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Poset pinball, GKM-compatible subspaces, and Hessenberg varieties

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Abstract. This paper has three main goals. First, we set up a general framework to address the problem of constructing module bases for the equivariant cohomology of certain subspaces of GKM spaces. To this end we introduce the notion of a GKM-compatible subspace of an ambient GKM space. We also discuss poset-upper-triangularity, a key combinatorial notion in both GKM theory and more generally in localization theory in equivariant cohomology. With a view toward other applications, we present parts of our setup in a general algebraic and combinatorial framework. Second, motivated by our central problem of building module bases, we introduce a combinatorial game which we dub poset pinball and illustrate with several examples. Finally, as first applications, we apply the perspective of GKM-compatible subspaces and poset pinball to construct explicit and computationally convenient module bases for the S^1-equivariant cohomology of all Peterson varieties of classical Lie type, and subregular Springer varieties of Lie type A. In addition, in the Springer case we use our module basis to lift the classical Springer representation on the ordinary cohomology of subregular Springer varieties to S^1-equivariant cohomology in Lie type A.

1. Introduction.

This manuscript has three main goals. First, we develop a general framework and perspective to construct computationally convenient module bases for the equivariant cohomology of certain spaces equipped with group actions. In particular, we introduce the notion of a GKM-compatible subspace of an ambient GKM space. (We recall some background on GKM theory below.) While not themselves GKM spaces, GKM-compatible subspaces allow us to exploit the combinatorial advantages of GKM theory applied to the ambient GKM space. We primarily use Borel-equivariant cohomology with field coefficients, but expect future applications in other generalized equivariant cohomology theories. For this reason, we present part of this framework in an abstract algebraic setting, formalizing algebraic properties of the equivariant cohomology of GKM spaces in the language of submodules of product modules indexed by a graded partially ordered set. We also discuss the crucial notion of poset-upper-triangular subsets of a module, and

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give one possible answer to a question of Billey's by providing examples of topological spaces with no combinatorially-natural poset-upper-triangular basis.

Second, we introduce a combinatorial game we call poset pinball. The game is designed to address some of the difficulties which arise in studying the topology of GKM-compatible subspaces, but the game itself is purely combinatorial and does not depend on the motivating geometry. A successful game of poset pinball often results in the construction of a module basis for the equivariant cohomology of a GKM-compatible subspace. We refer to any basis obtained in this manner as a poset pinball basis.

Third, as applications, we use the above theory to describe certain nilpotent Hessenberg varieties, which are a rich class of algebraic varieties arising in geometric representation theory. GKM theory does not directly apply to nilpotent Hessenberg varieties but our methods do. This provides motivation for the development of our theory. We first prove in Theorem 5.4 that given certain conditions, the nilpotent Hessenberg varieties $\mathcal{H}(N, H)$ are GKM-compatible subspaces of the flag variety. Then in Theorem 5.9 we use poset pinball to construct explicit module bases for the S^1-equivariant cohomology rings of Peterson varieties in all classical Lie types; this generalizes earlier work in the Lie type A case [**23**]. Similarly, we use poset pinball in Theorem 6.6 to construct module bases for the S^1-equivariant cohomology of subregular Springer varieties in Lie type A (also studied by Slodowy [**35**]). We then use this poset pinball basis and Kostant–Kumar's S_n-action on $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n); \mathbb{C})$ to explicitly construct in Corollary 6.11 a new geometric representation of S_n on $H^*_{S^1}(\mathcal{S}_N; \mathbb{C})$ and prove that it lifts the well-known Springer representation on the ordinary cohomology of subregular Springer varieties to S^1-equivariant cohomology.

