))^\perp \subset N_g L(p)$ holds up to parallel identification in $\mathbb{R}^{n+3}$. The vector bundle $\mathcal{V}$ can be orthogonally decomposed as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ where $\mathcal{V}^1$ denotes the plane bundle determined by

$$G_*(s, p, v)\mathcal{V}^1 = N_2^g(p).$$

Let $\{E_3, E_4\}$ and $\{E_5, \dots, E_n\}$ be local orthonormal frames of $\mathcal{V}^1$ and $\mathcal{V}^0$, respectively, such that

$$G_*E_j = e_{j+2}\ \text{ for }\ 3 \leq j \leq n.$$

LEMMA 8. *The vectors $X_1, X_2 \in T_{(p,v)}\Lambda_g$ defined as*

$$X_i = Y_i + G_iE_3 + H_iE_4 - \sum_{j\geq 7} \langle \nabla_{e_i}^\perp \delta_v, e_j\rangle E_{j-2} \tag{8}$$

satisfy that $X_1, X_2 \in \mathcal{H}^G(p, v)$ and that

$$G_*X_1 = sg_*e_1 - \varphi_1 e_3 - \frac{1}{\lambda}\varphi_2 e_4,\quad G_*X_2 = sg_*e_2 - \varphi_2 e_3 + \frac{1}{\lambda}\varphi_1 e_4$$

where $\varphi_j = t_1^0 a_j + t_2^0 b_j$ and $t_j^0 = t_j(p, v)$. Moreover, the space $N_G N(s, p, v)$ is spanned by

$$\xi = g_*(t_1^0 V(p) + t_2^0 W(p)) + se_3(p),\quad \eta = g_*(t_1^0 Y(p) + t_2^0 Z(p)) + se_4(p).$$

In particular, if g is 1-isotropic then

$$\|X_1\| = \Omega = \|X_2\| \ \ \textit{with} \ \ \langle X_1, X_2\rangle = 0 \ \ \textit{and} \ \ \|\xi\| = \Omega = \|\eta\| \ \ \textit{with} \ \ \langle \xi, \eta\rangle = 0$$

where $\Omega^2 = s^2 + \|t_1^0 V(p) + t_2^0 W(p)\|^2$.

Proof. On one hand,

$$G_* Y_i = s g_* e_i(p) + \sum_{j\geq 3} \langle \nabla^\perp_{e_i} \delta_v, e_j\rangle(p) e_j(p), \ \ 1 \leq i \leq 2,$$

gives

$$G_* Y_i - \sum_{j\geq 5} \langle \nabla^\perp_{e_i} \delta_v, e_j\rangle(p) G_* E_{j-2} = s g_* e_i(p) - \sum_{3\leq k\leq 4} \langle \nabla^\perp_{e_i} e_k, \delta_v\rangle(p) e_k(p).$$

On the other hand,

$$\begin{aligned}
\langle \nabla^\perp_{e_i} \delta_v, e_5\rangle(p) &= -t_2^0 \omega^i_{56}(p) - t_3^0 \omega^i_{57}(p) - t_4^0 \omega^i_{58}(p) = -G_i(p, v),\\
\langle \nabla^\perp_{e_i} \delta_v, e_6\rangle(p) &= t_1^0 \omega^i_{56}(p) - t_3^0 \omega^i_{67}(p) - t_4^0 \omega^i_{68}(p) = -H_i(p, v),\\
\langle \nabla^\perp_{e_i} e_3, \delta_v\rangle(p) &= t_1^0 \omega^i_{35}(p) + t_2^0 \omega^i_{36}(p) = t_1^0 a_i(p) + t_2^0 b_i(p),\\
\langle \nabla^\perp_{e_i} e_4, \delta_v\rangle(p) &= t_1^0 \omega^i_{45}(p) + t_2^0 \omega^i_{46}(p) = t_1^0 c_i(p) + t_2^0 d_i(p).
\end{aligned}$$

Hence,

$$G_* X_i = s g_* e_i - (t_1^0 a_i + t_2^0 b_i) e_3 - (t_1^0 c_i + t_2^0 d_i) e_4, \ \ 1 \leq i \leq 2.$$

