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Geometry of the Gromov product: Geometry at infinity of Teichmüller space

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Abstract. This paper is devoted to studying transformations on metric spaces. It is done in an effort to produce qualitative version of quasi-isometries which takes into account the asymptotic behavior of the Gromov product in hyperbolic spaces. We characterize a quotient semigroup of such transformations on Teichmüller space by use of simplicial automorphisms of the complex of curves, and we will see that such transformation is recognized as a "coarsification" of isometries on Teichmüller space which is rigid at infinity. We also show a hyperbolic characteristic that any finite dimensional Teichmüller space does not admit (quasi)-invertible rough-homothety.

1. Introduction.

1.1. Backgrounds.

Let (X, d_X) be a metric space. The *Gromov product* with reference point $x_0 \in X$ is defined by

$$\langle x_1 \,|\, x_2 \rangle^X_{x_0} = \langle x_1 \,|\, x_2 \rangle_{x_0} = \frac{1}{2}(d_X(x_0, x_1) + d_X(x_0, x_2) - d_X(x_1, x_2)). \tag{1}$$

We define the *Gromov product* of two sequences $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$, $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}}$ in X by

$$\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle_{x_0} = \liminf_{n,m\to\infty} \langle x_n \,|\, y_m \rangle_{x_0} \tag{2}$$

Convention 1.1. *When the metric space and the reference point in the discussion are clear in the context, we omit to specify them in denoting the Gromov product. We always assume in this paper that any metric space is of infinite diameter.*

Let $\mathrm{USq}(X) \subset X^{\mathbb{N}}$ is the set of unbounded sequences in X. We call a sequence $\boldsymbol{x} \in \mathrm{USq}(X)$ *convergent at infinity* if

$$\langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle = \infty \tag{3}$$

(cf. Section 8 in [**14**]). Any sequence satisfying (3) is contained in $\mathrm{USq}(X)$. The definition (3) is independent of the choice of the reference point. Let

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Table 1. Comparison with the coarse geometry for general metric spaces X. For details, see Section 1.2 and Section 3.

Coarse geometry	Geometry on the Gromov product
Quasi-isometries (qi)	Asymptotically conservative (ac) mappings
Coarsely Lipschitz	weakly ac
Coarsely co-Lipschitz	$\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x})) \Rightarrow \boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x}) \quad (\forall \boldsymbol{x}, \boldsymbol{z} \in \mathrm{USq}(X))$
Cobounded	Asymptotically surjective
Cobounded qi	Invertible ac
Quasi-inverse	Asymptotic quasi-inverse
Parallelism	Close at infinity
$\mathfrak{Q}I(X)$ (cf. (21))	$\mathfrak{AC}(X)$ (Section 1.3)

Table 2. Comparison with the coarse geometry for metric spaces which are WBGP. For details, see Section 4.

Coarse geometry	Geometry on the Gromov product
Quasi-isometries	Asymptotically conservative mappings
Coarsely Lipschitz	$\langle \boldsymbol{x} \,\vert\, \boldsymbol{z} \rangle = \infty \Rightarrow \langle \omega(\boldsymbol{x}) \,\vert\, \omega(\boldsymbol{z}) \rangle = \infty \quad (\forall \boldsymbol{x}, \boldsymbol{z} \in \mathrm{Sq}^{\infty}(X))$
Coarsely co-Lipschitz	$\langle \omega(\boldsymbol{x}) \,\vert\, \omega(\boldsymbol{z}) \rangle = \infty \Rightarrow \langle \boldsymbol{x} \,\vert\, \boldsymbol{z} \rangle = \infty \quad (\forall \boldsymbol{x}, \boldsymbol{z} \in \mathrm{Sq}^{\infty}(X))$

$$\mathrm{Sq}^{\infty}(X) = \{\boldsymbol{x} \in \mathrm{USq}(X) \mid \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle = \infty\}.$$

We say that a sequence $\boldsymbol{y} \in X^{\mathbb{N}}$ is *visually indistinguishable* from $\boldsymbol{x} \in X^{\mathbb{N}}$ if

$$\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = \infty. \tag{4}$$

For a sequence $\boldsymbol{x} \in X^{\mathbb{N}}$, we define

$$\mathrm{Vis}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathrm{Sq}^{\infty}(X) \mid \langle \boldsymbol{y} \,|\, \boldsymbol{x} \rangle = \infty\}.$$

Notice that $\mathrm{Vis}(\boldsymbol{x}) = \emptyset$ when a sequence $\boldsymbol{x}$ is bounded. Two sequences $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{USq}(X)$ are said to be *asymptotic* if $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. When X is a Gromov hyperbolic space, $\mathrm{Vis}(\boldsymbol{x})$ defines a point in the Gromov boundary (cf. [**6**]).

In this paper, aiming for developing the coarse geometry and the asymptotic geometry on Teichmüller space, we investigate the theory of mappings on metric spaces with respecting for asymptotic behavior of sequences converging at infinity (cf. Tables 1 and 2). Namely, we (pretend to) recognize that two unbounded sequences $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ determine the same ideal point at infinity if two sequences $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ converging at infinity are asymptotic. Intuitively, asymptotically conservative mappings given in this paper are mappings keeping the divergence conditions of the Gromov products of two sequences converging at infinity

1.2. Definitions.

Let X and Y be metric spaces. A mapping $\omega \in Y^X$ is said to be *asymptotically conservative with the Gromov product* (*asymptotically conservative* for short) if for any sequence $\boldsymbol{x} \in \mathrm{USq}(X)$, the following two conditions hold;

1. $\omega(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\omega(\boldsymbol{x}))$.
2. For any $\boldsymbol{z} \in \mathrm{USq}(X)$, if $\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$, then $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$.

We will call a map $\omega \in Y^X$ with the condition (1) above *weakly asymptotically conservative* (cf. Section 3.2).

Here, for a sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, $E \subset X^{\mathbb{N}}$ and a mapping $\omega \in Y^X$, we define usually

$$\omega(\boldsymbol{x}) = \{\omega(x_n)\}_{n\in\mathbb{N}} \in Y^{\mathbb{N}}, \quad \omega(E) = \{\omega(\boldsymbol{x}) \mid \boldsymbol{x} \in E\}.$$

Two mappings ω_1, $\omega_2 \in Y^X$ are said to be *close at infinity*, if for any $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^{\infty}(X)$, $\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^2))$ holds whenever $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. An asymptotically conservative mapping $\omega \in Y^X$ is said to be *invertible* if there is an asymptotically conservative mapping $\omega' \in X^Y$ such that $\omega' \circ \omega$ and $\omega \circ \omega'$ are close to the identity mappings on X and Y, respectively. We call such ω' an *asymptotic quasi-inverse* of ω. Let $\mathrm{AC}_{\mathrm{inv}}(X)$ be the set of invertible asymptotically conservative mappings on X to itself. For instance, any isometric isomorphism between metric spaces is invertible asymptotically conservative. The notions of mappings given above are stable under parallelism (cf. Proposition 3.1). In Section 3, we will give more discussion.

1.3. Results.

We first observe the following theorem (cf. Section 2.7).

Theorem A (The group $\mathfrak{AC}(X)$). *Let X be a metric space. The set $\mathrm{AC}_{\mathrm{inv}}(X)$ admits a monoid structure with respect to the composition of mappings. Furthermore, the relation "closeness at infinity" is a semigroup congruence on $\mathrm{AC}_{\mathrm{inv}}(X)$ and the quotient semigroup $\mathfrak{AC}(X)$ is a group.*

Large scale geometry of Teichmüller space. Our main interest is to clarify the large scale geometry of Teichmüller space $\mathcal{T}$ in respecting for asymptotic behaviors of sequences converging at infinity.

Rigidity theorem. Let S be a compact orientable surface. We denote the *complexity* of S by

$$\mathrm{cx}(S) = 3\,\mathrm{genus}(S) - 3 + \#\{\text{components of } \partial S\}.$$

The Euler characteristic of S is denoted by $\chi(S)$. Throughout this paper, we always assume that $\chi(S) < 0$. S is said to be in the *sporadic case* if $\mathrm{cx}(S) \leq 1$.

Let $\mathcal{T}$ be the Teichmüller space of S endowed with the Teichmüller distance. The extended mapping class group $\mathrm{MCG}^*(S)$ of S acts isometrically on $\mathcal{T}$ and we have a group homomorphism

$$\mathcal{I}_0 \colon \mathrm{MCG}^*(S) \to \mathrm{Isom}(\mathcal{T}).$$

We also have a monoid homomorphism

$$\mathcal{I}\colon \mathrm{Isom}(\mathcal{T}) \to \mathrm{AC}_{\mathrm{inv}}(\mathcal{T})$$

defined by the inclusion (see Section 9.3.1). We will prove the following rigidity theorem (cf. Theorem 9.2).

Theorem B (Rigidity). *Suppose* $\mathrm{cx}(S) \geq 2$. *Let* $\mathbb{X}(S)$ *be the complex of curves on* S. *Then, there is a monoid epimorphism*

$$\Xi\colon \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}) \to \mathrm{Aut}(\mathbb{X}(S))$$

which descends to an isomorphism

$$\mathfrak{AC}(\mathcal{T}) \to \mathrm{Aut}(\mathbb{X}(S))$$

satisfying the following commutative diagram

$$\begin{array}{ccccc} \mathrm{MCG}^*(S) & \xrightarrow{\mathcal{I}_0} & \mathrm{Isom}(\mathcal{T}) & \xrightarrow{\mathcal{I}} & \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}) \\ & & \searrow^{\text{group iso}} & \downarrow^{\text{proj}} & \searrow^{\Xi} \\ & & & \mathfrak{AC}(\mathcal{T}) & \xrightarrow{\text{group iso}} \mathrm{Aut}(\mathbb{X}(S)), \end{array}$$

where $\mathrm{Aut}(\mathbb{X}(S))$ *is the group of simplicial automorphisms of* $\mathbb{X}(S)$.

Relation to the coarse geometry. Recently, A. Eskin, H. Masur and K. Rafi [**9**] and B. Bowditch [**5**] independently observed a remarkable result that any cobounded quasi-isometry of $\mathcal{T}$ is parallel to an isometry, and the inclusion $\mathrm{Isom}(\mathcal{T}) \hookrightarrow \mathrm{QI}(\mathcal{T})$ induces an isomorphism

$$\mathrm{Isom}(\mathcal{T}) \cong \mathfrak{QI}(\mathcal{T}) = \mathrm{QI}(\mathcal{T})/(\text{parallelism})$$

when S is in the non-sporadic case. Especially, any self quasi-isometry on $\mathcal{T}$ is weakly asymptotically conservative (cf. Proposition 3.1). Hence, we have the following sequence of monoid homomorphisms

$$\mathrm{Isom}(\mathcal{T}) \hookrightarrow \mathrm{QI}(\mathcal{T}) \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}) \tag{5}$$

by inclusions (see Corollary 3.1). Theorem B implies the following.

Corollary 1.1 (Relation to the coarse geometry on $\mathcal{T}$). *For non-sporadic cases, a quotient set* $\mathrm{QI}(\mathcal{T})/(\textit{close at infinity})$ *admits a group structure equipped with the operation defined by composition, and the sequence* (5) *descends to the following sequence of isomorphisms*

$$\mathrm{Isom}(\mathcal{T}) \cong \mathfrak{QI}(\mathcal{T}) \cong \mathrm{QI}(\mathcal{T})/(\textit{close at infinity}) \cong \mathfrak{AC}(\mathcal{T}).$$

Corollary 1.2 (Criterion for parallelism). *Let* $\omega, \psi\colon \mathcal{T} \to \mathcal{T}$ *be cobounded quasi-isometries. The following are equivalent.*

1. *For any* $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Sq}^{\infty}(\mathcal{T})$ *with* $\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = \infty$*, it holds* $\langle \omega(\boldsymbol{x}) \,|\, \psi(\boldsymbol{y}) \rangle = \infty$*.*

2. ψ *is parallel to* ω*.*

Corollary 1.2 gives a hyperbolic nature and Teichmüller space. Indeed, the same conclusion holds when we consider the hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ instead of the Teichmüller space $\mathcal{T}$.

No-rough homothety. By applying the discussion in the proof of Theorem B, we also obtain a hyperbolic characteristic of Teichmüller space. In fact, we will give a proof of the following folklore result in Section 9.5.

Theorem C (No rough-homothety with $K \neq 1$). *There is no* (K, D)*-rough homothety with asymptotic quasi-inverse on the Teichmüller space of* S *unless* $K = 1$*.*

Here, a mapping $\omega\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is said to be a (K, D)-*rough homothety* if

$$|d_Y(\omega(x_1), \omega(x_2)) - K\, d_X(x_1, x_2)| \leq D \tag{6}$$

for $x_1, x_2 \in X$ (cf. Chapter 7 of [**7**]). Any rough-homothety is asymptotically conservative. Theorem C implies that there is no non-trivial similarity in Teichmüller space, like in hyperbolic spaces. Since rough homotheties are quasi-isometries, if ω in Theorem C is cobounded, the rigidity in the theorem follows from Eskin–Masur–Rafi–Bowditch's quasi-isometry rigidity theorem. However, the author does not know whether rough homotheties in the statement are always cobounded or not.

Recently, enormous hyperbolic characteristics are observed in Teichmüller space (e.g. [**4**], [**5**], [**9**]), though Teichmüller space is not Gromov hyperbolic (cf. [**30**]. See also Section 6.4). Some of these hyperbolic nature might naturally imply that a measurable (K, D)-homothety on Teichmüller space does not exist unless $K \neq 1$.

1.4. Plan of this paper.

This paper is organized as follows: In Section 2, we will introduce asymptotically conservative mappings on metric spaces. We first start with the basics for the Gromov product, and we next develop the properties of asymptotically conservative mappings. We will prove Theorem A in Section 2.7. In Section 3 and Section 4, we will discuss a relation between our geometry and the coarse geometry.

From Section 5 to Section 7, we devote to preparing for the proofs of Theorems B and C. In Section 5, we give basic notions of Teichmüller theory including the definitions of Teichmüller space, measured foliations and extremal length. In Section 6, we recall our unification theorem for extremal length geometry on Teichmüller space via intersection number. One of the key for proving our rigidity theorem is to characterize the null sets for points in the GM-cone (cf. Theorem 7.1) The characterization is also applied to proving a rigidity theorem of holomorphic disks in the Teichmüller space and to studying the null-set reductions of several compactifications of Teichmüller space (cf. [**3**] and [**37**]). In Section 8, We define the reduced Gardner–Masur compactification and study the action of asymptotically conservative mappings on the reduced Gardiner–Masur compactification. In Section 9, we will prove Theorems B and C.

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2. Asymptotically conservative with the Gromov product.

2.1. Basics of the Gromov product.

Let (X, d_X) be a metric space. The following is known for x_1, x_2, x_3, z_1, $w_1 \in X$:

$$\langle x_1 \,|\, x_2 \rangle_{z_1} \geq 0, \tag{7}$$

$$\langle x_1 \,|\, x_2 \rangle_{z_1} \leq \min\{d_X(z_1, x_1), d_X(z_1, x_2)\}, \tag{8}$$

$$\langle x_1 \,|\, x_1 \rangle_{z_1} = d_X(z_1, x_1), \tag{9}$$

$$|\langle x_1 \,|\, x_2 \rangle_{z_1} - \langle x_1 \,|\, x_2 \rangle_{w_1}| \leq d_X(z_1, w_1), \tag{10}$$

$$|\langle x_1 \,|\, x_2 \rangle_{z_1} - \langle x_1 \,|\, x_3 \rangle_{z_1}| \leq d_X(x_2, x_3). \tag{11}$$

2.2. Sequences converging at infinity.

We notice the following.

Remark 2.1 (Basic properties). Let X be a metric space. The following hold:

1. The relation "visually indistinguishable" is reflexive on $\mathrm{Sq}^\infty(X)$: $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$ if and only if $\boldsymbol{x} \in \mathrm{Vis}(\boldsymbol{x})$.
2. The relation "visually indistinguishable" is symmetric on $\mathrm{Sq}^\infty(X)$: If $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$, then $\boldsymbol{x} \in \mathrm{Vis}(\boldsymbol{z})$.
3. The relation "visually indistinguishable" is not transitive in general. Namely, it is possible that $\mathrm{Vis}(\boldsymbol{z}) \neq \mathrm{Vis}(\boldsymbol{x})$ for some unbounded sequences $\boldsymbol{x}$, $\boldsymbol{z}$ with $\mathrm{Vis}(\boldsymbol{x}) \cap \mathrm{Vis}(\boldsymbol{z}) \neq \emptyset$.
4. For $\boldsymbol{x} \in X^{\mathbb{N}}$, any subsequence $\boldsymbol{z}'$ of $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$ is also in $\mathrm{Vis}(\boldsymbol{x})$.
5. Any subsequence $\boldsymbol{x}'$ of $\boldsymbol{x} \in X^{\mathbb{N}}$ satisfies $\mathrm{Vis}(\boldsymbol{x}) \subset \mathrm{Vis}(\boldsymbol{x}')$.

Indeed, (1) and (2), follow from the definitions. Notice that for any subsequences $\boldsymbol{x}' \subset \boldsymbol{x}$ and $\boldsymbol{y}' \subset \boldsymbol{y}$, it holds

$$\langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle \leq \langle \boldsymbol{x}' \,|\, \boldsymbol{y}' \rangle. \tag{12}$$

In particular, any subsequence of a sequence converging at infinity also converges at infinity. (4) and (5) follow from (12). For (3), we will see that on Teichmüller space $\mathcal{T}$ equipped with the Teichmüller distance, the relation "visually indistinguishable" does not define an equivalence relation on $\mathrm{Sq}^\infty(\mathcal{T})$, when the base surface is neither a torus with one hole nor a sphere with four holes (cf. Section 6.4).

2.3. Asymptotically conservative.

For metric spaces X and Y, we define

$$\mathrm{AC}(X,Y) = \{\omega \in Y^X \mid \omega \text{ is asymptotically conservative}\}$$

(for the definition, see Section 1.2). Set $\mathrm{AC}(X) = \mathrm{AC}(X,X)$.

Proposition 2.1. *Let $\omega \in \mathrm{AC}(X,Y)$. For a sequence $\boldsymbol{x} \in \mathrm{USq}(X)$, $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$ if and only if $\omega(\boldsymbol{x}) \in \mathrm{Sq}^\infty(Y)$.*

Proof. Let $\boldsymbol{x} \in \mathrm{USq}(X)$. Suppose first that $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$. From (1) of Remark 2.1, $\boldsymbol{x} \in \mathrm{Vis}(\boldsymbol{x})$. Since ω is asymptotically conservative, $\omega(\boldsymbol{x}) \in \omega(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\omega(\boldsymbol{x}))$ and $\omega(\boldsymbol{x}) \in \mathrm{Sq}^\infty(Y)$.

Conversely, assume that $\omega(\boldsymbol{x}) \in \mathrm{Sq}^\infty(Y)$. Since $\omega(\boldsymbol{x}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$, from the definition of asymptotically conservative mappings, we have $\boldsymbol{x} \in \mathrm{Vis}(\boldsymbol{x})$ and hence $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$. □

Proposition 2.2 (Composition in AC). *Let X, Y and Z be metric spaces. For $\omega_1 \in \mathrm{AC}(Y,Z)$ and $\omega_2 \in \mathrm{AC}(X,Y)$, $\omega_1 \circ \omega_2 \in \mathrm{AC}(X,Z)$.*

Proof. Let $\boldsymbol{x} \in \mathrm{USq}(X)$. Then,

$$\omega_1 \circ \omega_2(\mathrm{Vis}(\boldsymbol{x})) \subset \omega_1(\mathrm{Vis}(\omega_2(\boldsymbol{x}))) \subset \mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x})).$$

Let $\boldsymbol{z} \in \mathrm{USq}(X)$ with $\omega_1 \circ \omega_2(\boldsymbol{z}) \in \mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x}))$. Since ω_1 is asymptotically conservative, $\omega_2(\boldsymbol{z}) \in \mathrm{Vis}(\omega_2(\boldsymbol{x}))$. Since ω_2 is also asymptotically conservative again, we have $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$. □

2.4. Remark on closeness.

Recall that two mappings $\omega_1, \omega_2 \in Y^X$ are close at infinity if for any $\boldsymbol{x}^1$, $\boldsymbol{x}^2 \in \mathrm{Sq}^\infty(X)$, it holds $\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^2))$ whenever $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. In particular, such ω_1 and ω_2 satisfy

$$\mathrm{Vis}(\omega_1(\boldsymbol{x})) = \mathrm{Vis}(\omega_2(\boldsymbol{x})) \tag{13}$$

for all $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$.

Proposition 2.3 (Composition and closeness). *Let X, Y and Z be metric spaces. Let $\omega_1, \omega_1' \in Z^Y$ and $\omega_2, \omega_2' \in Y^X$. If ω_i and ω_i' are close at infinity for $i = 1, 2$, $\omega_1 \circ \omega_2$ is close to $\omega_1' \circ \omega_2'$ at infinity.*

Proof. Let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^\infty(X)$ with $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. By definition, $\mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2'(\boldsymbol{x}^2))$, and hence $\mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_1' \circ \omega_2'(\boldsymbol{x}^2))$. □

2.5. Asymptotic surjectivity and closeness at infinity.

A mapping $\omega \in Y^X$ is said to be *asymptotically surjective* if for any $\boldsymbol{y} \in \mathrm{Sq}^\infty(Y)$, there is $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$ with $\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\omega(\boldsymbol{x}))$. Let

$$\mathrm{AC_{as}}(X,Y) = \{\omega \in \mathrm{AC}(X,Y) \mid \omega \text{ is asymptotically surjective}\}.$$

Proposition 2.4. *Let $\omega \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. For $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^{\infty}(X)$, if $\mathrm{Vis}(\boldsymbol{x}^2) \subset \mathrm{Vis}(\boldsymbol{x}^1)$, then $\mathrm{Vis}(\omega(\boldsymbol{x}^2)) \subset \mathrm{Vis}(\omega(\boldsymbol{x}^1))$. In particular, if $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are asymptotic, so are $\omega(\boldsymbol{x}^2)$ and $\omega(\boldsymbol{x}^1)$.*

Proof. Let $\boldsymbol{y} \in \mathrm{Vis}(\omega(\boldsymbol{x}^2))$. Since ω is asymptotically surjective, there is $\boldsymbol{z} \in \mathrm{Sq}^{\infty}(X)$ such that $\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\omega(\boldsymbol{z}))$. Since ω is asymptotically conservative and $\omega(\boldsymbol{x}^2) \in \mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\omega(\boldsymbol{z}))$, we have $\boldsymbol{x}^2 \in \mathrm{Vis}(\boldsymbol{z})$ and

$$\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x}^2) \subset \mathrm{Vis}(\boldsymbol{x}^1).$$

Therefore, $\boldsymbol{x}^1 \in \mathrm{Vis}(\boldsymbol{z})$ (cf. (2) of Remark 2.1). Hence we deduce

$$\omega(\boldsymbol{x}^1) \in \omega(\mathrm{Vis}(\boldsymbol{z})) \subset \mathrm{Vis}(\omega(\boldsymbol{z})) = \mathrm{Vis}(\boldsymbol{y}),$$

and $\boldsymbol{y} \in \mathrm{Vis}(\omega(\boldsymbol{x}^1))$. □

Proposition 2.5 (Composition of mappings in $\mathrm{AC}_{\mathrm{as}}$). *For $\omega_1 \in \mathrm{AC}_{\mathrm{as}}(Y, Z)$ and $\omega_2 \in \mathrm{AC}_{\mathrm{as}}(X, Y)$, we have $\omega_1 \circ \omega_2 \in \mathrm{AC}_{\mathrm{as}}(X, Z)$.*

Proof. Let $\boldsymbol{z} \in \mathrm{Sq}^{\infty}(Z)$. By definition, there are $\boldsymbol{y} \in \mathrm{Sq}^{\infty}(Y)$ and $\boldsymbol{x} \in \mathrm{Sq}^{\infty}(X)$ such that $\mathrm{Vis}(\boldsymbol{z}) = \mathrm{Vis}(\omega_1(\boldsymbol{y}))$ and $\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\omega_2(\boldsymbol{x}))$. Since ω_1 is asymptotically surjective again, from Proposition 2.4, we conclude

$$\mathrm{Vis}(\boldsymbol{z}) = \mathrm{Vis}(\omega_1(\boldsymbol{y})) = \mathrm{Vis}(\omega_1(\omega_2(\boldsymbol{x}))) = \mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x}))$$

and hence $\omega_1 \circ \omega_2$ is asymptotically surjective. □

Proposition 2.6 (Closeness is an equivalence relation on $\mathrm{AC}_{\mathrm{as}}$). *Let X and Y be metric spaces. The relation "closeness at infinity" is an equivalence relation on $\mathrm{AC}_{\mathrm{as}}(X, Y)$.*

Proof. Let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^{\infty}(X)$. Suppose $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are asymptotic.

