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Structure and equivalence of a class of tube domains with solvable groups of automorphisms

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Abstract. In the study of the holomorphic equivalence problem for tube domains, it is fundamental to investigate tube domains with polynomial infinitesimal automorphisms. To apply Lie group theory to the holomorphic equivalence problem for such tube domains T_Ω, investigating certain solvable subalgebras of $\mathfrak{g}(T_\Omega)$ plays an important role, where $\mathfrak{g}(T_\Omega)$ is the Lie algebra of all complete polynomial vector fields on T_Ω. Related to this theme, we discuss in this paper the structure and equivalence of a class of tube domains with solvable groups of automorphisms. Besides, we give a concrete example of a tube domain whose automorphism group is solvable and contains nonaffine automorphisms.

Introduction.

A tube domain T_Ω with polynomial infinitesimal automorphisms is a tube domain on which every complete holomorphic vector field is a polynomial vector field. We denote by $\mathfrak{g}(T_\Omega)$ the Lie algebra of all complete holomorphic vector fields on T_Ω. In the study of the holomorphic equivalence problem for tube domains, it is fundamental to investigate such tube domains. A Siegel domain of the first kind is a typical example of a tube domain with polynomial infinitesimal automorphisms, and then the structure of $\mathfrak{g}(T_\Omega)$ is clarified well. For example, it is known that $\mathfrak{g}(T_\Omega)$ has the direct sum decomposition as a graded Lie algebra, and so on. Furthermore, by using them, an affirmative answer to the holomorphic equivalence problem for Siegel domains of the first kind is given. But these results rely heavily on the peculiar own properties of Siegel domains of the first kind, and it is difficult to apply a similar argument or method directly to arbitrary tube domain T_Ω with polynomial infinitesimal automorphisms. In fact, even the direct sum decomposition of $\mathfrak{g}(T_\Omega)$ is not clear for such a case. Consequently, a new point of view is needed in order to deal with tube domains with polynomial infinitesimal automorphisms that are not necessarily Siegel domains of the first kind. The Prolongation Theorem given in [**6**] about complete polynomial vector fields on a tube domain assures the result that $\mathfrak{g}(T_\Omega)$ has some natural direct sum decomposition, and others, for aribitrary T_Ω, and gives a lead to our study.

In general, a well-known theorem of H. Cartan that the holomorphic automorphism group of a complex bounded domain has the structure of a Lie group enables us to apply the conjugacy theorems in Lie theory to the theory of complex bounded domains. To

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apply the conjugacy theorems to the holomorphic equivalence problem for tube domains T_Ω with polynomial infinitesimal automorphisms, investigating certain solvable subalgebras of $\mathfrak{g}(T_\Omega)$ plays an important role. A typical case is just the case where $\mathfrak{g}(T_\Omega)$ itself is solvable. In this paper, related to this theme, we discuss the structure and equivalence of a class of tube domains with solvable groups of automorphisms from the view point stated above.

This paper is organized as follows. In Section 1, we recall basic concepts and results on tube domains. In particular, we recall two important theorems called the Structure and Prolongation Theorems. Some consequences of the Prolongation Theorem are discussed in Section 2 together with lemmas on solvable subalgebras of $\mathfrak{g}(T_\Omega)$ for a tube domain T_Ω with polynomial infinitesimal automorphisms. In Section 3, we give a structure theorem for solvable $\mathfrak{g}(T_\Omega)$ as Theorem 3.1, which is a main result of this paper. More precisely speaking, let T_Ω be a tube domain in $\mathbf{C}^n$ with polynomial infinitesimal automorphisms and suppose that the base Ω of T_Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines. Then we clarify the structure of $\mathfrak{g}(T_\Omega)$ under the assumptions that the holomorphic automorphism group $\mathrm{Aut}(T_\Omega)$ of T_Ω is a solvable Lie group and has the orbit through some point of T_Ω with dimension $n+1$. Besides, as an application of Theorem 3.1, we give an affirmative answer to the holomorphic equivalence problem for such tube domains. Finally, Section 4 is devoted to a concrete example of a tube domain as in Theorem 3.1. Among tube domains with polynomial infinitesimal automorphisms, tube domains T_Ω whose bases Ω are convex cones are characteristic in the point that they have the property that if $\mathrm{Aut}(T_\Omega)$ is solvable, then it necessarily consists of affine transformations. The following example, given as Theorem 4.1, is an example of Theorem 3.1 as well as an example that there is a tube domain T_Ω such that $\mathrm{Aut}(T_\Omega)$ is solvable, but contains nonaffine automorphisms when Ω is not a convex cone.

Example. Let Ω_0 is a convex domain in $\mathbf{R}^3$ containing no complete straight lines given by

$$\Omega_0 = \{(y_1, y_2, y_3) \in \mathbf{R}^3 \mid y_2 > y_1^2 + e^{y_3^2} - 2\}.$$

Then T_{Ω_0} is a tube domain in $\mathbf{C}^3$ with polynomial infinitesimal automorphisms satisfying the assumptions of Theorem 3.1 that $\mathrm{Aut}(T_{\Omega_0})$ is a solvable Lie group and has the orbit through the origin of $\mathbf{C}^3$ in T_{Ω_0} with dimension 4. Moreover, Ω_0 is not a convex cone, and $\mathrm{Aut}(T_{\Omega_0})$ is solvable, but contains nonaffine automorphisms.

1. Preliminaries and background facts.

We first recall some notation and terminology. An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. The complex manifold M is called homogeneous if $\mathrm{Aut}(M)$ acts transitively on M. We denote by $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$ the subgroup of $\mathrm{Aut}(\mathbf{C}^n)$ consisting of all transformations of the form

$$\mathbf{C}^n \ni z \longmapsto Az + \beta \in \mathbf{C}^n,$$

where $A \in GL(n, \mathbf{R})$ and $\beta \in \mathbf{C}^n$. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them. For a Lie group G, we denote by G° the identity component of G and by $\operatorname{Lie} G$ the Lie algebra of G. If $E = \{\cdots\}$ is a subset of a vector space V over a field $\boldsymbol{F}$, the linear subspace of V spanned by E is denoted by $E_{\boldsymbol{F}} = \{\cdots\}_{\boldsymbol{F}}$. The symbol δ_{ij} denotes the Kronecker's delta.

We now recall basic concepts and results on tube domains. A tube domain T_Ω in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ given by $T_\Omega = \mathbf{R}^n + \sqrt{-1}\Omega$, where Ω is a domain in $\mathbf{R}^n$ and is called the base of T_Ω. Clearly, each element $\xi \in \mathbf{R}^n$ gives rise to an automorphism $\sigma_\xi \in \operatorname{Aut}(T_\Omega)$ defined by

$$\sigma_\xi(z) = z + \xi \quad \text{for } z \in T_\Omega.$$

Write $\Sigma = \mathbf{R}^n$. The additive group Σ acts as a group of automorphisms on T_Ω by

$$\xi \cdot z = \sigma_\xi(z) \quad \text{for } \xi \in \Sigma \text{ and } z \in T_\Omega.$$

The subgroup of $\operatorname{Aut}(T_\Omega)$ induced by Σ is denoted by Σ_{T_Ω}. Note that if $\varphi \in GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$, then $\varphi(T_\Omega)$ is a tube domain in $\mathbf{C}^n$, and we have $\varphi \Sigma_{T_\Omega} \varphi^{-1} = \Sigma_{T_\Xi}$, where $T_\Xi = \varphi(T_\Omega)$.

Consider a biholomorphic mapping $\varphi : T_{\Omega_1} \to T_{\Omega_2}$ between two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$. Then, by what we have noted above and [**3**, Section 1, Proposition], φ is given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$ if and only if φ is equivariant with respect to the Σ-actions. Biholomorphic mappings between tube domains equivariant with respect to the Σ-actions may be considered as natural isomorphisms in the category of tube domains. In view of this observation, we say that two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are affinely equivalent if there is a biholomorphic mapping between them given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$.

If the convex hull of the base Ω of a tube domain T_Ω in $\mathbf{C}^n$ contains no complete straight lines, then T_Ω is holomorphically equivalent to a bounded domain in $\mathbf{C}^n$ and, by a well-known theorem of H. Cartan, the group $\operatorname{Aut}(T_\Omega)$ of all automorphisms of T_Ω forms a Lie group with respect to the compact-open topology. The Lie algebra $\mathfrak{g}(T_\Omega)$ of the Lie group $\operatorname{Aut}(T_\Omega)$ can be identified canonically with the finite-dimensional real Lie algebra consisting of all complete holomorphic vector fields on T_Ω.

Let $z_j = x_j + \sqrt{-1}y_j$, $j = 1, \ldots, n$, be the complex coordinate functions of $\mathbf{C}^n$, where $x_j, y_j \in \mathbf{R}$, $j = 1, \ldots, n$. For $z = (z_1, \ldots, z_n)$, we write $\operatorname{Re} z = (x_1, \ldots, x_n)$ and $\operatorname{Im} z = (y_1, \ldots, y_n)$. For $j = 1, \ldots, n$, we write $\partial_j = \partial/\partial z_j$. Let D be a domain in $\mathbf{C}^n$. Then every holomorphic vector field Z on D can be written in the form

$$Z = \sum_{j=1}^{n} f_j(z)\partial_j,$$

where $f_1(z), \ldots, f_n(z)$ are holomorphic functions on D. The vector field Z is called a polynomial vector field if $f_1(z), \ldots, f_n(z)$ are polynomials in $z_1, \ldots, z_n$. The maximum value of the degrees of the polynomials $f_1(z), \ldots, f_n(z)$ is called the degree of Z. The following result is fundamental in our study.