The remaining of the proof is straightforward using (5). □

Lemma 9. *The following equations hold*:

$$\xi_* \partial/\partial s = e_3, \ \ \eta_* \partial/\partial s = e_4, \tag{9}$$

$$\xi_* E_3 = g_* V, \ \ \xi_* E_4 = g_* W \ \ \textit{and} \ \ \xi_* = 0 \ \textit{on} \ \mathcal{V}^0, \tag{10}$$

$$\eta_* E_3 = g_* Y, \ \ \eta_* E_4 = g_* Z \ \ \textit{and} \ \ \eta_* = 0 \ \textit{on} \ \mathcal{V}^0, \tag{11}$$

$$\begin{aligned}
\xi_* X_1 = g_*\big(&(e_1(\varphi_1) - s\kappa) e_1 + e_1(\varphi_2) e_2 + \omega^1_{12} J(t_1^0 V + t_2^0 W) + G_1 V + H_1 W\big)\\
&+\kappa\varphi_1 e_3 + (s\omega^1_{34} + \lambda\kappa\varphi_2) e_4 + s a_1 e_5 + s b_1 e_6 - \varphi_1 g,
\end{aligned} \tag{12}$$

$$\begin{aligned}
\xi_* X_2 = g_*\big(&e_2(\varphi_1) e_1 + (e_2(\varphi_2) + s\kappa) e_2 + \omega^2_{12} J(t_1^0 V + t_2^0 W) + G_2 V + H_2 W\big)\\
&-\kappa\varphi_2 e_3 + (s\omega^2_{34} + \lambda\kappa\varphi_1) e_4 + s a_2 e_5 + s b_2 e_6 - \varphi_2 g,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\eta_* X_1 = g_*\big(&e_1(\psi_1) e_1 + (e_1(\psi_2) - s\lambda\kappa) e_2 + \sigma\omega^1_{12}(t_1^0 V + t_2^0 W) - \sigma G_1 J V - \sigma H_1 J W\big)\\
&-(s\omega^1_{34} - \kappa\psi_1) e_3 + \lambda\kappa\psi_2 e_4 + s\sigma a_2 e_5 + s\sigma b_2 e_6 - \psi_1 g,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\eta_* X_2 = g_*\big(&(e_2(\psi_1) - s\lambda\kappa) e_1 + e_2(\psi_2) e_2 + \sigma\omega^2_{12}(t_1^0 V + t_2^0 W) - \sigma G_2 J V - \sigma H_2 J W\big)\\
&-(s\omega^2_{34} + \kappa\psi_2) e_3 + \lambda\kappa\psi_1 e_4 - s\sigma a_1 e_5 - s\sigma b_1 e_6 - \psi_2 g
\end{aligned} \tag{15}$$

where $\sigma = 1/\lambda$ *and* $\psi_j = t_1^0 c_j + t_2^0 d_j,\ j = 1, 2.$

PROOF. We compute at $(s, p, v) \in N^{n+1}$. Let $\gamma(t) = (s, p, v(t))$ be a curve in N^{n+1} such that $v(0) = v$, and thus $\gamma'(0) \in \mathcal{V}(p, v)$. We have that

$$\xi_* \gamma'(0) = \langle Dv/dt(0), e_5(p)\rangle g_* V(p) + \langle Dv/dt(0), e_6(p)\rangle g_* W(p),$$

or equivalently, that

$$\xi_* \gamma'(0) = \langle G_* \gamma'(0), e_5(p)\rangle g_* V(p) + \langle G_* \gamma'(0), e_6(p)\rangle g_* W(p).$$

From this we obtain (10). Similarly, we have (11).

To obtain (12) to (15) one has to use Lemma 8 and the Gauss and Weingarten formulas for g. We only argue for (12) since the proof of the other equations is similar. We have from (8) and (10) that

$$\xi_* X_i = \xi_* Y_i + G_i g_* V + H_i g_* W,\ \ 1 \le i \le 2.$$

In view of (7) and since

$$\xi \circ \beta_i(t) = t_1^0 g_* V(c_i(t)) + t_2^0 g_* W(c_i(t)) + s e_3(c_i(t)),$$

