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Hypergroup structures arising from certain dual objects of a hypergroup

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Abstract. In the present paper hypergroup structures are investigated on distinguished dual objects related to a given hypergroup K, especially to a semi-direct product hypergroup $K = H \rtimes_\alpha G$ defined by an action α of a locally compact group G on a commutative hypergroup H. Typical dual objects are the sets of equivalence classes of irreducible representations of K, of infinite-dimensional irreducible representations of type I hypergroups K, and of quasi-equivalence classes of type II_1 factor representations of non-type I hypergroups K. The method of proof relies on the notion of a character of a representation of $K = H \rtimes_\alpha G$.

1. Introduction.

To investigate hypergroup structures on spaces of representations of a hypergroup is a challenging but also promising task. In general not even the duals of arbitrary locally compact groups admit a hypergroup structure. There are, however, various classes of commutative hypergroups K such that the space $\hat{K}$ of hypergroup characters is again a commutative hypergroup. In particular the space G^B of $\overline{B}$-orbits, where G is a locally compact group and B is a relatively compact subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Int}(G)$ have a hypergroup structure, as Hartmann, Henrichs and Lasser have shown in [**6**]. Also, the double coset hypergroups $G//H$ for Riemanian symmetric pairs (G, H), where H is a compact subgroup of G, admit a dual hypergroup.

Dual hypergroup structures occur in a natural way within the classes of polynomial and of Sturm–Liouville hypergroups, for which double dual hypergroups have been studied by Zeuner in [**18**]. See also [**2**]. In the non-commutative situation we just know that the group dual $\hat{G}$, i.e. the space of equivalence classes of irreducible representations of a compact group G, is a discrete commutative hypergroup. In the present paper we establish hypergroup structures for dual objects within three different settings.

In Section 3 we discuss a hypergroup structure for the set $\hat{K}$ of equivalence classes of irreducible representations of a non-commutative finite hypergroup K. In fact we show that the character set $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ of a semi-direct product hypergroup $H \rtimes_\alpha G$ is a commutative hypergroup if and only if the action α defining $H \rtimes_\alpha G$ satisfies a certain regularity condition. One notes that there is a finite hypergroup K for which no convolution can be introduced in $\mathcal{K}(\hat{K})$.

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Section 4 is devoted to discussing a hypergroup structure on the set $\widehat{K^\infty}$ of equivalence classes of infinite-dimensional irreducible representations of a hypergroup K of type I. There are two situations, for which it can be shown that the space $\mathcal{K}(\widehat{K^\infty})$ is a hypergroup which coincides with an orbital hypergroup. In both situations K is of the form $H \rtimes_\alpha G$, where H is a commutative hypergroup. In the first situation a compact Abelian group G acts on H, in the second one G is chosen to be a locally compact Abelian group such that for the stabilizer $G(\chi)$ of $\chi \in \hat{H}$, $G/G(\chi)$ is compact.

Finally, in Section 5, we are concerned with the case of hypergroups of non-type I. The object of interest is the set $\widehat{K^{\mathrm{II}_1}}$ of quasi-equivalence classes of type II_1 factor representations of a non-type I hypergroup $K = H \rtimes_\alpha G$. Under the condition that H is a commutative hypergroup of strong type and that a countably infinite discrete Abelian group G acts on H, the character set $\mathcal{K}(\widehat{K^{\mathrm{II}_1}})$ can be identified with an orbital hypergroup. Through the three Sections we provide illuminating examples.

2. Technical preparations.

In order to make precise some of the notions on representation theory we choose the more general framework of normed involutive algebras preferred by Dixmier in [**3**]. As a necessary reference to von Neumann algebras the seminal monographs [**3**] of Dixmer and [**16**] of Takesaki are proposed.

2.1. Representations of normed involutive algebras.

Let A be a normed involutive algebra. By a representation π of A with representing Hilbert space $\mathcal{H}(\pi)$ we mean a continuous homomorphism into the algebra $B(\mathcal{H}(\pi))$ of bounded linear operators on $\mathcal{H}(\pi)$. Given representations π_1 and π_2 of A with representing Hilbert spaces $\mathcal{H}(\pi_1)$ and $\mathcal{H}(\pi_2)$ respectively, π_1 is said to be equivalent to π_2, in symbols $\pi_1 \cong \pi_2$ if there exists an isomorphism ϕ from $\mathcal{H}(\pi_1)$ onto $\mathcal{H}(\pi_2)$ such that

$$\phi(\pi_1(x))\phi^{-1} = \pi_2(x)$$

for all $x \in A$. This definition gives rise to the notion of equivalence classes of representations of A. In the sequel we apply the convention

$$\pi(A) := \{\pi(x) : x \in A\}$$

for a representation π of A.

A representation π of A with representing Hilbert space $\mathcal{H}(\pi)$ is said to be irreducible if one of the following equivalent conditions are satisfied:

(i) The only closed $\pi(A)$-invariant subspaces of $\mathcal{H}(\pi)$ are $\{0\}$ and $\mathcal{H}(\pi)$.

(ii) The commutant $\pi(A)'$ of $\pi(A)$ in $B(\mathcal{H}(\pi))$ reduces to scalars.

(iii) The double commutant $\pi(A)''$ of $\pi(A)$ coincides with $B(\mathcal{H}(\pi))$.

If a representation π of A is finite-dimensional in the sense that $\dim\mathcal{H}(\pi) < \infty$, then π admits a direct decomposition of the form

$$\pi = \sum_{k=1}^{n}{}^{\oplus} \pi_k,$$

where π_k is an irreducible representation of A for each $k \in \{1, 2, \ldots, n\}$.

Now let π_1 and π_2 be representations of A, $\mathcal{A}(\pi_1)$ and $\mathcal{A}(\pi_2)$ the von Neumann algebras generated by $\pi_1(A)$ and $\pi_2(A)$ respectively. π_1 and π_2 are said to be quasi-equivalent, in symbols $\pi_1 \simeq \pi_2$ if there exists an isomorphism ϕ from $\mathcal{A}(\pi_1)$ onto $\mathcal{A}(\pi_2)$ such that

$$\phi(\pi_1(x)) = \pi_2(x)$$

for all $x \in A$.