STRUCTURE THEOREM ([**3**, Section 2, Theorem]). *To each tube domain T_Ω in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines, there is associated a tube domain $T_{\tilde{\Omega}}$ which is affinely equivalent to T_Ω such that $\mathfrak{g}(T_{\tilde{\Omega}})$ has the direct sum decomposition*

$$\mathfrak{g}(T_{\tilde{\Omega}}) = \mathfrak{p} + \mathfrak{e}$$

for which

$$\mathfrak{p} = \{X \in \mathfrak{g}(T_{\tilde{\Omega}}) \,|\, X \text{ is a polynomial vector field}\},$$

$$\mathfrak{e} = \sum_{i=1}^{r} \left\{ e^{z_i}\left(\partial_i + \sum_{j=r+1}^{n} \sqrt{-1}a_i^j \partial_j\right),\, e^{-z_i}\left(\partial_i - \sum_{j=r+1}^{n} \sqrt{-1}a_i^j \partial_j\right) \right\}_{\mathbf{R}},$$

where r is an integer between 0 and n and a_i^j, $i = 1, \dots, r$, $j = r+1, \dots, n$, are real constants.

The integer r is called the exponential rank of the tube domain T_Ω, and is denoted by $e(T_\Omega)$. This is well-defined, because it is readily verified that if two tube domains T_{Ω_1} and T_{Ω_2} are affinely equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$. When a tube domain T_Ω satisfies $e(T_\Omega) = 0$, we call T_Ω a tube domain with polynomial infinitesimal automorphisms.

Our main theme in this paper is a study of tube domains with polynomial infinitesimal automorphisms. This is motivated by the holomorphic equivalence problem for tube domains, which we will explain below.

In terms of the notion of the affine equivalence of tube domains, the holomorphic equivalence problem for tube domains may be formulated as the problem of studying the relationship between the holomorphic equivalence of tube domains and the affine equivalence of tube domains. It is clear that if two tube domains in $\mathbf{C}^n$ are affinely equivalent, then they are holomorphically equivalent. What we have to ask is whether the converse assertion holds or not:

PROBLEM. If two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are holomorphically equivalent, then are they affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, an affirmative answer is given (see Matsushima [**1**]). On the other hand, when Ω_1 and Ω_2 are arbitrary domains in $\mathbf{R}^n$ whose convex hulls contain no complete straight lines, there is a simple counter example. In fact, consider the upper half plane

$$T_{(0,\infty)} = \{x + \sqrt{-1}y \in \mathbf{C} \,|\, x \in \mathbf{R},\, y > 0\}$$

and the strip

$$T_{(0,\pi)} = \{x + \sqrt{-1}y \in \mathbf{C} \,|\, x \in \mathbf{R},\, 0 < y < \pi\}$$

in the complex plane. Then the tube domains $T_{(0,\infty)}$ and $T_{(0,\pi)}$ in $\mathbf{C}$ are holomorphically equivalent, but not affinely equivalent. We can clarify what causes a phenomenon like this by making use of the Structure Theorem stated above.

Let T_{Ω_1} and T_{Ω_2} be tube domains in $\mathbf{C}^n$ whose bases Ω_1 and Ω_2 have the convex hulls containing no complete straight lines. Since the exponential rank of a tube domain is an affine invariant, it is natural to reformulate the holomorphic equivalence problem for tube domains as follows:

PROBLEM (∗). If $e(T_{\Omega_1}) = e(T_{\Omega_2})$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

The counter example shown above corresponds to the case where $e(T_{\Omega_1}) \neq e(T_{\Omega_2})$, because $e(T_{(0,\infty)}) = 0$ and $e(T_{(0,\pi)}) = 1$. On the other hand, when Ω_1 and Ω_2 are bounded domains in $\mathbf{R}^n$, it is shown ([**4**]) that if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$, and T_{Ω_1} and T_{Ω_2} are affinely equivalent.

Specifying Problem (∗), we consider the following problem which has fundamental importance:

PROBLEM (∗∗). If $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, we have $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ (see [**1**]), and an affirmative answer to Problem (∗∗) is given, as stated above. For an attempt to solve Problem (∗∗) in the case where T_{Ω_1} and T_{Ω_2} are arbitrary tube domains with polynomial infinitesimal automorphisms, we need a further study of the structure of $\mathfrak{g}(T_\Omega)$. The Prolongation Theorem stated below enables us to make a more detailed analysis of the structure of $\mathfrak{g}(T_\Omega)$.

Before stating the Prolongation Theorem, we recall some facts on the affine automorphism group of a tube domain. Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines. The group $\mathrm{Aff}(T_\Omega)$ of all complex affine transformations of $\mathbf{C}^n$ leaving T_Ω invariant may be viewed as a subgroup of $\mathrm{Aut}(T_\Omega)$, and is called the affine automorphism group of T_Ω. Note that $\mathrm{Aff}(T_\Omega)$ is a closed subgroup of the Lie group $\mathrm{Aut}(T_\Omega)$ and that Σ_{T_Ω} is a subgroup of $\mathrm{Aff}(T_\Omega)$. The subalgebra $\mathfrak{a}(T_\Omega)$ of $\mathfrak{g}(T_\Omega)$ corresponding to $\mathrm{Aff}(T_\Omega)$ is given by

$$\mathfrak{a}(T_\Omega) = \{X \in \mathfrak{g}(T_\Omega) \,|\, X \text{ is a polynomial vector field of degree at most one}\}$$

and the subalgebra $\mathfrak{s}(T_\Omega)$ of $\mathfrak{g}(T_\Omega)$ corresponding to Σ_{T_Ω} is given by

$$\mathfrak{s}(T_\Omega) = \{\partial_1, \ldots, \partial_n\}_{\mathbf{R}}.$$

Now, the group $\mathrm{Aff}(\Omega)$ of all affine transformations of $\mathbf{R}^n$ leaving Ω invariant has the structure of a Lie group in a natural manner. Let $y_1, \ldots, y_n$ be the coordinate functions of $\mathbf{R}^n$. We call a vector field Y on Ω an affine vector field if Y has the form

$$Y = \sum_{j=1}^{n} h_j(y) \frac{\partial}{\partial y_j},$$

where $h_1(y), \ldots, h_n(y)$ are polynomials in $y_1, \ldots, y_n$ of degree at most one. Then the Lie algebra $\mathfrak{a}(\Omega)$ of $\mathrm{Aff}(\Omega)$ can be identified canonically with the Lie algebra of all complete affine vector fields on Ω. By [**4**, Section 1, Lemma 3], there exists a Lie algebra isomorphism ι_* of $\mathfrak{a}(\Omega)$ into $\mathfrak{a}(T_\Omega)$ such that $\mathfrak{a}(T_\Omega)$ is decomposed as the direct sum

$$\mathfrak{a}(T_\Omega) = \mathfrak{s}(T_\Omega) + \iota_*(\mathfrak{a}(\Omega)) \tag{1.1}$$

of $\mathfrak{s}(T_\Omega)$ and $\iota_*(\mathfrak{a}(\Omega))$. In fact, $\iota_* : \mathfrak{a}(\Omega) \to \mathfrak{a}(T_\Omega)$ is given by

$$\iota_* : \mathfrak{a}(\Omega) \ni \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} y_i + b_j \right) \frac{\partial}{\partial y_j} \longmapsto \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} z_i + \sqrt{-1} b_j \right) \partial_j \in \mathfrak{a}(T_\Omega), \tag{1.2}$$

where a_{ji}, b_j, $j, i = 1, \ldots, n$, are real constants. As a consequence, note that $\mathfrak{s}(T_\Omega)$ is an abelian ideal in $\mathfrak{a}(T_\Omega)$.

To state the Prolongation Theorem, let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines. For a polynomial vector field Z on T_Ω of degree 2, we write

$$Z = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1} Y^{(k)} \right),$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components with respect to $\partial_1, \ldots, \partial_n$ are homogeneous polynomials in $z_1, \ldots, z_n$ with real coefficients of degree k, and set

$$\begin{aligned} Z_{[b]} &= X^{(2)} + \sqrt{-1} Y^{(1)}, \\ Z_{[a]} &= X^{(1)} + \sqrt{-1} Y^{(0)}, \\ Z_{[s]} &= X^{(0)}. \end{aligned}$$

Note that $Z = Z_{[s]} + Z_{[a]} + Z_{[b]} + \sqrt{-1} Y^{(2)}$. The following theorem gives a criterion on the completeness of Z.

PROLONGATION THEOREM ([**6**, Section 2], [**5**]). *Let Z be a polynomial vector field on T_Ω of degree 2. Then Z is complete on T_Ω if and only if one has $Y^{(2)} = 0$, and the vector fields $[\partial_i, Z]$, $i = 1, \ldots, n$, and $Z_{[a]}$ are all complete on T_Ω. Consequently, if Z is complete on T_Ω, then $Z_{[b]}$ is complete on T_Ω. Also, if $Z = Z_{[b]}$ and if the vector fields $[\partial_i, Z]$, $i = 1, \ldots, n$, are all complete on T_Ω, then Z is complete on T_Ω.*

2. Tube domains with polynomial infinitesimal automorphisms.

When we are discussing tube domains T_Ω with polynomial infinitesimal automorphisms, it is one of the key points that a polynomial gives the Taylor expansion around the origin of the function it represents. The purpose of this section is to give some fundamental results on $\mathfrak{g}(T_\Omega)$ obtained by combining the Prolongation Theorem with this fact.

2.1. General observations on an isotropy subalgebra of $\mathfrak{g}(T_\Omega)$.

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines. We may assume without loss of generality that T_Ω contains the origin of $\mathbf{C}^n$. Every element Z of $\mathfrak{g}(T_\Omega)$ has the Taylor expansion around the origin given as

$$Z = \sum_{k=0}^{\infty} Z^{((k))},$$

where $Z^{((k))}$ is a polynomial vector field whose components with respect to $\partial_1, \ldots, \partial_n$ are homogeneous polynomials in $z_1, \ldots, z_n$ of degree k. We write

$$Z^{((1))} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{ji}(Z) z_i \right) \partial_j,$$

where $c_{ji}(Z)$, $j, i = 1, \ldots, n$, are complex constants. Let $\mathfrak{k}$ denote the isotropy subalgebra of $\mathfrak{g}(T_\Omega)$ at the origin. Then $\mathfrak{k}$ consists of those elements Z of $\mathfrak{g}(T_\Omega)$ which satisfy $Z^{((0))} = 0$. An application of H. Cartan's uniqueness theorem [**2**, Chapter 5, Proposition 1] yields the following result.