we obtain in terms of the connection in L^2 that

$$\begin{aligned}\xi_* Y_i = {}& t_1^0\big(g_* \nabla_{e_i} V + \alpha_g(e_i, V)\big)(p) + t_2^0\big(g_* \nabla_{e_i} W + \alpha_g(e_i, W)\big)(p)\\ &+ (-1)^i s\kappa(p) g_* e_i(p) + s\nabla^{\perp}_{e_i} e_3(p),\end{aligned}$$

and (12) follows by a direct computation. $\square$

LEMMA 10. *The second fundamental form of G in terms of the orthonormal frame*

$$E_0 = \partial/\partial s,\ \ E_i = X_i/\Omega,\ i = 1, 2,\ \ \textit{and}\ \ G_* E_j = e_{j+2},\ 3 \le j \le n,$$

vanishes along $\mathcal{V}^0$ *and restricted to* $\operatorname{span}\{E_0\} \oplus \mathcal{H}^G \oplus \mathcal{V}^1$ *is given by*

$$A_\xi = \begin{bmatrix} 0 & \bar\varphi_1 & \bar\varphi_2 & 0 & 0\\ \bar\varphi_1 & h_1 + \kappa & h_2 & r_1 & s_1\\ \bar\varphi_2 & h_2 & -h_1 - \kappa & r_2 & s_2\\ 0 & r_1 & r_2 & 0 & 0\\ 0 & s_1 & s_2 & 0 & 0\end{bmatrix},\ \ A_\eta = \begin{bmatrix} 0 & \bar\varphi_2 & -\bar\varphi_1 & 0 & 0\\ \bar\varphi_2 & h_2 & \kappa - h_1 & r_2 & s_2\\ -\bar\varphi_1 & \kappa - h_1 & -h_2 & -r_1 & -s_1\\ 0 & r_2 & -r_1 & 0 & 0\\ 0 & s_2 & -s_1 & 0 & 0\end{bmatrix}$$

where $\bar\varphi_i \Omega = \varphi_i,\ r_i\Omega = -s a_i,\ s_i \Omega = -s b_i$ *and*

$$\begin{aligned}h_i = -\frac{s}{\Omega^2}\Big(& t_1(e_i(a_1) - a_2 B_i - b_1\omega_{56}^i) + t_2(e_i(b_1) - b_2 B_i + a_1\omega_{56}^i)\\ &+ t_3(a_1\omega_{57}^i + b_1\omega_{67}^i) + t_4(a_1\omega_{58}^i + b_1\omega_{68}^i)\Big)\end{aligned}$$

with $B_i = \omega_{12}^i + \omega_{34}^i,\ \ i = 1, 2.$

Proof. Since g is 1-isotropic, then (12) to (15) hold for $\psi_1 = \varphi_2$ and $\psi_2 = -\varphi_1$. On the other hand, a straightforward computation shows that the Ricci equations

$$\langle R^\perp(e_1, e_2)e_\alpha, e_\beta\rangle = 0$$

for $\alpha = 3, 4$ and $\beta = 5, 6$ are equivalent to

$$\begin{aligned}
e_1(a_2) - e_2(a_1) + a_1B_1 + a_2B_2 - b_2\omega_{56}^1 + b_1\omega_{56}^2 &= 0,\\
e_1(b_2) - e_2(b_1) + b_1B_1 + b_2B_2 + a_2\omega_{56}^1 - a_1\omega_{56}^2 &= 0,\\
e_1(a_1) + e_2(a_2) - a_2B_1 + a_1B_2 - b_1\omega_{56}^1 - b_2\omega_{56}^2 &= 0,\\
e_1(b_1) + e_2(b_2) - b_2B_1 + b_1B_2 + a_1\omega_{56}^1 + a_2\omega_{56}^2 &= 0,
\end{aligned}$$

and for $\alpha = 3, 4$ and $\beta = 7, 8$ are equivalent to

$$\begin{aligned}
a_2\omega_{57}^1 - a_1\omega_{57}^2 + b_2\omega_{67}^1 - b_1\omega_{67}^2 &= 0,\\
a_2\omega_{58}^1 - a_1\omega_{58}^2 + b_2\omega_{68}^1 - b_1\omega_{68}^2 &= 0,\\
a_1\omega_{57}^1 + a_2\omega_{57}^2 + b_1\omega_{67}^1 + b_2\omega_{67}^2 &= 0,\\
a_1\omega_{58}^1 + a_2\omega_{58}^2 + b_1\omega_{68}^1 + b_2\omega_{68}^2 &= 0.
\end{aligned}$$