There has been much interest in this circle of ideas, especially in relation to the study of Hessenberg varieties, since the authors first introduced them in [**23**]. Poset pinball module bases have been computed for various special cases of (type A) Hessenberg varieties by Dewitt and the first author, and Bayegan and the first author, in [**13**] and [**4**] respectively. Bayegan and the first author also found a Giambelli formula for the cohomology ring of type A Peterson varieties in [**5**], thus giving another explicit example where poset-pinball bases can give rise to a good notion of generalized Schubert calculus. Furthermore, Drellich recently generalized the results in [**23**] and [**5**] to obtain poset pinball bases, as well as Monk and Giambelli formulas, for Peterson varieties in general Lie type [**14**]. More recently, the first author, Abe, Horiguchi, and Masuda found generators and relations [**2**] of the equivariant cohomology ring $H^*_{S^1}(\mathrm{Hess}(h))$ for all type A regular nilpotent Hessenberg varieties $\mathrm{Hess}(h)$. As a corollary they conclude that the restriction map $H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathrm{Hess}(h))$ is surjective, so the theory of GKM-compatibility and poset-pinball bases applies. Similarly, the first author, Abe, and Horiguchi have work in progress which finds an upper-triangular poset-pinball basis for the (equivariant) cohomology of 2-step (type A) nilpotent Springer varieties [**1**]. The basis in [**1**] appears to be particularly well-suited for Schubert calculus-style calculations with combinatorially good properties; we hope to explore this further in future work.

We now give an overview of the history and background which informs the present paper. Our work develops out of GKM theory, named for the influential manuscript of Goresky–Kottwitz–MacPherson [**18**]. If X is a suitable G-space, GKM theory gives a combinatorial description of the generalized equivariant cohomology ring $E^*_G(X)$ via restriction to $E_G^*(X^G)$. If the G-space X has isolated fixed points, there often exists a computationally convenient module basis for $E_G^*(X)$; a classical example is the set of (equivariant) Schubert classes $\{\sigma_w\}_{w\in W}$ which form a basis for $H_T^*(\mathcal{G}/\mathcal{B})$ where $\mathcal{G}$ is a reductive complex algebraic group and $\mathcal{B}$ is a Borel subgroup. Current research in equivariant topology (especially Schubert calculus) frequently exploits combinatorial properties of these module bases to obtain topological information, e.g. the product structure of equivariant cohomology rings (see for instance [**25**], [**27**], [**30**], [**32**], [**43**]). However the conditions which guarantee that GKM theory applies to a G-space X are stringent (see Section 4.1), so the theory is restricted in scope. On the other hand, many topological spaces are subspaces of a G-space X for which the GKM package holds, since for instance we may take $X = \mathbb{P}^n$ with the standard torus action. Definition 4.4 introduces the notion of a GKM-compatible subspace Y of the G-space X, equipped with the action of a subgroup $G' \subseteq G$. We show that we can use GKM theory on the ambient space X in order to draw conclusions about the G'-equivariant topology of Y. For example, when $G = T$ is a torus and $G' = S$ is a subtorus, we present two concrete constructions of combinatorial bases for $H_S^*(Y)$ given a suitable basis for $H_T^*(X)$: Proposition 4.15 gives a poset pinball basis and Theorem 4.18 gives a matching basis.