Again one can speak of quasi-equivalence classes of representations of A. A representation π of A is called a factor representation provided $\mathcal{A}(\pi)$ is a factor in the sense of von Neumann algebras, i.e. provided $\mathcal{A}(\pi) \cap \mathcal{A}(\pi)' = \mathbb{C} \cdot 1$.

Clearly every irreducible representation of A is a factor representation, and on the space $\mathrm{Irr}(A)$ of all irreducible representations of A of type I the equivalence relations $\cong$ and $\simeq$ coincide. From the theory of von Neumann algebra we know the classification by types. In what follows we need to apply the types I and II_1. A representation π of A is said to be of type I if $\mathcal{A}(\pi)$ is of type I and to be of type II_1 if $\mathcal{A}(\pi)$ is of type II_1.

2.2. Representations of hypergroups.

We adopt the axiomatics of a (locally compact) hypergroup K from our previous publications as it has been set up in Jewett [**11**] and Bloom–Heyer [**2**]. For the reader's convenience we repeat a few basic definitions and facts from the analysis on K. Given a locally compact space X the space of bounded measures on X will be denoted by $M^b(X)$, its subspace of probability measures is symbolized by $M^1(X)$ and the space of bounded continuous functions on X is denoted by $C_b(X)$. The space $M^b(X)$ is equipped with the weak topology i.e. $\sigma(M^b(X), C_b(X))$-topology. For each $x \in X$, δ_x denotes the Dirac measure of X. There is a natural notion of homomorphism between hypergroups.

Let $\mathrm{Aut}(K) := \mathrm{Aut} M^b(X)$ stand for the set of automorphisms of K which together with the weak topology derived from the weak topology of $M^b(X)$ becomes a topological group. We call α an action of a locally compact group G on a given hypergroup K if α is a continuous homomorphism from G into $\mathrm{Aut}(K)$. We also need the notion of an action β of a hypergroup K on a locally compact space X defined by the following requirements:

1. β is a continuous Banach algebra homomorphism from $M^b(K)$ into the Banach algebra $B(M^b(X))$ of bounded operators on $M^b(X)$.
2. For $k \in K$ and $x \in X$, $\beta(\delta_k)\delta_x$ is a measure in $M^1(X)$ with compact support.

Occasionally we abbreviate the image $\beta(\delta_k)$ under β by $\beta(k)$ for $k \in K$. For every $x \in X$,

$$\mathrm{Orb}(x) := \bigcup_{k \in K} \mathrm{supp}(\beta(k)\delta_x)$$

denotes the orbit of x under the action β. We also write $O(x)$ instead of $\mathrm{Orb}(x)$. Moreover we apply the notions of smooth, irreducible and absorbing actions as introduced in [**9**].

Representations π of K with representing separable Hilbert space $\mathcal{H}(\pi)$ are introduced as $*$-homomorphism from $M^b(K)$ into $B(\mathcal{H}(\pi))$ such that $\pi(\delta_e) = 1$ and such that for ξ, $\eta \in \mathcal{H}(\pi)$ the mapping

$$\mu \mapsto \langle \pi(\mu)\xi, \eta \rangle$$

is continuous on $M^b(K)$.

A short excursion to the case of a locally compact group seems to be in order. We recall the profound work of Tatsuuma [**17**] and Takesaki [**15**] on a duality theorem for general locally compact groups. We recall that a locally compact group G is said to be of type I if the von Neumann algebra $\mathcal{A}(\pi) = \pi(G)''$ is of type I for every representation π of G. Among the groups of type I we just mention Abelian groups, compact groups, Heisenberg groups, connected semi-simple Lie groups, connected nilpotent groups. Discrete group are of type I if and only if they possess an Abelian normal subgroup of finite index. By the way, Glimm showed in [**4**] that a second countable locally compact group G is of type I if and only if the dual $\hat{G}$ of G is smooth with respect to the topology derived from the Jacobson topology of the primitive space $\mathrm{Prim}C^*(G)$. This fact indicates that studying duals for type I groups is more promising. See also Mackey [**14**].

A simple example of a non-type I group is a discrete Mautner group $G = \mathbb{C} \rtimes_\alpha \mathbb{Z}$, where α is the irrational rotation of $\mathbb{Z}$ on $\mathbb{C}$ (See Baggett [**1**]).

A class of hypergroups that will be central to our studying dual objects related to hypergroups will be the class of semi-direct product hypergroups introduced in [**9**]. Given a hypergroup $H = (H, \circ)$, a locally compact group G and action α of G on H. Let $K := H \times G$ be the set product of H and G such that

$$M^b(K) = M^b(H) \otimes M^b(G)$$

as Banach $*$-algebras, where the cross norm of the tensor product is given by

$$\| \mu \| = \sup \left\{ \left| \sum_{k=1}^{n} \mu_{1,k}(f)\mu_{2,k}(g) \right| : f \in C_c(H), \| f \|_\infty \leq 1; g \in C_c(G), \| g \|_\infty \leq 1 \right\}$$

for $\mu = \sum_{k=1}^{n} \mu_{1,k} \otimes \mu_{2,k} \in M^b(H) \otimes_{alg} M^b(G)$. One defines a convolution of Dirac measures in $M^b(K)$ by

$$\varepsilon_{(h_1,g_1)} *_\alpha \varepsilon_{(h_2,g_2)} := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)}) \otimes \delta_{g_1 g_2}$$

with unit element

$$\varepsilon_{(e,e)} := \varepsilon_e \otimes \delta_e,$$

where e denotes the unit element of H as well as of G, and an involution

$$(\mu \otimes \delta_g)^- := \alpha_g^{-1}(\mu^-) \otimes \delta_{g^{-1}} = \alpha_g^{-1}(\mu)^- \otimes \delta_{g^{-1}}$$

for $(h_1, g_1), (h_2, g_2) \in K$, $g \in G$, $\mu \in M^b(H)$. It turns out that $(K, *_\alpha)$ is in fact a hypergroup.

3. Duals related to finite hypergroups.

Let K be a finite hypergroup. For an irreducible representation π of K its character $ch(\pi)$ is given by

$$ch(\pi) := \frac{1}{\dim\pi}\mathrm{tr}(\pi(k))$$

for all $k \in K$. We consider the character set

$$\mathcal{K}(\hat{K}) := \{ch(\pi) : \pi \in \hat{K}\}.$$

We shall say, that the dual $\hat{K}$ of K *admits a hypergroup structure* if $\mathcal{K}(\hat{K})$ is a hypergroup with respect to the product of functions on K. Clearly, the dual $\hat{G}$ of a finite group G always admits a hypergroup structure.