(Reflexive law) This follows from Proposition 2.4.

(Symmetric law) Take two mappings $\omega_1, \omega_2 \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. Since ω_1 is close to ω_2 at infinity, $\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^2))$. By interchanging the roles of $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, $\mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_1(\boldsymbol{x}^2))$. This means that ω_2 is close to ω_1 at infinity.

(Transitive law) Take three mappings $\omega_1, \omega_2, \omega_3 \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. Suppose that ω_i is close to ω_{i+1} at infinity ($i = 1, 2$). Then, from (13),

$$\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_3(\boldsymbol{x}^2))$$

and hence, ω_1 is close to ω_3 at infinity. □

2.6. Invertibility and asymptotic quasi-inverse.

Define

$$\mathrm{AC}_{\mathrm{inv}}(X, Y) = \{\omega \in \mathrm{AC}(X, Y) \mid \omega \text{ is invertible}\}$$

(for the definition, see Section 1.2). Set $\mathrm{AC}_{\mathrm{inv}}(X) = \mathrm{AC}_{\mathrm{inv}}(X, X)$ as the introduction. Notice that any $\omega \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$ admits an asymptotic quasi-inverse $\omega' \in \mathrm{AC}_{\mathrm{inv}}(Y, X)$, and ω is also an asymptotic quasi-inverse of ω'.

Proposition 2.7 (Invertibility implies asymptotic-surjectivity). *For any metric spaces X and Y, $\mathrm{AC}_{\mathrm{inv}}(X, Y) \subset \mathrm{AC}_{\mathrm{as}}(X, Y)$.*

Proof. Let $\omega \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$. Let ω' be an asymptotic quasi-inverse of ω. Let $\boldsymbol{y} \in \mathrm{Sq}^{\infty}(Y)$. Set $\boldsymbol{x} = \omega'(\boldsymbol{y})$. Since ω' is asymptotically conservative, $\boldsymbol{x} \in \mathrm{Sq}^{\infty}(X)$. Since $\omega \circ \omega'$ is close to the identity mapping on Y at infinity, from (13),

$$\mathrm{Vis}(\omega(\boldsymbol{x})) = \mathrm{Vis}(\omega \circ \omega'(\boldsymbol{y})) = \mathrm{Vis}(\boldsymbol{y}).$$

Therefore, we conclude $\omega \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. □

Proposition 2.8 (Composition of mappings in $\mathrm{AC}_{\mathrm{inv}}$). *For $\omega_1 \in \mathrm{AC}_{\mathrm{inv}}(Y, Z)$ and $\omega_2 \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$, we have $\omega_1 \circ \omega_2 \in \mathrm{AC}_{\mathrm{inv}}(X, Z)$.*

Proof. Let ω_i' be an asymptotic quasi-inverse of ω_i for $i = 1, 2$. Suppose $\boldsymbol{z}^1, \boldsymbol{z}^2 \in \mathrm{Sq}^{\infty}(Z)$ are asymptotic. From Propositions 2.1 and 2.2, $\boldsymbol{x}^i = \omega_2' \circ \omega_1'(\boldsymbol{z}^i) \in \mathrm{Sq}^{\infty}(X)$ for $i = 1, 2$. Since $\omega_2 \circ \omega_2'$ is close to the identity mapping on Y,

$$\mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\omega_2' \circ \omega_1'(\boldsymbol{z}^1))) = \mathrm{Vis}(\omega_2 \circ \omega_2'(\omega_1'(\boldsymbol{z}^1))) = \mathrm{Vis}(\omega_1'(\boldsymbol{z}^1)).$$

From Proposition 2.7, ω_1' is asymptotically surjective, and from Proposition 2.4, $\omega_1'(\boldsymbol{z}^1)$ and $\omega_1'(\boldsymbol{z}^2)$ are asymptotic. Therefore, $\mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_1'(\boldsymbol{z}^2))$. Since ω_1 is also asymptotically surjective, by applying Proposition 2.4 again, we have

$$\begin{aligned}\mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x}^1)) &= \mathrm{Vis}(\omega_1(\omega_2(\boldsymbol{x}^1))) \\ &= \mathrm{Vis}(\omega_1(\omega_1'(\boldsymbol{z}^2))) = \mathrm{Vis}(\omega_1 \circ \omega_1'(\boldsymbol{z}^2)) = \mathrm{Vis}(\boldsymbol{z}^2)\end{aligned}$$

since $\omega_1 \circ \omega_1'$ is asymptotically close to the identity mapping on Z at infinity. Therefore,

$$\mathrm{Vis}((\omega_1 \circ \omega_2) \circ (\omega_2' \circ \omega_1')(\boldsymbol{z}^1)) = \mathrm{Vis}(\omega_1 \circ \omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\boldsymbol{z}^2),$$

which means that $(\omega_1 \circ \omega_2) \circ (\omega_2' \circ \omega_1')$ is close to the the identity mapping on Z at infinity. By the same argument, we can see that $(\omega_2' \circ \omega_1') \circ (\omega_1 \circ \omega_2)$ is close to the identity mapping on X. Therefore, $\omega_2' \circ \omega_1'$ is an asymptotic quasi-inverse of $\omega_1 \circ \omega_2$ and $\omega_1 \circ \omega_2 \in \mathrm{AC}_{\mathrm{inv}}(X, Z)$. □

Proposition 2.9 (Stability of $\mathrm{AC}_{\mathrm{inv}}$ in $\mathrm{AC}_{\mathrm{as}}$). *Let $\omega_1, \omega_2 \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. Suppose that ω_1 and ω_2 are close at infinity. If $\omega_1 \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$, so is ω_2. In addition, any asymptotic quasi-inverse of ω_1 is also that of ω_2.*

Proof. Let ω_1' be an asymptotic quasi-inverse of ω_1. Suppose $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^{\infty}(X)$ are asymptotic. Since ω_1 and ω_2 are close at infinity,

$$\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^2)).$$

Since ω_1' is asymptotically surjective, by Proposition 2.4, we have

$$\mathrm{Vis}(\omega_1' \circ \omega_2(\boldsymbol{x}^2)) = \mathrm{Vis}(\omega_1' \circ \omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\boldsymbol{x}^1). \tag{14}$$

Suppose that $\boldsymbol{y}^1$, $\boldsymbol{y}^2 \in \mathrm{USq}(Y)$ are asymptotic. Since ω_1' is asymptotically surjective again,

$$\mathrm{Vis}(\omega_1'(\boldsymbol{y}^1)) = \mathrm{Vis}(\omega_1'(\boldsymbol{y}^2)).$$

Since ω_1 and ω_2 are close at infinity, we deduce

$$\mathrm{Vis}(\omega_2 \circ \omega_1'(\boldsymbol{y}^1)) = \mathrm{Vis}(\omega_1 \circ \omega_1'(\boldsymbol{y}^2)) = \mathrm{Vis}(\boldsymbol{y}^2). \tag{15}$$

From (14) and (15), we conclude that ω_1' is an asymptotic quasi-inverse of ω_2, and hence $\omega_2 \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$. □

2.7. Monoids and Semigroup congruence.

We have defined three kinds of classes of mappings between metric spaces. By definition and Proposition 2.7, the relation of the classes is given as

$$\mathrm{AC}_{\mathrm{inv}}(X, Y) \subset \mathrm{AC}_{\mathrm{as}}(X, Y) \subset \mathrm{AC}(X, Y) (\subset Y^X) \tag{16}$$

for metric spaces X and Y.

The following theorem follows from Propositions 2.2, 2.5, and 2.8.

Theorem 2.1. $\mathrm{AC}(X)$ *admits a canonical monoid structure with respect to the composition of mappings. The identity element of* $\mathrm{AC}(X)$ *is the identity mapping on* X*. In addition,* $\mathrm{AC}_{\mathrm{as}}(X)$ *and* $\mathrm{AC}_{\mathrm{inv}}(X)$ *are submonoids of* $\mathrm{AC}(X)$.

Let G be a semigroup. A *semigroup congruence* is an equivalence relation $\sim$ on G with the property that for $x, y, z, w \in G$, $x \sim y$ and $z \sim w$ imply $xz \sim yw$. Then, the congruence classes

$$G/\sim = \{[g] \mid g \in G\}$$

is also a semigroup with the product $[g_1][g_2] = [g_1 g_2]$. We call $G/\sim$ the *quotient semigroup* of G with the semigroup congruence $\sim$.

We define a relation on $\mathrm{AC}_{\mathrm{as}}(X)$ by using the closeness at infinity. Namely, for two ω_1 and $\omega_2 \in \mathrm{AC}_{\mathrm{as}}(X)$, ω_1 is *equivalent* to ω_2 if ω_1 is close to ω_2 at infinity. From Propositions 2.3 and 2.6, this relation is a subgroup congruence on $\mathrm{AC}_{\mathrm{as}}(X)$. We define the quotient monoid by

$$\mathfrak{AC}_{\mathrm{as}}(X) = \mathrm{AC}_{\mathrm{as}}(X)/(\text{close at infinity}).$$

We also define the semigroup congruence on $\mathrm{AC}_{\mathrm{inv}}(X)$ in the same procedure, and obtain the quotient semigroup by

$$\mathfrak{AC}(X) = \mathrm{AC}_{\mathrm{inv}}(X)/(\text{close at infinity}).$$

As a result, we summarize as follows.

THEOREM 2.2 (Group $\mathfrak{AC}(X)$). *Let X be a metric space. Then, the quotient semigroup $\mathfrak{AC}(X)$ is a group. The identity element of $\mathfrak{AC}(X)$ is the congruence class of the identity mapping, and the inverse of the congruence class $[\omega]$ of $\omega \in \mathrm{AC}_{\mathrm{inv}}(X)$ is the congruence class of an asymptotic quasi-inverse of ω.*

COROLLARY 2.1. *Let X and Y be metric spaces. Let $\omega \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$ and ω' an asymptotic quasi-inverse of ω. Then, the mapping*

$$\mathrm{AC}_{\mathrm{as}}(X) \ni f \mapsto \omega \circ f \circ \omega' \in \mathrm{AC}_{\mathrm{as}}(Y),$$
$$\mathrm{AC}_{\mathrm{inv}}(X) \ni f \mapsto \omega \circ f \circ \omega' \in \mathrm{AC}_{\mathrm{inv}}(Y)$$

induces isomorphisms

$$\mathfrak{AC}_{as}(X) \ni [f] \mapsto [\omega \circ f \circ \omega'] \in \mathfrak{AC}_{as}(Y),$$
$$\mathfrak{AC}(X) \ni [f] \mapsto [\omega \circ f \circ \omega'] \in \mathfrak{AC}(Y).$$

Notice from Proposition 2.9 that any equivalence class in $\mathfrak{AC}(X)$ consists of elements in $\mathrm{AC}_{\mathrm{inv}}(X)$. Hence, we conclude the following.

THEOREM 2.3. *The inclusion $\mathrm{AC}_{\mathrm{inv}}(X) \hookrightarrow \mathrm{AC}_{\mathrm{as}}(X)$ induces a monoid monomorphism*

$$\mathfrak{AC}(X) \hookrightarrow \mathfrak{AC}_{as}(X). \tag{17}$$

In other words, $\mathfrak{AC}(X)$ is a subgroup of $\mathfrak{AC}_{as}(X)$.

The monomorphism (17) could be an isomorphism. The author does not know whether it is true or not in general.

3. Comparison with the coarse geometry.

3.1. Backgrounds from the coarse geometry.

3.1.1. Parallelism.

Two mappings $\omega, \xi \in Y^X$ between metric spaces are said to be *parallel* if and only if

$$\sup_{x \in X} d_Y(\omega(x), \xi(x)) < \infty$$

(cf. Section 1.A′ in [**15**]). The "parallelism" defines an equivalence relation on any subclass in Y^X. If two mappings $\omega, \xi \in Y^X$ are parallel,

$$\sup_{x,z \in X} |\langle \omega(x) \,|\, \omega(z) \rangle^Y_{y_0} - \langle \xi(x) \,|\, \xi(z) \rangle^Y_{y_0}| < \infty, \tag{18}$$
$$\sup_{x \in X, y \in Y} |\langle \omega(x) \,|\, y \rangle^Y_{y_0} - \langle \xi(x) \,|\, y \rangle^Y_{y_0}| < \infty. \tag{19}$$

From (19), for any sequence $\boldsymbol{x} \in \mathrm{USq}(X)$, it holds

$$\mathrm{Vis}(\omega(\boldsymbol{x})) = \mathrm{Vis}(\xi(\boldsymbol{x})). \tag{20}$$

3.1.2. Quasi-isometries.

A mapping $\omega \in Y^X$ satisfies

$$d_Y(\omega(x_1), \omega(x_2)) \leq K\, d_X(x_1, x_2) + D$$

for all $x_1, x_2 \in X$, we call ω a *coarsely* (K,D)*-Lipschitz*. If $\omega \in Y^X$ satisfies

$$\frac{1}{K}\, d_X(x_1, x_2) - D \leq d_Y(\omega(x_1), \omega(x_2))$$

for all $x_1, x_2 \in X$, we call ω a *coarsely co-*(K,D)*-Lipschitz*. A mapping $\omega \in Y^X$ is said to be (K,D)*-quasi-isometry* if ω is both coarsely (K,D)-Lipschitz and coarsely co-(K,D)-Lipschitz. A mapping $\omega \in Y^X$ is D*-cobounded* if the D-neighborhood of the image of X under ω coincides with Y. A *quasi-inverse* of a mapping $\omega \in Y^X$ is a mapping $\omega' \in X^Y$ such that $\omega' \circ \omega$ and $\omega \circ \omega'$ are parallel to the identity mappings. Usually, quasi-inverses are assumed to be quasi-isometry. However, we do not assume so for our purpose. Any quasi-inverse of a quasi-isometry is automatically a quasi-isometry. Any cobounded quasi-isometry admits a quasi-inverse.

Let $\mathrm{QI}(X,Y)$ be the set of cobounded quasi-isometries from X to Y. Then, $\mathrm{QI}(X) = \mathrm{QI}(X,X)$ admits a monoid structure defined by composition. One can easily check that the parallelism is a subgroup congruence on $\mathrm{QI}(X)$. Hence we have a quotient group defined by

$$\mathfrak{Q}I(X) = \mathrm{QI}(X)/(\text{parallelism}). \tag{21}$$

The group $\mathfrak{Q}I(X)$ is a central object in the coarse geometry (cf. Section I.8 in [**6**]).

3.2. Asymptotically conservative mappings in the coarse geometry.

Recall that a mapping $\omega \in Y^X$ is called *weakly asymptotically conservative* if $\omega(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\omega_1(\boldsymbol{x}))$ for all sequence $\boldsymbol{x} \in \mathrm{USq}(X)$. Let

$$\mathrm{AC}^w(X,Y) = \{\omega \in Y^X \mid \omega \text{ is weakly asymptotically conservative}\}.$$

Set $\mathrm{AC}^w(X) = \mathrm{AC}^w(X,X)$. From (16), we have

$$\mathrm{AC}_{\mathrm{inv}}(X,Y) \subset \mathrm{AC}_{\mathrm{as}}(X,Y) \subset \mathrm{AC}(X,Y) \subset \mathrm{AC}^w(X,Y) (\subset Y^X) \tag{22}$$

for metric spaces X and Y.

A class M of Y^X is said to be *stable under parallelism* if a mapping $\omega \in Y^X$ is parallel to some $\xi \in M$, then $\omega \in M$.

Proposition 3.1 (Stability under parallelism). *All classes* $\mathrm{AC}^w(X,Y)$*,* $\mathrm{AC}(X,Y)$*,* $\mathrm{AC}_{\mathrm{as}}(X,Y)$ *and* $\mathrm{AC}_{\mathrm{inv}}(X,Y)$ *are stable under the parallelism.*

Proof. Suppose that $\omega, \xi \in Y^X$ are parallel.

(i) Suppose $\xi \in \mathrm{AC}^w(X,Y)$. Let $\boldsymbol{x} \in \mathrm{USq}(X)$. Take $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$. Since ξ is weakly asymptotically conservative, $\xi(\boldsymbol{z}) \in \xi(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\xi(\boldsymbol{x}))$. From (18), we have $\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$ and $\omega(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\omega(\boldsymbol{x}))$. Hence $\omega \in \mathrm{AC}^w(X, Y)$.

(ii) Suppose $\xi \in \mathrm{AC}(X, Y)$. From the argument above, $\omega \in \mathrm{AC}^w(X, Y)$. Suppose a sequence $\boldsymbol{z}$ in X satisfies $\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$. From (18) again, $\xi(\boldsymbol{z}) \in \mathrm{Vis}(\xi(\boldsymbol{x}))$. Since ξ is asymptotically conservative, $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$ and $\omega \in \mathrm{AC}(X, Y)$.

(iii) Suppose $\xi \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. Let $\boldsymbol{y} \in \mathrm{Sq}^\infty(Y)$. Take $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$ such that $\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\xi(\boldsymbol{x}))$. From (20), we have

$$\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\xi(\boldsymbol{x})) = \mathrm{Vis}(\omega(\boldsymbol{x})),$$

which implies $\omega \in \mathrm{AC}_{\mathrm{as}}(X, Y)$.

(iv) Suppose $\xi \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$. Let $\xi' \in \mathrm{AC}_{\mathrm{inv}}(Y, X)$ be an asymptotic quasi-inverse of ξ. Let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^\infty(X)$ with $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. Since ξ' is asymptotically surjective, by Proposition 2.4 and (20), we have

$$\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\xi' \circ \xi(\boldsymbol{x}^2)) = \mathrm{Vis}(\xi' \circ \omega(\boldsymbol{x}^2)),$$

and hence $\xi' \circ \omega$ is close to the identity mapping on X.

To prove the converse, we notice from the above that ω is asymptotically surjective and asymptotically conservative from Proposition 2.7. Then, by Propositions 2.2 and 2.5, $\omega \circ \xi'$ is also asymptotically surjective and asymptotically conservative.

Let $\boldsymbol{y}^1, \boldsymbol{y}^2 \in \mathrm{Sq}^\infty(Y)$ with with $\mathrm{Vis}(\boldsymbol{y}^1) = \mathrm{Vis}(\boldsymbol{y}^2)$. Then, since $\xi'(\boldsymbol{y}^i) \in \mathrm{Sq}^\infty(X)$ for $i = 1, 2$, by Proposition 2.4 and (20) again, we have

$$\mathrm{Vis}(\xi'(\boldsymbol{y}^1)) = \mathrm{Vis}(\xi'(\boldsymbol{y}^2))$$

and

$$\mathrm{Vis}(\omega \circ \xi'(\boldsymbol{y}^1)) = \mathrm{Vis}(\xi \circ \xi'(\boldsymbol{y}^2)) = \mathrm{Vis}(\boldsymbol{y}^2),$$

and $\omega \circ \xi'$ is close to the identity mapping on Y. Therefore, we conclude that $\omega \in \mathrm{AC}_{\mathrm{inv}}(X, Y)$. □

The following proposition gives comparisons between items in rows of the correspondence table in the introduction (cf. Table 1).

PROPOSITION 3.2 (Comparison). 1. *A cobounded asymptotically conservative mapping is asymptotically surjective.*

2. *If two asymptotically surjective, asymptotically conservative mappings are parallel, they are close at infinity.*

3. *If $\omega \in Y^X$ admits a quasi-inverse $\omega' \in X^Y$ in the sense of the coarse geometry, $\omega' \circ \omega$ and $\omega \circ \omega'$ are close to the identity mappings on X and Y at infinity, respectively.*

PROOF. (1) Let $\boldsymbol{y} = \{y_n\}_{n \in \mathbb{N}} \in \mathrm{Sq}^\infty(Y)$. Take $x_n \in X$ such that $d_Y(\omega(x_n), y_n) \le D_0$ where $D_0 > 0$ is independent of n. Since

$$|\langle \omega(\boldsymbol{x}) \,|\, \omega(\boldsymbol{x}) \rangle^Y_{y_0} - \langle \boldsymbol{y} \,|\, \boldsymbol{y} \rangle^Y_{y_0}| \leq 2D_0$$

for $y_0 \in Y$, we have $\omega(\boldsymbol{x}) \in \mathrm{Sq}^\infty(Y)$ and $\boldsymbol{x} \in \mathrm{Sq}^\infty(X)$ from Proposition 2.1. Since

$$|\langle \omega(\boldsymbol{x}) \,|\, \boldsymbol{z} \rangle^Y_{y_0} - \langle \boldsymbol{y} \,|\, \boldsymbol{z} \rangle^Y_{y_0}| \leq D_0$$

for every sequence $\boldsymbol{z} = \{z_n\}_{n \in \mathbb{N}} \in \mathrm{USq}(Y)$, we obtain $\mathrm{Vis}(\boldsymbol{y}) = \mathrm{Vis}(\omega(\boldsymbol{x}))$. Thus, ω is asymptotically surjective.

(2) Let ω_1, $\omega_2 \in \mathrm{AC}_{\mathrm{as}}(X, Y)$. Suppose that ω_1 is parallel to ω_2. Take asymptotic sequences $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^\infty(X)$. Since ω_2 is asymptotically surjective, from Proposition 2.4 and (20), we conclude that

$$\mathrm{Vis}(\omega_1(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^1)) = \mathrm{Vis}(\omega_2(\boldsymbol{x}^2))$$

and ω_1 and ω_2 are close at infinity.

(3) Since the identity mapping is asymptotically surjective and asymptotically conservative, from Proposition 3.1, $\omega' \circ \omega$ and $\omega \circ \omega'$ are also asymptotically surjective and asymptotically conservative. Hence, from above (2), we conclude what we wanted. □

3.3. Criterion for subclasses to be compatible in $\mathfrak{AC}$.

Let M be a subclass of X^X. Consider the following conditions.

(S1) M is a monoid with the operation defined by composition, and the parallelism is a semigroup congruence on M.

(S2) Any element in M is cobounded.

(S3) Any element in M admits a quasi-inverse in M in the coarse geometry.

Notice that the condition (S3) implies (S2). Under the condition (S1), the quotient set $\mathcal{M} = M/(\text{parallelism})$ has a canonical monoid structure, and if M satisfies all conditions, $\mathcal{M}$ has a canonical group structure. For instance, the monoid $\mathrm{QI}(X)$ of cobounded self quasi-isometries on X satisfies all conditions above.

Proposition 3.3 (Criterion). *Let M be a subclass of X^X satisfying the condition* (S1) *posed above.*

1. *Suppose in addition that M satisfies the condition* (S2) *posed above. When $M \subset \mathrm{AC}(X)$, then $M \subset \mathrm{AC}_{\mathrm{as}}(X)$. The inclusion $M \hookrightarrow \mathrm{AC}_{\mathrm{as}}(X)$ induces maps* (*as sets*)

$$\mathcal{M} = M/(\textit{parallelism}) \to M/(\textit{close at infinity}) \to \mathfrak{AC}_{as}(X)$$

such that the composition of the maps $\mathcal{M} \hookrightarrow \mathfrak{AC}_{as}(X)$ is a monoid homomorphism.

2. *Suppose that M satisfies the condition* (S3) *posed above. When $M \subset \mathrm{AC}^w(X)$, then $M \subset \mathrm{AC}_{\mathrm{inv}}(X)$. The inclusion $M \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(X)$ induces maps* (*as sets*)

$$\mathcal{M} \to M/(\textit{close at infinity}) \to \mathfrak{AC}(X) \tag{23}$$

such that the composition of the maps $\mathcal{M} \to \mathfrak{AC}(X)$ is a group homomorphism.

Proof. (1) From (1) of Proposition 3.2, $M \subset \mathrm{AC}_{\mathrm{as}}(X)$, and the "closeness at infinity" is an equivalence relation on M by Proposition 2.6. Therefore, from (2) of Proposition 3.2, we have well-defined mappings between quotient sets

$$\begin{aligned}\mathcal{M} &\to M/(\text{close at infinity})\\ &\to \mathrm{AC}_{\mathrm{as}}(X)/(\text{close at infinity}) = \mathfrak{AC}_{as}(X).\end{aligned}$$

From (2) of Proposition 3.2 again, the composition

$$\mathcal{M} \to \mathfrak{AC}_{as}(X)$$

induces a monoid homomorphism.