One of the goals of this manuscript is to formalize some features of GKM theory and rephrase them in purely algebraic and combinatorial terms. We believe that this separation of the algebra and combinatorics from the underlying geometry serves to clarify the mathematical difficulties involved in finding a computationally convenient set of generators for GKM-compatible subspaces. We start with a poset $\mathcal{I}$ satisfying conditions arising naturally in geometric applications. We then place the GKM description of equivariant cohomology rings in the more general algebraic setting of a submodule M of a product module $\prod_{i\in\mathcal{I}} M_i$ whose factors are indexed by the poset $\mathcal{I}$. One of the core notions of this manuscript is poset-upper-triangularity, defined precisely in Definition 2.3. Roughly, a subset $\{x_\alpha\}_{\alpha\in A} \subset \prod_{i\in\mathcal{I}} M_i$ is poset-upper-triangular if for each α there exist distinct $i_\alpha \in \mathcal{I}$ such that $x_\alpha(j) = 0$ for all $j \not\geq i_\alpha$ in the poset. In many contexts, geometric classes in equivariant cohomology give rise to poset-upper-triangular subsets, like the equivariant Schubert classes for flag varieties, or cohomology classes obtained from Morse flows with respect to moment maps on a symplectic manifold. We present in Theorem 4.2 more general circumstances under which poset-upper-triangular module bases exist for Borel-equivariant cohomology. In this algebraic formalism, the analogue of a GKM-compatible subspace Y of X is a subset $\mathcal{J} \subseteq \mathcal{I}$ of the ambient poset $\mathcal{I}$ and a homomorphism $\prod_{i\in\mathcal{I}} M_i \to \prod_{j\in\mathcal{J}} M'_j$ which is zero on the factors $i \in \mathcal{I}$ with $i \notin \mathcal{J}$. Our central problem, recorded in this algebraic context in Question 2.8 and in a geometric context in Question 4.8, is that a poset-upper-triangular subset of $\prod_{i\in\mathcal{I}} M_i$ may not be poset-upper-triangular when restricted to the components indexed by $\mathcal{J}$. This is precisely the issue which our poset pinball construction and its variations are designed to address. While geometric in inspiration, we emphasize that poset pinball only requires the combinatorial data of a poset $(\mathcal{I}, <)$ and a choice of initial subset $\mathcal{J} \subseteq \mathcal{I}$.

As a consequence of the examples of poset pinball games computed in this manuscript, we also propose a perspective on poset-upper-triangularity which somewhat differs from that which may be most natural from the point of view of combinatorics (see

Remark 4.17). Combinatorists view poset-upper-triangularity as a key computational property; indeed, Billey suggests that it is one of the essential features of the Schubert basis and asks for constructions of such poset-upper-triangular bases in equivariant cohomology rings of other G-spaces [**7**]. On the other hand, in our poset pinball examples it can happen that we obtain subsets of modules that are *not* poset-upper-triangular with respect to the original partial order $<$ but are nevertheless poset-upper-triangular with respect to a total order $\prec$ compatible with the original partial order. Poset-upper-triangularity with respect to a total order often suffices to guarantee that a subset is linearly independent and hence a module basis, so in some geometric contexts, it may be more natural to require only that module bases be upper-triangular with respect to some choice of total order compatible with the original partial order.

We now present a concrete example of poset pinball in order to convey the flavor of the game. Let $\mathcal{I} = S_4$ denote the permutation group S_4. Elements of $\mathcal{I} = S_4$ are the vertices in Figure 1.1 and are labelled by a choice of reduced-word decomposition. (We omit some of the elements of S_4 in the figure because they are not relevant in this example.) The set $\mathcal{I}$ is partially ordered by Bruhat order, so we draw an edge between vertices $w, w' \in \mathcal{I}$ if and only if $w < w'$ and there is no $w'' \in \mathcal{I}$ with $w < w'' < w'$. The vertices are drawn so that the poset's minimal element e is at the bottom, and horizontal levels indicate Bruhat length. Let $\mathcal{J}$ be the subset of $\mathcal{I}$ indicated by the circled vertices in Figure 1.1, so there are 6 elements in the subset $\mathcal{J}$. To play poset pinball, we successively release a circled vertex, starting from the lowest vertex in $\mathcal{J}$ and then moving up; we imagine each circled vertex rolling down along the edges of the poset until it comes to rest in a lowest-possible unoccupied vertex, at which point the circle turns into a square. (See Section 3.1 for precise statements.) This results in a choice of 6 vertices of $\mathcal{I}$ corresponding to the original elements of $\mathcal{J}$ and indicated by the squared vertices in the figure below. Table (1.1) records the exact correspondence between the initial vertices $w \in \mathcal{J}$ and the squared vertices $v \in \mathcal{I}$.