LEMMA 2.1. *If Z is an element of $\mathfrak{k}$ and if $Z^{((1))} = 0$, then $Z = 0$.*

This result implies that the linear representation of $\mathfrak{k}$ given by

$$\mathfrak{k} \ni Z \longmapsto (c_{ji}(Z)) \in \mathfrak{gl}(n, \mathbf{C})$$

is faithful, where $\mathfrak{gl}(n, \mathbf{C})$ denotes the set of complex n by n matrices viewed as the Lie algebra of $GL(n, \mathbf{C})$. We recall here that T_Ω has the Bergman metric $ds^2_{T_\Omega}$. Using the invariance of $ds^2_{T_\Omega}$ under the action of Σ_{T_Ω}, after a suitable real linear change of coordinates we may assume that the holomorphic vector fields $\partial_1, \ldots, \partial_n$ form an orthonormal basis at the origin with respect to $ds^2_{T_\Omega}$. Then the matrix $(c_{ji}(Z))$ is a skew-Hermitian matrix for every element Z of $\mathfrak{k}$. Indeed, this follows from the fact that every automorphism of T_Ω is an isometry with respect to $ds^2_{T_\Omega}$.

2.2. Consequences of the prolongation theorem.

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines, and suppose further that $e(T_\Omega) = 0$, or $\mathfrak{g}(T_\Omega)$ consists of all polynomial vector fields which are complete on T_Ω. Then every element Z of $\mathfrak{g}(T_\Omega)$ can be written in the form

$$Z = \sum_{k=0}^{\infty} Z^{(k)}, \tag{2.1}$$

where $Z^{(k)}$ is a polynomial vector field whose components with respect to $\partial_1, \ldots, \partial_n$ are homogeneous polynomials in $z_1, \ldots, z_n$ of degree k. Note that, in (2.1), only finitely many $Z^{(k)}$'s are not equal to zero. We may assume without loss of generality that T_Ω contains the origin, and that $\partial_1, \ldots, \partial_n$ form an orthonormal basis at the origin with respect to the Bergman metric $ds^2_{T_\Omega}$. Then (2.1) gives the Taylor expansion of Z around the origin. For $k = 0, 1, 2, \ldots$, we write

$$Z^{(k)} = X^{(k)} + \sqrt{-1}Y^{(k)},$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components are homogeneous polynomials with real coefficients of degree k. We define real vector subspaces $\mathfrak{q}, \mathfrak{s}, \mathfrak{a}_*, \mathfrak{b}$ of $\mathfrak{g}(T_\Omega)$ by

$$\begin{aligned}
\mathfrak{q} &= \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = \sum_{k=0}^{2} Z^{(k)} = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1}Y^{(k)} \right) \right\}, \\
\mathfrak{s} &= \{\partial_1, \ldots, \partial_n\}_{\mathbf{R}}, \\
\mathfrak{a}_* &= \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(1)} + \sqrt{-1}Y^{(0)} \right\}, \\
\mathfrak{b} &= \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(2)} + \sqrt{-1}Y^{(1)} \right\}.
\end{aligned}$$

The Prolongation Theorem shows that $\mathfrak{q}$ has the direct sum decomposition

$$\mathfrak{q} = \mathfrak{s} + \mathfrak{a}_* + \mathfrak{b}.$$

Note that $\mathfrak{b}$ is contained in the isotropy subalgebra $\mathfrak{k}$ of $\mathfrak{g}(T_\Omega)$ at the origin. The following result on $\mathfrak{b}$ is useful for a further study of the structure of $\mathfrak{g}(T_\Omega)$.

LEMMA 2.2. *Let $Z = X^{(2)} + \sqrt{-1}Y^{(1)}$ be an element of $\mathfrak{b}$ and write*

$$Y^{(1)} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji}(Z) z_i \right) \partial_j,$$

where $b_{ji}(Z)$, $j, i = 1, \ldots, n$, are real constants. Then the following hold.

i) $X^{(2)} = 0$ *if and only if* $Y^{(1)} = 0$.

ii) *The real n by n matrix $(b_{ji}(Z))$ is symmetric for every element Z of $\mathfrak{b}$.*

PROOF. To prove i), suppose that $X^{(2)} = 0$. Then, for every $i = 1, \ldots, n$, we have

$$[\partial_i,\, Z] = \sqrt{-1} \sum_{j=1}^{n} b_{ji}(Z) \partial_j.$$

On the other hand, since $\partial_i, Z \in \mathfrak{g}(T_\Omega)$ and since $\mathfrak{g}(T_\Omega)$ forms a Lie algebra, it follows that $[\partial_i,\, Z] \in \mathfrak{g}(T_\Omega)$. Therefore we see by [**3**, Section 3, Lemma 5] that $b_{ji}(Z) = 0$ for all $j = 1, \ldots, n$. This implies that $Y^{(1)} = 0$, and the "only if" part of i) is proved. The "if" part of i) is an immediate consequence of Lemma 2.1, because we have $Z \in \mathfrak{k}$ and $Z^{((1))} = \sqrt{-1}Y^{(1)}$.

To prove ii), let $Z = X^{(2)} + \sqrt{-1}Y^{(1)}$ be any element of $\mathfrak{b}$. Then we have $c_{ji}(Z) = \sqrt{-1}b_{ji}(Z)$ for all $j, i = 1, \ldots, n$, or $(c_{ji}(Z)) = \sqrt{-1}(b_{ji}(Z))$ as n by n matrices. Since

$(c_{ji}(Z))$ is a skew-Hermitian matrix and $(b_{ji}(Z))$ is a real matrix, it follows that $(b_{ji}(Z))$ is a symmetric matrix, which proves ii). □

As a consequence of ii) of Lemma 2.2, it should be observed that, when $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}(T_\Omega)$, the matrices $(b_{ji}(Z))$, $Z \in \mathfrak{b}$, are simultaneously diagonalizable by a suitable orthogonal change of coordinates.

2.3. Lemmas on solvable subalgebras of $\mathfrak{g}(T_\Omega)$.

As is shown in Matsushima [**1**], in the study of tube domains T_Ω with polynomial infinitesimal automorphisms, investigating solvable subalgebras of $\mathfrak{g}(T_\Omega)$ plays an important role. In this subsection, we give a lemma useful in the investigation of solvable subalgebras of $\mathfrak{g}(T_\Omega)$ containing $\mathfrak{s}(T_\Omega)$.

Let T_Ω and $\mathfrak{q} = \mathfrak{s} + \mathfrak{a}_* + \mathfrak{b}$ be as in the preceding subsection. Let Z be an element of $\mathfrak{q}$. Then, with the notation of Section 1, we have

$$Z = Z_{[s]} + Z_{[a]} + Z_{[b]} \tag{2.2}$$

and

$$\begin{aligned} Z_{[s]} &= X^{(0)} \in \mathfrak{s}, \\ Z_{[a]} &= X^{(1)} + \sqrt{-1}Y^{(0)} \in \mathfrak{a}_*, \\ Z_{[b]} &= X^{(2)} + \sqrt{-1}Y^{(1)} \in \mathfrak{b}. \end{aligned}$$

We write

$$X^{(2)} = \sum_{j=1}^{n} f_j \partial_j \quad \text{and} \quad Y^{(1)} = \sum_{j=1}^{n} g_j \partial_j,$$

where f_j and g_j are homogeneous polynomials in $z_1, \dots, z_n$ with real coefficients of degrees 2 and 1, respectively.

LEMMA 2.3. *Let $\mathfrak{t}$ be a solvable subalgebra of $\mathfrak{g}(T_\Omega)$ containing $\mathfrak{s}$. If $Z \in \mathfrak{q} \cap \mathfrak{t}$ and if the polynomials g_j, $j = 1, \dots, n$, depend on only the variables $z_1, \dots, z_m$, then the polynomials f_j, $j = 1, \dots, n$, depend on only the variables $z_1, \dots, z_m$.*

PROOF. Let i be any index with $m + 1 \le i \le n$. Then we have $[\partial_i, Z_{[s]}] \in [\mathfrak{s}, \mathfrak{s}] = \{0\}$. Also, since $\partial_i g_j = 0$ for all $j = 1, \dots, n$ by assumption, it follows that

$$\begin{aligned} [\partial_i, Z_{[b]}] &= [\partial_i, X^{(2)}] + \sqrt{-1}[\partial_i, Y^{(1)}] \\ &= \sum_{j=1}^{n} \partial_i f_j \partial_j + \sqrt{-1} \sum_{j=1}^{n} \partial_i g_j \partial_j \\ &= \sum_{j=1}^{n} \partial_i f_j \partial_j. \end{aligned}$$

Therefore we see from (2.2) that

$$[\partial_i,\, Z] = [\partial_i,\, Z_{[a]}] + \sum_{j=1}^{n} \partial_i f_j \partial_j. \tag{2.3}$$

Note that we have

$$[\partial_i,\, Z] \in [\mathfrak{t},\, \mathfrak{t}] \subset \mathfrak{t}, \quad [\partial_i,\, Z_{[a]}] \in \mathfrak{s} \subset \mathfrak{t}. \tag{2.4}$$

Write $W = \sum_{j=1}^{n} \partial_i f_j \partial_j$. As a consequence of (2.4), W belongs to $\mathfrak{t}$.