Let α be an action of a finite Abelian group G on a finite commutative hypergroup H of strong type in the sense that the dual $\hat{H}$ of H has a hypergroup structure. Then a semi-direct product hypergroup $K = H \rtimes_\alpha G$ can be defined as in Heyer–Kawakami [**9**] (See also subsection 2.2 above). Let $\hat{\alpha}$ be the induced action of G on the dual $\hat{H}$ of H, given by

$$(\hat{\alpha}_g(\chi))(h) := \chi(\alpha_g^{-1}(h)),$$

where $g \in G$, $\chi \in \hat{H}$ and $h \in H$. Let

$$G(\chi) := \{g \in G : \hat{\alpha}_g(\chi) = \chi\}$$

be the stabilizer of $\chi \in \hat{H}$ under the action $\hat{\alpha}$. Now, let

$$\hat{H} := \{\chi_0, \chi_1, \dots, \chi_n\},$$

where χ_0 denotes the trivial character of H.

Definition. The action α is said to satisfy the *regularity condition* (or is called regular) provided

$$G(\chi_k) \supset G(\chi_i) \cap G(\chi_j)$$

for all $\chi_k \in \hat{H}$ such that

$$\chi_k \in \mathrm{supp}(\chi_i\chi_j) := \mathrm{supp}(\delta_{\chi_i} \hat{*} \delta_{\chi_j})$$

whenever $\chi_i, \chi_j \in \hat{H}$ and $\hat{*}$ symbolizes the convolution on $\hat{H}$, $k, i, j \in \{0, 1, \dots, n\}$.

Lemma 3.1. *If the action α satisfies the regularity condition, then the character set $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ of the semi-direct product hypergroup $H \rtimes_\alpha G$ is a commutative hypergroup, i.e. $\widehat{H \rtimes_\alpha G}$ admits a hypergroup structure.*

Proof. This fact is shown in [**12**] and [**7**], but we add another distinguished proof for consistency with the present paper. Let $\{O(\chi_0), O(\chi_1), \ldots, O(\chi_n)\}$ denote the set of orbits in $\hat{H}$ under the action $\hat{\alpha}$ of G such that $O(\chi_0) = \{\chi_0\}$ and $\chi_k \in O(\chi_k)$ for all $k = 0, 1, \ldots, n$. For each orbit $O(\chi_k)$ we set

$$\rho(O(\chi_k)) := \frac{1}{|G|} \sum_{g \in G} \hat{\alpha}_g(\chi_k) = \frac{1}{|O(\chi_k)|} \sum_{\sigma \in O(\chi_k)} \sigma.$$

Then the orbital hypergroup

$$K^{\hat{\alpha}}(\hat{H}) = \{\rho(O(\chi_0)), \rho(O(\chi_1)), \ldots, \rho(O(\chi_n))\}$$

carries the convolution

$$\rho(O(\chi_i))\rho(O(\chi_j)) = \sum_{k=0}^{n} a_{ij}^k \rho(O(\chi_k)),$$

where $a_{ij}^k \geq 0$ for all $i, j, k \in \{0, 1, \ldots, n\}$ and $\sum_{k=0}^{n} a_{ij}^k = 1$. Now we specify

$$\mathrm{supp}(\rho(O(\chi_i))\rho(O(\chi_j))) = \{O(\chi_k) : a_{ij}^k \neq 0 \text{ for } k = 0, 1, \ldots, n\}.$$

By the Mackey machine (Theorem 7.1 of [**9**]) each irreducible representation π of $H \rtimes_\alpha G$ is of the form

$$\pi = \pi^{(\chi,\tau)} = \mathrm{ind}_{H \rtimes_\alpha G(\chi)}^{H \rtimes_\alpha G} (\chi \odot \tau)$$

for some $\chi \in \hat{H}$ and $\tau \in \widehat{G(\chi)}$, where

$$(\chi \odot \tau)(h, g) = \chi(h)\tau(g)$$

for $h \in H$, $g \in G$. Moreover, from [**10**] we deduce that

$$ch(\pi^{(\chi,\tau)})(h, g) = \rho(O(\chi))(h) \cdot \tau(g) \cdot 1_{G(\chi)}(g).$$

The structure of the convolution of $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ is described as follows:

$$\begin{aligned} ch(\pi^{(\chi_i,\tau_i)})ch(\pi^{(\chi_j,\tau_j)}) &= \rho(O(\chi_i))\tau_i 1_{G(\chi_i)} \rho(O(\chi_j))\tau_j 1_{G(\chi_j)} \\ &= \rho(O(\chi_i))\rho(O(\chi_j))\tau_i\tau_j 1_{G(\chi_i)} 1_{G(\chi_j)} \\ &= \sum_{k=0}^{n} a_{ij}^k \rho(O(\chi_k))\tau_i\tau_j 1_{G(\chi_i)\cap G(\chi_j)}. \end{aligned}$$

For $O(\chi_k) \in \mathrm{supp}(\rho(O(\chi_i))\rho(O(\chi_j)))$ we put

$$A(k) := A(\chi_k, \tau_i\tau_j) = \{\tau \in \widehat{G(\chi_k)} : \tau(g) = \tau_i(g)\tau_j(g) \text{ for all } g \in G(\chi_i) \cap G(\chi_j)\}.$$

Now it is easy to see that the character $ch(\pi)$ of the representation

$$\sigma = \sum_{\tau \in A(k)}{}^{\oplus} \tau$$

takes on the form

$$ch(\sigma) = \frac{1}{|A(k)|} \sum_{\tau \in A(k)} \tau 1_{G(\chi_k)} = \tau_i \tau_j 1_{G(\chi_i) \cap G(\chi_j)}.$$