(2) We first check that $M \subset \mathrm{AC}(X)$. Let $\omega \in M$ and $\omega' \in M$ a quasi-inverse of ω. Let $\boldsymbol{z}$ be an unbounded sequence in X with $\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$. Since ω' is weakly asymptotically conservative, we have

$$\omega' \circ \omega(\boldsymbol{z}) \in \omega'(\mathrm{Vis}(\omega(\boldsymbol{x}))) \subset \mathrm{Vis}(\omega' \circ \omega(\boldsymbol{x})) = \mathrm{Vis}(\boldsymbol{x})$$

and

$$|\langle x \,|\, z\rangle^X_{x_0} - \langle \omega' \circ \omega(x) \,|\, \omega' \circ \omega(z)\rangle^X_{x_0}| = O(1)$$

for all $x \in \boldsymbol{x}$ and $z \in \boldsymbol{z}$, since is parallel to the identity mapping on X and infinity from Proposition 2.4, Proposition 3.1 and (2) of Proposition 3.2. Therefore, $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$, and ω is asymptotically conservative.

Then, by applying the same argument as above, we have a mappings

$$\mathcal{M} \to M/(\text{close at infinity}) \to \mathfrak{AC}(X)$$

such that the composition

$$\mathcal{M} \to \mathfrak{AC}(X) \tag{24}$$

is a monoid homomorphism. From (3) of Proposition 3.2, any quasi-inverse of $\omega \in M$ corresponds to a asymptotic quasi-inverse of ω in $\mathrm{AC}_{\mathrm{inv}}(X)$ under the inclusion $\mathcal{M} \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(X)$. Hence (24) is a group homomorphism. □

Corollary 3.1 (Criterion for quasi-isometries). *Let X be a metric space. If $\mathrm{QI}(X) \subset \mathrm{AC}^w(X)$, then the inclusion $\mathrm{QI}(X) \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(X)$ induces a group homomorphism*

$$\mathfrak{Q}I(X) \to \mathfrak{AC}(X).$$

3.4. Remarks.

The notions of quasi-isometries and the asymptotically conservation are independent in general:

1. An asymptotically conservative mapping need not be a quasi-isometry. Indeed, let

$X = [0, \infty)$ with $d_X(x_1, x_2) = |x_1 - x_2|$. Let $x_0 = 0$ be the reference point. Then, any increasing function $f\colon X \to X$ with $f(0) = 0$ is asymptotically conservative.

2. Meanwhile, little is known as to when quasi-isometries become asymptotically conservative. For instance, any rough homothety is asymptotically conservative (cf. (6)). Actually, it follows from the following fact that any rough homothety ω satisfies

$$|K\langle x_1 \,|\, x_2\rangle^X_{x_0} - \langle \omega(x_1) \,|\, \omega(x_2)\rangle^Y_{y_0}| \leq D' \quad (x_1, x_2 \in X) \tag{25}$$

for some $K, D' > 0$.

3. In general, the homomorphism (23) is not injective: Take an increasing sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{Z}$ such that $a_0 = 0$ and $a_{n+1} - a_n \to \infty$. Consider a graph in $\mathbb{R}^2$ defined by

$$X = [0, \infty) \times \{-1, 1\} \cup \bigcup\nolimits_{n=0}^{\infty} \{a_n\} \times [-1, 1] \subset \mathbb{R}^2.$$

Then, X is a metric space equipped with the graph metric such that the length of the edges are measured by the Euclidean metric. Let $x_0 = (0, 0) \in X$ as a base point. In this case, $\mathrm{Sq}^\infty(X) = \mathrm{USq}(X)$ and $\mathrm{Vis}(\boldsymbol{x}) = \mathrm{Sq}^\infty(X)$ for all $\boldsymbol{x} \in \mathrm{USq}(X)$. Hence, $\mathfrak{AC}(X)$ is the trivial group. Furthermore, X is WBGP in the sense of Section 4, and any quasi-isometry on X is weakly asymptotically conservative (cf. Proposition 4.2). Thus, we have a homomorphism (23) in this case. Define an isometry r on X by $r(x, y) = (x, -y)$. Then, r is not parallel to the identity mapping id_X, and id_X and r are contained in the different classes in $\mathfrak{Q}I(X)$.

4. In general, the homomorphism (23) is not surjective: When $X = Y = \mathbb{D}$ equipped with the Poincaré distance, $\mathfrak{AC}(\mathbb{D})$ is canonically isomorphic to the group of homeomorphism on $\partial\mathbb{D}$ via extensions. Hence, we can find an invertible asymptotically conservative mapping on $\mathbb{D}$ which is not parallel to any cobounded quasi-isometry.

4. Relaxation of the definition.

4.1. Metric spaces which are WBGP.

Let X be a metric space. For $\boldsymbol{x} \in \mathrm{USq}(X)$, we define

$$\mathrm{sub}^\infty(\boldsymbol{x}) = \{\boldsymbol{x}' \mid \text{subsequences of } \boldsymbol{x} \text{ with } \boldsymbol{x}' \in \mathrm{Sq}^\infty(X)\}.$$

A metric space X is called *well-behaved at infinity with respect to the Gromov product* (WBGP) if $\mathrm{sub}^\infty(\boldsymbol{x}) \neq \emptyset$ for all $\boldsymbol{x} \in \mathrm{USq}(X)$.

Examples. The following are metric spaces which are WBGP:

1. Proper geodesic spaces that are Gromov-hyperbolic (of infinite diameter).

2. Teichmüller space equipped with the Teichmüller distance.

3. The Cayley graphs for pairs (G, S) of finitely generated infinite group G and a finite system S of symmetric generators.

Indeed, (1) follows from the compactness of the Gromov's bordification (compactification) (e.g. Proposition 2.14 in [**20**]). (2) is proven at Proposition 6.1.

We check (3). Let $\Sigma(G,S)$ be the Caylay graph. Let F_S be the free group generated by S. There is a canonical surjection $\pi\colon \Sigma(F_S,S)\to\Sigma(G,S)$ induced by the quotient map $F_S\to G$. Let $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}}$ be an unbounded sequence in $\Sigma(G,S)$. Take $y_n\in\Sigma(F_S,S)$ such that $\pi(y_n)=x_n$ and $d_G(id,x_n)=d_{F_S}(id,y_n)$. Then $\boldsymbol{y}=\{y_n\}_{n\in\mathbb{N}}$ is an unbounded sequence in $G(F_S,S)$, and hence we can find a subsequence $\boldsymbol{y}'=\{y_{n_j}\}_j$ of $\boldsymbol{y}$ such that $\langle \boldsymbol{y}'\,|\,\boldsymbol{y}'\rangle=\infty$ from (1) above. Since the projection π is 1-Lipschitz, $d(x_{n_j},x_{n_k})\le d(y_{n_j},y_{n_k})$ and we have

$$\langle \pi(\boldsymbol{y}')\,|\,\pi(\boldsymbol{y}')\rangle_{id}\ge\langle \boldsymbol{y}'\,|\,\boldsymbol{y}'\rangle_{id}=\infty.$$

Thus $\pi(\boldsymbol{y}')$ is a desired subsequence of $\boldsymbol{x}$.

4.2. Properties.

We shall give a couple of properties of metric spaces which are WBGP.

Proposition 4.1. *Suppose a metric space X is WBGP. For any $\boldsymbol{x}\in\mathrm{USq}(X)$, we have*

$$\mathrm{Vis}(\boldsymbol{x})=\bigcap\nolimits_{\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})}\mathrm{Vis}(\boldsymbol{x}').$$

Proof. From (5) of Remark 2.1, we have

$$\mathrm{Vis}(\boldsymbol{x})\subset\bigcap\nolimits_{\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})}\mathrm{Vis}(\boldsymbol{x}').$$

Let $\boldsymbol{z}\in\mathrm{USq}(X)-\mathrm{Vis}(\boldsymbol{x})$. Then $\langle\boldsymbol{z}\,|\,\boldsymbol{x}\rangle<\infty$. Since X is WBGP, we can take subsequences $\boldsymbol{z}'\subset\boldsymbol{z}$ and $\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})$ such that $\langle\boldsymbol{z}'\,|\,\boldsymbol{x}'\rangle<\infty$. Hence $\boldsymbol{z}'\notin\mathrm{Vis}(\boldsymbol{x}')$ and $\boldsymbol{z}\notin\mathrm{Vis}(\boldsymbol{x}')$ from (4) of Remark 2.1. Therefore, $\boldsymbol{z}\notin\bigcap_{\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})}\mathrm{Vis}(\boldsymbol{x}')$. □

Proposition 4.2 (Relaxation of the definition). *Let X and Y be metric spaces which are WBGP.*

1. *A mapping $\omega\in Y^X$ is in $\mathrm{AC}^w(X,Y)$ if and only if for any $\boldsymbol{x},\boldsymbol{z}\in\mathrm{Sq}^\infty(X)$, $\langle\omega(\boldsymbol{x})\,|\,\omega(\boldsymbol{z})\rangle=\infty$ whenever $\langle\boldsymbol{x}\,|\,\boldsymbol{z}\rangle=\infty$.*
2. *A mapping $\omega\in Y^X$ is in $\mathrm{AC}(X,Y)$ if and only if for any $\boldsymbol{x},\boldsymbol{z}\in\mathrm{Sq}^\infty(X)$, $\langle\omega(\boldsymbol{x})\,|\,\omega(\boldsymbol{z})\rangle=\infty$ implies $\langle\boldsymbol{x}\,|\,\boldsymbol{z}\rangle=\infty$, and vice versa.*

Proof. (1) The condition is paraphrased that $\omega(\mathrm{Vis}(\boldsymbol{x}))\subset\mathrm{Vis}(\omega(\boldsymbol{x}))$ for all $\boldsymbol{x}\in\mathrm{Sq}^\infty(X)$. Hence, the "only if" part follows from the definition. We show the "if" part. Let $\boldsymbol{x}\in\mathrm{USq}(X)$. Suppose to the contrary that there is $\boldsymbol{z}\in\mathrm{Vis}(\boldsymbol{x})$ such that $\omega(\boldsymbol{z})\notin\mathrm{Vis}(\omega(\boldsymbol{x}))$. From Proposition 4.1, there is $\boldsymbol{y}\in\mathrm{sub}^\infty(\omega(\boldsymbol{x}))$ such that $\omega(\boldsymbol{z})\notin\mathrm{Vis}(\boldsymbol{y})$. By taking subsequences $\boldsymbol{z}'\in\mathrm{sub}^\infty(\boldsymbol{z})$ and $\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})$ respectively, we may assume that $\omega(\boldsymbol{x}')\in\mathrm{sub}^\infty(\boldsymbol{y})\subset\mathrm{sub}^\infty(\omega(\boldsymbol{x}))$ and $\omega(\boldsymbol{z}')\notin\mathrm{Vis}(\omega(\boldsymbol{x}'))$.

On the other hand, since $\boldsymbol{x}'\in\mathrm{sub}^\infty(\boldsymbol{x})\subset\mathrm{Sq}^\infty(X)$, from the condition (1), we have $\omega(\mathrm{Vis}(\boldsymbol{x}'))\subset\mathrm{Vis}(\omega(\boldsymbol{x}'))$. Hence $\boldsymbol{z}'\notin\mathrm{Vis}(\boldsymbol{x}')$, which implies $\boldsymbol{z}'\notin\mathrm{Vis}(\boldsymbol{x})$ because

$\mathrm{Vis}(\boldsymbol{x}) \subset \mathrm{Vis}(\boldsymbol{x}')$ from (5) of Remark 2.1. This contradicts to (4) of Remark 2.1. Thus, we conclude that $\omega(\mathrm{Vis}(\boldsymbol{x})) \subset \mathrm{Vis}(\omega(\boldsymbol{x}))$.

(2) We only show the "if" part. From (1) above, $\omega \in \mathrm{AC}^w(X, Y)$. Let $\boldsymbol{z} \in \mathrm{USq}(X)$ with $\omega(\boldsymbol{z}) \in \mathrm{Vis}(\omega(\boldsymbol{x}))$. Suppose $\boldsymbol{z} \notin \mathrm{Vis}(\boldsymbol{x})$. From the argument in Proposition 4.1, there is a subsequence $\boldsymbol{z}' \in \mathrm{sub}^\infty(\boldsymbol{z})$ with $\boldsymbol{z}' \notin \mathrm{Vis}(\boldsymbol{x})$. This means that $\omega(\boldsymbol{z}') \notin \mathrm{Vis}(\omega(\boldsymbol{x}))$ from the assumption, and hence $\omega(\boldsymbol{z}) \notin \mathrm{Vis}(\omega(\boldsymbol{x}))$ from (4) of Remark 2.1. This is a contradiction. □

From Proposition 3.3, we conclude the following.

Corollary 4.1. *Let X is a metric space which is WBGP. Let M be a subclass of X^X satisfying* (S1) *and* (S3) *in Section* 3.3. *Suppose that any $\omega \in M$ satisfies the condition that $\langle \omega(\boldsymbol{x}) \,|\, \omega(\boldsymbol{z}) \rangle = \infty$ whenever $\langle \boldsymbol{x} \,|\, \boldsymbol{z} \rangle = \infty$ for all $\boldsymbol{x}, \boldsymbol{z} \in \mathrm{Sq}^\infty(X)$. Then, $M \subset \mathrm{AC}_{\mathrm{inv}}(X)$ and the inclusion $M \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(X)$ induces a group homomorphism*

$$M/(\textit{parallelism}) \to \mathfrak{AC}(X).$$

5. Teichmüller theory.

In this section, we recall basics in the Teichmüller theory. For details, the readers can refer to [**1**], [**10**], [**17**] and [**18**].

5.1. Teichmüller space.

The *Teichmüller space* $\mathcal{T} = \mathcal{T}(S)$ of S is the set of equivalence classes of marked Riemann surfaces (Y, f) where Y is a Riemann surface of analytically finite type and $f : \mathrm{Int}(S) \to Y$ is an orientation preserving homeomorphism. Two marked Riemann surfaces (Y_1, f_1) and (Y_2, f_2) are *Teichmüller equivalent* if there is a conformal mapping $h : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$.

Teichmüller space $\mathcal{T}$ is topologized with a canonical complete distance, called the *Teichmüller distance* d_T (cf. (32)). It is known that the Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ of S is homeomorphic to $\mathbb{R}^{2\,\mathrm{cx}(S)}$.

Convention 5.1. *Throughout this paper, we fix a conformal structure X on S and consider $x_0 = (X, id)$ as the base point of the Teichmüller space $\mathcal{T}$ of S.*

5.2. Measured foliations.

Let $\mathcal{S}$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on S. Consider the set of weighted simple close curves $\mathcal{WS} = \{t\alpha \mid t \geq 0, \alpha \in \mathcal{S}\}$, where $t\alpha$ is the formal product between $t \geq 0$ and $\alpha \in \mathcal{S}$. We embed $\mathcal{WS}$ into the space $\mathbb{R}_+^{\mathcal{S}}$ of non-negative functions on $\mathcal{S}$ by

$$\mathcal{WS} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto t\, i(\alpha, \beta)] \in \mathbb{R}_+^{\mathcal{S}} \tag{26}$$

where $i(\cdot, \cdot)$ is the geometric intersection number on $\mathcal{S}$. The closure $\mathcal{MF}$ of the image of the mapping (26) is called the *space of measured foliations* on S. The space $\mathbb{R}_+^{\mathcal{S}}$ admits a canonical action of $\mathbb{R}_{>0}$ by multiplication. The quotient space

$$\mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0} \subset \mathrm{P}\mathbb{R}_+^{\mathcal{S}} = (\mathbb{R}_+^{\mathcal{S}} - \{0\})/\mathbb{R}_{>0}$$

is said to be the *space of projective measured foliations*. By definition, $\mathcal{MF}$ contains $\mathcal{WS}$ as a dense subset. The intersection number function on $\mathcal{WS}$ defined by

$$\mathcal{WS} \times \mathcal{WS} \ni (t\alpha, s\beta) \mapsto ts\, i(\alpha, \beta)$$

extends continuously on $\mathcal{MF}\times\mathcal{MF}$. It is known that $\mathcal{MF}$ and $\mathcal{PMF}$ are homeomorphic to $\mathbb{R}^{2\,\mathrm{cx}(S)}$ and $S^{2\,\mathrm{cx}(S)-1}$, respectively.

Normal forms. Any $G \in \mathcal{MF}$ is represented by a pair $(\mathcal{F}_G, \mu_G)$ of a singular foliation $\mathcal{F}_G$ and a transverse measure μ_G to $\mathcal{F}_G$. The intersection number $i(G, \alpha)$ with $\alpha \in \mathcal{S}$ is obtained as

$$i(G, \alpha) = \inf_{\alpha' \sim \alpha} \int_{\alpha'} d\mu_G.$$

The *support* $\mathrm{Supp}(G)$ of a measured foliation G is, by definition, the minimal essential subsurface containing the underlying foliation. A measured foliation is said to be *minimal* if it intersects any curves in $\mathcal{S}$ in its support.

According to the structure of the underlying foliation, any $G \in \mathcal{MF}$ has the *normal form*: Any measured foliation $G \in \mathcal{MF}$ is decomposed as

$$G = G_1 + G_2 + \cdots + G_{m_1} + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3} \tag{27}$$

where G_i is a minimal foliation in its support $X_i = \mathrm{Supp}(G_i)$, β_j and γ_k are simple closed curves such that each β_j cannot be deformed into any X_i and γ_k is homotopic to a component of ∂X_i for some i (cf. Section 2.4 of [**18**]). In this paper, we call G_i, β_j and γ_k a *minimal component*, an *essential curve*, and a *peripheral curve* of G respectively.

5.3. Null sets of measured foliations.

For a measured foliation G, we define the *null set* of G by

$$\mathcal{N}_{MF}(G) = \{F \in \mathcal{MF} \mid i(F, G) = 0\}. \tag{28}$$

We denote by G° the measured foliation defined from G by deleting the foliated annuli associated to the peripheral curves in G. We here call G° the *distinguished part of G on nullity*. Notice that $(G^\circ)^\circ = G^\circ$. The following might be well-known. However, we give a proof of the proposition for completeness.

Proposition 5.1 (Null sets and topologically equivalence). *Let $G, H \in \mathcal{MF}$. Then, the following are equivalent.*

(1) $\mathcal{N}_{MF}(G) = \mathcal{N}_{MF}(H)$.

(2) *G° is topologically equivalent to H°.*

In particular, $\mathcal{N}_{MF}(G) = \mathcal{N}_{MF}(G^\circ)$.

We say that two measured foliations F_1 and F_2 are *topologically equivalent* if the underlying foliations of F_1 and F_2 are modified by Whitehead operations to foliations with trivalent singularities such that the resulting foliations (without transversal measures) are isotopic (cf. Section 3.1 of [**19**]).

Proof. Suppose (1) holds. We decompose G as (27):

$$G=\sum_{i=1}^{m_1}G_i+\sum_{i=1}^{m_2}\beta_i+\sum_{i=1}^{m_3}\gamma_i.$$

Since $i(G,H)=0$, the decomposition of H is represented as

$$H=\sum_{i=1}^{m_1}H_i+\sum_{i=1}^{m_2}a_i\beta_i+\sum_{i=1}^{m_1}\sum_{\gamma\subset\partial X_i}b_\gamma\gamma+H_0 \tag{29}$$

where H_i is either topologically equivalent to G_i or is 0, $a_i,b_\gamma\geq 0$ and $\mathrm{Supp}(H_0)\subset X-\mathrm{Supp}(G)$. In the summation $\sum_{\gamma\subset\partial X_i}$ in (29), γ runs over all component of ∂X_i. See Proposition 3.2 of Ivanov [**18**] or Lemma 3.1 of Papadopoulos [**38**]. Indeed, Ivanov in [**18**] works under the assumption that each G_i is a stable lamination for some pseudo-Anosov mapping on X_i. However, the discussion of his proof can be applied to our case.

If $H_0\neq 0$, there is an $\alpha\subset\mathcal{S}$ with $i(G,\alpha)=0$ but $i(H_0,\alpha)\neq 0$. Since $\alpha\in\mathcal{N}_{MF}(G)=\mathcal{N}_{MF}(H)$ from the assumption, this is a contradiction. Hence $H_0=0$.

Suppose $a_i=0$ for some i. Since β_i is essential, we can find an $\alpha\in\mathcal{S}$ such that $i(G,\alpha)=i(\beta_i,\alpha)\neq 0$. Such an α satisfies $i(H,\alpha)=a_i\,i(\beta_i,\alpha)=0$, which is a contradiction. With the same argument, we can see that $H_i\neq 0$. Thus,

$$G^\circ=\sum_{i=1}^{m_1}G_i+\sum_{i=1}^{m_2}\beta_i,$$
$$H^\circ=\sum_{i=1}^{m_1}H_i+\sum_{i=1}^{m_2}a_i\beta_i$$

are topologically equivalent.

Suppose (2) holds. Let $F\in\mathcal{N}_{MF}(G)$. Consider the decomposition (29) for F instead of H, one can easily deduce that $F\in\mathcal{N}_{MF}(H)$. □

5.4. Extremal length.

Let X be a Riemann surface and let A be a doubly connected domain on X. If A is conformally equivalent to a round annulus $\{1<|z|<R\}$, we define the *modulus* of A by

$$\mathrm{Mod}(A)=\frac{1}{2\pi}\log R.$$

Extremal length of a simple closed curve α on X is defined by

$$\mathrm{Ext}_X(\alpha)=\inf\left\{\frac{1}{\mathrm{Mod}(A)}\;\middle|\;\text{the core curve of } A\subset X \text{ is homotopic to } \alpha\right\}. \tag{30}$$

In [**23**], Kerckhoff showed that if we define the extremal length of $t\alpha\in\mathcal{WS}$ by

$$\mathrm{Ext}_X(t\alpha) = t^2\,\mathrm{Ext}_X(\alpha),$$

then the extremal length function Ext_X on $\mathcal{WS}$ extends continuously to $\mathcal{MF}$. For $y = (Y, f) \in \mathcal{T}$ and $G \in \mathcal{MF}$, we define

$$\mathrm{Ext}_y(G) = \mathrm{Ext}_Y(f(G)).$$

We define the *unit sphere* in $\mathcal{MF}$ by

$$\mathcal{MF}_1 = \{F \in \mathcal{MF} \mid \mathrm{Ext}_{x_0}(F) = 1\}.$$

The projection $\mathcal{MF} - \{0\} \to \mathcal{PMF}$ induces a homeomorphism $\mathcal{MF}_1 \to \mathcal{PMF}$.

It is known that for any $G \in \mathcal{MF}$ and $y = (Y, f) \in \mathcal{T}$, there is a unique holomorphic quadratic differential $J_{G,y}$ such that

$$i(G,\alpha) = \inf_{\alpha' \sim f(\alpha)} \int_{\alpha'} \left|\mathrm{Re}\sqrt{J_{G,y}}\right|.$$

Namely, the *vertical foliation* of $J_{G,y}$ is equal to G. We call $J_{G,y}$ the *Hubbard–Masur differential* for G on y (cf. [**16**]). The Hubbard–Masur differential $J_{G,y} = J_{G,y}(z)\,dz^2$ for G on $y = (Y, f)$ satisfies

$$\mathrm{Ext}_y(G) = \|J_{G,y}\| = \iint_Y |J_{G,y}(z)|\,dx\,dy.$$

In particular, it is known that

$$\mathrm{Ext}_y(\alpha) = \|J_{\alpha,y}\| = \frac{\ell_{J_{G,y}}(\alpha)^2}{\|J_{\alpha,y}\|} \tag{31}$$

where $\ell_{J_{G,y}}(\alpha)$ is the length of the geodesic representative homotopic to $f(\alpha)$ with respect to the singular flat metric $|J_{\alpha,y}| = |J_{\alpha,y}(z)||dz|^2$.

Kerckhoff's formula. The Teichmüller distance d_T is expressed by extremal length, which we call *Kerckhoff's formula*:

$$d_T(y_1, y_2) = \frac{1}{2}\log\sup_{\alpha\in\mathcal{S}} \frac{\mathrm{Ext}_{y_2}(\alpha)}{\mathrm{Ext}_{y_1}(\alpha)} \tag{32}$$

(see [**23**]).

Minsky's inequality. Minsky [**31**] observed the following inequality, which we recently call *Minsky's inequality*:

$$i(F,G)^2 \le \mathrm{Ext}_y(F)\,\mathrm{Ext}_y(G) \tag{33}$$

for $y \in \mathcal{T}$ and $F, G \in \mathcal{MF}$. Minsky's inequality is sharp in the sense that for any $y \in \mathcal{T}$ and $F \in \mathcal{MF}$, there is a unique $G \in \mathcal{MF}$ up to multiplication by a positive constant such that $i(F,G)^2 = \mathrm{Ext}_y(F)\,\mathrm{Ext}_y(G)$ (cf. [**13**]).