We thus have that

$$\langle A_\xi E_i, E_j\rangle = -\langle G_*E_i, \xi_*E_j\rangle \ \text{ and } \ \langle A_\eta E_i, E_j\rangle = -\langle G_*E_i, \eta_*E_j\rangle,\ 0 \le i, j \le n,$$

and the result follows by a straightforward computation. $\square$

Proof of Theorem 1. We first prove the converse. If $F\colon M^n \to \mathbb{S}^{n+2}$, $n \ge 4$, is an $(n-2)$-ruled minimal immersion with rank $\rho = 4$ everywhere, then the tangent bundle splits as $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is orthogonal to the rulings. Moreover, we have that $\mathcal{V}$ splits as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ with the fibers of $\mathcal{V}^0$ being the relative nullity leaves.

The normal space of the surface $g = F \circ j$ at $x \in L^2$ is given by

$$N_gL(x) = F_*(j(x))\mathcal{V} \oplus N_FM(j(x)).$$

Being j totally geodesic, we have

$$\alpha_g(X, Y) = \alpha_F(j_*X, j_*Y) \tag{16}$$

for all $X, Y \in TL$. This and our assumptions imply that g is minimal.

Let $\pi\colon \Lambda_g \to L^2$ denote the subbundle of the normal bundle of g whose fiber at $x \in L^2$ is $F_*(j(x))\mathcal{V}$. We consider the cone $\mathcal{C}F\colon \mathbb{R} \times M^n \to \mathbb{R}^{n+3}$ given by

$$\mathcal{C}F(t, p) = tF(p).$$

Observe that

$$\mathcal{C}F(t, p) - \mathcal{C}F(u(t, p), j(x)) = \mathcal{C}F(t, p) - u(t, p)g \circ \pi(p) \in F_*(j(x))\mathcal{V}$$

for any $p \in M^n$, where $x = \pi(p)$, since p and $j(x)$ belong to the same leaf of $\mathcal{V}$ and

$$u(t,p) = t/\langle F(p), g \circ \pi(p)\rangle.$$

Since $\mathcal{C}F$ maps locally diffeomorphically the leaves of $\mathcal{V}$ onto affine subspaces, it follows that the map $T\colon \mathbb{R} \times M^n \to \mathbb{R} \times \Lambda_g$ given by

$$T(t,p) = (u(t,p), \pi(p), \mathcal{C}F(t,p) - u(t,p)g \circ \pi(p))$$

is a local diffeomorphism. Clearly the immersion $\tilde{G} = \mathcal{C}F \circ T^{-1}$ satisfies

$$\tilde{G}(s,x,v) = sg(x) + v,$$

i.e., $\tilde{G} = G_g$ is of the form (2). Identifying locally $\mathbb{R} \times M^n$ with $\mathbb{R} \times \Lambda_g$ via T, we have that $\mathcal{C}F = G_g = G$ and j is the zero section of Λ_g, i.e., we have the parametrization given by (2). The horizontal and the vertical bundles satisfy

$$G_*(s,p,v)\mathcal{V} = (N_1^g(p))^\perp \subset N_g L(p), \;\; G_*(s,p,v)\mathcal{H}^G \subset g_* T_p L \oplus (\Lambda_g(p))^\perp,$$

$$N_G N(s,p,v) \subset g_* T_p L \oplus (\Lambda_g(p))^\perp$$

and now (16) yields $N_1^g = \Lambda_g^\perp$.

It remains to see that g is 1-isotropic. For an adapted frame $\{e_1, \ldots, e_{n+2}\}$ of g set

$$g_{ij} = \langle G_* X_i, G_* X_j\rangle$$

and

$$b_{ij}^\xi = \langle \xi_* X_i, G_* X_j\rangle, \quad b_{ij}^\eta = \langle \eta_* X_i, G_* X_j\rangle, \;\; i,j = 1,2.$$