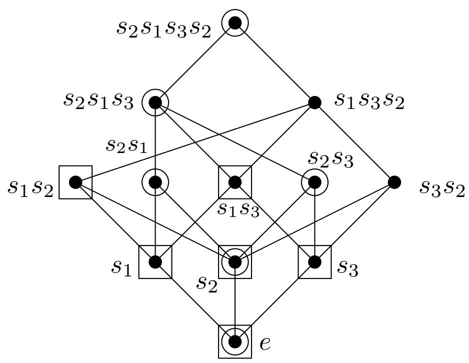

Figure 1.1. An instance of poset pinball, for the Springer variety in $\mathcal{F}\ell ags(\mathbb{C}^4)$ specified by a nilpotent operator N corresponding to the partition $(2, 2)$.

pinball step	w_k	v_k
1	$w_1 = e$	$v_1 = e$
2	$w_2 = s_2$	$v_2 = s_2$
3	$w_3 = s_2s_3$	$v_3 = s_3$
4	$w_4 = s_2s_1$	$v_4 = s_1$
5	$w_5 = s_2s_1s_3$	$v_5 = s_1s_3$
6	$w_6 = s_2s_1s_3s_2$	$v_6 = s_1s_2$

(1.1)

Let $\mathcal{S}_N \subseteq \mathcal{F}\ell ags(\mathbb{C}^4) \cong GL(4,\mathbb{C})/\mathcal{B}$ denote the Springer variety of type A corresponding to a nilpotent endomorphism N with two Jordan blocks of size 2. Let the maximal torus T act in the standard way on $\mathcal{F}\ell ags(\mathbb{C}^4)$. An S^1 subtorus of T preserves $\mathcal{S}_N$ and the S^1-fixed points $\mathcal{S}_N^{S^1} = \mathcal{J}$ are exactly the subset of $\mathcal{I} = S_4 \cong \mathcal{F}\ell ags(\mathbb{C}^4)^T$ indicated in Figure 1.1. (See Section 6.) In this case the choices $\{v_i\}_{i=1}^6$ obtained via poset pinball give rise to an $H^*_{S^1}(\mathrm{pt};\mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{S}_N;\mathbb{F})$. Specifically, consider the ring homomorphism

$$H^*_T(\mathcal{F}\ell ags(\mathbb{C}^4);\mathbb{F}) \to H^*_{S^1}(\mathcal{S}_N;\mathbb{F}) \tag{1.2}$$

obtained by composing the map $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^4);\mathbb{F}) \to H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^4);\mathbb{F})$ with the map $H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^4);\mathbb{F}) \to H^*_{S^1}(\mathcal{S}_N;\mathbb{F})$ induced by inclusion of groups $S^1 \hookrightarrow T$ and spaces $\mathcal{S}_N \hookrightarrow \mathcal{F}\ell ags(\mathbb{C}^4)$ respectively. Denote the image of an equivariant Schubert class $\sigma_w \in H^*_T(\mathcal{F}\ell ags(\mathbb{C}^4);\mathbb{F})$ under the map (1.2) by p_w. Then the 6 classes $p_{v_1}, \ldots, p_{v_6}$ which correspond to the outcome of the previous poset pinball game form an $H^*_{S^1}(\mathrm{pt};\mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{S}_N;\mathbb{F})$. Moreover, the set of images of the p_{v_i} under the natural restriction map

$$\iota^* : H^*_{S^1}(\mathcal{S}_N;\mathbb{F}) \hookrightarrow H^*_{S^1}(\mathcal{S}_N^{S^1};\mathbb{F}) \cong \bigoplus_{i=1}^{6} H^*_{S^1}(\mathrm{pt};\mathbb{F}) \cong \bigoplus_{i=1}^{6} \mathbb{F}[t] \tag{1.3}$$

is poset-upper-triangular with respect to the partial order on the fixed points $\mathcal{S}_N^{S^1} = \{w_i\}_{i=1}^6$ induced from Bruhat order. In other words, for each $p \in H^*_{S^1}(\mathcal{S}_N;\mathbb{F})$ let $p(w_i)$ denote the component of $\iota^* p$ in the w_i-th summand of the right side of Equation (1.3). Then for each $i = 1, 2, \ldots, 6$ we have

$$p_{v_i}(w_j) = 0 \text{ for } w_j \not\geq w_i \tag{1.4}$$

where the inequality indicates the partial order on $\mathcal{S}_N^{S^1} \subseteq S_4$ induced from Bruhat order on S_4. These vanishing properties allow us to do explicit computations in $H^*_{S^1}(\mathcal{S}_N;\mathbb{F})$ (see Section 6 for applications).