5.5. Teichmüller rays.

Let $x = (X, f) \in \mathcal{T}$ and $[G] \in \mathcal{PMF}$. By the Ahlfors–Bers theorem, we can define an isometric embedding

$$[0, \infty) \ni t \mapsto R_{G,x}(t) \in \mathcal{T}$$

with respect to the Teichmüller distance by assigning the solution of the Beltrami equation defined by the Teichmüller Beltrami differential

$$\tanh(t)\frac{|J_{G,x}|}{J_{G,x}} \tag{34}$$

for $t \geq 0$. We call $R_{G,x}$ the *Teichmüller (geodesic) ray* associated to $[G] \in \mathcal{PMF}$. Notice that the differential (34) depends only on the projective class of G. The *exponential map*

$$\mathcal{PMF} \times [0, \infty)/(\mathcal{PMF} \times \{0\}) \ni ([G], t) \mapsto R_{G,t}(t) \in \mathcal{T} \tag{35}$$

which is a homeomorphism (see also [**17**]).

6. Thurston theory with extremal length.

In this section, we recall the unification of extremal length geometry via intersection number developed in [**36**].

6.1. Gardiner–Masur closure.

Consider a mapping

$$\begin{aligned}
\tilde{\Phi}_{GM} &: \mathcal{T} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \mathbb{R}_+^{\mathcal{S}}, \\
\Psi_{GM} &: \mathcal{T} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto e^{-d_T(x_0,y)}\mathrm{Ext}_y(\alpha)^{1/2}] \in \mathbb{R}_+^{\mathcal{S}}.
\end{aligned}$$

Let $\mathrm{proj}\colon \mathbb{R}_+^{\mathcal{S}} - \{0\} \to \mathrm{P}\mathbb{R}_+^{\mathcal{S}}$ be the quotient mapping of the action. In [**13**], Gardiner and Masur showed that the mapping

$$\Phi_{GM} = \mathrm{proj} \circ \Psi_{GM} = \mathrm{proj} \circ \tilde{\Phi}_{GM} : \mathcal{T} \to \mathrm{P}\mathbb{R}_+^{\mathcal{S}}$$

is an embedding with compact closure. The closure $\mathrm{cl}_{GM}(\mathcal{T})$ of the image is called the *Gardiner–Masur closure* or the *Gardiner–Masur compactification*, and the complement $\partial_{GM}\mathcal{T} = \mathrm{cl}_{GM}(\mathcal{T}) - \Phi_{GM}(\mathcal{T})$ is called the *Gardiner–Masur boundary*. They also observed that the space $\mathcal{PMF}$ of projective measured foliaitons is contained in $\partial_{GM}\mathcal{T}$.

6.2. Cones, the intersection number and the Gromov product.

We define

$$\begin{aligned}
\mathcal{C}_{GM} &= \mathrm{proj}^{-1}(\mathrm{cl}_{GM}(\mathcal{T})) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}}, \\
\mathcal{T}_{GM} &= \mathrm{proj}^{-1}(\Phi_{GM}(\mathcal{T})) \subset \mathbb{R}_+^{\mathcal{S}}, \\
\tilde{\partial}_{GM} &= \mathrm{proj}^{-1}(\partial_{GM}\mathcal{T}) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}}.
\end{aligned}$$

Since $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}$, $\mathcal{MF} \subset \tilde{\partial}_{GM} \subset \mathcal{C}_{GM}$. From Proposition 1 of [**36**], $\Psi_{GM}\colon \mathcal{T} \to \mathcal{C}_{GM}$ extends to an injective continuous mapping on $\mathrm{cl}_{GM}(\mathcal{T})$.

CONVENTION 6.1. *We denote by* $[\mathfrak{a}] \in \mathrm{cl}_{GM}(\mathcal{T})$ *the projective class of* $\mathfrak{a} \in \mathcal{C}_{GM}$. *Unless otherwise stated, we always identify* $y \in \mathcal{T}$ *with the projective class* $\Phi_{GM}(y) = [\tilde{\Phi}_{GM}(y)] = [\Psi_{GM}(y)]$.

In [**36**], the author established the following *unification* of extremal length geometry via the intersection number.

THEOREM 6.1 (Theorem 1.1 in [**36**]). *Let* $x_0 \in \mathcal{T}$ *be the base point taken as above. There is a unique continuous function*

$$i(\,\cdot\,,\,\cdot\,)\colon \mathcal{C}_{GM} \times \mathcal{C}_{GM} \to \mathbb{R}$$

with the following properties.

(i) $i(\tilde{\Phi}_{GM}(y), F) = \mathrm{Ext}_y(F)^{1/2}$ *for any* $y \in \mathcal{T}$ *and* $F \in \mathcal{MF}$.

(ii) *For* $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$, $i(\mathfrak{a}, \mathfrak{b}) = i(\mathfrak{b}, \mathfrak{a})$.

(iii) *For* $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$ *and* $t, s \geq 0$, $i(t\mathfrak{a}, s\mathfrak{b}) = ts\, i(\mathfrak{a}, \mathfrak{b})$.

(iv) *For any* $y, z \in \mathcal{T}$,

$$i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z)) = \exp(d_T(y,z)),$$
$$i(\Psi_{GM}(y), \Psi_{GM}(z)) = \exp(-2\langle y \,|\, z\rangle_{x_0}).$$

(v) *For* $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$, *the value* $i(F,G)$ *is equal to the geometric intersection number* $I(F,G)$ *between* F *and* G.

We define the *extremal length* of $\mathfrak{a} \in \mathcal{C}_{GM}$ on $y \in \mathcal{T}$ by

$$\mathrm{Ext}_y(\mathfrak{a}) = \sup_{F \in \mathcal{MF}-\{0\}} \frac{i(\mathfrak{a}, F)^2}{\mathrm{Ext}_y(F)} \tag{36}$$

(cf. Corollary 4 in [**36**]). One see that

$$e^{-2d_T(x,y)}\mathrm{Ext}_x(\mathfrak{a}) \leq \mathrm{Ext}_y(\mathfrak{a}) \leq e^{2d_T(x,y)}\mathrm{Ext}_x(\mathfrak{a}) \tag{37}$$

from (32) (see also (5.6) in [**36**]). From (33) and Gardiner–Masur's work in [**13**], (36) coincides with the original extremal length when $\mathfrak{a} \in \mathcal{MF}$. The extremal length Ext_y is continuous on $\mathcal{C}_{GM}$ and satisfies

$$e^{-d_T(x_0,y)}\mathrm{Ext}_y(\Psi_{GM}(z))^{1/2} = \exp(-2\langle y \,|\, z\rangle_{x_0}) = i(\Psi_{GM}(y), \Psi_{GM}(z)), \tag{38}$$
$$e^{-d_T(x_0,y)}\mathrm{Ext}_y(\mathfrak{a})^{1/2} = i(\Psi_{GM}(y), \mathfrak{a}) \tag{39}$$

for $y, z \in \mathcal{T}$ and $\mathfrak{a} \in \mathcal{C}_{GM}$ (cf. Theorem 4 and Proposition 7 in [**36**]). The extremal length (36) also satisfies the following *generalized Minsky inequality*:

$$i(\mathfrak{a},\mathfrak{b})^2 \leq \mathrm{Ext}_y(\mathfrak{a})\,\mathrm{Ext}_y(\mathfrak{b}) \tag{40}$$

for all $y \in \mathcal{T}$ and $\mathfrak{a},\mathfrak{b} \in \mathcal{C}_{GM}$ (cf. Corollary 3 in [**36**]).

6.3. Intersection number with base point.

We define the *intersection number with base point* x_0 by

$$i_{x_0}(p,q) = i(\Psi_{GM}(p), \Psi_{GM}(q)) \tag{41}$$

for $p,q \in \mathrm{cl}_{GM}(\mathcal{T})$ (cf. Section 8.2 in [**36**]). Since the intersection number is continuous, so is i_{x_0} on the product $\mathrm{cl}_{GM}(\mathcal{T}) \times \mathrm{cl}_{GM}(\mathcal{T})$. From Theorem 6.1, the Gromov product

$$\langle y \,|\, z\rangle_{x_0} = -\frac{1}{2}\log i_{x_0}(y,z) \tag{42}$$

extends continuously to $\mathrm{cl}_{GM}(\mathcal{T}) \times \mathrm{cl}_{GM}(\mathcal{T})$ with values in the closed interval $[0,\infty]$ (cf. Corollary 1 in [**36**]).

Proposition 6.1. *Teichmüller space $(\mathcal{T}, d_T)$ is WBGP.*

Proof. Let $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in \mathrm{USq}(\mathcal{T})$. Since the Gardiner–Masur closure is compact, we find a subsequence $\boldsymbol{x}' = \{x_{n(k)}\}_{k\in\mathbb{N}}$ and $p \in \partial_{GM}\mathcal{T}$ such that $x_{n(k)} \to p$ as $k \to \infty$. Since

$$i_{x_0}(x_{n(k)}, x_{n(l)}) \to i_{x_0}(p,p) = 0 \quad (k,l\to\infty),$$

we have $\boldsymbol{x}' \in \mathrm{Sq}^\infty(\mathcal{T})$ from (42). □

Proposition 6.2 (Intersection number with base point). *For any $[\mathfrak{a}],[\mathfrak{b}] \in \mathrm{cl}_{GM}(\mathcal{T})$, it holds*

$$i_{x_0}([\mathfrak{a}],[\mathfrak{b}]) = \frac{i(\mathfrak{a},\mathfrak{b})}{\mathrm{Ext}_{x_0}(\mathfrak{a})^{1/2}\mathrm{Ext}_{x_0}(\mathfrak{b})^{1/2}}. \tag{43}$$

Notice that the intersection number in the right-hand side of (43) is the original intersection number on $\mathcal{MF}$ (cf. (v) of Theorem 6.1).

Proof of Proposition 6.2. Let $y \in \mathcal{T}$. Notice that

$$\mathrm{Ext}_{x_0}(\Psi_{GM}(y)) = \exp(-2\langle x_0 \,|\, y\rangle_{x_0}) = 1.$$

Since Ext_{x_0} is continuous on $\mathcal{C}_{GM}$, we have

$$\Psi_{GM}([\mathfrak{a}]) = \frac{\mathfrak{a}}{\mathrm{Ext}_{x_0}(\mathfrak{a})^{1/2}}. \tag{44}$$

Therefore,

$$i_{x_0}([\mathfrak{a}],[\mathfrak{b}]) = i(\Psi_{GM}([\mathfrak{a}]), \Psi_{GM}([\mathfrak{b}])) = \frac{i(\mathfrak{a},\mathfrak{b})}{\mathrm{Ext}_{x_0}(\mathfrak{a})^{1/2}\mathrm{Ext}_{x_0}(\mathfrak{b})^{1/2}}$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$. □

6.4. A short proof for non-Gromov hyperbolicity.

We check that the relation "visually indistinguishable" is not an equivalence relation on $\mathrm{Sq}^{\infty}(\mathcal{T})$ when $\mathrm{cx}(S) \geq 2$. This also implies that Teichmüller space $(\mathcal{T}, d_T)$ is not Gromov hyperbolic.

Indeed, let $\alpha, \beta, \gamma \in \mathcal{S}$ with $i(\alpha, \beta) = i(\alpha, \gamma) = 0$, but $i(\beta, \gamma) \neq 0$. Consider sequences $\boldsymbol{x} = \{x_n\}_{n \in \mathbb{N}}$, $\boldsymbol{y} = \{y_n\}_{n \in \mathbb{N}}$ and $\boldsymbol{z} = \{z_n\}_{n \in \mathbb{N}}$ in $\mathcal{T}$ with $x_n \to [\alpha]$, $y_n \to [\beta]$ and $z_n \to [\gamma]$ in $\mathrm{cl}_{GM}(\mathcal{T})$, where the projective classes $[\alpha]$, $[\beta]$ and $[\gamma]$ are recognized as points in $\partial_{GM}\mathcal{T}$. Then,

$$i_{x_0}(x_n, y_n) \to i_{x_0}([\alpha], [\beta]) = 0,$$
$$i_{x_0}(x_n, z_n) \to i_{x_0}([\alpha], [\gamma]) = 0,$$

but

$$i_{x_0}(y_n, z_n) \to i_{x_0}([\beta], [\gamma]) \neq 0.$$

From (42), these observations imply that $\boldsymbol{y}, \boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$ but $\boldsymbol{y} \notin \mathrm{Vis}(\boldsymbol{z})$.

6.5. Subadditivity of the intersection number.

The intersection number has the following *subadditive property.*

LEMMA 6.1 (Subadditivity). *Let $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$ with $i(F, G) = 0$. Then, for any $\mathfrak{a} \in \mathcal{C}_{GM}$ we have*

$$(i(\mathfrak{a}, F)^2 + i(\mathfrak{a}, G)^2)^{1/2} \leq i(\mathfrak{a}, F + G) \leq i(\mathfrak{a}, F) + i(\mathfrak{a}, G). \tag{45}$$

PROOF. Let $y \in \mathcal{T}$. Then, we have

$$\begin{aligned} i(\tilde{\Phi}_{GM}(y), F + G) &= \mathrm{Ext}_y(F + G)^{1/2} \\ &= \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F + G)}{\mathrm{Ext}_y(H)^{1/2}} = \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F) + i(H, G)}{\mathrm{Ext}_y(H)^{1/2}} \\ &\leq \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, F)}{\mathrm{Ext}_y(H)^{1/2}} + \sup_{H \in \mathcal{MF} - \{0\}} \frac{i(H, G)}{\mathrm{Ext}_y(H)^{1/2}} \\ &= \mathrm{Ext}_y(F)^{1/2} + \mathrm{Ext}_y(G)^{1/2} = i(\tilde{\Phi}_{GM}(y), F) + i(\tilde{\Phi}_{GM}(y), G). \end{aligned}$$

Hence, the right-hand side of (45) follows from the density of $\mathcal{T}_{GM}$ in $\mathcal{C}_{GM}$.

We prove the left-hand side of (45). We first show the case where F and G are rational. Let $F = \sum_{i=1}^{N_1} t_i \alpha_i + \sum_{j=1}^{N_3} u_j \gamma_j$ and $G = \sum_{i=1}^{N_2} s_i \beta_i + \sum_{j=1}^{N_3} v_j \gamma_j$ where $\alpha_i, \beta_i, \gamma_j$ are mutually disjoint and distinct simple closed curves and $t_i, s_i > 0$ and $u_j, v_j \geq 0$. Let $y \in \mathcal{T}$ and A_{α_i}, A_{β_i}, A_{γ_j} be the characteristic annuli for α_i, β_i and γ_j of $J_{F+G,y}$ (cf. [**43**]). From (30) and Theorem 20.5 in [**43**], we have

$$i(\tilde{\Phi}_{GM}(y), F + G)^2 = \mathrm{Ext}_y(F + G) = \|J_{F+G,y}\|$$

$$
\begin{aligned}
&= \sum_{i=1}^{N_1} \frac{t_i^2}{\mathrm{Mod}(A_{\alpha_i})} + \sum_{i=1}^{N_2} \frac{s_i^2}{\mathrm{Mod}(A_{\beta_i})} + \sum_{j=1}^{N_3} \frac{(u_j+v_j)^2}{\mathrm{Mod}(A_{\gamma_j})} \\
&\geq \left(\sum_{i=1}^{N_1} \frac{t_i^2}{\mathrm{Mod}(A_{\alpha_i})} + \sum_{j=1}^{N_3} \frac{u_j}{\mathrm{Mod}(A_{\gamma_j})} \right) \\
&\quad + \left(\sum_{i=1}^{N_2} \frac{s_i^2}{\mathrm{Mod}(A_{\beta_i})} + \sum_{j=1}^{N_3} \frac{v_j^2}{\mathrm{Mod}(A_{\gamma_j})} \right) \\
&\geq \|J_{F,y}\| + \|J_{G,y}\| = \mathrm{Ext}_y(F) + \mathrm{Ext}_y(G) \\
&= i(\tilde{\Phi}_{GM}(y), F)^2 + i(\tilde{\Phi}_{GM}(y), G)^2. \qquad (46)
\end{aligned}
$$

Since $\mathcal{T}_{GM}$ is dense in $\mathcal{C}_{GM}$, the above calculation implies

$$i(\mathfrak{a}, F)^2 + i(\mathfrak{a}, G)^2 \leq i(\mathfrak{a}, F+G)^2$$

for all $\mathfrak{a} \in \mathcal{C}_{GM}$. Hence, the left-hand side of (45) also follows by approximating arrational components by weighted multicurves (cf. Theorem C of [**24**]). □

7. Structure of the null sets.

We define the *null set* for $\mathfrak{a} \in \mathcal{C}_{GM}$ by

$$\mathcal{N}(\mathfrak{a}) = \{\mathfrak{b} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{b}) = 0\}.$$

This section is devoted to showing the following theorem.

Theorem 7.1 (Structure of the null set). *For any $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$, any associated foliation $[G] \in \mathcal{PMF}$ for $\mathfrak{a}$ satisfies $\mathcal{N}(\mathfrak{a}) = \mathcal{N}(G) = \mathcal{N}(G^\circ)$.*

The associated foliation for $\mathfrak{a}$ in Theorem 7.1 is defined in Section 7.1. We will see that the associated foliations for $\mathfrak{a}$ are essentially uniquely determined from $\mathfrak{a}$ (cf. Theorem 7.2). The following is known (cf. Proposition 9.1 in [**36**]).

Proposition 7.1. *For $\mathfrak{a} \in \mathcal{C}_{GM} - \{0\}$, $\mathcal{N}(\mathfrak{a}) \neq \{0\}$ if and only if $\mathfrak{a} \in \tilde{\partial}_{GM}$. In any case, we have $\mathcal{N}(\mathfrak{a}) \subset \tilde{\partial}_{GM}$, and $\mathfrak{a} \in \mathcal{N}(\mathfrak{a})$ if $\mathfrak{a} \in \tilde{\partial}_{GM}$.*

7.1. Associated foliations.

Let $[\mathfrak{a}] \in \partial_{GM}\mathcal{T}$ and $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$. A projective measured foliation $[G] \in \mathcal{PMF}$ is said to be an *associated foliation* for $[\mathfrak{a}] \in \partial_{GM}\mathcal{T}$ if there exist $x \in \mathcal{T}$, a sequence $[G_n] \in \mathcal{PMF}$ and $t_n > 0$ such that $R_{G_n,x}(t_n) \to [\mathfrak{a}]$ and $[G_n] \to [G]$ as $n \to \infty$. We call the point x the *base point* for the associated foliation $[G]$. We denote by $\mathcal{AF}([\mathfrak{a}])$ the set of associated foliations for $[\mathfrak{a}]$.

In this section, we prove the following.

Proposition 7.2 (Uniqueness of vanishing curves). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$. For any $[G] \in \mathcal{AF}([\mathfrak{a}])$, we have*

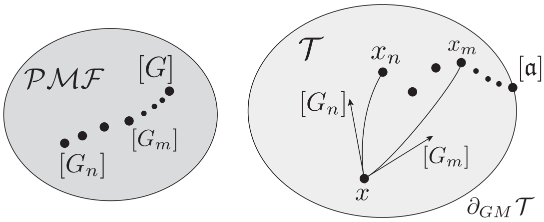

Figure 1. Associated foliation $[G]$: In the figure, we set $x_n = R_{G_n,x}(t_n)$. x is the base point for $[G]$.

$$\mathcal{N}(G) \cap \mathcal{S} = \mathcal{N}(\mathfrak{a}) \cap \mathcal{S}.$$

7.1.1. Lemmas.
Let

$$\mathcal{N}_{MF}(\mathfrak{a}) = \mathcal{N}(\mathfrak{a}) \cap \mathcal{MF}. \tag{47}$$

When $\mathfrak{a} \in \mathcal{MF}$, the set (47) coincides with the set defined as (28).

Lemma 7.1. *The following hold*:

1. $\{G \in \mathcal{MF} \mid [G] \in \mathcal{AF}([\mathfrak{a}])\} \subset \mathcal{N}_{MF}(\mathfrak{a})$.
2. *For* $[G] \in \mathcal{AF}([\mathfrak{a}])$, *we have* $\mathcal{N}(\mathfrak{a}) \subset \mathcal{N}(G)$ *and* $\mathcal{N}_{MF}(\mathfrak{a}) \subset \mathcal{N}_{MF}(G)$.

In particular $i(G_1, G_2) = 0$ *for* $[G_1], [G_2] \in \mathcal{AF}([\mathfrak{a}])$.

Proof. (1) Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. Take $x \in \mathcal{T}$, $\{[G_n]\}_{n\in\mathbb{N}} \subset \mathcal{PMF}$, and $t_n > 0$ such that $R_{G_n,x}(t_n) \to [\mathfrak{a}]$ and $G_n \to G$ as $n \to \infty$. From (44), $\Psi_{GM} \circ R_{G_n,x}(t_n) = e^{-t_n}\tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n)$ converges to $\mathfrak{a}' \in \mathcal{C}_{GM} - \{0\}$ which is projectively equivalent to $\mathfrak{a}$. Therefore

$$\begin{aligned}
i(\mathfrak{a}', G) &= \lim_{n\to\infty} i(e^{-t_n}\tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n), G_n) \\
&= \lim_{n\to\infty} e^{-t_n}\mathrm{Ext}_{R_{G_n,x}(t_n)}(G_n)^{1/2} \\
&= \lim_{n\to\infty} e^{-2t_n}\mathrm{Ext}_x(G_n)^{1/2} = 0
\end{aligned}$$

and $G \in \mathcal{N}_{MF}(\mathfrak{a})$.

(2) Let $\mathfrak{b} \in \mathcal{N}(\mathfrak{a})$. Take $x \in \mathcal{T}$, $\{[G_n]\}_{n\in\mathbb{N}} \subset \mathcal{PMF}$, $t_n > 0$, and $\mathfrak{a}'$ as above. From (39) and (40), we have

$$\begin{aligned}
i(G, \mathfrak{b}) &= \lim_{n\to\infty} i(G_n, \mathfrak{b}) \\
&\leq \lim_{n\to\infty} \mathrm{Ext}_{R_{G_n,x}(t_n)}(G_n)^{1/2}\mathrm{Ext}_{R_{G_n,x}(t_n)}(\mathfrak{b})^{1/2} \\
&= \lim_{n\to\infty} e^{-t_n}\mathrm{Ext}_x(G_n)^{1/2}\mathrm{Ext}_{R_{G_n,x}(t_n)}(\mathfrak{b})^{1/2} \\
&= \lim_{n\to\infty} \mathrm{Ext}_x(G_n)^{1/2} i(e^{-t_n}\tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n), \mathfrak{b})
\end{aligned}$$

$$= \mathrm{Ext}_x(G)^{1/2} i(\mathfrak{a}', \mathfrak{b}) = 0$$

and $\mathfrak{b} \in \mathcal{N}(G)$. From the definition,

$$\mathcal{N}_{MF}(\mathfrak{a}) = \mathcal{N}(\mathfrak{a}) \cap \mathcal{MF} \subset \mathcal{N}(G) \cap \mathcal{MF} = \mathcal{N}_{MF}(G).$$

And we are done. □

For $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$, we define

$$\mathcal{AN}(\mathfrak{a}) = \bigcup_{[G] \in \mathcal{AF}([\mathfrak{a}])} \mathcal{N}_{MF}(G) \subset \mathcal{MF}.$$

Lemma 7.2. *$\mathcal{AN}(\mathfrak{a}) \cap \mathcal{S} \subset \mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$ for all $\mathfrak{a} \in \mathcal{C}_{GM} - \{0\}$.*

Proof. Let $\alpha \in \mathcal{AN}(\mathfrak{a}) \cap \mathcal{S}$. Let $[G] \in \mathcal{AF}([\mathfrak{a}])$ with $i(G, \alpha) = 0$. Then, there are $x \in \mathcal{T}$, a sequence $\{[G_n]\}_{n \in \mathbb{N}}$ converging to $[G]$ and $t_n > 0$ such that $R_{G_n,x}(t_n)$ tends to $[\mathfrak{a}]$ as $n \to \infty$. Let $y_t = (Y_t, f_t) = R_{G,x}(t)$.