We show that the endomorphism $ad\, W$ of $\mathfrak{g}(T_\Omega)$ is zero. It is sufficient to show that the endomorphism $ad\, W$ is nilpotent and semisimple. We put

$$\widetilde{ad}\,\mathfrak{t} = \left\{ \widetilde{ad\, T} : \mathfrak{g}(T_\Omega)^{\mathbf{C}} \to \mathfrak{g}(T_\Omega)^{\mathbf{C}} \,\middle|\, T \in \mathfrak{t} \right\},$$

where $\mathfrak{g}(T_\Omega)^{\mathbf{C}}$ denotes the complexification of $\mathfrak{g}(T_\Omega)$ and $\widetilde{ad\, T}$ denotes the complex linear extension of $ad\, T : \mathfrak{g}(T_\Omega) \to \mathfrak{g}(T_\Omega)$ to $\mathfrak{g}(T_\Omega)^{\mathbf{C}}$. Since $\mathfrak{t}$ is solvable, Lie's theorem shows that, after a suitable choice of basis of $\mathfrak{g}(T_\Omega)^{\mathbf{C}}$, every endomorphism belonging to $\widetilde{ad}\,\mathfrak{t}$ is represented by an upper triangular matrix. As a consequence, $\widetilde{ad\,[\partial_i,\, Z]} = \left[\widetilde{ad\,\partial_i},\, \widetilde{ad\, Z}\right]$ is a nilpotent endomorphism of $\mathfrak{g}(T_\Omega)^{\mathbf{C}}$. On the other hand, $\widetilde{ad\,[\partial_i,\, Z_{[a]}]}$ is a nilpotent endomorphism of $\mathfrak{g}(T_\Omega)^{\mathbf{C}}$ in view of the fact that $ad\, X$, and hence $\widetilde{ad\, X}$ is a nilpotent endomorphism for every element X of $\mathfrak{s}$. Therefore we conclude by (2.3) and what Lie's theorem has shown that

$$\widetilde{ad\, W} = \widetilde{ad\,[\partial_i,\, Z]} - \widetilde{ad\,[\partial_i,\, Z_{[a]}]},$$

and hence $ad\, W$ is nilpotent. It remains to show that the endomorphism $ad\, W$ is semisimple. To see this, note that the components of W with respect to $\partial_1, \ldots, \partial_n$ are homogeneous polynomials of degree 1. Therefore the value of W at the origin is equal to zero. This implies that $W \in \mathfrak{k}$. Since $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}(T_\Omega)$, we see that the endomorphism $ad\, W$ is semisimple, and our assertion is shown.

The result of the preceding paragraph implies that $[\partial_i,\, W] = 0$ for all $i = 1, \ldots, n$. Therefore we have $W \in \mathfrak{s}$. Since the components of W with respect to $\partial_1, \ldots, \partial_n$ must be homogeneous polynomials of degree 1, it follows that

$$0 = W = \sum_{j=1}^{n} \partial_i f_j \partial_j,$$

or $\partial_i f_j = 0$ for all $j = 1, \ldots, n$. Since this holds for every $i = m+1, \ldots, n$, we conclude that the polynomials f_j, $j = 1, \ldots, n$, depend on only the variables $z_1, \ldots, z_m$, and the lemma is proved. □

In the next section, we need the following lemma.

Lemma 2.4. *Let $\mathfrak{t}$ be a solvable subalgebra of $\mathfrak{g}(T_\Omega)$ containing $\mathfrak{s}$. If $Z = Z_{[b]} \in \mathfrak{b} \cap \mathfrak{t}$ and if $Y^{(1)}$ has the form $Y^{(1)} = b_1 z_i \partial_1 + \cdots + b_n z_i \partial_n$, where $b_1, \ldots, b_n$ are real constants, then $X^{(2)}$ has the form $X^{(2)} = a_1 z_i^2 \partial_1 + \cdots + a_n z_i^2 \partial_n$, where $a_1, \ldots, a_n$ are real constants.*

Moreover, the constant a_i *is equal to* 0.

PROOF. The fact that $X^{(2)}$ has the form $X^{(2)} = a_1 z_i^2 \partial_1 + \cdots + a_n z_i^2 \partial_n$ is an immediate consequence of Lemma 2.3. We show that $a_i = 0$. Suppose contrarily that $a_i \neq 0$. For convenience, we denote by the notation "$\cdots$" a vector field of the form $h_1\partial_1 + \cdots + h_{i-1}\partial_{i-1} + h_{i+1}\partial_{i+1} + \cdots + h_n\partial_n$, where $h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n$ are functions. Then $a_i^{-1} Z$ is written as $a_i^{-1} Z = (z_i^2 + \sqrt{-1}\lambda z_i)\partial_i + \cdots$, where λ is a real constant. Applying if necessary a change of coordinates given by the translation that replaces z_i by $z_i - (\sqrt{-1}/2)\lambda$, we have $a_i^{-1} Z - (\lambda^2/4)\partial_i = z_i^2 \partial_i + \cdots$, which is an element of $\mathfrak{t}$ and denoted by W. Since $\mathfrak{t}$ contains the element $(1/2)[\partial_i, W] = z_i \partial_i + \cdots$, it follows that

$$\mathfrak{t} \supset \{\partial_i, z_i\partial_i + \cdots, z_i^2\partial_i + \cdots\}_{\mathbf{R}}. \tag{2.5}$$

Now, we denote by $\mathfrak{D}^m \mathfrak{t}$ the m-th derived algebra of the Lie algebra $\mathfrak{t}$. Then we see from (2.5) that $\mathfrak{D}^m \mathfrak{t}$ contains a nonzero vector subspace $\{\partial_i, z_i\partial_i + \cdots, z_i^2\partial_i + \cdots\}_{\mathbf{R}}$ for every $m = 0, 1, 2, \ldots$. This contradicts the assumption that $\mathfrak{t}$ is solvable, and our assertion is proved. □

3. A class of tube domains with solvable groups of automorphisms.

Among tube domains with polynomial infinitesimal automorphisms, tube domains T_Ω whose bases Ω are convex cones are characteristic in the point that they have the property that if $\mathrm{Aut}(T_\Omega)$ is solvable, then $\mathrm{Aut}(T_\Omega)$ necessarily consists of affine transformations. On the other hand, when Ω is an arbitrary convex domain in $\mathbf{R}^n$ containing no complete straight lines, there is a tube domain T_Ω in $\mathbf{C}^n$ such that $\mathrm{Aut}(T_\Omega)$ is solvable, but contains nonaffine automorphisms, as is shown in the next section. More generally, we have the following structure theorem on a class of tube domains with solvable groups of automorphisms.

THEOREM 3.1. *Let* T_Ω *be a tube domain in* $\mathbf{C}^n$ *whose base* Ω *is a convex domain in* $\mathbf{R}^n$ *containing no complete straight lines and let* $n \geq 2$. *Assume that*:

i) T_Ω *is a tube domain with polynomial infinitesimal automorphisms*;

ii) $\mathrm{Aut}(T_\Omega)$ *is a solvable Lie group*;

iii) T_Ω *contains the origin* o *of* $\mathbf{C}^n$ *and the orbit* $G(T_\Omega) \cdot o$ *of* $G(T_\Omega)$ *through* o *has dimension* $n+1$, *where* $G(T_\Omega)$ *denotes the identity component of* $\mathrm{Aut}(T_\Omega)$.

Then, in the notation of Subsection 2.2, $\mathfrak{g}(T_\Omega)$ *coincides with* $\mathfrak{q}$. *Moreover, according to the cases of* a) $\mathfrak{b} \neq \{0\}$ *and* b) $\mathfrak{b} = \{0\}$, *the following hold.*

a) *One has* $n \geq 3$ *and, after a real linear change of coordinates in* $\mathbf{C}^n$, $\mathfrak{a}_*$, $\mathfrak{b}$ *and the nilradical* $\mathfrak{n}$ *of* $\mathfrak{g}(T_\Omega)$ *are given by*

$$\mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}} + \mathfrak{k} \cap \mathfrak{a}_* \quad (\textit{direct sum}),$$
$$\mathfrak{b} = \{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}},$$
$$\mathfrak{n} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}.$$

Also, any n-dimensional abelian ideal in $\mathfrak{n}$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$.

b) *The nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ contains $\mathfrak{s}$ and has dimension less than or equal to $n+1$. Also, any n-dimensional abelian ideal in $\mathfrak{n}$ coincides with $\mathfrak{s}$.*

Proof. The condition iii) implies that, after a real linear change of coordinates in $\mathbf{C}^n$, we may assume that

$$T_o(G(T_\Omega)\cdot o) = \{\partial_1, \ldots, \partial_n\}_{\mathbf{R}} + \{\sqrt{-1}\partial_1\}_{\mathbf{R}}, \tag{3.1}$$

where $T_o(G(T_\Omega)\cdot o)$ denotes the tangent space to $G(T_\Omega)\cdot o$ at o.

Consider first the case where $\mathfrak{b} \neq \{0\}$. Take a nonzero element $Z = X^{(2)} + \sqrt{-1}Y^{(1)}$ of $\mathfrak{b}$. Since the value of the vector field $[\partial_i, Z]$ at o is in $T_o(G(T_\Omega)\cdot o)$ for every $i = 1, \ldots, n$, it follows from (3.1) that $Y^{(1)}$ has the form $Y^{(1)} = (c_1 z_1 + \cdots + c_n z_n)\partial_1$, where $c_1, \ldots, c_n$ are real constants.