We now briefly outline the contents of the paper. In Section 2 we present the combinatorial and algebraic preliminaries for the pinball game. Poset pinball itself is described in detail in Section 3. We give two concrete examples of poset pinball in Section 3.2 and make initial observations concerning the role played by principal order ideals in pinball theory in Section 3.3. We then explain the geometric motivation and context in Section 4. We begin with a brief review of relevant GKM theory in Section 4.1, and then in Section 4.2 define GKM-compatible subspaces of GKM spaces. With a view toward future work, we keep the discussion in Sections 2, 4.1, and 4.2 as general as possible. Section 4.3 discusses the case of Borel-equivariant cohomology, which is the main focus of this manuscript. The construction of poset pinball bases for the S^1-equivariant cohomology of Peterson varieties in classical Lie type occupies Section 5. In Section 6, we construct a pinball basis for the equivariant cohomology $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ of the subregular Springer variety of type A, and lift the usual Springer action on $H^*(\mathcal{S}_N;\mathbb{C})$ to $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$.

Open questions and avenues for future work are mentioned throughout the manuscript.

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2. Poset-upper-triangular bases for submodules of product modules.

Two central goals of this manuscript are to construct module bases for equivariant cohomology rings, and to use them in applications like geometry (see Section 5) and representation theory (see Section 6). We make the case here and below that poset-upper-triangularity is a useful criterion to apply to module bases for both theoretical and computational purposes. Indeed, we shall see that poset-upper-triangularity can help us to prove that a set of module generators are linearly independent, thus letting us conclude that the generators in fact form a basis. Also, it is conceptually simple to compute structure constants in an equivariant cohomology ring with respect to a poset-upper-triangular basis, as the experience with Schubert classes suggests. For these reasons, poset-upper-triangularity plays a central role in the exposition below.

Although our motivation comes from geometry, the issues we face in constructing module bases are purely algebraic. We highlight this fact in this section by phrasing the problem in the general algebraic language of certain submodules of product modules over a graded poset. More specifically, fix a commutative ring R and a partially ordered set $(\mathcal{I},<)$. Let M be an R-module that can be realized as a submodule of a product $\prod_{i\in\mathcal{I}} M_i$ of R-modules M_i for each $i\in\mathcal{I}$. (In our motivating examples, the ring $R=E^*_G(\mathrm{pt})$ is the equivariant cohomology ring of a point, the module $M=E^*_G(X)$ is that of a G-space X, and M injects as a submodule into the product $\prod_{i\in\mathcal{I}} M_i=\prod_{i\in\mathcal{I}} E^*_G(\mathrm{pt})$.) In this language our main problem is: given two modules $M\subseteq\prod_{i\in\mathcal{I}} M_i$ and $M'\subseteq\prod_{j\in\mathcal{J}} M'_j$ for

$\mathcal{J} \subseteq \mathcal{I}$, together with a homomorphism $M \longrightarrow M'$ and a poset-upper-triangular basis for M as defined below, construct a computationally-convenient module basis for M'.

We now briefly outline this section. Section 2.1 establishes notation and terminology for posets and product modules indexed by posets. In particular, we give a precise definition of a poset-upper-triangular set of module elements. As mentioned above, a module basis must both generate the module and be linearly independent. Accordingly, in Section 2.2 we prove Proposition 2.4 (which deals with the issue of generation) and take some steps towards the more difficult issue of linear independence in Proposition 2.6. The central algebraic question is formulated precisely in Section 2.3. In Section 2.4 we briefly recall a result which addresses a variant on the central question posed in Section 2.3, namely, that of building a module basis which is not necessarily poset-upper-triangular. (Section 2.4 is used in a geometric application of our poset pinball game in Section 4.)