We refer to the argument in Section 5.3 of [**34**] (see also [**19**] and [**29**]). Let Γ_G be the critical vertical graph of the holomorphic quadratic differential of $J_{G,x}$. We add mutually disjoint critical vertical segments to Γ_G emanating from critical points to get a graph Γ_G^0 whose edges are all vertical. The degree of a vertex Γ_G^0 is one-prong if it is one of endpoints of an added vertical segment. Take $\epsilon > 0$ sufficiently small such that the ϵ-neighborhood $C(\epsilon)$ (with respect to the $|J_{G,x}|$-metric) is embedded in X. Then, as the argument in the proof of Theorem 3.1 in [**19**], by shrinking with a factor e^{-t}, we get a canonical conformal embedding $g_t : C(\epsilon) \to Y_t$ such that $g_t(\Gamma_G) = f_t(\Gamma_G)$. Since $i(\alpha, G) = 0$, α can be deformed into $C(\epsilon)$. Hence, by from the geometric definition (30) of extremal length, the conformal embedding $g_t : C(\epsilon) \to Y_t$ induces

$$\mathrm{Ext}_{y_t}(\alpha) \le \mathrm{Ext}_{C(\epsilon)}(\alpha) =: c_0 \tag{48}$$

for some $c_0 > 0$ independent of t.

Let $\epsilon > 0$. Take $T > 0$ such that $2c_0 e^{-T} < \epsilon$. Since $[G_n] \to [G]$, by (35), there exists an $n_0 > 0$ such that $d(R_{G,x}(T), R_{G_n,x}(T)) \le (\log 2)/2$ and $t_n \ge T$ for $n \ge n_0$. It has shown from Lemma 1 of [**34**] that a function

$$[0, \infty) \ni t \mapsto e^{-t} \mathrm{Ext}_{y_t}(F)^{1/2}$$

is a non-increasing function for any $F \in \mathcal{MF}$. Hence, from (48), we have

$$\begin{aligned} i(e^{-t_n} \tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n), \alpha) &= e^{-t_n} \mathrm{Ext}_{R_{G_n,x}(t_n)}(\alpha)^{1/2} \\ &\le e^{-T} \mathrm{Ext}_{R_{G_n,x}(T)}(\alpha)^{1/2} \\ &\le 2e^{-T} \mathrm{Ext}_{y_T}(\alpha)^{1/2} \le 2c_0 e^{-T} < \epsilon. \end{aligned}$$

Since $|t_n - d_T(x_0, R_{G_n,x}(t_n))| \le d_T(x, x_0)$, by taking a subsequence,

$$e^{-t_n} \tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n) = e^{t_n - d_T(x_0, R_{G_n,x}(t_n))} \cdot \Psi_{GM} \circ R_{G_n,x}(t_n)$$

converges to $\mathfrak{a}' \in \mathcal{C}_{GM} - \{0\}$ with $[\mathfrak{a}'] = [\mathfrak{a}]$. Therefore, we get

$$i(\mathfrak{a}', \alpha) = \lim_{n\to\infty} i(e^{-t_n}\tilde{\Phi}_{GM} \circ R_{G_n,x}(t_n), \alpha) = 0$$

and $\alpha \in \mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$. □

7.1.2. Proof of Proposition 7.2.

Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. From (2) of Lemma 7.1 and Lemma 7.2, we have

$$\begin{aligned}\mathcal{N}(\mathfrak{a}) \cap \mathcal{S} \subset \mathcal{N}(G) \cap \mathcal{S} &= \mathcal{N}_{MF}(G) \cap \mathcal{S} \\ &\subset \left(\bigcup_{[G]\in\mathcal{AF}([\mathfrak{a}])} \mathcal{N}_{MF}(G) \right) \cap \mathcal{S} = \mathcal{AN}(\mathfrak{a}) \cap \mathcal{S} \subset \mathcal{N}(\mathfrak{a}) \cap \mathcal{S}.\end{aligned}$$ □

7.2. Vanishing surface.

The aim of this section is to define the *vanishing surface* for $\mathfrak{a}$, which is used for proving Theorem 7.2 stated in the next section.

7.2.1. Minimal vanishing surfaces.

Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$. Let $Z^0_{\mathfrak{a}}$ be the minimal essential subsurface of X which contains all simple closed curve in $\mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$. We call $Z^0_{\mathfrak{a}}$ the *minimal vanishing surface* for $\mathfrak{a}$. By definition, any component Z_i of $Z^0_{\mathfrak{a}}$ contains a collection of curves in $\mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$ which fills up Z_i. It is possible that either $\mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$ or $Z^0_{\mathfrak{a}}$ is empty.

7.2.2. Properties of minimal vanishing surfaces.

From Lemma 10.1 in Appendix, if $\alpha \in \mathcal{S}$ can be deformed into $Z^0_{\mathfrak{a}}$, then $i(\mathfrak{a}, \alpha) = 0$ (see also Theorem 6.1 of [**13**]).

PROPOSITION 7.3. *Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. For $\alpha \in \mathcal{S}$, the following are equivalent.*

1. *α is homotopic to a component of $\partial Z^0_{\mathfrak{a}}$.*

2. *α is homotopic to either an essential curve or a peripheral curve of G.*

PROOF. (1) ⇒ (2). Since $i(\mathfrak{a}, \alpha) = 0$, from Proposition 7.2, we have $i(\alpha, G) = 0$. Suppose that α is non-peripheral in a component W of $X - \mathrm{Supp}(G)$. Then, there is an $\alpha' \in \mathcal{S}$ which is non-peripheral in W satisfying $i(\alpha, \alpha') \neq 0$. Since $i(\alpha', G) = 0$, $i(\alpha', \mathfrak{a}) = 0$ by Proposition 7.2. This means that α cannot be homotopic to a component of $\partial Z^0_{\mathfrak{a}}$ because $Z^0_{\mathfrak{a}}$ contains a regular neighborhood of $\alpha \cup \alpha'$. This contradicts our assumption.

(2) ⇒ (1). Since $i(\alpha, G) = 0$, by Proposition 7.2, α can be deformed into the vanishing surface $Z^0_{\mathfrak{a}}$. Suppose to the contrary that α is non-peripheral in $Z^0_{\mathfrak{a}}$. Then, there is a non-peripheral curve δ in a component of $Z^0_{\mathfrak{a}}$ with $i(\alpha, \delta) \neq 0$. Since $\delta \in \mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$, we have $i(\delta, G) = 0$ by Proposition 7.2 again.

If δ is a component of some ∂X_i, $i(\alpha, G) \geq i(\alpha, G_i) \neq 0$ by Lemma 2.14 of [**18**]. This contradicts that $\alpha \subset Z^0_{\mathfrak{a}}$. If δ is non-peripheral in a component of $X - \mathrm{Supp}(G)$, so is α since $i(\alpha, \delta) \neq 0$. This contradicts to the assumption. □

Proposition 7.4. *For $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$, none of components of $Z^0_{\mathfrak{a}}$ are pairs of pants.*

Proof. Let Z be a component of $Z^0_{\mathfrak{a}}$. Suppose to the contrary that Z is a pair of pants. Since any simple closed curve in Z is homotopic to a component of ∂Z, $(Z^0_{\mathfrak{a}} - Z) \cup N(\partial Z)$ contains all curves in $\mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$, where $N(\partial Z)$ is the regular neighborhood of ∂Z. This contradicts the minimality of $Z^0_{\mathfrak{a}}$. □

7.2.3. Vanishing surface.

We define a subsurface $Z_{\mathfrak{a}}$ of X as follows:

1. Remove annular components of $Z^0_{\mathfrak{a}}$ whose core is homotopic to a component of ∂W, where W runs components of $X - Z^0_{\mathfrak{a}}$ which are pairs of pants.
2. To the resulting surface, add components of $X - Z^0_{\mathfrak{a}}$ which are pairs of pants.

See Figure 2. We call $Z_{\mathfrak{a}}$ the *vanishing surface* for $\mathfrak{a}$. Notice from definition that $i(\partial Z, \mathfrak{a}) = 0$ for every component Z of $Z_{\mathfrak{a}}$, and none of the components of $X - Z_{\mathfrak{a}}$

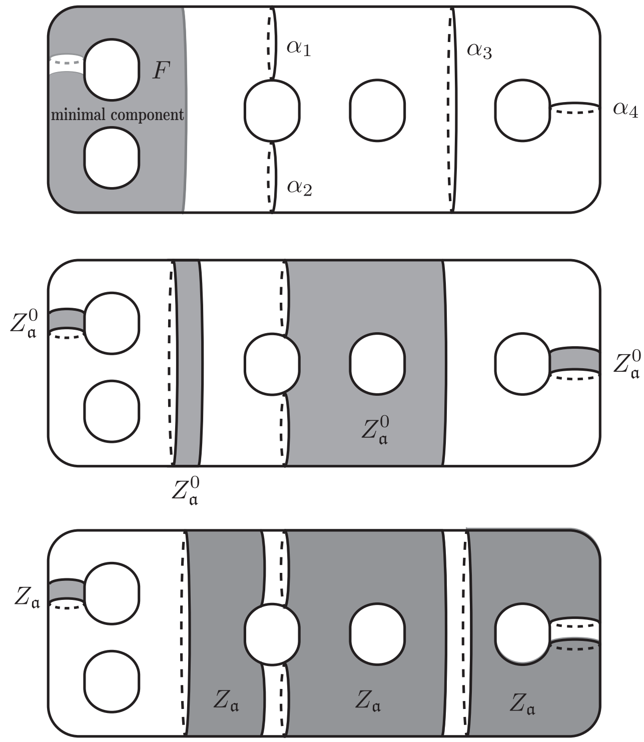

Figure 2. Case of $G = F + \sum_{i=1}^{4} \alpha_i$. In this case, $G^\circ = G$. $Z^0_{\mathfrak{a}}$ has three annular components. Two have the core curves which are homotopic to a peripheral curve. The other comes from an essential curve α_4 of G. The complement $X - Z^0_{\mathfrak{a}}$ has two components which are pairs of pants.

are pairs of pants. Recall that G° denotes the distinguished part of $G \in \mathcal{MF}$ on nullity (cf. Section 5.3).

7.3. Uniqueness of the underlying foliations.

The following uniqueness theorem implies that the underlying foliations of associated foliations for $\mathfrak{a}$ is essentially determined from $\mathfrak{a}$.

THEOREM 7.2 (Uniqueness of the underlying foliations). *For any $[G_1], [G_2] \in \mathcal{AF}([\mathfrak{a}])$, ${G_1}^\circ$ and ${G_2}^\circ$ are topologically equivalent.*

The above uniqueness theorem follows from Proposition 7.5 below.

PROPOSITION 7.5 (Decomposition associated to $\mathfrak{a}$). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$ and $Z_{\mathfrak{a}}$ the vanishing surface for $\mathfrak{a}$. Then, the reference surface X is decomposed into a union of compact essential surfaces with mutually disjoint interiors as*

$$Z_{\mathfrak{a}} \cup X_1 \cup \cdots \cup X_{m_1} \cup B_1 \cup \cdots \cup B_{m_2} \tag{49}$$

such that for all $[G] \in \mathcal{AF}([\mathfrak{a}])$, the following properties hold.

1. *The family $\{X_i\}_{i=1}^{m_1}$ consists of all components of $X - Z_{\mathfrak{a}}$ whose complexities are at least 1. The support of any minimal component of G is some X_i. For any for $i = 1, \ldots, m_1$, X_i contains arrational foliation F_i such that the minimal component of G whose support is X_i is topologically equivalent to F_i. Conversely, for any i, G contains an arrational component whose support is isotopic to X_i.*
2. *The family $\{B_i\}_{i=1}^{m_2}$ consists of all annular components of $X - Z_{\mathfrak{a}}$. any essential curve of G is homotopic to the core curve of some B_i. Conversely, the core of any B_i is homotopic to some essential curve of G.*
3. *Any curve $\alpha \in \mathcal{S}$ deformed into $Z_{\mathfrak{a}}$ satisfies $i(\alpha, \mathfrak{a}) = i(\alpha, G) = 0$.*

Proof of Proposition 7.5. Proposition 7.5 follows from the combination of the following four lemmas given below.

LEMMA 7.3 (Non annular components of $Z_{\mathfrak{a}}$). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$ and $[G] \in \mathcal{AF}([\mathfrak{a}])$. Every non-annular component of $Z_{\mathfrak{a}}$ is isotopic to a non-annular component of $X - \mathrm{Supp}(G^\circ)$, and vice versa.*

PROOF. Let Z be a non-annular component of $Z_{\mathfrak{a}}$. Suppose first that Z is not a pair of pants. Then, Z is also a component of $Z^0_{\mathfrak{a}}$ and Z contains a finite family of curves in $\mathcal{N}(\mathfrak{a})$ which fills up. By Proposition 7.2, there is a component W of the component of $X - \mathrm{Supp}(G)$ such that $Z \subset W$ in homotopy sense. Since $\mathrm{cx}(W) \geq 1$, W is also a component of $X - \mathrm{Supp}(G^\circ)$ in homotopy sense. From Proposition 7.3, all components of ∂Z are peripheral curves in W. Hence, $\overline{Z}$ is isotopic to $\overline{W}$.

Suppose Z is a pair of pants. By definition, $i(\partial Z, \mathfrak{a}) = 0$ and $i(\partial Z, G) = 0$. Since Z does not contain any minimal component of G, Z is contained in a component W of $X - \mathrm{Supp}(G^\circ)$. By the same argument as above, we obtain that $\overline{Z}$ is isotopic to $\overline{W}$.

The converse follows from the same argument. However, let us give a sketch for the completeness. Let W be a non-annular component of $X - \mathrm{Supp}(G^\circ)$. If $\mathrm{cx}(W) \geq 1$, by Proposition 7.2 again, W is contained in $Z^0_{\mathfrak{a}}$ in homotopy sense. From Proposition 7.3 again, W is isotopic to the component Z of $Z^0_{\mathfrak{a}}$ containing W. Since $\mathrm{cx}(W) \geq 1$, Z is also a component of $Z_{\mathfrak{a}}$ in homotopy sense. Since $i(\partial W, G) = 0$, $i(\partial W, \mathfrak{a}) = 0$ and hence $\overline{Z}$ is isotopic to $\overline{W}$. If W is a pair of pants, since $i(\partial W, \mathfrak{a}) = 0$ again, we also conclude that $\overline{Z}$ is isotopic to $\overline{W}$. □

Lemma 7.4 (Non annular components of $X - Z_{\mathfrak{a}}$). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$.*

1. *Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. Let W be a component of $X - Z_{\mathfrak{a}}$ with $\mathrm{cx}(W) \geq 1$. There is a minimal component G_i of G such that $\overline{W} = \mathrm{Supp}(G_i)$ in homotopy sense. Conversely, the support of any arrational component of G is isotopic to the closure of a component W of $X - Z_{\mathfrak{a}}$ with $\mathrm{cx}(W) \geq 1$.*

2. *For $[G_1], [G_2] \in \mathcal{AF}([\mathfrak{a}])$, any arrational component of G_1 is topologically equivalent to that of G_2.*

Proof. (1) Let W be a component of $X - Z_{\mathfrak{a}}$ with $\mathrm{cx}(W) \geq 1$. By definition, W is also a component of $X - Z^0_{\mathfrak{a}}$. From Proposition 7.2, we have $i(\alpha, G) \neq 0$ for every curve α which is non-peripheral in W. From Proposition 7.3 essential curves and peripheral curves of G are deformed into $Z^0_{\mathfrak{a}}$. Hence α intersects some minimal component G_i of G.

We check that $\overline{W} = \mathrm{Supp}(G_i)$ in homotopy sense. We first check $\mathrm{Supp}(G_i) \subset W$. Otherwise, there is a component γ of $\partial W \subset \partial Z^0_{\mathfrak{a}}$ which intersects non-trivially to $\mathrm{Supp}(G_i)$. This means that $i(\gamma, G) \geq i(\gamma, G_i) \neq 0$ and hence $i(\gamma, \mathfrak{a}) \neq 0$ from Proposition 7.2, which is a contradiction. If a component γ of $\partial\,\mathrm{Supp}(G_i)$ is non-peripheral in W, γ cannot be deformed into $Z^0_{\mathfrak{a}}$ and hence $i(\gamma, \mathfrak{a}) \neq 0$. Therefore, $i(\gamma, G) \neq 0$, as we checked in the previous paragraph. Thus, we conclude that $\mathrm{Supp}(G_i) \hookrightarrow \overline{W}$ is a deformation retract.

Let G_i be a minimal component of G. Since any simple closed curve which is non-peripheral in $\mathrm{Supp}(G_i)$ satisfies $i(\alpha, G) = i(\alpha, G_i) \neq 0$, we have $i(\alpha, \mathfrak{a}) \neq 0$. Therefore, $\mathrm{Supp}(G_i)$ is disjoint from $Z^0_{\mathfrak{a}}$ (in homotopy sense). Let W be a component of $Z - Z^0_{\mathfrak{a}}$ with $\mathrm{Supp}(G_i) \subset W$ in homotopy sense. Since $i(\partial\,\mathrm{Supp}(G_i), G) = 0$, from Proposition 7.2, we can deduce that $\mathrm{Supp}(G_i)$ is isotopic to $\overline{W}$.

(2) Let H_1 be a minimal component of G_1. From (1) above, there is a minimal component H_2 of G_2 such that $\mathrm{Supp}(H_2) = \mathrm{Supp}(H_1)$. Since $i(H_1, H_2) \leq i(G_1, G_2) = 0$ from Lemma 7.1. Hence H_1 is topologically equivalent to H_2 (e.g. Theorem 1.1 in [**42**]). □

Lemma 7.5 (Annular components of $X - Z_{\mathfrak{a}}$). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$ and $[G] \in \mathcal{AF}([\mathfrak{a}])$. The core curve of any annular component of $X - Z_{\mathfrak{a}}$ is homotopic to an essential curve of G, and vice versa.*

Proof. Let W be an annular component of $X - Z_{\mathfrak{a}}$. Let Z_1 and Z_2 be components of $Z_{\mathfrak{a}}$ adjacent to W. Possibly $Z_1 = Z_2$. Suppose some Z_i is an annulus. Then, Z_i is also a component of $Z_{\mathfrak{a}}$. Since W is also an annulus, Z_i is absorbed into the regular neighborhood of ∂Z_j where $\{i, j\} = \{1, 2\}$. This contradicts to the minimality of $Z^0_{\mathfrak{a}}$, because each component of ∂Z_j is in $\mathcal{N}(\mathfrak{a}) \cap \mathcal{S}$ and the regular neighborhood of ∂Z_j is contained in $Z^0_{\mathfrak{a}}$. Hence, the core curve δ of W is not a peripheral curve of G from Lemma 7.3.

Since the core δ of W is non-peripheral in $Z_1 \cup W \cup Z_2$, we can take a curve $\beta \in \mathcal{S}$ such that $\beta \subset Z_1 \cup W \cup Z_2$ and $i(\delta, \beta) \neq 0$. If δ is not an essential curve of G, $i(\beta, G) = 0$ and hence $i(\beta, \mathfrak{a}) = 0$. Therefore, $Z_1 \cup W \cup Z_2$ is a non-annular component of $X - \mathrm{Supp}(G^\circ)$, since each component of $X - \mathrm{Supp}(G^\circ)$ is incompressible. This is a contradiction because W can be deformed into the outside of $Z_{\mathfrak{a}}$ (cf. Lemma 7.3).

Conversely, let δ be a essential curve of G. Let W_1 and W_2 be components of $X - \mathrm{Supp}(G^\circ)$ which are adjacent to the annular component N_δ of $\mathrm{Supp}(G^\circ)$ whose core is δ. Since neither W_1 nor W_2 is not annulus, from Lemma 7.3, each W_i is a component of $Z_{\mathfrak{a}}$. Therefore, N_δ is a component of $X - Z_{\mathfrak{a}}$. □

Lemma 7.6 (Annular component of $Z_{\mathfrak{a}}$). *Let $\mathfrak{a} \in \tilde{\partial}_{GM} - \{0\}$ and $[G] \in \mathcal{AF}([\mathfrak{a}])$. The core curve of any annular component of $Z_{\mathfrak{a}}$ is homotopic to a component of the boundary of the support of a minimal component of G.*

Proof. Let Z be an annular component of $Z_{\mathfrak{a}}$. Then, Z is also a component of $Z^0_{\mathfrak{a}}$. Hence the core curve δ of Z is not peripheral in X. Let $\partial Z = \gamma_1 \cup \gamma_2$ Let W_1 and W_2 be the closures of components of $X - Z_{\mathfrak{a}}$ such that $\gamma_i \subset \partial W_i$ $(i = 1, 2)$. Possibly $W_1 = W_2$. Since each W_i is not a pair of pants, if some W_i is an annulus, Z is absorbed in the component of $Z_{\mathfrak{a}}$ which is on the opposite side of W_i to Z. This contradicts to the minimality of $Z^0_{\mathfrak{a}}$. Hence, each W_i satisfies $\mathrm{cx}(W_i) \geq 1$, from Lemma 7.4, we conclude that δ is homotopic to a component of the boundary of some minimal component of G. □

7.4. Intersection number lemma.

The following intersection number lemma encodes the intersection number for two points in $\partial_{GM}\mathcal{T}$ to that of those associated foliations up to multiple by positive constant.

Lemma 7.7 (Intersection number lemma). *Let $\mathfrak{a}, \mathfrak{b} \in \tilde{\partial}_{GM} - \{0\}$ and $[G] \in \mathcal{AF}([\mathfrak{a}])$ and $[H] \in \mathcal{AF}([\mathfrak{b}])$. Then, there is an $[F_\infty] \in \overline{\mathcal{AF}([\mathfrak{a}])}$ in $\mathcal{PMF}$ such that*

$$D_0\, i_{x_0}([G], [H]) \leq i_{x_0}([\mathfrak{a}], [\mathfrak{b}]) \leq i_{x_0}([F_\infty], [\mathfrak{b}]) \tag{50}$$

where $D_0 = e^{-d_T(x_0,x) - d_T(x_0,y)}$ and x and y are base points for the associated foliations $[G]$ and $[H]$ respectively.

Proof. By definition, there are $\{[G_n]\}_{n\in\mathbb{N}}$, $\{[H_n]\}_{n\in\mathbb{N}} \subset \mathcal{PMF}$ and $t_n, s_n > 0$ such that

- $R_{G_n,x}(t_n) \to [\mathfrak{a}]$ and $R_{H_n,y}(s_n) \to [\mathfrak{b}]$ as $n \to \infty$, and
- $G_n \to G$ and $H_n \to H$ as $n \to \infty$.

For simplicity, let $x_n = R_{G_n,x}(t_n)$ and $y_n = R_{H_n,y}(s_n)$.

Since $d_T(x_0, x_n) \leq t_n + d_T(x_0, x)$ and $d_T(x_0, y_n) \leq s_n + d_T(x_0, y)$, from Proposition 6.2, we deduce

$$
\begin{aligned}
i_{x_0}(x_n, y_n) &= \exp(-2\langle x_n \,|\, y_n\rangle_{x_0}) \\
&= \exp(d_T(x_n, y_n) - d_T(x_0, x_n) - d_T(x_0, y_n)) \\
&\geq D_0 \exp(d_T(x_n, y_n)) e^{-t_n} e^{-s_n} \\
&= D_0 \exp(d_T(x_n, y_n)) \frac{\mathrm{Ext}_{x_n}(G_n)^{1/2}}{\mathrm{Ext}_{x_0}(G_n)^{1/2}} \frac{\mathrm{Ext}_{y_n}(H_n)^{1/2}}{\mathrm{Ext}_{x_0}(H_n)^{1/2}} \\
&= D_0 \frac{\mathrm{Ext}_{x_n}(G_n)^{1/2}}{\mathrm{Ext}_{x_0}(G_n)^{1/2}} \frac{\exp(d_T(x_n, y_n))\mathrm{Ext}_{y_n}(H_n)^{1/2}}{\mathrm{Ext}_{x_0}(H_n)^{1/2}} \\
&\geq D_0 \frac{\mathrm{Ext}_{x_n}(G_n)^{1/2}}{\mathrm{Ext}_{x_0}(G_n)^{1/2}} \frac{\mathrm{Ext}_{x_n}(H_n)^{1/2}}{\mathrm{Ext}_{x_0}(H_n)^{1/2}} \\
&\geq D_0 \frac{i(H_n, G_n)}{\mathrm{Ext}_{x_0}(G_n)^{1/2}\mathrm{Ext}_{x_0}(H_n)^{1/2}} = D_0 i_{x_0}([G_n], [H_n]).
\end{aligned}
$$

By letting $n \to \infty$, we obtain the left-hand side of (50).

Fix $n \in \mathbb{N}$. Let $F_{m,n} \in \mathcal{MF}_1$ with $x_m = R_{F_{m,n}, y_n}(u_{m,n})$, where $u_{m,n} = d_T(x_n, y_m)$. Notice that

$$\mathrm{Ext}_{x_m}(F_{m,n}) = e^{-2u_{m,n}} \mathrm{Ext}_{y_n}(F_{m,n}). \tag{51}$$

By taking a subsequence (or by the diagonal argument), we may assume that $F_{m,n} \to F_{\infty,n} \in \mathcal{MF}_1$ as $m \to \infty$ for each n, and $F_{\infty,n}$ converges to $F_\infty \in \mathcal{MF}_1$. Since $x_m \to [\mathfrak{a}]$, $[F_{\infty,n}]$ is an associated foliation for $\mathfrak{a}$ with base point y_n. Therefore, the limit $[F_\infty]$ is contained in the closure of $\mathcal{AF}([\mathfrak{a}])$ in $\mathcal{PMF}$.