We show that $c_1 \neq 0$. Suppose the contrary. Note that $c_2, \ldots, c_n$ are not all 0. Indeed, otherwise, by i) of Lemma 2.2, Z must be 0. By a permutation of the coordinates $z_2, \ldots, z_n$, we may assume that $c_2 \neq 0$. Applying if necessary a change of coordinates

$$\mathbf{C}^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in \mathbf{C}^n,$$

$$\begin{cases} w_1 = z_1, \\ w_2 = c_2 z_2 + \cdots + c_n z_n, \\ w_i = z_i, \quad i = 3, \ldots, n, \end{cases}$$

we see that $Y^{(1)}$ has the form $Y^{(1)} = w_2(\partial/\partial w_1)$. For simplicity, write $z_1, \ldots, z_n$ as $w_1, \ldots, w_n$ again. Then it follows from Lemma 2.4 that $X^{(2)}$ has the form

$$X^{(2)} = a_1 z_2^2 \partial_1 + a_3 z_2^2 \partial_3 + \cdots + a_n z_2^2 \partial_n,$$

where $a_1, a_3, \ldots, a_n$ are real constants. Therefore we have

$$Z = (a_1 z_2^2 + \sqrt{-1} z_2)\partial_1 + a_3 z_2^2 \partial_3 + \cdots + a_n z_2^2 \partial_n. \tag{3.2}$$

We recall here the general result ([**3**, Section 3, Lemma 6]) that if T_Ω is a tube domain in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines and if a complete holomorphic vector field X on T_Ω is of the form

$$X = \sum_{j=k+1}^{n} f_j(z_1, \ldots, z_k)\partial_j,$$

then $f_j(z_1, \ldots, z_k)$, $j = k+1, \ldots, n$, are real constants. Combining (3.2) with this result, we obtain $Z = 0$, which is a contradiction and our assertion is shown.

Replacing Z by $c_1^{-1}Z$, we may assume that $Y^{(1)} = (z_1 + c_2 z_2 + \cdots + c_n z_n)\partial_1$. Applying if necessary a change of coordinates

$$\mathbf{C}^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in \mathbf{C}^n,$$

$$\begin{cases} w_1 = z_1 + c_2 z_2 + \cdots + c_n z_n, \\ w_i = z_i, \quad i = 2, \dots, n, \end{cases}$$

we see that $Y^{(1)}$ has the form $Y^{(1)} = w_1(\partial/\partial w_1)$. For simplicity, write $z_1, \dots, z_n$ as $w_1, \dots, w_n$ again. Then it follows from Lemma 2.4 that $X^{(2)}$ has the form

$$X^{(2)} = a_2 z_1^2 \partial_2 + \cdots + a_n z_1^2 \partial_n,$$

where $a_2, \dots, a_n$ are real constants. Therefore we have

$$Z = \sqrt{-1} z_1 \partial_1 + a_2 z_1^2 \partial_2 + \cdots + a_n z_1^2 \partial_n = \sqrt{-1} z_1 \partial_1 + z_1^2 (a_2 \partial_2 + \cdots + a_n \partial_n).$$

Note that $a_2, \dots, a_n$ are not all 0. Indeed, otherwise, by i) of Lemma 2.2, Z must be 0. Hence, by a suitable real linear change of the coordinates $z_2, \dots, z_n$, Z has the form $Z = \sqrt{-1} z_1 \partial_1 + z_1^2 \partial_2$.

We show that $\mathfrak{b} = \{\sqrt{-1} z_1 \partial_1 + z_1^2 \partial_2\}_{\mathbf{R}}$. Note that T_Ω contains the origin o. Since $\mathfrak{b}$ is contained in a compact subalgebra of $\mathfrak{g}(T_\Omega)$ given as the isotropy subalgebra of $\mathfrak{g}(T_\Omega)$ at o and since $\mathfrak{g}(T_\Omega)$ is solvable, we see that $\mathfrak{b}$ is abelian. Take any element $W = U^{(2)} + \sqrt{-1} V^{(1)}$ of $\mathfrak{b}$. Since the value of the vector field $[\partial_i, W]$ at o is in $T_o(G(T_\Omega) \cdot o)$ for every $i = 1, \dots, n$, it follows from (3.1) that $V^{(1)}$ has the form $V^{(1)} = (c_1 z_1 + \cdots + c_n z_n)\partial_1$, where $c_1, \dots, c_n$ are real constants. The fact that $[Z, W] = 0$ implies that

$$[Z, U^{(2)}] + (c_2 z_2 + \cdots + c_n z_n)\partial_1 + \sqrt{-1} c_2 z_1^2 \partial_1 - 2\sqrt{-1} z_1 (c_1 z_1 + \cdots + c_n z_n)\partial_2 = 0.$$

Here the coefficient functions of the vector field $[Z, U^{(2)}]$ are polynomials of degree greater than or equal to 2. Therefore we have $c_2 = \cdots = c_n = 0$, which shows that $V^{(1)} = c_1 z_1 \partial_1$. From this, we see that $W - c_1 Z = U^{(2)} - c_1 z_1^2 \partial_2$. By i) of Lemma 2.2, we obtain $W - c_1 Z = 0$, or $W = c_1 Z$. We thus conclude that $\mathfrak{b} = \{\sqrt{-1} z_1 \partial_1 + z_1^2 \partial_2\}_{\mathbf{R}}$.

We show that $\mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2 z_1 \partial_2\}_{\mathbf{R}} + \mathfrak{k} \cap \mathfrak{a}_*$ (direct sum). Since

$$[\partial_1, \sqrt{-1} z_1 \partial_1 + z_1^2 \partial_2] = \sqrt{-1}\partial_1 + 2 z_1 \partial_2$$

is an element of $\mathfrak{g}(T_\Omega)$, we see that $\sqrt{-1}\partial_1 + 2 z_1 \partial_2 \in \mathfrak{a}_*$. We note here that $n \geq 3$. Indeed, if $n = 2$, then Ω is given by $\Omega = \{(y_1, y_2) \in \mathbf{R}^2 \mid y_2 > y_1^2 + c\}$ for some constant $c \in \mathbf{R}$, because the vector field $\sqrt{-1}\partial_1 + 2 z_1 \partial_2$ is complete on T_Ω and Ω is a convex domain in $\mathbf{R}^2$ containing no complete straight lines. Therefore T_Ω must be homogeneous, which contradicts the condition iii). Now, take any element $Z = X^{(1)} + \sqrt{-1} Y^{(0)}$ of $\mathfrak{a}_*$. Since the value of the vector field Z at o is in $T_o(G(T_\Omega) \cdot o)$, it follows from (3.1) that $Y^{(0)}$ has the form $Y^{(0)} = \lambda \partial_1$, where λ is a real constant. Put $W = Z - \lambda(\sqrt{-1}\partial_1 + 2 z_1 \partial_2)$. Then W takes the value 0 at the origin o, becuase $Y^{(0)} = \lambda \partial_1$. Therefore we have $W \in \mathfrak{k} \cap \mathfrak{a}_*$ and $Z = \lambda(\sqrt{-1}\partial_1 + 2 z_1 \partial_2) + W$. This concludes that $\mathfrak{a}_*$ is the direct sum of $\{\sqrt{-1}\partial_1 + 2 z_1 \partial_2\}_{\mathbf{R}}$ and $\mathfrak{k} \cap \mathfrak{a}_*$.

We show that $\mathfrak{n} \cap \mathfrak{q} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2 z_1 \partial_2\}_{\mathbf{R}}$. For brevity, write $Z_0 = \sqrt{-1} z_1 \partial_1 + z_1^2 \partial_2$ and $W_0 = \sqrt{-1}\partial_1 + 2 z_1 \partial_2$. Then $ad\, W_0 = [ad\, \partial_1, ad\, Z_0]$ is nilpotent in view of Lie's theorem. Also, it is obvious that $ad\, \partial_i$ is nilpotent for every $i = 1, \dots, n$. Therefore $\mathfrak{n} \cap \mathfrak{q}$ contains $\mathfrak{s} + \{W_0\}_{\mathbf{R}}$. Let Z be any element of $\mathfrak{q}$ such that $ad\, Z$ is nilpotent. We can write $Z = \lambda Z_0 + \mu W_0 + U + T$, where λ, μ are real contants and $U \in \mathfrak{k} \cap \mathfrak{a}_*$, $T \in \mathfrak{s}$. Then $ad\,(\lambda Z_0 + U)$ is semisimple, because $\lambda Z_0 + U$ belongs to the isotropy subalgebra $\mathfrak{k}$. On the other hand, $ad\,Z$, $ad\,\mu W_0$, $ad\,T$ are all nilpotent. Since $ad\,(\lambda Z_0 + U) = ad\,Z - ad\,\mu W_0 - ad\,T$, we see that $ad\,(\lambda Z_0 + U)$ is nilpotent and semisimple, and hence $ad\,(\lambda Z_0 + U) = 0$. As a consequence, we have

$$0 = (ad\,(\lambda Z_0 + U))\partial_i = [\lambda Z_0 + U, \partial_i] \quad \text{for every } i = 1, \dots, n,$$

which implies that $\lambda Z_0 + U = 0$. This concludes that $\mathfrak{n} \cap \mathfrak{q} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}$.

We show that $\mathfrak{g}(T_\Omega) = \mathfrak{q}$. Suppose that $\mathfrak{g}(T_\Omega) \neq \mathfrak{q}$, or that there exists a nonzero element Z of $\mathfrak{g}(T_\Omega)$ of degree greater than or equal to 3. Then we can choose suitable nonnegative integers $\nu_1, \dots, \nu_n$ with some $\nu_i > 0$ such that $(ad\,\partial_1)^{\nu_1} \cdots (ad\,\partial_n)^{\nu_n} Z$ is an element of $\mathfrak{g}(T_\Omega)$ of degree just 2, which we denote by W. In view of Lie's theorem, $ad\,W$ is nilpotent, so that $W \in \mathfrak{n} \cap \mathfrak{q}$. But, as is shown above, we must have $\mathfrak{n} \cap \mathfrak{q} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}$. This is a contradiction, because the degree of W is 2. We thus conclude that $\mathfrak{g}(T_\Omega) = \mathfrak{q}$.