2.1. Combinatorial preliminaries.

Let $(\mathcal{I}, <)$ be a partially ordered set. For $i, j \in \mathcal{I}$, we say that i *covers* j if $j < i$ and, in addition, there is no $i' \in \mathcal{I}$ with $j < i' < i$. A *rank function* $\rho : \mathcal{I} \to \mathbb{N}$ is an $\mathbb{N}$-valued function on the poset such that if i covers j then $\rho(i) = \rho(j) + 1$. In the case where $\mathcal{I}$ is infinite, we also require $\rho(i) > \rho(j)$ whenever $i > j$ since chains of covering relations are not guaranteed to be finite. For $i \in \mathcal{I}$, we call $\rho(i)$ the *rank* of i. A poset $(\mathcal{I}, <, \rho)$ equipped with a rank function is called a *graded poset.*

All partially ordered sets $\mathcal{I}$ in this manuscript are graded. Moreover they satisfy the following conditions:

- $\mathcal{I}$ is countable and
- for any $d \in \mathbb{N}$ the set $\{i \in \mathcal{I} : \rho(i) \leq d\}$ is finite.

We also recall the following [**38**, Chapter 3].

Definition 2.1. Given a poset $(\mathcal{I}, <)$ and an element $i \in \mathcal{I}$, the *principal order ideal* $\mathcal{L}_{\mathcal{I}}(i)$ *of* i is the subset of elements $i' \in \mathcal{I}$ less than or equal to i with respect to $<$. In other words

$$\mathcal{L}_{\mathcal{I}}(i) := \{i' \in \mathcal{I} \mid i' \leq i\}.$$

Similarly, the *principal order filter* $\mathcal{U}_{\mathcal{I}}(i)$ *of* i is the subset of elements $i' \in \mathcal{I}$ greater than or equal to i with respect to $<$. In other words

$$\mathcal{U}_{\mathcal{I}}(i) := \{i' \in \mathcal{I} \mid i \leq i'\}.$$

The posets appearing in this manuscript arise as indexing sets for products of modules, so we introduce some terminology for this situation. Let R be a commutative ring. Suppose $\mathcal{I}$ is a poset as above and M_i an R-module for each $i \in \mathcal{I}$. Let M be a submodule of the product module $\prod_{i \in \mathcal{I}} M_i$. For $x \in M \subseteq \prod_{i \in \mathcal{I}} M_i$ we denote by $x(i) \in M_i$ the component of x in the i-th factor of the direct product. For $x \in M$, let

$$\mathrm{supp}(x) := \{i \in \mathcal{I} : x(i) \neq 0 \in M_i\}$$

denote the *support of* x, i.e. the indices $i \in \mathcal{I}$ on which the component of x in the i-th factor does not vanish.

Definition 2.2. An element $x \in M$ is a *poset-flow-up* (*with respect to* $<$) if the support of x contains an element i whose principal order filter contains $\mathrm{supp}(x)$, in other words $i \in \mathrm{supp}(x) \subseteq \mathcal{U}_{\mathcal{I}}(i)$. We denote the (unique) element i by $min(x)$ and call it the *minimum nonzero coordinate of* (*the poset-flow-up*) x.

Note that if $x \in M$ is a poset-flow-up, then

$$x(j) = 0 \text{ for all } j \not\geq min(x). \tag{2.1}$$

Throughout the manuscript, we consider collections of elements in a module $M \subseteq \prod_{i \in \mathcal{I}} M_i$ with vanishing properties similar to those in (2.1). We have the following.

Definition 2.3. Suppose $\mathcal{B} = \{x_\alpha\} \subseteq M$. The set $\mathcal{B}$ is *poset-upper-triangular* if

- each $x_\alpha \in \mathcal{B}$ is a poset-flow-up, and
- for all $x_\alpha, x_\beta \in \mathcal{B}$ with $\alpha \neq \beta$, we have $min(x_\alpha) \neq min(x_\beta)$.