Using Lemma 8 and Lemma 9, we find that

$$g_{11} = s^2 + \varphi_1^2 + \sigma^2\varphi_2^2, \;\; g_{12} = (1-\sigma^2)\varphi_1\varphi_2, \;\; g_{22} = s^2 + \varphi_2^2 + \sigma^2\varphi_1^2,$$

$$\begin{aligned}
b_{11}^\xi &= s(e_1(\varphi_1) - s\kappa - \omega_{12}^1\varphi_2 + G_1a_1 + H_1b_1) - \kappa\varphi_1^2 - \sigma\varphi_2(s\omega_{34}^1 + \mu\varphi_2),\\
b_{12}^\xi &= s(e_1(\varphi_2) + \omega_{12}^1\varphi_1 + G_1a_2 + H_1b_2) - \kappa\varphi_1\varphi_2 + \sigma\varphi_1(s\omega_{34}^1 + \mu\varphi_2),\\
b_{21}^\xi &= s(e_2(\varphi_1) - \omega_{12}^2\varphi_2 + G_2a_1 + H_2b_1) + \kappa\varphi_1\varphi_2 - \sigma\varphi_2(s\omega_{34}^2 + \mu\varphi_1),\\
b_{22}^\xi &= s(e_2(\varphi_2) + s\kappa + \omega_{12}^2\varphi_1 + G_2a_2 + H_2b_2) + \kappa\varphi_2^2 + \sigma\varphi_1(s\omega_{34}^2 + \mu\varphi_1)
\end{aligned}$$

and

$$\begin{aligned}
b_{11}^\eta &= s(e_1(\psi_1) - \omega_{12}^1\psi_2 + \sigma G_1a_2 + \sigma H_1b_2) + s\omega_{34}^1\varphi_1 - \kappa(\varphi_1\psi_1 + \varphi_2\psi_2),\\
b_{12}^\eta &= s(e_1(\psi_2) - \mu + \omega_{12}^1\psi_1 - \sigma G_1a_1 - \sigma H_1b_1) + s\omega_{34}^1\varphi_2 + \kappa(\varphi_1\psi_2 - \varphi_2\psi_1),\\
b_{21}^\eta &= s(e_2(\psi_1) - \mu - \omega_{12}^2\psi_2 + \sigma G_2a_2 + \sigma H_2b_2) + s\omega_{34}^2\varphi_1 + \kappa(\varphi_1\psi_2 - \varphi_2\psi_1),\\
b_{22}^\eta &= s(e_2(\psi_2) + \omega_{12}^2\psi_1 - \sigma G_2a_1 - \sigma H_2b_1) + s\omega_{34}^2\varphi_2 + \kappa(\varphi_1\psi_1 + \varphi_2\psi_2).
\end{aligned}$$

From our assumptions, we have

$$g_{11}b^{\xi}_{22} - g_{12}(b^{\xi}_{12} + b^{\xi}_{21}) + g_{22}b^{\xi}_{11} = 0 \text{ and } g_{11}b^{\eta}_{22} - g_{12}(b^{\eta}_{12} + b^{\eta}_{21}) + g_{22}b^{\eta}_{11} = 0.$$

Viewing these as polynomials where the coefficients of t_1^4, t_2^4 and $t_1^2t_2^2$ must vanish gives

$$(\lambda^2 - 1)(a_1^2 + a_2^2)(a_1^2 - a_2^2) = 0 = (\lambda^2 - 1)(b_1^2 + b_2^2)(b_1^2 - b_2^2)$$

and

$$(\lambda^2 - 1)a_1a_2(a_1^2 + a_2^2) = 0 = (\lambda^2 - 1)b_1b_2(b_1^2 + b_2^2).$$

Hence $\lambda = 1$ since, otherwise, we would have that $\omega_{35} = \omega_{36} = \omega_{45} = \omega_{46} = 0$, and that is a contradiction.

We now prove the direct statement. Since $F_g = G|_M$, we obtain that $g = F_g \circ j$ where $j\colon L^2 \to M^n$ is given by $j(x) = (\pm 1, x, 0)$. Clearly, we have that j is an integral surface of the distribution orthogonal to the rulings that is totally geodesic and a global cross section to the rulings. Up to uniqueness of the integral surface the proof follows from Lemma 10.