Since $\mathfrak{g}(T_\Omega) = \mathfrak{q}$, we have $\mathfrak{n} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}$. We show that any n-dimensional abelian ideal $\mathfrak{s}_0$ in $\mathfrak{n}$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$. Write $Z_0 = \sqrt{-1}z_1\partial_1 + z_1^2\partial_2$ and $W_0 = \sqrt{-1}\partial_1 + 2z_1\partial_2$ as above. If $\mathfrak{s}_0 = \mathfrak{s}$, then our assertion is obvious. Suppose that $\mathfrak{s}_0 \neq \mathfrak{s}$. Then we see that $\mathfrak{s}_0$ has the form $\mathfrak{s}_0 = \{\lambda W_0 + \mu\partial_1, \partial_2, \cdots, \partial_n\}_{\mathbf{R}}$, where λ, μ are real constants and $\lambda \neq 0$. Now, we have the relations

$$(ad\,Z_0)W_0 = \partial_1,\ (ad\,Z_0)\partial_1 = -W_0,\ (ad\,Z_0)\partial_i = 0,\ i = 2, \dots, n.$$

From these, it follows that, when $\alpha, \beta \in \mathbf{R}$,

$$\begin{aligned} &Ad(\mathrm{Exp}\,tZ_0)(\alpha W_0 + \beta\partial_1) = (\alpha\cos t - \beta\sin t)W_0 + (\alpha\sin t + \beta\cos t)\partial_1, \\ &Ad(\mathrm{Exp}\,tZ_0)\partial_i = \partial_i, \quad i = 2, \dots, n, \end{aligned}$$

for all $t \in \mathbf{R}$, where $Ad : G(T_\Omega) \to \mathrm{Int}(\mathfrak{g}(T_\Omega))$ is the adjoint representation of the Lie group $G(T_\Omega)$. If $\mu \neq 0$, then, for $t_0 \in \mathbf{R}$ with $\tan t_0 = \lambda/\mu$, we have

$$Ad(\mathrm{Exp}\,t_0 Z_0)(\lambda W_0 + \mu\partial_1) = \gamma\partial_1 \quad \text{for some real constant } \gamma,$$

and therefore $Ad(\mathrm{Exp}\,t_0 Z_0)\mathfrak{s}_0 = \mathfrak{s}$. On the other hand, if $\mu = 0$, then, for $t_0 = \pi/2$, we have

$$Ad(\mathrm{Exp}\,t_0 Z_0)(\lambda W_0) = \lambda\partial_1,$$

and therefore $Ad(\mathrm{Exp}\,t_0 Z_0)\mathfrak{s}_0 = \mathfrak{s}$. These show that $\mathfrak{s}_0$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$, as desired.

Consider next the case where $\mathfrak{b} = \{0\}$. We show that $\mathfrak{g}(T_\Omega) = \mathfrak{q} = \mathfrak{s} + \mathfrak{a}_*$. Suppose that $\mathfrak{g}(T_\Omega) \neq \mathfrak{q}$, or that there exists a nonzero element Z of $\mathfrak{g}(T_\Omega)$ of degree greater than or equal to 3. Then we can choose suitable nonnegative integers $\nu_1, \dots, \nu_n$ such that $(ad\,\partial_1)^{\nu_1} \cdots (ad\,\partial_n)^{\nu_n} Z$ is an element of $\mathfrak{g}(T_\Omega)$ of degree just 2. By the Prolongation Theorem, this yields that $\mathfrak{b} \neq \{0\}$, which is a contradiction. We thus conclude that

$\mathfrak{g}(T_\Omega) = \mathfrak{q} = \mathfrak{s} + \mathfrak{a}_*$.

We show that $\mathfrak{n}$ contains $\mathfrak{s}$ and has dimension less than or equal to $n+1$. The fact that $\mathfrak{n} \supset \mathfrak{s}$ follows by a similar way to the case where $\mathfrak{b} \neq \{0\}$. Assume that $\mathfrak{n} \neq \mathfrak{s}$. Since $\mathfrak{g}(T_\Omega) = \mathfrak{s} + \mathfrak{a}_*$ and $\mathfrak{n} \supset \mathfrak{s}$, there exists a nonzero element $Z = X^{(1)} + \sqrt{-1}Y^{(0)}$ of $\mathfrak{a}_*$ such that $ad\,Z$ is nilpotent. Since the value of the vector field Z at o is in $T_o(G(T_\Omega)\cdot o)$, it follows from (3.1) that $Y^{(0)}$ has the form $Y^{(0)} = \lambda\partial_1$, where λ is a real constant. Here we have $\lambda \neq 0$. Indeed, otherwise, Z belongs to $\mathfrak{k}$. From this, we see that $ad\,Z$ is nilpotent and semisimple, and hence $ad\,Z = 0$. This implies that $Z = 0$, which is a contradiction. Now, let W be any element of $\mathfrak{g}(T_\Omega)$ such that $ad\,W$ is nilpotent. Write $W = U^{(1)} + \sqrt{-1}V^{(0)} + T$, where $U^{(1)} + \sqrt{-1}V^{(0)} \in \mathfrak{a}_*$ and $T \in \mathfrak{s}$. It follows again from (3.1) that $V^{(0)}$ has the form $V^{(0)} = \mu\partial_1$, where μ is a real constant. Therefore we have $W - (\mu/\lambda)Z = U^{(1)} - (\mu/\lambda)X^{(1)} + T$, or

$$U^{(1)} - \frac{\mu}{\lambda}X^{(1)} = W - \frac{\mu}{\lambda}Z - T. \tag{3.3}$$

Since W, $(\mu/\lambda)Z$, $T \in \mathfrak{g}(T_\Omega)$, this shows that $U^{(1)} - (\mu/\lambda)X^{(1)}$ is an element of $\mathfrak{g}(T_\Omega)$ which takes the value 0 at the origin o, so that $U^{(1)} - (\mu/\lambda)X^{(1)}$ belongs to $\mathfrak{k}$. On the other hand, since $ad\,(U^{(1)} - (\mu/\lambda)X^{(1)}) = ad\,W - ad\,(\mu/\lambda)Z - ad\,T$ by (3.3) and since $ad\,W$, $ad\,((\mu/\lambda)Z)$, $ad\,T$ are all nilpotent, $ad\,(U^{(1)} - (\mu/\lambda)X^{(1)})$ is nilpotent. From these, we see that $ad\,(U^{(1)} - (\mu/\lambda)X^{(1)})$ is semisimple and nilpotent, and hence $ad\,(U^{(1)} - (\mu/\lambda)X^{(1)}) = 0$, which implies that $U^{(1)} - (\mu/\lambda)X^{(1)} = 0$. By (3.3), we have $W = (\mu/\lambda)Z + T$, and $\mathfrak{n} = \mathfrak{s} + \{Z\}_{\mathbf{R}}$ is shown. We thus conclude that $\mathfrak{n}$ has dimension less than or equal to $n+1$.

We show that any n-dimensional abelian ideal $\mathfrak{s}_0$ in $\mathfrak{n}$ coincides with $\mathfrak{s}$. Suppose that $\mathfrak{s}_0 \neq \mathfrak{s}$. Then, since $\mathfrak{s}_0 \subset \mathfrak{n} \subset \mathfrak{g}(T_\Omega) = \mathfrak{s} + \mathfrak{a}_* = \mathfrak{a}(T_\Omega)$ and since $\mathfrak{n}$ contains $\mathfrak{s}$, we can apply [**6**, Lemma 4.2] to $\mathfrak{s}_0$ by noting the proof of it. Therefore there exists a nonzero complete polynomial vector field on T_Ω of degree 2. By the Prolongation Theorem, this yields that $\mathfrak{b} \neq \{0\}$, which is a contradiction. We thus conclude that $\mathfrak{s}_0 = \mathfrak{s}$, and the proof of the theorem is completed. $\square$

Remark 3.2. Theorem 3.1 asserts that, under the assumption of the theorem, any n-dimensional abelian ideal in the nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$. The result like this plays a key role on the study of the holomorphic equivalence probelem for tube domains.

Using Theorem 3.1, we can give an answer to the holomorphic equivalence problem for a class of tube domains with solvable groups of automorphisms.

Theorem 3.3. *Let T_Ω and $T_{\Omega'}$ be two tube domains in $\mathbf{C}^n$ whose bases Ω and Ω' are convex domains in $\mathbf{R}^n$ containing no complete straight lines and let $n \geq 2$. Assume that*:

i) *T_Ω and $T_{\Omega'}$ are tube domains with polynomial infinitesimal automorphisms*;

ii) *$\mathrm{Aut}(T_\Omega)$ is a solvable Lie group*;

iii) *There exists a point z_0 of T_Ω such that the orbit of $G(T_\Omega)$ through z_0 has dimension $n+1$.*

Under these assumptions, if T_Ω and $T_{\Omega'}$ are holomorphically equivalent, then they are affinely equivalent.

PROOF. Let $\varphi : T_\Omega \to T_{\Omega'}$ be a biholomorphic mapping between T_Ω and $T_{\Omega'}$. Since $\varphi \mathrm{Aut}(T_\Omega)\varphi^{-1} = \mathrm{Aut}(T_{\Omega'})$ and $\varphi G(T_\Omega)\varphi^{-1} = G(T_{\Omega'})$, we see from the assumption that $\mathrm{Aut}(T_{\Omega'})$ is solvable and the orbit of $G(T_{\Omega'})$ through $\varphi(z_0)$ has dimension $n+1$. Note that, replacing if necessary T_Ω and $T_{\Omega'}$ by $T_\Omega - z_0$ and $T_{\Omega'} - \varphi(z_0)$, respectively, we may assume that z_0 and $\varphi(z_0)$ are the origin.