As a consequence we obtain that

$$\begin{aligned} ch(\pi^{(\chi_i,\tau_i)})ch(\pi^{(\chi_j,\tau_j)}) &= \sum_{k=0}^{n} a_{ij}^k \rho(O(\chi_k)) \frac{1}{|A(k)|} \sum_{\tau \in A(k)} \tau 1_{G(\chi_k)} \\ &= \sum_{k=0}^{n} \sum_{\tau \in A(k)} \frac{a_{ij}^k}{|A(k)|} \rho(O(\chi_k)) \tau 1_{G(\chi_k)} \\ &= \sum_{k=0}^{n} \sum_{\tau \in A(k)} \frac{a_{ij}^k}{|A(k)|} ch(\pi^{(\chi_k,\tau)}), \end{aligned}$$

and this shows the existence of a convolution on $\mathcal{K}(\widehat{H \rtimes_\alpha G})$. Among the axioms of a hypergroup for $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ we only mention the existence of an involution, namely

$$ch(\pi^{(\chi,\tau)})^- := ch(\pi^{(\chi^-,\tau^-)})$$

for all $\chi \in \hat{H}$, $\tau \in \widehat{G(\chi)}$. □

The converse of the statement in Lemma 3.1 is the content of

Lemma 3.2. *Let $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ be a commutative hypergroup, where α is any action of G on H. Then this action α satisfies the regularity condition.*

Proof. Let ι symbolize the trivial character of $G(\chi)$ for $\chi \in \hat{H}$. Given $\chi_i, \chi_j \in \hat{H}$ there exists a decomposition

$$\pi^{(\chi_i,\iota)} \otimes \pi^{(\chi_j,\iota)} \cong \sum_{k=1}^{\ell}{}^{\oplus} \pi_k,$$

where $\pi^{(\chi_i,\iota)}$, $\pi^{(\chi_j,\iota)}$ are the irreducible representations introduced in the proof of the previous Lemma and $\pi_k \in \widehat{H \rtimes_\alpha G}$. Since $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ is assumed to be a hypergroup, the equality

$$ch(\pi^{(\chi_i,\iota)})ch(\pi^{(\chi_j,\iota)}) = \sum_{k=1}^{\ell} a_{ij}^k ch(\pi_k) \qquad (*)$$

holds, with $a_{ij}^k > 0$ and $\sum_{k=1}^{\ell} a_{ij}^k = 1$, $i,j \in \{0,1,\dots,n\}$. For each $\chi_k \in \operatorname{supp}(\chi_i\chi_j)$ we have

$$O(\chi_k) \in \operatorname{supp}(\rho(O(\chi_i))\rho(O(\chi_j))),$$

hence there exists π_k such that

$$\pi_k = \pi^{(\chi_k, \tau)}$$

for some $\tau \in \widehat{G(\chi_k)}$. Now we assume that

$$G(\chi_k) \supset G(\chi_i) \cap G(\chi_j)$$

does not hold. Then there is a $g \in G(\chi_i) \cap G(\chi_j)$, but $g \notin G(\chi_k)$. For this element g of G and the unit $h_0 \in H$ we obtain

$$ch(\pi^{(\chi_i,\iota)})ch(\pi^{(\chi_j,\iota)})(h_0, g) = \rho(O(\chi_i))(h_0)\rho(O(\chi_j))(h_0)1_{G(\chi_i)\cap G(\chi_j)}(g) = 1.$$

On the other hand,

$$\sum_{k=1}^{\ell} a_{ij}^k ch(\pi_k)(h_0, g) < 1,$$

since

$$\begin{aligned} ch(\pi_k)(h_0, g) &= ch(\pi^{(\chi_k,\tau)})(h_0, g) \\ &= \rho(O(\chi_k)) \cdot \tau(g) \cdot 1_{G(\chi_k)}(g) \\ &= 0. \end{aligned}$$

This contradicts the above equality $(*)$, and the proof is complete. $\square$

As a summary we state

THEOREM 3.3. *The character set $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ of the semi-direct product hypergroup $H \rtimes_\alpha G$ is a commutative hypergroup if and only if the action α of G on H satisfies the regularity condition.*

Now we look at the semi-direct products

$$D(4) = \mathbb{Z}_4 \rtimes_\alpha \mathbb{Z}_2,$$

and

$$W(4) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2$$

with β being the flip action. We remark that $D(4) \cong W(4)$ as a group. Considering q-deformations of $D(4)$ and of $W(4)$ we have the following.

EXAMPLE 1. $D_q(4) = \mathbb{Z}_q(4) \rtimes_\alpha \mathbb{Z}_2 = \{h_0, h_1, h_2, h_3, g, h_1 g, h_2 g, h_3 g\}$ $(0 < q \le 1)$. The structure of $D_q(4)$ is given by the equalities

$$
\begin{aligned}
\delta_g \circ \delta_g &= \delta_{h_2 g} \circ \delta_{h_2 g} = \delta_{h_0}, \\
\delta_{h_1 g} \circ \delta_{h_1 g} &= \delta_{h_3 g} \circ \delta_{h_3 g} = \frac{1-q}{2}\delta_{h_1} + q\delta_{h_2} + \frac{1-q}{2}\delta_{h_3}, \\
\delta_{h_1 g} \circ \delta_{h_3 g} &= \delta_{h_3 g} \circ \delta_{h_1 g} = \frac{1-q}{2}\delta_{h_1} + q\delta_{h_2} + \frac{1-q}{2}\delta_{h_3}, \\
\delta_{h_1 g} \circ \delta_{h_2 g} &= \delta_{h_2 g} \circ \delta_{h_3 g} = \delta_{h_3 g}, \quad \delta_{h_3 g} \circ \delta_{h_2 g} = \delta_{h_2 g} \circ \delta_{h_1 g} = \delta_{h_1 g}.
\end{aligned}
$$

EXAMPLE 2. $W_q(4) = (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2 = \{h_0, h_1, h_2, h_3, g, h_1 g, h_2 g, h_3 g\}$ $(0 < q \leq 1)$. The structure of $W_q(4)$ is given by the following equalities.