Assume that there exists a second integral surface $\tilde{j}\colon L^2 \to M^n$. Set $\tilde{g} = F_g \circ \tilde{j}$ and let $\tilde{T}\colon \mathbb{R} \times M^n \to \mathbb{R} \times \Lambda_{\tilde{g}}$ be the local diffeomorphism given by

$$\tilde{T}(t,p) = (\tilde{u}(t,p), \pi(p), \mathcal{C}F(p) - \tilde{u}(t,p)\tilde{g} \circ \pi(p))$$

where

$$\tilde{u}(t,p) = t/\langle F(p), \tilde{g} \circ \pi(p)\rangle.$$

Then $\tilde{T} \circ T^{-1}\colon \mathbb{R} \times \Lambda_g \to \mathbb{R} \times \Lambda_{\tilde{g}}$ is given by

$$\tilde{T} \circ T^{-1}(s,x,v) = (\tilde{s}, x, v + sg(x) - \tilde{s}\tilde{g}(x)),$$

where $T^{-1}(s,x,v) = (t,p)$ and $\tilde{s} = \tilde{u}(t,p)$. Hence Λ_g and $\Lambda_{\tilde{g}}$ can be identified by parallel translation. Using that $sg(x) - \tilde{s}\tilde{g}(x) \in \Lambda_g(x)$, we obtain that $\tilde{g} = \pm g$. $\square$

The vertical bundle $\mathcal{V}$ of the submersion π given by $\mathcal{V} = \ker \pi_*$ can be orthogonally decomposed as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ on an open dense subset of L^2, where $\mathcal{V}^1$ denotes the plane bundle determined by N_2^g. In fact, this holds if N_1^g and N_2^g are subbundles, which we can assume without loss of generality. In the sequel, we consider the orthogonal decomposition of the tangent bundle of N^{n+1} given by

$$TN = \operatorname{span}\{\partial/\partial s\} \oplus \mathcal{H}^G \oplus \mathcal{V}$$

where we identify isometrically (and use the same notation) the subbundle $\mathcal{V}$ tangent to the rulings with the corresponding normal subbundle to g. Then, it follows from the proof that the relative nullity leaves of G are identified with the fibers of $\operatorname{span}\{\partial/\partial s\} \oplus \mathcal{V}^0$.

Let $\mathcal{J}$ denote the endomorphism such that $\mathcal{J}|_{\mathcal{H}^G}\colon \mathcal{H}^G \to \mathcal{H}^G$ is the almost complex structure in $\mathcal{H}^G$ determined by the orientation and restricted to $\mathrm{span}\{\partial/\partial s\} \oplus \mathcal{V}$ is the identity and set

$$\mathcal{J}_\theta = \cos\theta I + \sin\theta \mathcal{J}.$$

Proof of Theorem 3. For each $\theta \in \mathbb{S}^1$ consider the submanifold $G_\theta\colon N^{n+1} \to \mathbb{R}^{n+3}$ defined by

$$G_\theta(s,p,v) = s g_\theta(p) + \phi_\theta v$$

where $\phi_\theta\colon N_g L \to N_{g_\theta} L$ is the parallel vector bundle isometry that identifies the normal subbundles of g and of g_θ.

In the sequel, corresponding quantities of G_θ are denoted by the same symbol used for G marked with θ. That G_θ is isometric to G is immediate. Since the tangent frame $\{e_1, e_2\}$ has been fixed, we have for the adapted frames of g_θ that

$$e_3^\theta = \phi_\theta \circ R_\theta^1 e_3 \ \text{ and } \ e_4^\theta = \phi_\theta \circ R_\theta^1 e_4$$

where R_θ^1 is the rotation of angle θ on N_1^g. We complete the adapted frame choosing

$$e_j^\theta = \phi_\theta e_j, \quad 5 \le j \le n+2.$$

Clearly, it holds that $\omega_{34}^\theta = \omega_{34}$ and $\omega_{ij}^\theta = \omega_{ij}$ for $i,j \ge 5$. Moreover,

$$\omega_{35}^\theta = \cos\theta\omega_{35} - \sin\theta * \omega_{35} \ \text{ and } \ \omega_{36}^\theta = \cos\theta\omega_{36} - \sin\theta * \omega_{36}.$$