2.2. Upper-triangularity for module generators and bases.

In this section, we construct poset-upper-triangular module generators and bases of R-modules $M \subseteq \prod_{i \in \mathcal{I}} M_i$ from a purely algebraic viewpoint. An R-module basis must both generate the module and be R-linearly independent; we address the two conditions separately. We keep the assumptions of Section 2.1. Throughout, we think of M as a topological R-module as in [**21**], considered as the (inverse) limit of the submodules $M \cap \prod_{j \leq i} M_j$. In particular, if the poset $\mathcal{I}$ is infinite then the terms 'generator' and 'basis' are understood in the topological sense; if $\mathcal{I}$ is finite, then the topological definitions agree with the usual definitions.

Suppose $\prec$ is a total ordering on $\mathcal{I}$ compatible with the given partial order $<$ on $\mathcal{I}$. We begin by inductively constructing a set of generators of $M \subseteq \prod_{i \in \mathcal{I}} M_i$ that consists of poset-flow-ups with respect to $\prec$. Poset-upper-triangularity with respect to the total order $\prec$ is a weaker condition than that with respect to the original partial order $<$ but we will see later in the manuscript that upper-triangularity with respect to $\prec$ suffices for many computational purposes.

Proposition 2.4. *Let $(\mathcal{I}, <, \rho)$ be a countable graded poset with a finite number of elements of each rank. Let R be a commutative ring, M_i an R-module for each $i \in \mathcal{I}$, and M an R-submodule, $M \subseteq \prod_{i \in \mathcal{I}} M_i$. Suppose $\prec$ is a total ordering compatible with the partial order $<$ on $\mathcal{I}$. For each $i \in \mathcal{I}$, define*

$$\mathcal{V}^i := \{x \in M \mid x(j) = 0 \text{ for all } j \in \mathcal{I} \text{ with } j \prec i\} \tag{2.2}$$

and denote by $\mathcal{V}^i(i)$ the image of $\mathcal{V}^i$ in M_i under the natural projection $M \to M_i$. Then for each $i \in \mathcal{I}$ there exist sets $\mathcal{K}_i \subseteq \mathbb{N}$ and nonzero elements $\{x_{i,k}\}_{k \in \mathcal{K}_i} \subseteq M$ such that

(1) *$x_{i,k}(j) = 0$ for all $k \in \mathcal{K}_i$ and $j \in \mathcal{I}$ with $j \prec i$, namely $x_{i,k} \in \mathcal{V}^i$ for all k, and*

(2) *the set* $\{x_{i,k}(i)\}_{k\in\mathcal{K}_i} \subseteq M_i$ *generates* $\mathcal{V}^i(i)$ *as an* R*-module.*

Moreover, the union $\mathcal{B} := \bigcup_{i\in\mathcal{I}}\{x_{i,k}\}_{k\in\mathcal{K}_i}$ *is a set of* R*-module generators of* M *that is poset upper-triangular with respect to* $<$.

PROOF. Note first that $\mathcal{V}^i$ and $\mathcal{V}^i(i)$ are R-submodules of M and M_i respectively. For each $i \in \mathcal{I}$ choose a set of nonzero generators $\{y_{i,k}\}_{k\in\mathcal{K}_i}$ of $\mathcal{V}^i(i)$ indexed by a (possibly infinite) set $\mathcal{K}_i$. (In principle $\mathcal{K}_i$ could be all of $\mathcal{V}^i(i)$.) For each $i \in \mathcal{I}$ and $k \in \mathcal{K}_i$ choose an element $x_{i,k} \in \mathcal{V}^i$ that projects to $y_{i,k}$ under the natural map, namely $x_{i,k}(i) = y_{i,k}$. By construction, the sets $\mathcal{K}_i$ and elements $x_{i,k}$ for $i \in \mathcal{I}, k \in \mathcal{K}_i$ satisfy Conditions (1) and (2) of the proposition.