$$
\begin{aligned}
\delta_g \circ \delta_g &= \delta_{h_0}, \\
\delta_{h_1 g} \circ \delta_{h_1 g} &= \delta_{h_2 g} \circ \delta_{h_2 g} = \delta_{h_3}, \\
\delta_{h_3 g} \circ \delta_{h_3 g} &= q^2 \delta_{h_0} + q(1-q)\delta_{h_1} + q(1-q)\delta_{h_2} + (1-q)^2 \delta_{h_3}, \\
\delta_{h_1 g} \circ \delta_{h_2 g} &= q\delta_{h_0} + (1-q)\delta_{h_1}, \\
\delta_{h_2 g} \circ \delta_{h_1 g} &= q\delta_{h_0} + (1-q)\delta_{h_2}, \\
\delta_{h_1 g} \circ \delta_{h_3 g} &= \delta_{h_3 g} \circ \delta_{h_2 g} = q\delta_{h_2} + (1-q)\delta_{h_3}, \\
\delta_{h_2 g} \circ \delta_{h_3 g} &= \delta_{h_3 g} \circ \delta_{h_1 g} = q\delta_{h_1} + (1-q)\delta_{h_3}.
\end{aligned}
$$

The quaternion group $Q(4)$ is interpreted as a twisted semi-direct product group as follows:

$Q(4) = \mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$, where the 2-cocycle $c : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ is given by

$$c(e,e) = c(e,g) = c(g,e) = h_0, \ c(g,g) = h_2.$$

We define the q-deformation $Q_q(4)$ of $Q(4)$ as a twisted semi-direct product hypergroup.

EXAMPLE 3. $Q_q(4) = \mathbb{Z}_q(4) \rtimes_\alpha^c \mathbb{Z}_2 = \{h_0, h_1, h_2, h_3, g, h_1 g, h_2 g, h_3 g\}$. The structure of $Q_q(4)$ is given by

$$
\begin{aligned}
\delta_{h_2 g} \circ \delta_{h_2 g} &= \delta_{h_2}, \\
\delta_g \circ \delta_g &= \delta_{h_2}, \\
\delta_{h_1 g} \circ \delta_{h_1 g} &= \delta_{h_3 g} \circ \delta_{h_3 g} = \frac{1-q}{2}\delta_{h_1} + q\delta_{h_2} + \frac{1-q}{2}\delta_{h_3}, \\
\delta_{h_1 g} \circ \delta_{h_3 g} &= \delta_{h_3 g} \circ \delta_{h_1 g} = \frac{1-q}{2}\delta_{h_1} + q\delta_{h_2} + \frac{1-q}{2}\delta_{h_3}, \\
\delta_{h_1 g} \circ \delta_{h_2 g} &= \delta_{h_2 g} \circ \delta_{h_3 g} = \delta_{h_1 g}, \\
\delta_{h_3 g} \circ \delta_{h_2 g} &= \delta_{h_2 g} \circ \delta_{h_1 g} = \delta_{h_3 g}.
\end{aligned}
$$

RESULTS.

1. $\widehat{W_q(4)}$ does not admit a hypergroup structure if $q \neq 1$, since the action β of $\mathbb{Z}_2$ on $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ does not satisfy the regularity condition.

2. $\widehat{D_q(4)}$ and $\widehat{Q_q(4)}$ admit hypergroup structures in the sense that $\mathcal{K}(\widehat{D_q(4)})$ and $\mathcal{K}(\widehat{Q_q(4)})$ are hypergroups respectively. Moreover we see that

$$\mathcal{K}(\widehat{D_q(4)}) \cong \mathcal{K}(\widehat{Q_q(4)}),$$

although $D_q(4)$ is not isomorphic to $Q_q(4)$ as a hypergroup. For more details we refer to [**13**].

4. Duals related to hypergroups of type I.

Let $K = H \rtimes_\alpha G$ be a semi-direct product hypergroup of a commutative hypergroup H of strong type by a smooth action α of a locally compact Abelian group G. There is the action $\hat{\alpha} : G \to \mathrm{Aut}(\hat{H})$ induced on $\hat{H}$ by

$$(\hat{\alpha}_g(\chi))(h) := \chi(\alpha_g^{-1}(h))$$

for all $\chi \in \hat{H}$, $g \in H$, $h \in H$. For $\chi \in \hat{H}$ let

$$G(\chi) := \{g \in G : \hat{\alpha}_g(\chi) = \chi\}$$

be the stabilizer of χ.

Assume that for $\chi \in \hat{H}$ such that χ is not the trivial character χ_0 of H

$$G(\chi) = D,$$

where G/D is compact.

Given an irreducible (infinite-dimensional) representation π of K which by the Mackey machine (Theorem 7.1 of [**9**]) is in the form

$$\pi = \pi^{(\chi,\tau)} = \mathrm{ind}_{H \rtimes_\alpha D}^{H \rtimes_\alpha G}(\chi \odot \tau)$$

for some $\chi \in \hat{H}$ and $\tau \in \hat{D}$, where

$$(\chi \odot \tau)(h, g) = \chi(h)\tau(g)$$

whenever $h \in H$, $g \in G$. Under the assumptions made there exists an $\hat{\alpha}$-invariant probability measure μ on $\hat{H}$ supported by the orbit $O(\chi) = \mathrm{Orb}(\chi)$ of χ. We note that the orbit $O(\chi)$ is compact and $O(\chi) \cong G/G(\chi)$.

Let $\tilde{\tau} \in \hat{G}$ be the extension of $\tau \in \widehat{G(\chi)}$ to G. Then the representation π is realized on $L^2(O(\chi), \mu)$ as follows:

For $\xi \in L^2(O(\chi), \mu)$

$$(\pi(h, g)\xi)(\sigma) = \sigma(h)\tilde{\tau}(g)\xi(\hat{\alpha}_g^{-1}(\sigma))$$

for all $\sigma \in O(\chi)$, $h \in H$, $g \in G$. Now we choose $\xi_0 \in L^2(O(\chi), \mu)$ such that $\xi_0(\sigma) = 1$ for all $\sigma \in O(\chi)$ and consider the spherical function $\psi := \psi^{(\chi,\tau)}$ on K associated with ξ_0, i.e.