Hence, the dual vector fields of ω_{36}^θ and ω_{36}^θ are given, respectively, by

$$V_\theta = J_{-\theta} V \ \text{ and } \ W_\theta = J_{-\theta} W.$$

Thus,

$$\begin{aligned} a_1^\theta &= a_1 \cos\theta + a_2 \sin\theta, \ \ a_2^\theta = a_2 \cos\theta - a_1 \sin\theta,\\ b_1^\theta &= b_1 \cos\theta + b_2 \sin\theta, \ \ b_2^\theta = b_2 \cos\theta - b_1 \sin\theta. \end{aligned}$$

It follows from (6), (7) and (8) that

$$X_i^\theta = X_i, \ \ i = 1,2.$$

By Lemma 8, the normal bundle of G_θ is spanned by

$$\xi_\theta = g_{\theta_*} J_{-\theta}(t_1 V + t_2 W) + s\phi_\theta \circ R_\theta^1 e_3, \quad \eta_\theta = -g_{\theta_*} J_{\pi/2-\theta}(t_1 V + t_2 W) + s\phi_\theta \circ R_\theta^1 e_4.$$

A straightforward computation yields that the map $\Psi_\theta\colon N_G N \to N_{G_\theta} N$ given by

$$\Psi_\theta \xi = \xi_\theta \ \text{ and } \ \Psi_\theta \eta = \eta_\theta$$

is a parallel vector bundle isometry. The shape operators of G_θ vanish on $\mathcal{V}^0$ and restricted to $\operatorname{span}\{\partial/\partial s\} \oplus \mathcal{H}^G \oplus \mathcal{V}^1$ and with respect to $\{E_1, \ldots, E_n\}$ they are given by

$$A^\theta_{\xi_\theta} = \begin{bmatrix} 0 & \bar\varphi^\theta_1 & \bar\varphi^\theta_2 & 0 & 0 \\ \bar\varphi^\theta_1 & h^\theta_1 + \kappa & h^\theta_2 & r^\theta_1 & s^\theta_1 \\ \bar\varphi^\theta_2 & h^\theta_2 & -h^\theta_1 - \kappa & r^\theta_2 & s^\theta_2 \\ 0 & r^\theta_1 & r^\theta_2 & 0 & 0 \\ 0 & s^\theta_1 & s^\theta_2 & 0 & 0 \end{bmatrix}, \quad A^\theta_{\eta_\theta} = \begin{bmatrix} 0 & \bar\varphi^\theta_2 & -\bar\varphi^\theta_1 & 0 & 0 \\ \bar\varphi^\theta_2 & h^\theta_2 & \kappa - h^\theta_1 & r^\theta_2 & s^\theta_2 \\ -\bar\varphi^\theta_1 & \kappa - h^\theta_1 & -h^\theta_2 & -r^\theta_1 & -s^\theta_1 \\ 0 & r^\theta_2 & -r^\theta_1 & 0 & 0 \\ 0 & s^\theta_2 & -s^\theta_1 & 0 & 0 \end{bmatrix}$$

where $\bar\varphi^\theta_i \Omega = \varphi^\theta_i$, $r^\theta_i \Omega = -s a^\theta_i$, $s^\theta_i \Omega = -s b^\theta_i$, $i = 1, 2$, and

$$\varphi^\theta_1 = \varphi_1 \cos\theta + \varphi_2 \sin\theta, \quad \varphi^\theta_2 = -\varphi_1 \sin\theta + \varphi_2 \cos\theta,$$
$$h^\theta_1 = h_1 \cos\theta + h_2 \sin\theta, \quad h^\theta_2 = -h_1 \sin\theta + h_2 \cos\theta.$$

Let $L_\theta \colon TN \to TN$ be such that $L_\theta|_{\operatorname{span}\{\partial/\partial s\}\oplus\mathcal{V}} = 0$ and $L_\theta|_{\mathcal{H}^G} \colon \mathcal{H}^G \to \mathcal{H}^G$ is the reflection given by

$$L_\theta|_{\mathcal{H}^G} = \begin{bmatrix} -\sin(\theta/2) & \cos(\theta/2) \\ \cos(\theta/2) & \sin(\theta/2) \end{bmatrix}$$

with respect to the tangent frame $\{E_1, E_2\}$. It follows easily that

$$A^\theta_{\Psi_\theta \xi} = A_{R_\theta \xi} - 2\kappa \sin(\theta/2) L_\theta \ \text{ and } \ A^\theta_{\Psi_\theta \eta} = A_{R_\theta \eta} - 2\kappa \sin(\theta/2) \mathcal{J} \circ L_\theta.$$