Now, let $\Phi : \mathrm{Aut}(T_\Omega) \to \mathrm{Aut}(T_{\Omega'})$ be a Lie group isomorphism between $\mathrm{Aut}(T_\Omega)$ and $\mathrm{Aut}(T_{\Omega'})$ given by $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ for $f \in \mathrm{Aut}(T_\Omega)$. We denote by Φ_* the differential of Φ, which is regarded as a Lie algebra isomorphism between $\mathfrak{g}(T_\Omega)$ and $\mathfrak{g}(T_{\Omega'})$. Let $\mathfrak{n}$ and $\mathfrak{n}'$ be the nilradicals of $\mathfrak{g}(T_\Omega)$ and $\mathfrak{g}(T_{\Omega'})$, respectively. Then we have $\Phi_*(\mathfrak{n}) = \mathfrak{n}'$. Since $\mathfrak{s}(T_\Omega)$ is an n-dimensional abelian ideal in $\mathfrak{n}$, we see that $\Phi_*(\mathfrak{s}(T_\Omega))$ is an n-dimensional abelian ideal in $\mathfrak{n}'$. By Theorem 3.1 and the remark after it, there exists an inner automorphism τ_* of $\mathfrak{g}(T_{\Omega'})$ such that $\tau_*(\Phi_*(\mathfrak{s}(T_\Omega))) = \mathfrak{s}(T_{\Omega'})$. Here τ_* is the differential of a Lie group automorphism τ of $G(T_{\Omega'})$ given by $\tau(h) = g \circ h \circ g^{-1}$ for $h \in G(T_{\Omega'})$, where g is some element of $G(T_{\Omega'})$. Therefore we have $(g \circ \varphi)\Sigma_{T_\Omega}(g \circ \varphi)^{-1} = \Sigma_{T_{\Omega'}}$. It follows from [**3**, Section 1, Proposition] that $g \circ \varphi$ is given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$. This shows that T_Ω and $T_{\Omega'}$ are affinely equivalent, and our theorem is proved. □

4. An example of a tube domain whose automorphism group is solvable and contains nonaffine automorphisms.

In this section, we give a concrete example of a tube domain in Theorem 3.1, which is an example of a tube domain whose automorphism group is solvable and contains non-affine automorphisms as well. In what follows, we use the same notation as in Theorem 3.1.

THEOREM 4.1. *Let Ω_0 is a convex domain in $\mathbf{R}^3$ containing no complete straight lines given by*

$$\Omega_0 = \{(y_1, y_2, y_3) \in \mathbf{R}^3 \mid y_2 > y_1^2 + e^{y_3^2} - 2\}.$$

Then T_{Ω_0} is a tube domain with polynomial infinitesimal automorphisms, and $\mathfrak{g}(T_{\Omega_0})$ is given by

$$\mathfrak{g}(T_{\Omega_0}) = \mathfrak{s} + \mathfrak{a}_* + \mathfrak{b},$$
$$\mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}},$$
$$\mathfrak{b} = \{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}}.$$

To prove Theorem 4.1, it suffices to prove the following:
1) Every element of $\mathfrak{g}(T_{\Omega_0})$ is a polynomial vector field;
2) $\mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}$;

3) $\mathfrak{b} = \{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}}$;
4) $\mathfrak{g}(T_{\Omega_0}) = \mathfrak{q}$.

We prove 1). For this, we need a lemma. Before stating the lemma, we fix notation. Let r be an integer between 0 and n. Let $\pi' : \mathbf{C}^n \to \mathbf{C}^r$ be the projection given by $\pi'(z_1, \dots, z_n) = (z_1, \dots, z_r)$. For $k = (k_1, \dots, k_r) \in \mathbf{Z}^r$, we define a tube domain $D(k_1, \dots, k_r)$ in $\mathbf{C}^r$ by

$$D(k_1, \dots, k_r) = T_{(k_1\pi,(k_1+1)\pi)} \times \cdots \times T_{(k_r\pi,(k_r+1)\pi)}.$$

The following lemma gives a useful criterion for a given tube domain to be a tube domain with polynomial infinitesimal automorphisms.

Lemma 4.2. *Let $T_{\tilde{\Omega}}$ be a tube domain in $\mathbf{C}^n$ whose base $\tilde{\Omega}$ has the convex hull containing no complete straight lines. Assume that $\mathfrak{g}(T_{\tilde{\Omega}})$ contains a subspace $\mathfrak{e}$ given by*

$$\mathfrak{e} = \sum_{i=1}^{r} \left\{ e^{z_i} \left(\partial_i + \sum_{j=r+1}^{n} \sqrt{-1}a_i^j\partial_j \right), e^{-z_i} \left(\partial_i - \sum_{j=r+1}^{n} \sqrt{-1}a_i^j\partial_j \right) \right\}_{\mathbf{R}},$$

where r is an integer between 0 and n and a_i^j, $i = 1, \dots, r$, $j = r+1, \dots, n$, are real constants. If $T_{\tilde{\Omega}'}$ is the tube domain in $\mathbf{C}^r$ given as the image of the domain $T_{\tilde{\Omega}}$ under the projection π', then $T_{\tilde{\Omega}'} = D(k_1, \dots, k_r)$ for some $(k_1, \dots, k_r) \in \mathbf{Z}^r$. As a consequence, one has

$$\tilde{\Omega} \subset (k_1\pi, (k_1+1)\pi) \times \cdots \times (k_r\pi, (k_r+1)\pi) \times \mathbf{R}^{n-r}.$$

Proof. By [**3**, Section 3, Lemma 4], the holomorphic vector fields $e^{z_1}\partial_1, \dots, e^{z_r}\partial_r$ are complete on $T_{\tilde{\Omega}'}$. Therefore it follows from [**4**, Section 2, Lemma 3] that $T_{\tilde{\Omega}'} = D(k_1, \dots, k_r)$ for some $(k_1, \dots, k_r) \in \mathbf{Z}^r$. $\square$

We turn to the proof of 1). By the Structure Theorem in Section 1, there exists a tube domain $T_{\tilde{\Omega}_0}$ which is affinely equivalent to T_{Ω_0} such that $\mathfrak{g}(T_{\tilde{\Omega}_0})$ has the direct sum decomposition

$$\mathfrak{g}(T_{\tilde{\Omega}_0}) = \mathfrak{p} + \mathfrak{e}$$

for which

$$\mathfrak{p} = \{X \in \mathfrak{g}(T_{\tilde{\Omega}_0}) \,|\, X \text{ is a polynomial vector field}\},$$
$$\mathfrak{e} = \sum_{i=1}^{r} \left\{ e^{z_i} \left(\partial_i + \sum_{j=r+1}^{3} \sqrt{-1}a_i^j\partial_j \right), e^{-z_i} \left(\partial_i - \sum_{j=r+1}^{3} \sqrt{-1}a_i^j\partial_j \right) \right\}_{\mathbf{R}},$$

where r is an integer between 0 and 3 and a_i^j, $i = 1, \dots, r$, $j = r+1, \dots, 3$, are real constants. Suppose here that $\mathfrak{e} \neq \{0\}$, or $r \geq 1$. An application of Lemma 4.2 to $T_{\tilde{\Omega}_0}$, $\mathfrak{e}$ yields that

$$\tilde{\Omega}_0 \subset (k_1\pi, (k_1+1)\pi) \times \cdots \times (k_r\pi, (k_r+1)\pi) \times \mathbf{R}^{3-r}.$$

Since $\tilde{\Omega}_0$ is affinely equivalent to Ω_0, it follows that this can not occur. Therefore we obtain $\mathfrak{e} = \{0\}$, and hence $\mathfrak{g}(T_{\tilde{\Omega}_0}) = \mathfrak{p}$, which implies that every element of $\mathfrak{g}(T_{\Omega_0})$ is a polynomial vector field.

We prove 2). The defining function ρ of the boundary of Ω_0 is given by $\rho(y_1, y_2, y_3) = y_1^2 - y_2 + e^{y_3^2} - 2$. Let Y be an affine vector field on Ω_0 and write

$$\begin{aligned} Y = &(a_1y_1 + a_2y_2 + a_3y_3 + a_0)\frac{\partial}{\partial y_1} + (b_1y_1 + b_2y_2 + b_3y_3 + b_0)\frac{\partial}{\partial y_2} \\ &+ (c_1y_1 + c_2y_2 + c_3y_3 + c_0)\frac{\partial}{\partial y_3}, \end{aligned}$$

where a_i, b_i, c_i, $i = 0, 1, 2, 3$, are real constants. In view of (1.1) and (1.2), to prove 2), it is sufficient to show that

$$(Y\rho)(y_1, y_2, y_3) = 0 \quad \text{for all } (y_1, y_2, y_3) \in \mathbf{R}^3 \text{ with } y_2 = y_1^2 + e^{y_3^2} \tag{4.1}$$

if and only if $b_1 = 2a_0$, and the other coefficients are all 0, that is, Y has the form

$$Y = \lambda\frac{\partial}{\partial y_1} + 2\lambda y_1\frac{\partial}{\partial y_2},$$

where λ is a real constant. The "if" part is immediate. We show the "only if" part. (4.1) is written as

$$\begin{aligned} 0 = &\,2a_1y_1^2 + 2a_2y_1^3 + 2a_2y_1e^{y_3^2} + 2a_3y_1y_3 + 2a_0y_1 \\ &- b_1y_1 - b_2y_1^2 - b_2e^{y_3^2} - b_3y_3 - b_0 \\ &+ 2c_1y_1y_3e^{y_3^2} + 2c_2y_1^2y_3e^{y_3^2} + 2c_2y_3e^{2y_3^2} + 2c_3y_3^2e^{y_3^2} + 2c_0y_3e^{y_3^2}. \end{aligned} \tag{4.2}$$

By letting $y_3 = 0$ in (4.2), it follows that

$$a_2 = 0,\ b_2 = 2a_1. \tag{4.3}$$

On the other hand, putting $y_1 = 0$ in (4.2), we have

$$0 = -b_2e^{y_3^2} - b_3y_3 - b_0 + 2c_2y_3e^{2y_3^2} + 2c_3y_3^2e^{y_3^2} + 2c_0y_3e^{y_3^2}. \tag{4.4}$$