$$\psi(h, g) = \langle \pi(h, g)\xi_0, \xi_0 \rangle$$

$$= \int_{O(\chi)} \sigma(h)\tilde{\tau}(g)\xi_0(\alpha_g^{-1}(\sigma))\overline{\xi_0(\chi)}\mu(d\sigma)$$
$$= \left(\int_{O(\chi)} \sigma(h)\mu(d\sigma)\right) \cdot \tilde{\tau}(g)$$

whenever $h \in H$, $g \in G$. With the definition

$$\rho(O(\chi)) := \int_{O(\chi)} \sigma \ \mu(d\sigma)$$

we see that

$$\psi(h, g) = \rho(O(\chi))(h)\tilde{\tau}(g)$$

and

$$\psi(h, g)|_{H \rtimes_\alpha D} = \rho(O(\chi))(h)\tau(g)$$

for all $h \in H$, $g \in G$. We denote the set $\{\psi^{(\chi,\tau)} : \chi \in \hat{H}, \tau \in \hat{D}\}$ by $\mathcal{K}(\widehat{K^\infty})$. Since the orbital hypergroup $K^{\hat{\alpha}}(\hat{H})$ can be written as

$$K^{\hat{\alpha}}(\hat{H}) = \{\rho(O(\chi)) : \chi \in \hat{H}\},$$

we obtain

Theorem 4.1.

$$\mathcal{K}(\widehat{K^\infty}) \cong K^{\hat{\alpha}}(\hat{H}) \times \hat{D}.$$

In the special case that $D = \{e\}$ *one has*

$$\mathcal{K}(\widehat{K^\infty}) \cong K^{\hat{\alpha}}(\hat{H}).$$

Example 1. Let $K = M(2) = \mathbb{C} \rtimes_\alpha \mathbb{T}$ be the two-dimensional motion group, where the action α of $\mathbb{T}$ on $\mathbb{C}$ is given by

$$\alpha_\zeta(z) := \zeta \cdot z$$

for all $\zeta \in \mathbb{T}$, $z \in \mathbb{C}$. Any infinite-dimensional representation of K has the form

$$\pi^\lambda = \mathrm{ind}_{\mathbb{C}}^K \chi^\lambda \ (\lambda > 0),$$

where

$$\chi^\lambda(z) := e^{i\mathrm{Re}(\lambda z)},$$

the reason for this being the fact that

$$G(\chi^\lambda) = \{1\}$$

for any $\lambda > 0$. It is easy to see that in this situation

$$\rho(O(\chi^\lambda))(z) = J_0(\lambda|z|)$$

for all $z \in \mathbb{C}$, where J_0 denotes the Bessel function of order 0. Hence

$$\mathcal{K}(\widehat{K^\infty}) \cong K^{\hat{\alpha}}(\mathbb{C})$$

coincides with the Bessel–Kingman hypergroup $BK(J_0)$ of order 0 (See [**2**]).

Example 2. Let $K = \mathbb{C} \rtimes_\alpha \mathbb{R}$, where the action α of $\mathbb{R}$ on $\mathbb{C}$ is given by

$$\alpha_\theta(z) := e^{i\theta} \cdot z$$

for all $\theta \in \mathbb{R}$, $z \in \mathbb{C}$. In this case

$$G(\chi^\lambda) = D = 2\pi\mathbb{Z}$$

for χ^λ $(\lambda > 0)$,

$$\hat{D} \cong \mathbb{T}$$

and

$$K^{\hat{\alpha}}(\mathbb{C}) = BK(J_0),$$

as we saw in Example 1. Consequently

$$\mathcal{K}(\widehat{K^\infty}) \cong K^{\hat{\alpha}}(\mathbb{C}) \times \hat{D} \cong BK(J_0) \times \mathbb{T}.$$

5. Duals related to hypergroups of non-type I.

In this section we assume given a countably infinite discrete Abelian group G, a commutative hypergroup H of strong type and an action α of G on H. By Heyer–Kawakami [**9**] the semi-direct product hypergroup $K = H \rtimes_\alpha G$ is defined. As before we have the induced action $\hat{\alpha}$ of G on the hypergroup dual $\hat{H}$ of H given by

$$(\hat{\alpha}_g(\chi))(h) = \chi(\alpha_g^{-1}(h))$$

for all $g \in G$, $\chi \in \hat{H}$ and $h \in H$. The stabilizer of $\chi \in \hat{H}$ under the action $\hat{\alpha}$ of G is again symbolized by $G(\chi)$. For the subsequent discussion the following *Assumptions* are made:

1. The action $\hat{\alpha}$ of G on $\hat{H}$ is free, i.e. $G(\chi) = \{e\}$ for all $\chi \in \hat{H}$ except the trivial character χ_0.
2. Every orbit in $\hat{H}$ under the action $\hat{\alpha}$ of G is relatively compact.

We note that in this case the action $\hat{\alpha}$ of G on $\hat{H}$ is non-smooth.

Under these assumptions $K = H \rtimes_\alpha G$ has a type II_1 factor representation and represents a hypergroup of non-type I.

Now, let Γ denote the set of closures of orbits in $\hat{H}$ under the action $\hat{\alpha}$ of G. For $O \in \Gamma$ such that $O \neq O(\chi_0)$ and an $\hat{\alpha}$-invariant ergodic probability measure μ on O such that $\mathrm{supp}(\mu) = O$, *the canonical type* II_1 *factor representation* π^O of $K = H \rtimes_\alpha G$ is defined on the space $L^2(O, \mu) \otimes \ell^2(G)$ in the following way:
For $\xi \in L^2(O, \mu) \otimes \ell^2(G)$

$$(\pi^O(h, g)\xi)(\chi, k) := \chi(h)\xi(\hat{\alpha}_g^{-1}(\chi), kg)$$

whenever $h \in H$, $g \in G$ and $\chi \in \hat{H}$, $k \in G$. Since

$$\{\pi^O(h, g) : h \in H, g \in G\}'' \cong L^2(O, \mu) \rtimes_{\hat{\alpha}} G,$$

π^O is a type II_1 factor representation of K.

Given a type II_1 factor representation π of K we introduce the character $ch(\pi)$ of π by

$$ch(\pi)(k) := \tau(\pi(k))$$

for all $k \in K$, where τ denotes the unique trace of the type II_1 factor $\pi(K)''$ (See [**3**] or [**16**]).