By direct computation, we obtain

$$\alpha_{G_\theta}(X,Y) = \Psi_\theta\Big(R_{-\theta}\alpha_G(X,Y) - \frac{2\kappa}{\Omega^2}\sin(\theta/2)(\langle L_\theta X, Y\rangle \xi + \langle L_\theta \mathcal{J} X, Y\rangle \eta)\Big).$$

Now let β be the symmetric section of $Hom(TN \times TN, N_G N)$ with nullity $\mathcal{V}$ given by

$$\beta(E_1, E_1) = \frac{1}{\Omega^2}\xi = -\beta(E_2, E_2), \ \ \beta(E_1, E_2) = -\frac{1}{\Omega^2}\eta, \tag{17}$$

and the proof of (3) follows easily.

Finally, that the isometric deformations G_θ of G are genuine is immediate from Lemma 10 since the shape operators of G have rank four for any normal direction along an open dense subset of N^{n+1}. □

Proof of Theorem 2. Given $\theta \in \mathbb{S}^1$, denote $F_\theta = G_\theta|_M$ where

$$G_\theta(s,p,v) = s g_\theta(p) + \phi_\theta v.$$

That F_g allows a one-parameter family of minimal isometric immersions $F_\theta \colon M^n \to \mathbb{S}^{n+2}$, $\theta \in \mathbb{S}^1$, such that $F_0 = F_g$ and each F_θ carries the same ruling and relative nullity leaves as F_g is a consequence of Proposition 3. □

Proof of Theorem 4. It is completely analogous to the proof of Theorem 6 in [**8**]. □

Proof of Theorem 5. Let $\bar{F}\colon M^3 \to \mathbb{S}^5$ be a ruled isometric minimal immersion with the same rulings as F_g and set $\bar{g} = \bar{F} \circ j$. From the proof of Theorem 1, we have that the surface $\bar{g}$ is isometric to g and isotropic. Hence, the set of all minimal isometric immersions of M^3 into $\mathbb{S}^5$ with the same rulings as F_g can be identified with the set of all isotropic immersions of L^2 into $\mathbb{S}^5$. The proof now follows from the results in [**9**]. $\square$

Proof of Theorem 6. Using Lemma 8 and Lemma 10, we have that the squared length of the second fundamental form of G is given by

$$\|\alpha_G\|^2 = \frac{4}{\Omega^4}\Big((1-K)\Omega^2 + \varphi_1^2 + \varphi_2^2 + \Omega^2 \sum_{i=1,2}(h_i^2 + r_i^2 + s_i^2)\Big).$$

It follows that

$$\|\alpha_G\|^2(s,p,v) = \frac{4}{\Omega^2}\Big(2 - K + h_1^2 + h_2^2 + \frac{s^2}{\Omega^2}(\|V\|^2 + \|W\|^2 - 1)\Big). \tag{18}$$

By Corollary 4 in [**15**] any 1-isotropic torus in $\mathbb{S}^5$ is regular, hence M^3 is compact. On the other hand, we have that g is $O(6)$-congruent to a holomorphic curve in the nearly Kaehler sphere $\mathbb{S}^6$; see [**10**] or [**15**]. Choose local orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that

$$\begin{aligned} &\alpha_g(e_1,e_1) = \sqrt{1/2}e_3, \quad \alpha_g(e_1,e_2) = \sqrt{1/2}e_4,\\ &\alpha_g^3(e_1,e_1,e_1) = \kappa_1 e_5, \quad \alpha_g^3(e_1,e_1,e_2) = 0, \end{aligned}$$

where $\kappa_1 = \sqrt{1/2}$ by Theorem 5 in [**15**]. Hence, we have that $V = e_1$. From Lemma 6 in [**14**] we obtain $h_1 = h_2 = 0$. Now (18) gives

$$\|\alpha_G\|^2(s,p,v) = \frac{8}{\Omega^2} = \frac{8}{s^2 + t_1^2},$$

and hence $\|\alpha_F\|^2 = 8$. $\square$

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