Since $F_{m,n} \in \mathcal{MF}_1$, from Theorem 6.1, (39), (44) and (51), we deduce

$$
\begin{aligned}
i_{x_0}(y_n, x_m) &= \exp(-2\langle y_n \,|\, x_m\rangle_{x_0}) \\
&= \exp(u_{m,n} - d_T(x_0, x_m) - d_T(x_0, y_n)) \\
&= \exp(-d_T(x_0, y_n)) \frac{\mathrm{Ext}_{y_n}(F_{m,n})^{1/2}}{\exp(d_T(x_0, x_m))\mathrm{Ext}_{x_m}(F_{m,n})^{1/2}} \\
&\leq \exp(-d_T(x_0, y_n)) \mathrm{Ext}_{y_n}(F_{m,n})^{1/2} \\
&= i(\Psi_{GM}(y_n), F_{m,n}) = i(\Psi_{GM}(y_n), \Psi_{GM}(F_{m,n})) \\
&= i_{x_0}(y_n, [F_{m,n}]).
\end{aligned}
$$

Letting $m \to \infty$, we conclude

$$i_{x_0}(y_n, [\mathfrak{a}]) \leq i_{x_0}(y_n, [F_{\infty,n}]). \tag{52}$$

Thus, if $n \to \infty$ in (52), we obtain what we wanted. □

7.5. Proof of structure theorem.

We first check the following.

Lemma 7.8. *Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. Let $F \in \mathcal{MF}$ be a measured foliation which is topologically equivalent to a minimal component of G. Then, $i(F, \mathfrak{a}) = 0$.*

PROOF. Take $x \in \mathcal{T}$, $[G_n] \in \mathcal{PMF}$, and $t_n > 0$ such that $R_{G_n,x}(t_n) \to [\mathfrak{a}]$ and $G_n \to G$ as $n \to \infty$. Let $y_n = R_{G_n,x}(t_n)$. Let L_{F,y_n} be the geodesic current associated to the singular flat structure defined as $Q_n := J_{F,y_n}/\|J_{F,y_n}\|$ given by Duchin, Leininger and Rafi in [**8**].

Suppose on the contrary that $i(\mathfrak{a}, F) \neq 0$. Then, by Proposition 4 in [**34**], $\{Q_n\}_{n\in\mathbb{N}}$ is a stable sequence in the sense that the set of accumulation points of $\{e^{-t_n} L_{F,y_n}\}_{n\in\mathbb{N}}$ in the space of geodesic currents is contained in $\mathcal{MF} - \{0\}$ (as geodesic currents). In addition, any accumulation point $L_\infty \in \mathcal{MF} - \{0\}$ satisfies

$$i(L_\infty, F) = t_0 i(\mathfrak{a}, F) \neq 0, \tag{53}$$

$$i(L_\infty, H) \leq t_0 i(\mathfrak{a}, H) \tag{54}$$

for some $t_0 > 0$ and any $H \in \mathcal{MF}$ (see Proposition 5 in [**34**]).

Let G_0 be a minimal component of G which is topologically equivalent to F and X_0 be the support of G_0. From (54), $i(L_\infty, G) = 0$. Hence, if L_∞ has a component L_0 whose support intersects X_0, then L_0 is topologically equivalent to G_0 (cf. [**18**]). This means that $i(L_\infty, F) = 0$, which contradicts to (53). □

PROOF OF THEOREM 7.1. We are ready to prove Theorem 7.1. Let $[G] \in \mathcal{AF}([\mathfrak{a}])$. From (2) of Lemma 7.1, we need to show the converse $\mathcal{N}(\mathfrak{a}) \supset \mathcal{N}(G)$.

We first claim that $\mathcal{N}_{MF}(\mathfrak{a}) = \mathcal{N}_{MF}(G)$ for $[G] \in \mathcal{AF}([\mathfrak{a}])$. We decompose G as (27):

$$G = G'_1 + G'_2 + \cdots + G'_{m_1} + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3}.$$

Let $H \in \mathcal{N}_{MF}(G)$. Then, H can be decomposed as

$$H = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i \beta_i + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X_i} b_\gamma \gamma + F_0 \tag{55}$$

where $a_i, b_\gamma \geq 0$, H_i is a measured foliation topologically equivalent to G'_i (possibly $H_i = 0$), and F_0 is a measured foliation whose support is contained in the complement of $\mathrm{Supp}(G)$ (cf. [**18**]). From Proposition 7.5, the support of F_0 is contained in the vanishing surface $Z_\mathfrak{a}$. Therefore, $i(F_0, \mathfrak{a}) = 0$. Since any component of ∂X_i is deformed into $Z^0_\mathfrak{a}$, from Lemma 6.1 and Lemma 7.8, we have

$$i(H, \mathfrak{a}) \leq \sum_{i=1}^{m_1} i(H_i, \mathfrak{a}) + \sum_{i=1}^{m_2} a_i i(\beta_i, \mathfrak{a}) + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial X_i} b_\gamma i(\gamma, \mathfrak{a}) + i(F_0, \mathfrak{a}) = 0 \tag{56}$$

and hence $\mathcal{N}_{MF}(G) \subset \mathcal{N}_{MF}(\mathfrak{a})$.

Let $\mathfrak{b} \in \mathcal{N}(G)$ and take $\{y_n\}_{n=1}^\infty$ such that $y_n \to [\mathfrak{b}]$ as $n \to \infty$. Let $H_n \in \mathcal{MF}_1$, $s_n > 0$ such that $y_n = R_{H_n,x_0}(s_n)$. By taking a subsequence, we may assume that $H_n \to H_\infty$. Then, $[H_\infty] \in \mathcal{AF}([\mathfrak{b}])$. Let $[F_\infty] \in \overline{\mathcal{AF}([\mathfrak{a}])}$ as Lemma 7.7 for $[G]$, $[H_\infty]$, $\mathfrak{a}$ and $\mathfrak{b}$. To show that $\mathfrak{b} \in \mathcal{N}(\mathfrak{a})$, it suffices to show that $i(\mathfrak{b}, F_\infty) = 0$ from Lemma 7.7.

Since $\mathfrak{b} \in \mathcal{N}(G)$ and $\mathcal{N}_{MF}(H_\infty) = \mathcal{N}_{MF}(\mathfrak{b})$, we have $i(G, H_\infty) = 0$. Therefore, H_∞ is decomposed as

$$H_\infty = \sum_{i=1}^{m_1} H_i' + \sum_{i=1}^{m_2} a_i\beta_i + \sum_{i=1}^{m_1}\sum_{\gamma\subset\partial X_i} b_\gamma\gamma + H_0 \tag{57}$$

where H_i' is topologically equivalent to G_i', $a_i, b_\gamma \geq 0$, H_0 is a measured foliation whose support is contained in the complement of $\mathrm{Supp}(G)$. Since $[F_\infty] \in \overline{\mathcal{AF}([\mathfrak{a}])}$, from Theorem 7.2, F_∞ is decomposed as

$$F_\infty = \sum_{i=1}^{m_1} F_i' + \sum_{i=1}^{m_2} a_i\beta_i + \sum_{i=1}^{m_1}\sum_{\gamma\subset\partial X_i} b_\gamma\gamma, \tag{58}$$

where F_∞' is topologically equivalent to G_i' (possibly $F_i' = 0$) and $a_i, b_\gamma \geq 0$. From (57) and (58), we have $i(F_\infty, H_\infty) = 0$. Since $\mathcal{N}_{MF}(H_\infty) = \mathcal{N}_{MF}(\mathfrak{b})$ again, we conclude that $i(\mathfrak{b}, F_\infty) = 0$ as desired. □

7.6. Topological equivalence revisited.

Before closing this section, we notice the following expected property.

Corollary 7.1 (Topological equivalence and null sets). *For $G, H \in \mathcal{MF}$, the following are equivalent*:

1. *G° and H° are topologically equivalent*;
2. *$\mathcal{N}_{MF}(G) = \mathcal{N}_{MF}(H)$*;
3. *$\mathcal{N}(G) = \mathcal{N}(H)$.*

In particular, $\mathcal{N}(G) = \mathcal{N}(G^\circ)$ for any $G \in \mathcal{MF}$.

Proof. From Proposition 5.1, the conditions (1) and (2) are equivalent. Since $\mathcal{N}_{MF}(G) = \mathcal{N}(G)\cap\mathcal{MF}$, (2) follows from (3). Hence, we need to show that (1) implies (3). From the symmetry of the topological equivalence, it suffices to show that $\mathcal{N}(G) \subset \mathcal{N}(H)$.

Let $\mathfrak{a} \in \mathcal{N}(G)$ and $[F] \in \mathcal{AF}([\mathfrak{a}])$. Then, $i(G, F) = 0$ from Theorem 7.1. Since H° is topologically equivalent to G°, by Proposition 5.1, we have $i(H, F) = 0$. Hence, by applying Theorem 7.1 again, we have $i(H, \mathfrak{a}) = 0$ and $\mathfrak{a} \in \mathcal{N}(H)$. □

8. Action on the Reduced boundary.

Let S and S' be compact orientable surfaces of non-sporadic type. In this section, we study maps in $\mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$.

8.1. Null sets and accumulation sets.

For $p \in \mathrm{cl}_{GM}(\mathcal{T}(S))$, we define the *null set* for p by

$$\mathfrak{N}_S(p) = \{q \in \mathrm{cl}_{GM}(\mathcal{T}(S)) \mid i_{x_0}(p, q) = 0\}.$$

For $\boldsymbol{x} \in \mathrm{Sq}^\infty(\mathcal{T}(S))$, we define

$$\mathcal{ACM}_S(\boldsymbol{x}) = \bigcup\{\overline{\boldsymbol{z}} \cap \partial_{GM}\mathcal{T}(S) \mid \boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})\}, \tag{59}$$

where $\overline{\boldsymbol{z}}$ is the closure of $\boldsymbol{z}$ in $\mathrm{cl}_{GM}(\mathcal{T}(S))$. The following proposition follows from (42).

PROPOSITION 8.1. *Let $p, p^1, p^2 \in \partial_{GM}\mathcal{T}(S)$ and $\boldsymbol{x}, \boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{USq}(\mathcal{T}(S))$.*

1. *If $\boldsymbol{x}$ converges to p, $\mathfrak{N}_S(p) = \mathcal{ACM}_S(\boldsymbol{x})$.*
2. *Suppose each $\boldsymbol{x}^i$ converges to p^i for $i = 1, 2$. Then, $\mathfrak{N}_S(p^2) \subset \mathfrak{N}_S(p^1)$ if and only if $\mathrm{Vis}(\boldsymbol{x}^2) \subset \mathrm{Vis}(\boldsymbol{x}^1)$.*

PROPOSITION 8.2 (Structure of accumulation points). *Let $\boldsymbol{x} \in \mathrm{Sq}^\infty(\mathcal{T}(S))$. Then, there is $G \in \mathcal{MF}$ such that*

$$\mathcal{ACM}_S(\boldsymbol{x}) = \mathfrak{N}_S([G]).$$

Furthermore, the following are equivalent for $q \in \partial_{GM}\mathcal{T}(S)$:

1. *$q \in \mathfrak{N}_S([G])$;*
2. *for any $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$ and $[G_p] \in \mathcal{AF}(p)$, $q \in \mathfrak{N}_S([G_p])$;*
3. *for any $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$ $q \in \mathfrak{N}_S(p)$.*

PROOF. For $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$, fix $[G_p] \in \mathcal{AF}(p)$. From (42) and Theorem 7.1, $i(G_{p^1}, G_{p^2}) = 0$ for $p^1, p^2 \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$. Hence, we can find $G \in \mathcal{MF}$ such that

(1) for any $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$, ${G_p}^\circ$ is topologically equivalent to a subfoliation of G, and

(2) any component of G° is topologically equivalent to a component of some G_p, $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$.

We check that G satisfies the desired property. Let $q \in \mathcal{ACM}_S(\boldsymbol{x})$ be an accumulation point of $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$. Let $[H] \in \mathcal{AH}(q)$. Since $i(H, G_p) = 0$ for all $p \in \boldsymbol{x} \cap \partial_{GM}\mathcal{T}(S)$, from the condition (2) of G, we have $i(G, H) = 0$ and hence $\Psi_{GM}(q) \in \mathcal{N}(G)$ by Theorem 7.1. This means that $q \in \mathfrak{N}_S([G])$ and $\mathcal{ACM}_S(\boldsymbol{x}) \subset \mathfrak{N}_S([G])$.

Conversely, let $q \in \mathfrak{N}_S([G])$. Take a sequence $\boldsymbol{z}$ in X converging to q. By the condition (1) of G above, $\mathfrak{N}_S([G]) \subset \mathfrak{N}_S([G_p])$ for all $p \in \overline{\boldsymbol{x}} \cap \partial_{GM}\mathcal{T}(S)$. In other words, any subsequence of $\boldsymbol{x}$ contains a subsequence which is visually indistinguishable from $\boldsymbol{z}$. Therefore, we have $\boldsymbol{x} \in \mathrm{Vis}(\boldsymbol{z})$ and hence $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x})$.

The last statement follows from the construction of G and Theorem 7.1. □

PROPOSITION 8.3. *Let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathrm{Sq}^\infty(\mathcal{T}(S))$. The following are equivalent:*

(1) $\mathcal{ACM}_S(\boldsymbol{x}^1) \subset \mathcal{ACM}_S(\boldsymbol{x}^2)$;

(2) $\mathrm{Vis}(\boldsymbol{x}^1) \subset \mathrm{Vis}(\boldsymbol{x}^2)$.

PROOF. From the definition (59), the condition (2) implies (1).

Suppose the condition (1). Assume to the contrary that there is $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x}^1) \setminus \mathrm{Vis}(\boldsymbol{x}^2)$. Take subsequences $\boldsymbol{z}' = \{z'_n\}_{n \in \mathbb{N}}$ of $\boldsymbol{z}$ and ${\boldsymbol{x}'}^2 = \{x'_n\}_{n \in \mathbb{N}}$ of $\boldsymbol{x}^2$ such that

$$\langle x'_n \,|\, z'_n \rangle_{x_0} < M_1$$

for all $n \in \mathbb{N}$. Then, any $q' \in \overline{\boldsymbol{z}'} \cap \partial_{GM}\mathcal{T}(S)$ ($\subset \overline{\boldsymbol{z}} \cap \partial_{GM}\mathcal{T}(S)$) and $p' \in \overline{\boldsymbol{x}'^2} \cap \partial_{GM}\mathcal{T}(S)$ ($\subset \overline{\boldsymbol{x}^2} \cap \partial_{GM}\mathcal{T}(S)$) satisfy $i_{x_0}(p', q') \neq 0$ (cf. (42)). By Proposition 8.2, $q' \notin \mathcal{ACM}_S(\boldsymbol{x}^2)$. Since $\mathcal{ACM}_S(\boldsymbol{x}^1) \subset \mathcal{ACM}_S(\boldsymbol{x}^2)$ from the assumption, $q' \notin \mathcal{ACM}_S(\boldsymbol{x}^1)$. On the other hand, since $\boldsymbol{z} \in \mathrm{Vis}(\boldsymbol{x}^1)$, $q' \in \overline{\boldsymbol{z}} \cap \partial_{GM}\mathcal{T}(S) \subset \mathcal{ACM}_S(\boldsymbol{x}^1)$. This is a contradiction. □

8.2. Accumulation sets.

Let $\omega \in \mathrm{AC}(\mathcal{T}(S), \mathcal{T}(S'))$, For $p \in \mathrm{cl}_{GM}(\mathcal{T}(S))$, we define the *accumulation set* by

$$\mathcal{A}(\omega\colon p) = \{q \in \mathrm{cl}_{GM}(\mathcal{T}(S')) \mid \exists \{y_n\}_{n\in\mathbb{N}} \in \mathrm{Sq}^{\infty}(\mathcal{T}(S)) \text{ s.t. } y_n \to p \text{ and } \omega(y_n) \to q\}.$$

The following lemma will be applied for defining the extension to the reduced Gardiner–Masur closure in Section 8.4.

Lemma 8.1 (Null sets and accumulation points). *Let $\omega \in \mathrm{AC}_{\mathrm{as}}(\mathcal{T}(S), \mathcal{T}(S'))$. Let $p^1, p^2 \in \partial_{GM}\mathcal{T}(S)$ and $q^i \in \mathcal{A}(\omega\colon p_i)$ for $i = 1, 2$. If $\mathfrak{N}_S(p^2) \subset \mathfrak{N}_S(p^1)$, then $\mathfrak{N}_{S'}(q^2) \subset \mathfrak{N}_{S'}(q^1)$. Especially, $\mathfrak{N}_{S'}(q^2) = \mathfrak{N}_{S'}(q^1)$ for $p \in \partial_{GM}\mathcal{T}(S)$ and $q^1, q^2 \in \mathcal{A}(\omega\colon p)$.*

Proof. For $i = 1, 2$, let $\boldsymbol{x}^i$ be a sequence converging to p_i such that $\omega(\boldsymbol{x}^i)$ converges to q_i. From Proposition 8.1, the assumption $\mathfrak{N}_S(p_2) \subset \mathfrak{N}_S(p_1)$ implies $\mathrm{Vis}(\boldsymbol{x}^2) \subset \mathrm{Vis}(\boldsymbol{x}^1)$. By Propositions 2.4, we have $\mathrm{Vis}(\omega(\boldsymbol{x}^2)) \subset \mathrm{Vis}(\omega(\boldsymbol{x}^1))$. Therefore, by applying Proposition 8.1 again, we obtain $\mathfrak{N}_{S'}(q_2) \subset \mathfrak{N}_{S'}(q_1)$. □

8.3. Reduced Gardiner–Masur closure and boundary.

We say two points $p, q \in \mathrm{cl}_{GM}(\mathcal{T}(S))$ are equivalent if one of the following holds:

(1) $p, q \in \mathcal{T}(S)$ and $p = q$;

(2) $p, q \in \partial_{GM}\mathcal{T}(S)$ and $\mathfrak{N}_S(p) = \mathfrak{N}_S(q)$.

We denote by $[[p]]$ the equivalence class of $p \in \mathrm{cl}_{GM}(\mathcal{T}(S))$. We abbreviate the equivalence class $[[[G]]]$ of the projective class $[G] \in \mathcal{PMF} \subset \partial_{GM}\mathcal{T}(S)$ as $[[G]]$. We denote by $\mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S))$ the quotient of $\mathrm{cl}_{GM}(\mathcal{T}(S))$ under this equivalence relation. Let $\pi_{GM}\colon \mathrm{cl}_{GM}(\mathcal{T}(S)) \to \mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S))$ be the quotient map. We always identify $\pi_{GM}(\mathcal{T}(S))$ with $\mathcal{T}(S)$. We call $\mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S))$ the *reduced Gardiner–Masur closure* of $\mathcal{T}(S)$. From the definition, the space $\mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S))$ contains $\mathcal{T}(S)$ canonically. We call the complement

$$\partial^{\mathrm{red}}_{GM}\mathcal{T}(S) = \mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S)) - \mathcal{T}(S)$$

the *reduced Gardiner–Masur boundary* of $\mathcal{T}(S)$.

The reduced Gardiner–Masur closure is a variation of the reduced compactifications of Teichmüller space. See [**40**].

8.4. Boundary extension.

For $\omega \in \mathrm{AC}_{\mathrm{as}}(\mathcal{T}(S), \mathcal{T}(S'))$, we define the *boundary extension* $\partial_\infty(\omega)\colon \mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S)) \to \mathrm{cl}^{\mathrm{red}}_{GM}(\mathcal{T}(S'))$ by

$$\partial_\infty(\omega)([[p]]) = \begin{cases} [[\omega(p)]] & (p \in \mathcal{T}(S)), \\ [[q]] & (q \in \mathcal{A}(\omega\colon p) \text{ if } p \in \partial_{GM}\mathcal{T}(S)). \end{cases} \tag{60}$$

From Lemma 8.1, the extension $\partial_\infty(\omega)$ is well-defined.

LEMMA 8.2 (Composition). *For* $\omega_1, \omega_2 \in \mathrm{AC}_{\mathrm{as}}(\mathcal{T}(S), \mathcal{T}(S'))$*, the extensions satisfy*

$$\partial_\infty(\omega_1 \circ \omega_2) = \partial_\infty(\omega_1) \circ \partial_\infty(\omega_2)$$

on $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$.

PROOF. Let $[[p]] \in \partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$. Take $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{T}(S)$ such that $x_n \to p$ and $\omega_1 \circ \omega_2(x_n) \to p' \in \partial_{GM}\mathcal{T}(S')$. By definition,

$$\partial_\infty(\omega_1 \circ \omega_2)([[p]]) = [[p']].$$

On the other hand, from Proposition 8.1, we may assume that $\omega_2(\boldsymbol{x})$ converges to $q \in \mathcal{A}(\omega_2 \colon p)$. From the definition, we have $\partial_\infty(\omega_2)([[p]]) = [[q]]$. Since $\omega_1 \circ \omega_2(\boldsymbol{x}) = \omega_1(\omega_2(\boldsymbol{x}))$, $p' \in \mathcal{A}(\omega_1 \colon q)$ and hence

$$[[p']] = \partial_\infty(\omega_1)([[q]]) = \partial_\infty(\omega_1) \circ \partial_\infty(\omega_2)([[p]]). \qquad \square$$

LEMMA 8.3 (Close at infinity). *Let* $\omega_1, \omega_2 \in \mathrm{AC}_{\mathrm{as}}(\mathcal{T}(S), \mathcal{T}(S'))$*. If* ω_1 *is close to* ω_2 *at infinity,* $\partial_\infty(\omega_1) = \partial_\infty(\omega_2)$ *on* $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$.

PROOF. Let $p \in \partial_{GM}\mathcal{T}(S)$. Take $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in \mathrm{USq}(\mathcal{T}(S))$ with $x_n \to p$ as $n \to \infty$ such that $\omega_i(x_n) \to q^i \in \mathcal{A}(\omega_i \colon p)$ for $i = 1, 2$. Since $\mathrm{Vis}(\omega_1(\boldsymbol{x})) = \mathrm{Vis}(\omega_2(\boldsymbol{x}))$, by Proposition 8.1,

$$\mathfrak{N}_{S'}(q^1) = \mathcal{ACM}_{S'}(\omega_1(\boldsymbol{x})) = \mathcal{ACM}_{S'}(\omega_2(\boldsymbol{x})) = \mathfrak{N}_{S'}(q^2).$$

Hence

$$\partial_\infty(\omega_1)([[p]]) = [[q^1]] = [[q^2]] = \partial_\infty(\omega_2)([[p]])$$

and $\partial_\infty(\omega_1) = \partial_\infty(\omega_2)$ on $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$. $\square$

COROLLARY 8.1 (Inverse). *Let* $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$ *and* ω' *be an asymptotic quasi-inverse of* ω*. Then,* $\partial_\infty(\omega') \circ \partial_\infty(\omega)$ *and* $\partial_\infty(\omega) \circ \partial_\infty(\omega')$ *are identity mappings on* $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$ *and* $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S')$*, respectively.*

9. Rigidity of asymptotically conservative mappings.

9.1. Heights of reduced boundary points.

An ordered sequence $\{[[p_k]]\}_{k=1}^m$ in $\partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$ is said to be an *adherence tower* starting at $[[p_1]]$ if

$$\mathfrak{N}_S(p_1) \supsetneqq \mathfrak{N}_S(p_2) \supsetneqq \cdots \supsetneqq \mathfrak{N}_S(p_m).$$

The adherence tower is named with referring Ohshika's paper [**39**]. See also Papadopoulos's paper [**38**]. We call the number m the *length* of the adherence tower. Let $[[p]] \in \partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$. We define the *height* $\mathrm{ht}([[p]])$ of $[[p]]$ by

$$\mathrm{ht}([[p]]) = \sup\{\text{lengths of adherence towers starting } [[p]]\}.$$

For a measured foliation G, we set $\{X_i\}_{i=1}^{m_1}$ be the supports of the minimal components of G. We define the complexity of G by

$$\xi_0(G) = \left(-\sum_{i=1}^{m_1} \mathrm{cx}(X_i), {}^{\#}\{\text{essential curves in } G\}\right) \in \mathbb{Z} \times \mathbb{Z}. \tag{61}$$

(cf. Theorem 1 in [**39**]).