The dual object to be considered in this section will be the set $\widehat{K^{\mathrm{II}_1}}$ of quasi-equivalence classes of type II_1 factor representations of the (non-type I) hypergroup K. We are interested in studying the character set

$$\mathcal{K}(\widehat{K^{\mathrm{II}_1}}) := \{ch(\pi) : \pi \in \widehat{K^{\mathrm{II}_1}}\} \cup \{1_H\}.$$

For a type II_1 factor representation π of the hypergroup $K = H \times_\alpha G$ we write

$$\rho := \mathrm{Res}_H \pi$$

and

$$u := \mathrm{Res}_G \pi.$$

Indeed, for the canonical type II_1 factor representation π^O we have for the restrictions ρ^O and u^O that

$$(\rho^O(h)\xi)(\chi, k) = \chi(h)\xi(\chi, k)$$

and

$$(u^O\xi)(\chi, k) = \xi(\hat{\alpha}_g^{-1}(\chi), kg) \ (g \in G, \chi \in \hat{H}, k \in H)$$

respectively.

The von Neumann algebra $\rho(H)''$ generated by $\rho(H)$ is commutative, hence isomorphic to $L^\infty(O, \mu)$ for $O \in \Gamma$, $O \neq O_0$ and an $\hat{\alpha}$-invariant ergodic probability measure

$\mu =: \mu^{\pi}$ with $\mathrm{supp}(\mu^{\pi}) = O$. The corresponding imprimitivity relation

$$u_g \rho(f) u_g^* = \rho(\hat{\alpha}_g^{-1}(f))$$

holds, whenever $f \in L^{\infty}(O, \mu^{\pi})$.

LEMMA 5.1. *Let π be a type II_1 factor representation of the hypergroup $K = H \rtimes_{\alpha} G$ with representing Hilbert space $\mathcal{H}$ such that $\mu^{\pi} = \mu^{\pi^O}$. Then π is quasi-equivalent to π^O.*

PROOF. For any Hilbert space $\mathcal{H}$ we denote by $B(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the spaces of bounded and unitary operators on $\mathcal{H}$ respectively. Since the measure μ^{π} is ergodic, we may assume that

$$\mathcal{H} = L^2(O, \mu) \otimes \mathcal{H}_1, \ \ \rho^{\pi}(H)'' = L^{\infty}(O, \mu) \otimes \mathbb{C}$$

and

$$\rho^{\pi}(H)' = L^{\infty}(O, \mu) \otimes B(\mathcal{H}_1),$$

where $\mu = \mu^{\pi} = \mu^{\pi^O}$.

Since the von Neumann algebra $\pi(K)''$ is not type I, $\mathcal{H}_1$ must be infinite-dimensional. Then we may further assume that

$$\mathcal{H}_1 = \ell^2(G)$$

and

$$\mathcal{H} = L^2(O, \mu) \otimes \ell^2(G).$$

By applying

$$u_g \rho(f) u_g^* = \rho(\hat{\alpha}_g^{-1}(f))$$

and

$$u_g^O \rho(f) (u_g^O)^* = \rho(\hat{\alpha}_g^{-1}(f))$$

for $f \in L^{\infty}(O, \mu)$ we see that

$$u_g^O (u_g^O)^* \in L^{\infty}(O, \mu)' = L^{\infty}(O, \mu) \otimes B(\ell^2(G)).$$

Therefore there exists a $\mathcal{U}(\ell^2(G))$-valued 1-cocycle $c = c(\chi, k)$ of G on $\hat{H}$ satisfying

$$u_g \xi(\chi, k) = c(\chi, k) u_g^O \xi(\chi, k) = c(\chi, k) \xi(\hat{\alpha}_g^{-1}(\chi), kg).$$

The cocycle condition reads as

$$c(\chi, k_1 k_2) = c(\chi, k_1) c(\hat{\alpha}_{k_1}^{-1}(\chi), k_2)$$

for $\chi \in \hat{H}$, $k_1, k_2 \in G$. If we take $k_1 = k$, $k_2 = k^{-1}$, then

$$c(\chi, kk^{-1}) = c(\chi, k)c(\hat{\alpha}_k^{-1}(\chi), k^{-1})$$

and

$$c(\chi, kk^{-1}) = c(\chi, e) = I,$$

where I denotes the identity operator. Consequently,

$$c(\chi, k)c(\hat{\alpha}_k^{-1}(\chi), k^{-1}) = I$$

and

$$c(\hat{\alpha}_k(\chi), k)c(\chi, k^{-1}) = I.$$

Let W be an operator in $\mathcal{U}(L^2(O, \mu) \otimes \ell^2(G))$ given by

$$(W\xi)(\chi, k) := c(\hat{\alpha}_k(\chi), k)\xi(\chi, k)$$

for $\xi \in L^2(O, \mu) \otimes \ell^2(G)$. Clearly,

$$(W^*\xi)(\chi, k) = c(\chi, k^{-1})\xi(\chi, k)$$

for all $\chi \in \hat{H}$, $k \in G$. Now we obtain that

$$Wu_gW^* = u_g^O.$$

Indeed, for $\xi \in L^2(O, \mu) \otimes \ell^2(G)$, $\chi \in \hat{H}$, $k \in G$

$$\begin{aligned}
(Wu_gW^*\xi)(\chi, k) &= W(u_gW^*\xi)(\chi, k) \\
&= c(\hat{\alpha}_k(\chi), k)u_g(W^*\xi)(\chi, k) \\
&= c(\hat{\alpha}_k(\chi), k)c(\chi, g)(W^*\xi)((\hat{\alpha}_g^{-1}(\chi), kg) \\
&= c(\hat{\alpha}_k(\chi), k)c(\chi, g)c(\hat{\alpha}_g^{-1}(\chi), g^{-1}k^{-1})\xi(\hat{\alpha}_g^{-1}(\chi), kg) \\
&= c(\hat{\alpha}_k(\chi), k)c(\chi, g)c(\hat{\alpha}_g^{-1}(\chi), g^{-1})c(\chi, k^{-1})\xi(\hat{\alpha}_g^{-1}(\chi), kg) \\
&= c(\hat{\alpha}_k(\chi), k)Ic(\chi, k^{-1})\xi(\hat{\alpha}_g^{-1}(\chi), kg) \\
&= I\xi(\hat{\alpha}_g^{-1}(\chi), kg) \\
&= (u_g^O\xi)(\chi, k).
\end{aligned}$$

This means that the representation u of G is unitarily equivalent to u^O, i.e. the representation π of K is quasi-equivalent to π^O. □

Finally, let L be a compact Abelian group and G a countably infinite discrete subgroup of L which is dense in L. Suppose that the action $\hat{\alpha}$ of L on $\hat{H}$ is free.