By substituting $y_3 = 0$ into (4.4), we see that

$$0 = -b_2 - b_0. \tag{4.5}$$

Also, by differentiating the both sides of (4.4) with respect to y_3 and substituting $y_3 = 0$ into it, we see that

$$0 = -b_3 + 2c_2 + 2c_0. \tag{4.6}$$

Moreover, by differentiating the both sides of (4.4) twice with respect to y_3, it follows that

$$
\begin{aligned}
0 = {} & b_2 e^{y_3^2}(\alpha y_3^2 + (\text{terms of degree} \leq 1)) \\
& + c_2 e^{2y_3^2}(\beta y_3^3 + (\text{terms of degree} \leq 2)) + c_3 e^{y_3^2}(\gamma y_3^4 + (\text{terms of degree} \leq 3)) \\
& + c_0 e^{y_3^2}(\delta y_3^3 + (\text{terms of degree} \leq 2)),
\end{aligned} \tag{4.7}
$$

where α, β, γ, δ are some nonzero constants and (terms of degree $\leq k$) denotes a polynomial in y_3 of degree less than or equal to k. Multiplying the both sides of (4.7) by $e^{-y_3^2}$, we obtain

$$
\begin{aligned}
0 = {} & b_2(\alpha y_3^2 + (\text{terms of degree} \leq 1)) \\
& + c_2 e^{y_3^2}(\beta y_3^3 + (\text{terms of degree} \leq 2)) + c_3(\gamma y_3^4 + (\text{terms of degree} \leq 3)) \\
& + c_0(\delta y_3^3 + (\text{terms of degree} \leq 2)).
\end{aligned} \tag{4.8}
$$

Since $e^{y_3^2}$ is not a polynomial in y_3, this shows that $c_2 = 0$. As a result, we have $c_3 = 0$. Indeed, the right hand side of (4.8) is a polynomial in y_3 and the coefficient of y_3^4 is $c_3\gamma$. A similar argument shows that $c_0 = b_2 = 0$. Combining these with (4.3), (4.5) and (4.6), we have $a_1 = b_0 = b_3 = 0$ as well. To sum up so far, we obtain

$$
a_1 = a_2 = 0,\ b_2 = b_3 = b_0 = 0,\ c_2 = c_3 = c_0 = 0. \tag{4.9}
$$

Now, substituting (4.9) into (4.2) yields that

$$
0 = 2a_3 y_1 y_3 + 2a_0 y_1 - b_1 y_1 + 2c_1 y_1 y_3 e^{y_3^2}. \tag{4.10}
$$

Letting $y_3 = 0$ in (4.10), we see that

$$
b_1 = 2a_0. \tag{4.11}
$$

By substituting (4.11) into (4.10), it follows that

$$
a_3 = c_1 = 0. \tag{4.12}
$$

(4.9), (4.11), and (4.12) show the "only if" part, and the proof of 2) is completed.

We prove 3). Note first that, by 2), the vector field $\sqrt{-1}\partial_1 + 2z_1\partial_2$ is complete on T_{Ω_0}. Now, set $Z = \sqrt{-1}z_1\partial_1 + z_1^2\partial_2$. Since $[\partial_1, Z] = \sqrt{-1}\partial_1 + 2z_1\partial_2$ and $[\partial_2, Z] = [\partial_3, Z] = 0$ are all complete on T_{Ω_0}, it follows from the Prolongation Theorem that Z is complete on T_{Ω_0}, so that $\{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}} \subset \mathfrak{b}$. Let $W = U^{(2)} + \sqrt{-1}V^{(1)}$ be any element of $\mathfrak{b}$. Then we have

$$
[\partial_i, W] \in \mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}} \quad \text{for every } i = 1, 2, 3. \tag{4.13}
$$

Write

$$
U^{(2)} = \sum_{j=1}^{3} f_j \partial_j \quad \text{and} \quad V^{(1)} = \sum_{j=1}^{3} g_j \partial_j,
$$

where f_j and g_j are homogeneous polynomials in z_1, z_2, z_3 with real coefficients of degrees 2 and 1, respectively. By (4.13), for $j = 1, 3$, we have $\partial_i f_j = 0$ for every $i = 1, 2, 3$, which implies that $f_1 = f_3 = 0$. On the other hand, if we write $f_2(z) = az_1^2 + bz_2^2 + cz_3^2 + dz_1z_2 + ez_2z_3 + fz_3z_1$, where a, b, c, d, e, f are real constants, then we see from (4.13) that $\partial_2 f_2(z) = 2bz_2 + dz_1 + ez_3$ and $\partial_3 f_2(z) = 2cz_3 + ez_2 + fz_1$ are constant multiples of z_1, respectively. Therefore we have $b = c = e = 0$, so that $f_2(z) = az_1^2 + dz_1z_2 + fz_3z_1$. Furthermore, since $\partial_1 f_2(z) = 2az_1 + dz_2 + fz_3$ is a constant multiple of z_1 again by (4.13), it follows that $d = f = 0$. We thus obtain

$$U^{(2)} = az_1^2\partial_2. \tag{4.14}$$

Now, By (4.13), for $j = 2, 3$, we have $\partial_i g_j = 0$ for every $i = 1, 2, 3$, which implies that $g_2 = g_3 = 0$. From this and (4.14), we see that W has the form $W = \sqrt{-1}(pz_1 + qz_2 + rz_3)\partial_1 + az_1^2\partial_2$, where p, q, r are real constants. Since $[\partial_2, W] = \sqrt{-1}q\partial_1$ and $[\partial_3, W] = \sqrt{-1}r\partial_1$, it follows from [**3**, Section 3, Lemma 5] that $q = r = 0$, so that $W = \sqrt{-1}pz_1\partial_1 + az_1^2\partial_2$. Therefore we have $W - aZ = \sqrt{-1}(p - a)z_1\partial_1$. By i) of Lemma 2.2, this shows that $0 = \sqrt{-1}(p - a)z_1\partial_1 = W - aZ$, or $W = aZ$. We thus conclude 3).

Finally, we prove 4). Note first that, by 2) and 3), $\mathfrak{q}$ is given by

$$\mathfrak{q} = \{(\sqrt{-1}\lambda z_1 + \sqrt{-1}\mu + \alpha)\partial_1 + (\lambda z_1^2 + 2\mu z_1 + \beta)\partial_2 + \gamma\partial_3 \,|\, \alpha, \beta, \gamma, \lambda, \mu \in \mathbf{R}\}. \tag{4.15}$$

Now, suppose that $\mathfrak{g}(T_{\Omega_0}) \neq \mathfrak{q}$, or that there exists a nonzero element W of $\mathfrak{g}(T_{\Omega_0})$ of degree greater than or equal to 3. Then we can choose suitable nonnegative integers $\nu_1, \ldots, \nu_n$ such that $(ad\,\partial_1)^{\nu_1} \cdots (ad\,\partial_n)^{\nu_n} W$ is an element of $\mathfrak{g}(T_{\Omega_0})$ of degree just 3, which we denote again by W. Write $W = \sum_{j=1}^{3} f_j\partial_j$, where f_j, $j = 1, 2, 3$, are polynomials in z_1, z_2, z_3 of degree less than or equal to 3. For the polynomial f_j, we denote by $f_j^{(3)}$ and $f_j^{(2)}$ its homogeneous parts of degrees 3 and 2, respectively. Then, using (4.15) and the fact that $[\partial_i, W] \in \mathfrak{q}$ for every $i = 1, 2, 3$, we can show the following:

a) $f_1^{(3)}(z) = 0$ and $f_1^{(2)}(z) = pz_1^2$, where p is a complex constant;

b) $f_2^{(3)}(z) = rz_1^3$ and $f_2^{(2)}(z) = qz_1^2$, where r is a nonzero real constant and q is a complex constant;

c) $f_3^{(3)}(z) = 0$ and $f_3^{(2)}(z) = 0$.

Consequently, replacing $r^{-1}W, r^{-1}p, r^{-1}q$ by W, p, q if necessary, we can write

$$\begin{aligned} W = {} & (pz_1^2 + a_1z_1 + a_2z_2 + a_3z_3 + a_0)\partial_1 \\ & + (z_1^3 + qz_1^2 + b_1z_1 + b_2z_2 + b_3z_3 + b_0)\partial_2 + (c_1z_1 + c_2z_2 + c_3z_3 + c_0)\partial_3, \end{aligned} \tag{4.16}$$

where p, q, a_i, b_i, c_i, $i = 0, 1, 2, 3$, are complex constants. Since

$$[\partial_1, W] = (2pz_1 + a_1)\partial_1 + (3z_1^2 + 2qz_1 + b_1)\partial_2 + c_1\partial_3$$

coincides with

$$(\sqrt{-1}\lambda z_1 + \sqrt{-1}\mu + \alpha)\partial_1 + (\lambda z_1^2 + 2\mu z_1 + \beta)\partial_2 + \gamma\partial_3 \quad \text{for } \lambda = 3 \text{ and } \mu = q,$$

we see that

$$p = \frac{3}{2}\sqrt{-1},\ a_1 = \sqrt{-1}q + \alpha. \tag{4.17}$$

Set $Z = \sqrt{-1}\partial_1 + 2z_1\partial_2$. Then, substituting (4.17) into (4.16), we have

$$\begin{aligned}[Z, W] = {} & \{(2a_2 - 3)z_1 + (\sqrt{-1}\alpha - q)\}\partial_1 \\ & + \{2(b_2 - \alpha)z_1 - 2a_2z_2 - 2a_3z_3 + \sqrt{-1}b_1 - 2a_0\}\partial_2 + (2c_2z_1 + \sqrt{-1}c_1)\partial_3.\end{aligned} \tag{4.18}$$

This shows that $[Z, W]$ is an element of

$$\mathfrak{s} + \mathfrak{a}_* = \{(\sqrt{-1}\mu + \alpha)\partial_1 + (2\mu z_1 + \beta)\partial_2 + \gamma\partial_3 \,|\, \alpha, \beta, \gamma, \mu \in \mathbf{R}\}$$

and has the form (4.18). As a consequence, we must have $a_2 = 3/2$ and $a_2 = 0$ simultaneouly, which is a contradiction. We thus conclude 4), and the proof of the theorem is completed. □

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