Lemma 9.1 (Heights of boundary points). *The height of any* $[[p]] \in \partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$ *is at most* $\mathrm{cx}(S)$. *The equality* $\mathrm{ht}([[p]]) = \mathrm{cx}(S)$ *holds if and only if the support of any* $[G] \in \mathcal{AF}(p)$ *is a simple closed curve.*

Proof. We first discuss the associated foliations of points in an adherence tower of length two. Let $[[p_1]], [[p_2]] \in \partial_{GM}^{\mathrm{red}}\mathcal{T}(S)$. Let $[G_i] \in \mathcal{AF}(p_i)$ for $i = 1, 2$. Suppose that $\{[[p_1]], [[p_2]]\}$ is an adherence tower. From the definition, $\mathcal{N}(\Psi_{GM}(p_1)) \supsetneqq \mathcal{N}(\Psi_{GM}(p_2))$. From Corollary 7.1, we see

$$\mathcal{N}_{MF}({G_1}^{\circ}) = \mathcal{N}_{MF}(\Psi_{GM}(p_1)) \supsetneqq \mathcal{N}_{MF}(\Psi_{GM}(p_2)) = \mathcal{N}_{MF}({G_2}^{\circ}).$$

We decompose ${G_1}^{\circ}$ as in (27):

$${G_1}^{\circ} = \sum_{i=1}^{m_1} G_i' + \sum_{i=1}^{m_2} \beta_i \tag{62}$$

where G_i' is a minimal component, and β_i is a (weighted) essential curve of G_1. Since $G_2 \in \mathcal{N}_{MF}(G_1)$, the decomposition of ${G_2}^{\circ}$ is represented as

$${G_2}^{\circ} = \sum_{i=1}^{m_1} H_i' + \sum_{i=1}^{m_2} a_i\beta_i + G_3 \tag{63}$$

where H_i' is topologically equivalent to G_i', $a_i \geq 0$ and the support of G_3 is contained in the complement of the support of ${G_1}^{\circ}$ (at this moment, ${G_2}^{\circ}$ may contain curves homotopic to boundary components of arrational components of G_1 as essential curves). Since $\mathcal{N}_{MF}(G_2) \subset \mathcal{N}_{MF}(G_1)$, $H_i' \neq 0$ and $a_i \neq 0$. Moreover, from the assumption $\mathcal{N}_{MF}(G_2) \neq \mathcal{N}_{MF}(G_1)$ implies that $G_3 \neq 0$. Therefore, from (62) and (63), we have

$$\xi_0(G_1) < \xi_0(G_2)$$

in the lexicographical order in $\mathbb{Z}\times\mathbb{Z}$, since G_3 in (63) contains either a minimal component or an essential curve of G_2.

Let us return to the proof of the lemma. Let $\{[[p_i]]\}_{i=1}^{m}$ be an adherence tower of length m. Let $[G_i] \in \mathcal{AF}(p_i)$. From the above argument, we have

$$\xi_0(G_1) < \xi_0(G_2) < \cdots < \xi_0(G_m). \tag{64}$$

Since the number of essential curves is at most $\mathrm{cx}(S)$ and the sum of the first and second coordinates of $\xi_0(G)$ is at most $\mathrm{cx}(S)$ minus the number of boundary components of minimal foliations of G which are non-periperal in S, we have $m \le \mathrm{cx}(S)$. In addition, if $m = \mathrm{cx}(S)$, each G_i consists of essential curves. Hence, in this case, the adherence tower starts with a simple closed curve. □

9.2. Induced isomorphism.

Let $\mathbb{X}_0(S)$ be the 0-skeleton of $\mathbb{X}(S)$. We identify each vertex of $\mathbb{X}_0(S)$ with its projective class in $\partial_{GM}\mathcal{T}(S)$.

Theorem 9.1 (Induced isomorphism). *Let S and S' be compact orientable surfaces of non-sporadic type. For $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$, there is a simplicial isomorphism $h_\omega \colon \mathbb{X}(S) \to \mathbb{X}(S)$ such that for any $\alpha \in \mathbb{X}_0(S)$, and any sequence $\{x_n\}_n \subset \mathcal{T}(S)$ with $x_n \to [\alpha]$, we have $\omega(x_n) \to [h_\omega(\alpha)]$. Furthermore, When ω and ω' are close at infinity, $h_\omega = h_{\omega'}$.*

Proof. Let $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S), \mathcal{T}(S'))$ and $\alpha \in \mathbb{X}_0(S)$. From Lemma 9.1, there is an adherence tower $\{[[p_i]]\}_{i=1}^{\mathrm{cx}(S)}$ with $[[p_1]] = [[\alpha]]$. From Lemma 8.1, $\{\partial_\infty(\omega)([[p_i]])\}_{i=1}^{\mathrm{cx}(S)}$ is also an adherence tower starting $\partial_\infty(\omega)([[p_1]]) = \partial_\infty(\omega)([[\alpha]])$. Applying the above argument for asymptotic quasi-inverse of ω, we see that the adherence tower $\{\partial_\infty(\omega)([[p_i]])\}_{i=1}^{\mathrm{cx}(S)}$ has the maximal height. From Lemma 9.1 and Corollary 8.1, we obtain a bijection $h_\omega \colon \mathbb{X}_0(S) \to \mathbb{X}_0(S')$ such that

$$\partial_\infty(\omega)([[\alpha]]) = [[h_\omega(\alpha)]]. \tag{65}$$

Let $\alpha, \beta \in \mathbb{X}_0(S)$ with $i(\alpha, \beta) = 0$. Then, $G = \alpha + \beta \in \mathcal{MF}$ and $\mathcal{N}(\alpha) \cap \mathcal{N}(\beta) \supset \mathcal{N}(G)$. Therefore, $\{[[\alpha]], [[G]]\}$ and $\{[[\beta]], [[G]]\}$ are adherence towers. From Lemma 8.1, $\{\partial_\infty(\omega)([[\alpha]]), \partial_\infty(\omega)([[G]])\}$ and $\{\partial_\infty(\omega)([[\beta]]), \partial_\infty(\omega)([[G]])\}$ are also adherence towers. From Theorem 7.1, there is an $H \in \mathcal{MF}$ such that $\partial_\infty(\omega)([[G]]) = [[H]]$. Since h_ω is bijective, $h_\omega(\alpha)$ and $h_\omega(\beta)$ represent different components of H. Therefore, $i(h_\omega(\alpha), h_\omega(\beta)) = 0$. This means that h_ω extends a simplicial isomorphism from $\mathbb{X}(S)$ to $\mathbb{X}(S')$. From Lemma 8.3, one can easily see that $h_{\omega'} = h_\omega$ when ω' is close to ω at infinity.

Let $\boldsymbol{x} = \{x_n\}_n$ be a sequence in $\mathcal{T}(S)$ converging to a simple closed curve $[\alpha] \in \mathrm{cl}_{GM}(\mathcal{T}(S))$. By (60) and (65), any accumulation point $q \in \partial_{GM}\mathcal{T}(S')$ of a sequence $\omega(\boldsymbol{x})$ satisfies $\mathfrak{N}_{S'}(q) = \mathfrak{N}_{S'}([h_\omega(\alpha)])$ from Lemma 8.1. Hence q satisfies $i_{\omega(x_0)}(F, q) = 0$ for all $F \in \mathcal{N}_{MF}(h_\omega(\alpha))$ $(\subset \mathcal{MF}(S'))$. From Theorem 3 in [**34**], we conclude that $q = [h_\omega(\alpha)]$ in $\partial_{GM}\mathcal{T}(S')$. This means that $\omega(\boldsymbol{x})$ converges to $[h_\omega(\alpha)]$ in $\mathrm{cl}_{GM}(\mathcal{T}(S'))$. □

9.3. Rigidity theorem.

9.3.1. Actions of extended mapping class group.

The *extended mapping class group* $\mathrm{MCG}^*(S)$ of S is the group of all isotopy classes of homeomorphisms on S. The extended mapping class group $\mathrm{MCG}^*(S)$ acts on $\mathcal{T}(S)$ isometrically by

$$\mathcal{T}(S) \ni y = (Y, f) \mapsto [h]_*(y) = (Y, f \circ h^{-1}) \in \mathcal{T}(S)$$

for $[h] \in \mathrm{MCG}^*(S)$. Hence, we have a group homomorphism

$$\mathcal{I}_0 \colon \mathrm{MCG}^*(S) \ni [h] \to [h]_* \in \mathrm{Isom}(\mathcal{T}(S)), \tag{66}$$

where $\mathrm{Isom}(\mathcal{T}(S))$ is the group of all isometries of $\mathcal{T}(S)$.

Let $\mathbb{X}(S)$ be the complex of curves of S and $\mathrm{Aut}(\mathbb{X}(S))$ be the simplicial automorphisms on $\mathbb{X}(S)$. Since $\mathrm{MCG}^*(S)$ acts on $\mathbb{X}(S)$ canonically, we have a (group) homomorphism

$$\mathcal{J} \colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathbb{X}(S)). \tag{67}$$

It is known that $\mathcal{J}$ is an isomorphism if S is neither a torus with two holes nor a closed surface of genus 2, and an epimorphism if S is not a torus with two holes (cf. Ivanov [**19**], Korkmaz [**22**] and Luo [**26**]).

The action of any isometry on $\mathcal{T}(S)$ extends homeomorphically to the Gardiner–Masur boundary (cf. [**25**]). We can observe that the extension of the action leaves $\mathcal{S} \subset \partial_{GM}\mathcal{T}(S)$ invariant, and it induces a canonical homomorphism $\mathcal{J}_1 \colon \mathrm{Isom}(\mathcal{T}(S)) \to \mathrm{Aut}(\mathbb{X}(S))$ such that the diagram

$$\begin{array}{ccc} \mathrm{MCG}^*(S) & \xrightarrow{\mathcal{I}_0} & \mathrm{Isom}(\mathcal{T}(S)) \\ & \searrow^{\mathcal{J}} & \downarrow{\scriptstyle \mathcal{J}_1} \\ & & \mathrm{Aut}(\mathbb{X}(S)) \end{array}$$

is commutative (cf. [**36**]). The homomorphism $\mathcal{J}_1$ is an isomorphism for any S with $\mathrm{cx}(S) \geq 2$ (cf. [**19**]). The reason why (67) is not surjective when S is a torus with two holes is that there is no homeomorphism on S which sends a non-null-homologous curve to a null-homologous curve, while each curve on S' is null-homologous. Thus, in any case, the homomorphism $\mathcal{J}_1$ is surjective (cf. [**26**]).

9.3.2. Rigidity theorem.

Recall that any isometry is an invertible asymptotically conservative mapping. Hence, we have a monoid monomorphism

$$\mathcal{I} \colon \mathrm{Isom}(\mathcal{T}(S)) \hookrightarrow \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S)).$$

Our rigidity theorem is given as follows.

Theorem 9.2 (Rigidity theorem). *There is a monoid epimorphism*

$$\Xi \colon \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S)) \to \mathrm{Aut}(\mathbb{X}(S))$$

with the following properties:

(1) *If* $\omega' \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$ *is an asymptotic quasi-inverse of* $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$*,* $\Xi(\omega') = \Xi(\omega)^{-1}$;

(2) $\mathcal{J}_1 = \Xi \circ \mathcal{I}$ *as monoid homomorphisms.*

In addition, Ξ *descends to a group isomorphism*

$$\mathfrak{AC}(\mathcal{T}(S)) \to \mathrm{Aut}(\mathbb{X}(S)). \tag{68}$$

which satisfies the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{MCG}^*(S) & \xrightarrow{\mathcal{I}_0} & \mathrm{Isom}(\mathcal{T}(S)) & \xrightarrow{\mathcal{I}} & \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S)) & & \\ & & & \searrow^{\text{group iso}} & \downarrow{\scriptstyle proj} & \searrow^{\Xi} & \\ & & & & \mathfrak{AC}(\mathcal{T}(S)) & \xrightarrow{\text{group iso}} & \mathrm{Aut}(\mathbb{X}(S)). \end{array}$$

Proof. When S is a torus with two holes, the quotient map $S \to S'$ by the hyperelliptic action induces an isometry between the Teichmüller spaces of S and S' and an isomorphism between $\mathbb{X}(S)$ and $\mathbb{X}(S')$, where S' is a sphere with five holes (cf. [**11**] and [**26**]). Hence, we may assume that S is not a torus with two holes. For $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$, we take $h_\omega \in \mathrm{Aut}(\mathbb{X}(S))$ as Theorem 9.1. Define a homomorphism Ξ by $\Xi(\omega) = h_\omega$. Theorem 9.1 asserts that Ξ satisfies the condition (1) in the statement and descends to a homomorphism

$$\mathfrak{AC}(\mathcal{T}(S)) \ni [\omega] \mapsto \Xi(\omega) = h_\omega \in \mathrm{Aut}(\mathbb{X}(S)). \tag{69}$$

We next check the condition (2) in the statement. Since $\omega \in \mathrm{Isom}(\mathcal{T}(S))$ preserves $\mathcal{S}$ in $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}(S)$, from the definition of h_ω, for any $\alpha \in \mathcal{S}$, $\Xi(\omega)(\alpha)$ coincides with $\omega(\alpha)$ (cf. Section 9 in [**36**]). This means that $\mathcal{J}(\omega) = \Xi \circ \mathcal{I}(\omega)$.

We here check (68) is an epimorphism. Since S is not a torus with two holes, $\mathcal{J}$ is an epimorphism, and so are Ξ and (69). The injectivity of (68) (or (69)) is proven in the next section. □

9.4. Injectivity of homomorphism.

In this section, we shall show that the epimorphism (68) is an isomorphism. We first check the following.

Proposition 9.1. *Suppose that* S *is not a torus with two holes. For* $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$, *there is a homeomorphism* f_ω *of* S *with the following property*: *For any* $p \in \partial_{GM}\mathcal{T}(S)$, $[G] \in \mathcal{AF}(p)$ *and* $q \in \mathcal{A}(\omega\colon p)$, *we have* $\mathfrak{N}_S(q) = \mathfrak{N}_S([f_\omega(G)])$.

Proof. From the assumption and Theorem 9.2, there is a homeomorphism f_ω of S such that $h_\omega(\alpha) = f_\omega(\alpha)$. From Theorem 7.1, if we take $[H] \in \mathcal{AF}(q)$, then

$$\mathfrak{N}_S(q) = \mathfrak{N}_S([H]).$$

From Theorem 9.1, for any $\alpha \in \mathcal{S}$, $i(G, \alpha) = 0$ if and only if $[f_\omega(\alpha)] = [h_\omega(\alpha)] \in \mathfrak{N}_S(q)$. Hence, we deduce that

$$\mathfrak{N}_S([H]) \cap \mathcal{S} = \mathfrak{N}_S(q) \cap \mathcal{S} = \mathfrak{N}_S([f_\omega(G)]) \cap \mathcal{S} \tag{70}$$

where $\mathcal{S}$ stands for a subset of $\partial_{GM}\mathcal{T}(S)$ in (70). Therefore, the support of H° coincides with the support of $f_\omega(G)^\circ$. In particular, any essential curve of H is also that of $f_\omega(G)$, and vice versa. As (27), we decompose G as

$$G = G_1 + G_2 + \cdots + G_{m_1} + \beta_1 + \cdots + \beta_{m_2} + \gamma_1 + \cdots + \gamma_{m_3}.$$

Let X_i be the support of a minimal component G_i of G.

It is known that $\mathrm{cl}_{GM}(\mathcal{T}(S))$ is metrizable. For instance

$$d_\infty(p^1, p^2) = \sup_{p \in \partial_{GM}\mathcal{T}(S)} \left| i_{x_0}(p^1, p) - i_{x_0}(p^2, p) \right| \tag{71}$$

is a metric on $\mathrm{cl}_{GM}(\mathcal{T}(S))$ since $\mathcal{S} \subset \partial_{GM}\mathcal{T}(S)$ (cf. Theorem 1.2 in [**33**]).

Fix $i = 1, \ldots, k$. Take a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\alpha_n \subset X_i$ and

$$d_\infty([\alpha_n], [G_i]) < 1/n.$$

Since f_ω is a homeomorphism, $[f_\omega(\alpha_n)]$ tends to $[f_\omega(G_i)]$ in $\mathcal{PMF}$ (and hence in $\mathrm{cl}_{GM}(\mathcal{T}(S))$) as $n \to \infty$. By taking a subsequence, we may assume that

$$d_\infty([f_\omega(\alpha_n)], [f_\omega(G_i)]) < 1/n$$

for all $n \in \mathbb{N}$.

Let $\boldsymbol{x}^n = \{x^n_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathcal{T}(S)$ converging to $[\alpha_n]$ in $\mathrm{cl}_{GM}(\mathcal{T}(S))$. Since $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$, by Theorem 9.1, $\omega(\boldsymbol{x}^n)$ converges to $[f_\omega(\alpha_n)]$ in $\mathrm{cl}_{GM}(\mathcal{T}(S))$ for all n. By applying the diagonal argument and taking a subsequence if necessary, we can take $m(n) \in \mathbb{N}$ such that if we put $z_n = x^n_{m(n)}$ and $\boldsymbol{z} = \{z_n\}_{n\in\mathbb{N}}$, then

$$\max\{d_\infty(z_n, [G_i]), d_\infty(\omega(z_n), [f_\omega(\alpha_n)])\} < 2/n \tag{72}$$

in $\mathrm{cl}_{GM}(\mathcal{T}(S))$. Since f_ω is a homeomorphism of S, $[f_\omega(\alpha_n)]$ tends to $[f_\omega(G_i)]$ in $\mathcal{PMF}$ and hence in $\mathrm{cl}_{GM}(\mathcal{T}(S))$. From (72), we have

$$d_\infty(\omega(z_n), f_\omega([G_i])) \le d_\infty(\omega(z_n), f_\omega([\alpha_n])) + d_\infty(f_\omega([\alpha_n]), [f_\omega(G_i)]) \to 0$$

as $n \to \infty$. Therefore $\omega(\boldsymbol{z})$ converges to $[f_\omega(G_i)]$. This means that $[f_\omega(G_i)] \in \mathcal{A}(\omega\colon [G_i])$. Since G_i is a minimal component of G, $\mathfrak{N}_S(p) = \mathfrak{N}_S([G]) \subset \mathfrak{N}_S([G_i])$. Therefore, by Lemma 8.1, we conclude

$$\mathfrak{N}_S([H]) = \mathfrak{N}_S(q) \subset \mathfrak{N}_S([f_\omega(G_i)])$$

since $q \in \mathcal{A}(\omega\colon p)$. Therefore, H contains a minimal component H_i which is topologically equivalent to $f_\omega(G_i)$. Since the support of H° coincides with that of G°, minimal components of H are contained in $\bigcup f_\omega(X_i)$. Hence, the normal form of H should be

$$H = \sum_{i=1}^{m_1} H_i + \sum_{i=1}^{m_2} a_i f_\omega(\beta_i) + \sum_{i=1}^{m_1} \sum_{\gamma \subset \partial f_\omega(X_i)} b_\gamma \gamma$$

where $a_i > 0$ and $b_\gamma \geq 0$. Thus, H° is topologically equivalent to $f_\omega(G)^\circ$. Hence by Corollary 7.1, we deduce

$$\mathfrak{N}_S([f_\omega(G)]) = \mathfrak{N}_S([H]) = \mathfrak{N}_S(q),$$

which is what we desired. □

PROPOSITION 9.2 (Induced isometry). *For any $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$, there is a unique isometry ξ_ω on $\mathcal{T}(S)$ which is close to ω at infinity.*

PROOF. The case where S is a torus with two holes follows from the fact that the Teichmüller space of S is isometric to the Teichmüller space of a sphere with five holes. Hence, we may suppose that S is not a torus with two holes.

Take f_ω as in Proposition 9.1. Since f_ω is a homeomorphism of S, f_ω induces an isometry ξ_ω on $\mathcal{T}(S)$. When S is a closed surface of genus 2, there is an ambiguity of the choice of f_ω which is caused by the hyperelliptic involution. However, the isometry ξ_ω is independent of the choice.

Let $\boldsymbol{x}^1$, $\boldsymbol{x}^2 \in \mathrm{Sq}^\infty(\mathcal{T}(S))$ satisfying $\mathrm{Vis}(\boldsymbol{x}^1) = \mathrm{Vis}(\boldsymbol{x}^2)$. From Proposition 8.2, there are G, H_1, $H_2 \in \mathcal{MF}$ such that

$$\begin{aligned}
\mathfrak{N}_S([G]) &= \mathcal{ACM}_S(\boldsymbol{x}^1) = \mathcal{ACM}_S(\boldsymbol{x}^2),\\
\mathfrak{N}_S([H_1]) &= \mathcal{ACM}_S(\omega(\boldsymbol{x}^1)),\\
\mathfrak{N}_S([H_2]) &= \mathcal{ACM}_S(\xi_\omega(\boldsymbol{x}^2)).
\end{aligned}$$

Hence, our assertion follows from Proposition 8.3 and the following lemma. □

LEMMA 9.2. *It holds*

$$\mathfrak{N}_S([H_1]) = \mathfrak{N}_S([f_\omega(G)]) = \mathfrak{N}_S([H_2]).$$

PROOF. Let $\overline{\mathrm{w}} \in \mathrm{Vis}(\omega(\boldsymbol{x}^1))$ and $q \in \overline{\mathrm{w}} \cap \partial_{GM}\mathcal{T}(S)$. Let $p \in \overline{\boldsymbol{x}^1} \cap \partial_{GM}\mathcal{T}(S)$ and fix $[G_p] \in \mathcal{AF}(p)$. From Proposition 9.1, $i_{x_0}(q, [f_\omega(G_p)]) = 0$. Since p is taken arbitrarily in $\overline{\boldsymbol{x}^1} \cap \partial_{GM}\mathcal{T}(S)$, from the proof of Proposition 8.2, we have $i_{x_0}(q, [f_\omega(G)]) = 0$. Hence

$$\mathfrak{N}_S([H_1]) \subset \mathfrak{N}_S([f_\omega(G)]). \tag{73}$$

Let $q' \in \mathfrak{N}_S([f_\omega(G)])$. For $q \in \overline{\omega(\boldsymbol{x}^1)} \cap \partial_{GM}\mathcal{T}(S)$, we take a subsequence $\boldsymbol{z}$ of $\boldsymbol{x}^1$ such that $\omega(\boldsymbol{z})$ converges to q and $\boldsymbol{z}$ converges to some $p \in \overline{\boldsymbol{x}^1} \cap \partial_{GM}\mathcal{T}(S)$. Fix $[G_p] \in \mathcal{AF}(p)$. From Proposition 9.1, we have $\mathfrak{N}_S(q) = \mathfrak{N}_S([f_\omega(G_p)])$. From the construction of G, ${G_p}^\circ$ is topologically equivalent to a subfoliation of G. Hence, $q' \in \mathfrak{N}_S([f_\omega(G_p)]) = \mathfrak{N}_S(q)$. Since q is taken arbitrarily from $\overline{\omega(\boldsymbol{x}^1)} \cap \partial_{GM}\mathcal{T}(S)$, from (73), we deduce that $q' \in \mathfrak{N}_S([H_1])$ and

$$\mathfrak{N}_S([H_1]) = \mathfrak{N}_S([f_\omega(G)]). \tag{74}$$

Since ξ_ω is an isometry,

$$\mathrm{Vis}(\xi_\omega(\boldsymbol{x}^2)) = \xi_\omega(\mathrm{Vis}(\boldsymbol{x}^2)).$$

Since ξ_ω extends to $\mathrm{cl}_{GM}(\mathcal{T}(S))$ homeomorphically and coincides with the action of f_ω on $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}(S)$, we have

$$\begin{aligned}\mathfrak{N}_S([H_2]) \cap \mathcal{PMF} &= \mathcal{ACM}_S(\xi_\omega(\boldsymbol{x}^2)) \cap \mathcal{PMF} = \xi_\omega(\mathcal{ACM}_S(\boldsymbol{x}^2)) \cap \mathcal{PMF} \\ &= \xi_\omega(\mathfrak{N}_S([G])) \cap \mathcal{PMF} = \mathfrak{N}_S([f_\omega(G)]) \cap \mathcal{PMF}.\end{aligned}$$

This equality means that $\mathcal{N}_{MF}(H_2) = \mathcal{N}_{MF}(f_\omega(G))$. By Corollary 7.1, we have $\mathcal{N}(H_2) = \mathcal{N}(f_\omega(G))$ and $\mathfrak{N}_S([H_2]) = \mathfrak{N}_S([f_\omega(G)])$. □

Proof of the injectivity of the homomorphism (68). Let $\omega \in \mathrm{AC}_{\mathrm{as}}(\mathcal{T}(S))$ be in the kernel of Ξ. From Proposition 9.2, there is an isometry ξ_ω which is close to ω. Since $\Xi(\xi_\omega) = \Xi(\omega) = id$, ξ_ω is the identity mapping on $\mathcal{T}(S)$, and hence ω is close to the identity. □

9.5. Rough homotheties on Teichmüller space.

In this section, we shall prove Theorem C.