LEMMA 5.2. *Under the assumption just stated any G-invariant ergodic probability* *measure μ with supp$(\mu) = \overline{\mathrm{Orb}_G(\chi)} = \mathrm{Orb}_L(\chi)$ exists and it is unique.*

Proof. Since the action $\hat{\alpha}$ of L on the orbit $O(\chi)$ of χ in $\hat{H}$ is smooth and free, there exists uniquely the ergodic probability measure μ on $O(\chi)$ under the action $\hat{\alpha}$ of L, which is equivariant to the normalized Haar measure of the compact group L. The action $\hat{\alpha}$ of L on the orbit $O(\chi)$ induces the continuous action $\hat{\alpha}$ of L on the C*-algebra $C(O(\chi))$ consisting of continuous functions on $O(\chi)$ with the uniform topology. For $f \in C(O(\chi))$, suppose that $\mu(\hat{\alpha}_g(f)) = \mu(f)$ for all $g \in G$. Then for any $\ell \in L$ one can take a sequence $g_1, g_2, \ldots, g_n, \ldots$ in G such that

$$\mu(\hat{\alpha}_\ell(f)) = \lim_{n\to\infty} \mu(\hat{\alpha}_{g_n}(f)),$$

since G is dense in L. Hence we have

$$\mu(\hat{\alpha}_\ell(f)) = \mu(f)$$

for all $\ell \in L$. This implies $f = c \cdot 1_{O(\chi)}$, $c \in \mathbb{C}$. Therefore the measure μ is also ergodic under the action $\hat{\alpha}$ of G. Let ν be an arbitrary ergodic probability measure under the action of G supporting the orbit $O(\chi)$. Then this measure ν is ergodic under the action of L so that $\nu = \mu$.

Hence the desired conclusion is obtained. □

In summary we obtain

Theorem 5.3. *Keeping the assumptions preceding Lemma 5.2 and considering the semi-direct product hypergroup $K = H \rtimes_\alpha G$ the character set $\mathcal{K}(\widehat{K^{II_1}})$ becomes a commutative hypergroup isomorphic to the orbital hypergroup $\mathcal{K}^L(\hat{H})$ of $\hat{H}$ under the action $\hat{\alpha}$ of L.*

Example 1 (Discrete Mautner group). Let $K = \mathbb{C} \rtimes_\alpha \mathbb{Z} \subset \mathbb{C} \rtimes_\alpha \mathbb{T}$. Then

$$\Gamma(\hat{\mathbb{C}}) = \{O^\lambda : \lambda \in \mathbb{R}_+\}$$

where $O^\lambda = \{z \in \mathbb{C} : |z| = \lambda\}$. For each $\lambda \in \mathbb{R}_+$ a type II_1 factor representation π^λ is defined and

$$\pi^\lambda(K)'' = L^\infty(O^\lambda) \rtimes_\alpha \mathbb{Z}$$

is a type II_1 factor whenever $\lambda \neq 0$. Moreover,

$$ch(\pi^\lambda)(z, n) = J_0(\lambda|z|) \cdot 1_{\{0\}}(n)$$

for all $z \in \mathbb{C}$, $n \in \mathbb{Z}$, and $\mathcal{K}(\widehat{K^{\mathrm{II}_1}})$ is isomorphic to the Bessel–Kingman hypergroup $BK(J_0)$ of order 0 (See [**2**]).

References

[1] L. Baggett, Representations of the Mautner group I, Pacific J. Math., **77** (1978), 7–22.

[2] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, De Gruyter Studies in Math., **20**, Walter de Gruyter, 1995.

[3] J. Dixmier, Les C^*-algèbras et leurs représentations, Gauthier-Villars, Paris 1957.

[4] J. G. Glimm, Locally compact transformation groups, Trans. Amer. Math. Soc., **101** (1961), 124–138.

[5] W. Hauenschild, E. Kaniuth and A. Kumar, Harmonic analysis on central hypergroups and induced representations, Pacific J. Math., **110** (1984), 83–112.

[6] K. Hartmann, W. Henrichs and R. Lasser, Duals of orbit spaces in groups with relatively compact inner automorphism groups are hypergroups, Monatsh. Math., **88** (1979), 229–238.

[7] H. Heyer, T. Jimbo, S. Kawakami and K. Kawasaki, Finite commutative hypergroups associated with actions of finite abelian groups, Bull. Nara Univ. Educ., **54** (2005), 23–29.

[8] H. Heyer and S. Kawakami, A cohomology approach to the extension problem for commutative hypergroups, Semigroup Forum, **83** (2011), 371–394.

[9] H. Heyer and S. Kawakami, An Imprimitivity theorem for representations of a semi-direct product hypergroup, J. of Lie Theory, **24** (2014), 159–178.

[10] H. Heyer, S. Kawakami and S. Yamanaka, Characters of induced representations of a compact hypergroups, Monatsh. Math., published online, 2015.

[11] R. I. Jewett, Spaces with an abstract convolution of measures, Adv. in Math., **18** (1975), 1–101.

[12] S. Kawakami and W. Ito, Crossed Products of Commutative Finite Hypergroups, Bull. Nara Univ. Educ., **48** (1999), 1–6.

[13] S. Kawakami, T. Tsurii and S. Yamanaka, Deformations of finite hypergroups, Sci. Math. Jap., published online, 2015.

[14] G. W. Mackey, Borel structures in groups and their duals, Trans. Amer. Math. Soc., **85** (1957), 134–165.

[15] M. Takesaki, A characterization of group algebras as a converse of Tannaka–Stinespring–Tatsuuma duality theorem, American. J. Math., **91** (1969), 529–564.

[16] M. Takesaki, Theory of Operator Algebra I, Springer Verlag, 1979.

[17] N. Tatsuuma, A duality theorem for locally compact groups, J. Math. Kyoto Univ., **6** (1967), 373–393.

[18] Hm. Zeuner, Duality of commutative hypergroups, In: Probability Measures on Groups, X (ed. H. Heyer), Plenum Press, New York, London, 1991, 467-488.

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