Suppose first that $\mathrm{cx}(S) \geq 2$. We may assume that S is not a torus with two holes. Suppose to the contrary that there is a (K, D)-rough homothety ω with asymptotic quasi-inverse for some $K \neq 1$. Notice that $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathcal{T}(S))$. Take a homeomorphism f_ω on S as Proposition 9.1.

Let $\alpha, \beta \in \mathcal{S}$. Consider the projective classes $[\alpha]$ and $[\beta]$ as points in $\partial_{GM}\mathcal{T}(S)$. Then, from (25), Theorem 9.1 and Proposition 9.1, we have

$$e^{-D_0} i_{x_0}([\alpha], [\beta])^K \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq e^{D_0} i_{x_0}([\alpha], [\beta])^K \tag{75}$$

where D_0 is a constant depending only on D and $d_T(x_0, \omega(x_0))$. On the other hand, let $K_0 = e^{2d_T(x_0, \xi_\omega(x_0))}$, where ξ_ω is an isometry associated to ω taken as Proposition 9.2. From the definition, $\mathrm{Ext}_{x_0}(f_\omega(G)) = \mathrm{Ext}_{\xi_\omega^{-1}(x_0)}(G)$ for $G \in \mathcal{MF}$. By the quasiconformal invariance of extremal length, we obtain

$$K_0^{-1} i_{x_0}([\alpha], [\beta]) \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq K_0 i_{x_0}([\alpha], [\beta])$$

since f_ω is a homeomorphism on S and $i(f_\omega(\alpha), f_\omega(\beta)) = i(\alpha, \beta)$. Therefore, we deduce

$$i_{x_0}([\alpha], [\beta])^{1-K} \leq K_0 e^{D_0}, \tag{76}$$

$$i_{x_0}([\alpha], [\beta])^{K-1} \leq K_0 e^{D_0}, \tag{77}$$

for any $\alpha, \beta \in \mathcal{S}$ with $i_{x_0}([\alpha], [\beta]) \neq 0$. Since the left-hand sides of (76) and (77) are projectively invariant, when the projective classes $[\alpha]$, $[\beta]$ tend together to some projective measured foliation $[G] \in \mathcal{PMF}$ with keeping satisfying $i(\alpha, \beta) \neq 0$, the left-hand side in (76) diverges if $K > 1$, otherwise the left-hand side in (77) diverges. In any case, we get a contradiction.

We now consider the case where $\mathrm{cx}(S) = 1$. This case is indeed a prototype of our study. In this case, there is an isometry $\mathcal{T}(S) \to \mathbb{D}$ sending x_0 to the origin 0. Furthermore, the Gromov product $\langle x_1 \,|\, x_2 \rangle_0$ for $x_1, x_2 \in \mathbb{D}$ satisfies

$$|\langle x_1 \,|\, x_2\rangle_0 - d_{\mathbb{D}}(0, [x_1, x_2])| \leq D_1 \tag{78}$$

for some universal constant $D_1 > 0$, where $[x_1, x_2]$ is the geodesic connecting between x_1 and x_2 (cf. Section 2.33 in [**44**]).

Suppose on the contrary that there is a (K, D)-rough homothety ω with $K \neq 1$. Notice from the definition that any $\omega \in \mathrm{AC}_{\mathrm{inv}}(\mathbb{D})$ extends to a bijective mapping on $\partial\mathbb{D}$. We can easily see that the extension is continuous, and hence, ω extends to a self-homeomorphism on $\partial\mathbb{D}$. We may assume that $\omega(0) = 0$.

From (78), for $x_1, x_2 \in \mathbb{D}$,

$$|d_{\mathbb{D}}(0, [\omega(x_1), \omega(x_2)]) - K\, d_{\mathbb{D}}(0, [x_1, x_2])| \leq D_2$$

for some constant $D_2 > 0$. Therefore, for any $p_1, p_2 \in \partial\mathbb{D}$, we have

$$C_1|p_1 - p_2|^K \leq |\omega(p_1) - \omega(p_2)| \leq C_2|p_1 - p_2|^K \tag{79}$$

with positive constants C_1, C_2. If $K > 1$, ω is differentiable and the derivative is zero at any $\partial\mathbb{D}$. Hence, ω should be a constant on $\partial\mathbb{D}$, which is a contradiction. Suppose $K < 1$. Since the lift of a self-homeomorphism on $\partial\mathbb{D}$ to $\mathbb{R}$ is a monotone function, the extension of ω to $\partial\mathbb{D}$ is differentiable almost everywhere on $\partial\mathbb{D}$. However, from (79), ω is not differentiable any point on $\partial\mathbb{D}$. This is also a contradiction.

10. Appendix.

The main result of this section is Lemma 10.1. The estimates in the lemma look similar to those in Theorem 6.1 of [**13**]. However, our advantage here is that we treat the extremal lengths of all non-trivial (possibly peripheral) curves of subsurfaces and give a constant C_γ concretely (cf. (81) and (85)).

10.1. Measured foliations and intersection numbers.

Let Q be a holomorphic quadratic differential on X. The differential $|\mathrm{Re}\sqrt{Q}|$ defines a measured foliation on X. We say that such a measured foliation the *vertical foliation* of Q. The vertical foliation of $-Q$ is called the *horizontal foliation* of Q.

By a *step curve*, we mean a geodesic polygon in X the sides of which are horizontal and vertical arcs of Q (cf. Figure 3). For the intersection number functions defined by the vertical foliations of holomorphic quadratic differentials, it is known the following.

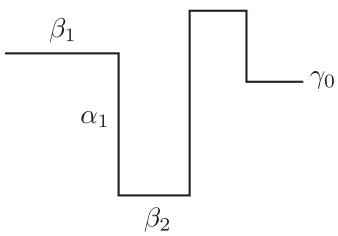

Figure 3. A step curve with the property stated in Proposition 10.1.

Proposition 10.1 (Theorem 24.1 of [**43**]). *Let Q be a quadratic differential and F the vertical foliation of Q. Let γ_0 be a simple closed step curve with the additional property that for any vertical side α_1 of γ_0 the two neighboring horizontal sides β_1 and β_2 are on different sides of α_1 (there are no zeros of Q on γ_0). Then,*

$$i(\gamma, F) = \int_{\gamma_0} |\mathrm{Re}\sqrt{Q}|,$$

where γ is the homotopy class containing γ_0.

It can be also observed that a step curve with the property stated in Theorem 10.1 is quasi-transversal. For instance, see the proof of Proposition II.6 or the curve (4) of Figure 10 of Exposé 5 in [**10**].

10.2. Filling curves and Extremal length.

Let X_0 be an essential subsurface of X. Denote by $\mathcal{S}(X_0)$ a subset of $\mathcal{S}$ consisting of curves which are non-peripheral in X_0. Let $\mathcal{S}_\partial(X_0)$ be a subset of $\mathcal{S}$ consisting of curves which can be deformed into X_0.

Lemma 10.1. *Let X_0 be a connected, compact and essential subsurface of X with negative Euler characteristic. Let $\{\alpha_i\}_{i=1}^m \subset \mathcal{S}(X_0)$ be a system of curves which fills up X_0. Then, for $\gamma \in \mathcal{S}_\partial(X_0)$, we have*

$$\mathrm{Ext}_X(\gamma) \le C_\gamma \max_{1\le i\le m} \mathrm{Ext}_X(\alpha_i), \tag{80}$$

where

$$C_\gamma = C(g,n,m)\left(\sum_{i=1}^m i(\alpha_i,\gamma)\right)^2 + 4(6g-6+n)^2 \tag{81}$$

and $C(g,n,m)$ depends only on the topological type (g,n) of X and the number m of the system $\{\alpha_i\}_{i=1}^m$. In particular, we have

$$\mathrm{Ext}_X(F) \le C(g,n,m)\left(\sum_{i=1}^m i(\alpha_i,F)\right)^2 \max_{1\le i\le m} \mathrm{Ext}_X(\alpha_i), \tag{82}$$

for all $F \in \mathcal{MF}(X_0) \subset \mathcal{MF}$.

Proof. Let $\gamma \in \mathcal{S}_\partial(X_0)$. We divide the proof into two cases.

Case 1 : γ is peripheral in X_0. Suppose first that γ is represented by a component of ∂X_0. When γ is homotopic to a puncture of X, $\mathrm{Ext}_X(\gamma) = 0$ since X contains an arbitrary wide annulus whose core is homotopic to γ. Hence we have nothing to do (in fact, we can set $C_\gamma = 0$).

Suppose that γ is not peripheral in X. Let J_γ be a Jenkins–Strebel differential for γ on X. Let A_γ be the characteristic annulus of J_γ. We consider a "compactification" $\overline{A_\gamma}$ by attaching two copies of circles as its boundaries. The induced flat structure on A_γ from J_γ canonically extends to the compactification $\overline{A_\gamma}$ and components of the boundary $\partial\overline{A_\gamma}$

are closed regular trajectories under this flat structure. There is a canonical surjection $I_\gamma : \overline{A_\gamma} \to \overline{X}$ (the completion of X at the punctures). Namely, $\overline{X}$ is reconstructed by identifying disjoint vertical straight arcs in $\partial\overline{A_\gamma}$ along vertical saddle connections of J_γ. (In this sense, I_γ is a quotient map). Without any confusion, we may recognize the characteristic annulus A_γ itself as a subset of X.

Let γ^* and α_i^* be the core trajectory in A_γ and the geodesic representative of α_i with respect to J_γ respectively. Since γ is parallel to ∂X_0, by taking an isotopy, we may assume that γ^* is a component of ∂X_0. Furthermore, since $\alpha_i \in \mathcal{S}(X_0)$, γ does not intersect any α_i for all i. Hence, each α_i^* consists of vertical saddle connections. In other words, α_i^* is contained in the critical graph $\Sigma_\gamma = I_\gamma(\partial\overline{A_\gamma})$ of J_γ in X, which consists of vertical saddle connections of J_γ.

Let γ_1 and γ_2 be components of $\partial\overline{A_\gamma}$. Each $\gamma_i^* := I_\gamma(\gamma_i)$ is canonically recognized as a path in Σ_γ consisting of vertical saddle connections. We claim:

Claim 1. One of γ_i^*, say γ_1^*, is contained in the union $\bigcup_{i=1}^m \alpha_i^*$.

Proof of Claim 1. Suppose $\gamma_1^* \cap \alpha_i^* \neq \emptyset$ for some i and γ_1^* contains a vertical saddle connection s_0 such that $s_0 \not\subset \alpha_i^*$ for all i. Then, s_0 intersects all α_i^* at most at endpoints (critical points of J_γ). Let $\mathrm{Int}(s_0) = s_0 \setminus \partial s_0$. Let h_1 be a horizontal arc in A_γ starting at $p_1 \in \gamma^*$ and terminating at a point of $\mathrm{Int}(s_0)$. Since the both side of s_0 is in A_γ, after h_1 passes through s_0, h_1 terminates at a point $p_2 \in \gamma^*$. Let γ_0^* be a segment of γ^* connecting p_1 and p_2 (cf. Figure 4). Set $\beta = \gamma_0^* \cup h_1$. By definition, β does not intersects any α_i^* and hence $i(\beta, \alpha_i) = 0$ for all i.

Suppose first that h_1 arrived p_2 from the different side from that where h_1 departed at p_1 (cf. (1) of Figure 4). Then, we have

$$i(\gamma, \beta) = \int_\beta |\mathrm{Re}\sqrt{J_\gamma}| = 1,$$

since the width of A_γ is one and β is a step curve with the property stated in Proposition 10.1. Hence β is non-trivial and non-peripheral simple closed curve in X. However, this contradicts that $\{\alpha_i\}_{i=1}^m$ fills up X_0, since such a $\beta \cap X_0$ contains homotopically non-trivial arc connecting ∂X_0 because γ is parallel to a component of ∂X_0.

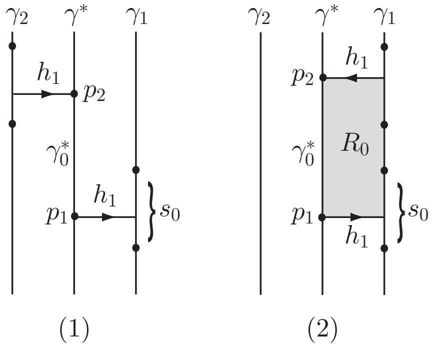

Figure 4. Trajectories in A_γ.

Suppose h_1 arrived at p_2 from the same side as that where h_1 departed (cf. (2) of Figure 4). We may also assume that h_1 departs from p_1 into X_0. Indeed, suppose we cannot assume so. Then, the component of ∂A_γ that lies on the same side as that of X_0 (near γ^*) is covered by $\{\alpha_i^*\}_{i=1}^m$, which contradicts what we assumed first.

Then, there is an open rectangle R_0 in A_γ such that β and a segment in γ_1 surround R_0 in A_γ. From the assumption, we may assume that the closure of $I_\gamma(R_0)$, say X_1, intersects some α_i^*. Suppose that β is trivial. Then, X_1 is a disk in X surrounded by β, since γ^* can be homotopic to the outside of X_1. This means that α_i^* is contained in a disk X_1 because α_i^* does not intersect β, which is a contradiction. By the same argument, we can see that β is non-peripheral (otherwise, α_i^* were peripheral). Since h_1 departs into X_0 at p_1 and returns to γ^* on the side where X_0 lies, after taking an isotopy if necessary, we can see that h_1 contains a subsegment which is nontrivial in X_0 and connecting ∂X_0, which contradicts again that $\{\alpha_i\}_{i=1}^m$ fills X_0 up. □

Let us continue to prove Lemma 10.1 for peripheral $\gamma \in \mathcal{S}_\partial(X_0)$. We take γ_1^* as in Claim 1. Since both sides of every vertical saddle connection face A_γ, γ_1^* visits each vertical saddle connection at most twice. Notice that the number of vertical saddle connections is at most $6g-6+n$. Since each vertical saddle connection in γ_1^* is contained in some α_i^*, we have

$$\begin{aligned}\ell_{J_\gamma}(\gamma) = \ell_{J_\gamma}(\gamma_1^*) &\leq 2(6g-6+n)\max\{\ell_{J_\gamma}(\alpha_i^*) \mid i=1,\ldots,n\}\\ &= 2(6g-6+n)\max\{\ell_{J_\gamma}(\alpha_i) \mid i=1,\ldots,n\},\end{aligned}$$

since α_i^* is the geodesic representative of α_i. Since the width of A_γ is one, from (31), we conclude

$$\begin{aligned}\mathrm{Ext}_X(\gamma) &= \ell_{J_\gamma}(\gamma)^2/\|J_\gamma\|\\ &\leq 4(6g-6+n)^2 \max_{1\leq i\leq m}\{\ell_{J_\gamma}(\alpha_i)^2/\|J_\gamma\|\}\\ &\leq 4(6g-6+n)^2 \max_{1\leq i\leq m}\mathrm{Ext}_X(\alpha_i). \qquad (83)\end{aligned}$$

Case 2 : $\gamma \in \mathcal{S}(X_0)$. We next assume that γ is not parallel to any component of ∂X_0. Let $\{\beta_i\}_{i=1}^s$ be components of ∂X_0 each of which is non-peripheral in X. Let $\epsilon > 0$ and set

$$F_\epsilon = \gamma + \epsilon \sum_{i=1}^{s} \beta_s \qquad (84)$$

(cf. [**18**]). It is possible that two curves β_{i_1} and β_{i_2} are homotopic in X. In this case, we recognize $\beta_{i_1}+\beta_{i_2}=\beta_{i_1}$ in (84). However, for the simplicity of the discussion, we shall assume that any two of $\{\beta_i\}_{i=1}^s$ are not isotopic. The general case can be treated in a similar way.

Let J_γ^ϵ be the holomorphic quadratic differential on X whose vertical foliation is F_ϵ. Since $F_\epsilon \to \gamma$ in $\mathcal{MF}$, J_γ^ϵ tends to J_γ in $\mathcal{Q}_X$ (cf. [**16**]. See also Theorem 21.3 in [**43**]). Let A_γ^ϵ and A_i^ϵ denote the characteristic annuli of J_γ^ϵ for γ and β_i, respectively.

Set $\gamma^{\epsilon,*}$ and $\beta_i^{\epsilon,*}$ to be closed trajectories in homotopic to γ and β_i, respectively. Let Y_0^ϵ be the closure of the component of $\epsilon/4$-neighborhood of the cores $\beta_i^{\epsilon,*}$, containing A_γ^ϵ. By definition, we may identify Y_0^ϵ with X_0. Let $\alpha_i^{\epsilon,*}$ be the geodesic representation of α_i with respect to J_γ^ϵ.

We fix an orientation on $\gamma^{\epsilon,*}$. Let ξ be a component of $\gamma^{\epsilon,*} \setminus \bigcup_{i=1}^m \alpha_i^{\epsilon,*}$. Let $I_0(\xi)$ be the set of points $p \in \xi$ such that the horizontal ray r_p departing at p from the right of ξ terminates at a curve in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$ before intersecting ξ twice. Let $C_0(\xi)$ be the set of $p \in \xi$ such that r_p terminates at a critical point of J_γ^ϵ. Then, we claim

Claim 2. $\xi \setminus I_0(\xi) \subset C_0(\xi)$, and $I_0(\xi) \setminus C_0(\xi)$ is open in ξ.

Proof of Claim 2. Let $p \in \xi \setminus I_0(\xi)$. Suppose $p \notin C_0(\xi)$. Since the completion $\overline{X}$ with respect to the punctures is closed, r_p is recurrent (cf. Section 10 of Chapter IV in [**43**]). By the definition of $I_0(\xi)$ and $p \notin I_0(\xi)$, r_p intersects ξ at least twice before intersecting curves in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$. Hence, r_p contains a consecutive horizontal segments h_1 and h_2 such that each h_i intersects ξ only at its endpoints, and does not intersect any curves in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$.

When one of the segments, say h_1, connects both sides of ξ, ξ contains a vertical segment v_1 connecting endpoints of h_i, and two trajectories h_1 and v_1 make a closed curve δ on X. Since the two ends of h_i terminate at ξ from different sides, the intersection number satisfies

$$i(F_\epsilon, \delta) = \int_\delta \left|\mathrm{Re}\sqrt{J_\gamma^\epsilon}\right|$$

and is greater than or equal to the width of A_γ^ϵ, by Proposition 10.1. Therefore δ is non-trivial and non-peripheral in X. Since h_i does not intersect $\beta_i^{\epsilon,*}$, δ is contained in Y_0^ϵ, where we have identified with X_0. Furthermore, δ is not peripheral in X_0 because δ has non-trivial intersection with γ. By definition, δ does not intersect all α_i, which is a contradiction because $\{\alpha_i\}_{i=1}^m$ fills X_0 up.

We assume that two ends of each h_i terminate at ξ from the same side. In this case, we can also construct a simple closed step curve δ with the property stated in Proposition 10.1 from h_1, h_2 and a subsegment of ξ (cf. Figure 5). This is a contradiction as above. Thus we conclude that $\xi \setminus I_0(\xi) \subset C_0(\xi)$.

We show that $I_0(\xi) \setminus C_0(\xi)$ is open in ξ. Let $p \in I_0(\xi) \setminus C_0(\xi)$ such that the horizontal ray r_p defined above does not terminate at critical points of J_γ^ϵ. By definition, the horizontal ray r_p terminate the interior of a straight arc contained in either $\alpha_i^{\epsilon,*}$ or $\beta_j^{\epsilon,*}$. Hence, when $p' \in \xi$ is in some small neighborhood of p, $r_{p'}$ also terminates at such a straight arc, and hence $p' \in I_0(\xi)$ for all point p' in a small neighborhood of p. □

Let us return to the proof of Case 2 of the lemma. Let ξ be a component of $\gamma^\epsilon \setminus \bigcup_{i=1}^m \alpha_i^{\epsilon,*}$. By definition, for $p \in I_0(\xi)$, r_p terminates at ξ at most once before intersecting curves in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$. Since any horizontal ray r_p with $p \notin C_0(\xi)$ can terminate at a curve in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$ from at most two sides. Hence for almost all point q in a curve in $\{\alpha_i^{\epsilon,*}, \beta_j^{\epsilon,*}\}_{i,j}$, there are at most 4 points in $I_0(\xi)$ such that the horizontal rays emanating there land at q. From Claim 2, we get

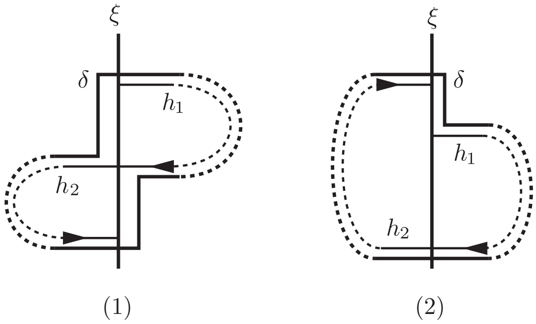

Figure 5. How to get a closed curve δ : There are two cases. In the case (1), the initial point of h_1 and the terminal point of h_2 are separated by the terminal point of h_1. The case (2) describes the other case.

$$|\xi| = |I_0(\xi) \setminus C_0(\xi)| \leq 4\left(\sum_{i=1}^{m} \ell_{J_\gamma^\epsilon}(\alpha_i) + \sum_{i=1}^{s} \ell_{J_\gamma^\epsilon}(\beta_i)\right),$$

where $|\cdot|$ means linear measure. Since $\alpha_i^{\epsilon,*}$ is the geodesic representative of α_i, the number of components of $\gamma^{\epsilon,*} \setminus \bigcup_{i=1}^{m} \alpha_i^{\epsilon,*}$ is

$$\sum_{i=1}^{m} i(\alpha_i, \gamma).$$

Therefore, we obtain

$$\ell_{J_\gamma^\epsilon}(\gamma) \leq 4\left(\sum_{i=1}^{m} i(\alpha_i,\gamma)\right)\left(\sum_{i=1}^{m} \ell_{J_\gamma^\epsilon}(\alpha_i) + \sum_{i=1}^{s} \ell_{J_\gamma^\epsilon}(\beta_i)\right).$$

Since J_γ^ϵ tends to J_γ as $\epsilon \to 0$, for all $\eta > 0$ we find an $\epsilon > 0$ such that

$$\begin{aligned}
\mathrm{Ext}_X(\gamma) &= \frac{\ell_{J_\gamma}(\gamma)^2}{\|J_\gamma\|} \leq \frac{\ell_{J_\gamma^\epsilon}(\gamma)^2}{\|J_\gamma^\epsilon\|} + \eta \\
&\leq 16\left(\sum_{i=1}^{m} i(\alpha_i,\gamma)\right)^2 (m+s)^2 \left(\sum_{i=1}^{m} \frac{\ell_{J_\gamma^\epsilon}(\alpha_i)^2}{\|J_\gamma^\epsilon\|} + \sum_{i=1}^{s} \frac{\ell_{J_\gamma^\epsilon}(\beta_i)^2}{\|J_\gamma^\epsilon\|}\right) + \eta \\
&\leq C'_\gamma \max_{1\leq i\leq m} \mathrm{Ext}_X(\alpha_i) + \eta
\end{aligned}$$

by Corollary 21.2 in [**43**] and the Cauchy–Schwartz inequality, where

$$C'_\gamma = 16(m+s)^2(m+4s(6g-6+n)^2)\left(\sum_{i=1}^{m} i(\alpha_i,\gamma)\right)^2$$

from (83).

Notice that the number s of components of ∂X_0 satisfies $s \leq 2g+n$. Indeed, we fix a hyperbolic metric on X and realize X_0 as a convex hyperbolic subsurface of X. Let

g' and n' be the genus and the number of punctures in X_0. Since $X_0 \subset X$ and X_0 is essential, by comparing to the hyperbolic area, we have

$$\begin{aligned} 2\pi(s-2) \leq 2\pi(2g'-2+s+n') &= \mathrm{Area}(X_0) \\ &\leq \mathrm{Area}(X) = 2\pi(2g-2+n), \end{aligned}$$

and hence $s \leq 2g+n$.

Thus, by (83), we conclude that (80) holds with

$$C(g,n,m) := 16(m+2g+n)^2(m+4(2g+n)(6g-6+n)^2), \tag{85}$$

which implies what we wanted. □

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