It remains to show that the union $\mathcal{B} := \bigcup_{i\in\mathcal{I}}\{x_{i,k}\}_{k\in\mathcal{K}_i}$ forms a set of R-module generators for M. We first claim that for any $i \in \mathcal{I}$, the module M is generated as an R-module by the elements $\mathcal{B}_i := \bigcup_{j\prec i}\bigcup_{k\in\mathcal{K}_j}\{x_{j,k}\}$, together with the submodule $\mathcal{V}^i$. We proceed by induction. For the base case, let $i_0 \in \mathcal{I}$ be the (unique) minimal element in $\mathcal{I}$ with respect to the total order $\prec$. In this case $\{j \in \mathcal{I} : j \prec i\}$ is empty, so $\mathcal{V}^{i_0} = M$ and the claim holds. Now let $i \in \mathcal{I}$ and suppose by induction that the claim holds for all $j \prec i$. By our assumptions on $\mathcal{I}$ in Section 2.1, the set $\{j \in \mathcal{I} : j \prec i\}$ is finite for any total ordering $\prec$ compatible with the partial order $<$. Hence there exists $i' \in \mathcal{I}$ which is maximal with respect to $\prec$ in $\{j \in \mathcal{I} : j \prec i\}$. Now let $x \in M$. By the inductive hypothesis there exist $x'' \in \mathcal{V}^{i'}$ and $a_{j,k} \in R$ for $j \prec i', k \in \mathcal{K}_j$ such that

$$x = x'' + \sum_{j\prec i'}\sum_{k\in\mathcal{K}_j} a_{j,k}x_{j,k}. \tag{2.3}$$

(Only finitely many coefficients $a_{j,k}$ are non-zero, so the second expression on the right-hand side of (2.3) is in fact a finite sum. A similar statement holds for the sums appearing in the arguments that follow.) Since the $\{x_{i',k}(i') = y_{i',k}\}_{k\in\mathcal{K}_{i'}}$ generate $\mathcal{V}^{i'}(i')$ there exist $a_{i',k} \in R$ for $k \in \mathcal{K}_{i'}$ such that

$$x''(i') = \sum_{k\in\mathcal{K}_{i'}} a_{i',k}x_{i',k}(i'). \tag{2.4}$$

Now define

$$x' := x'' - \sum_{k\in\mathcal{K}_{i'}} a_{i',k}x_{i',k}. \tag{2.5}$$

Then by construction

$$x = x'' + \sum_{j\prec i'}\sum_{k\in\mathcal{K}_j} a_{j,k}x_{j,k} = x' + \sum_{j\preceq i'}\sum_{k\in\mathcal{K}_j} a_{j,k}x_{j,k}. \tag{2.6}$$

To prove the claim, we show that $x' \in \mathcal{V}^i$. Suppose $j \prec i$. Then either $j \prec i'$ or $j = i'$. First suppose $j \prec i'$. Since $x'' \in \mathcal{V}^{i'}$ we have $x''(j) = 0$. Similarly, each $x_{i',k} \in \mathcal{V}^{i'}$ so $x_{i',k}(j) = 0$. Projection to M_j is R-linear, so

$$x'(j) = x''(j) - \sum_{k \in \mathcal{K}_{i'}} a_{i',k} x_{i',k}(j) = 0$$

as desired. Now suppose $j = i'$. Then by definition of $x_{i',k}$ and by Equation (2.4) we see

$$x'(i') = x''(i') - \sum_{k \in \mathcal{K}_{i'}} a_{i',k} x_{i',k}(i') = x''(i') - \sum_{k \in \mathcal{K}_{i'}} a_{i',k} y_{i',k} = 0. \tag{2.7}$$

Together these equations imply $x' \in \mathcal{V}^i$. Hence M is generated by the elements in $\mathcal{B}_i$ together with $\mathcal{V}^i$. The result now follows by induction since $M = \varprojlim M \cap \left(\prod_{j \leq i} M_j \right)$. □

Remark 2.5. (1) It is sometimes impossible to find a set of generators of a module M which is poset-upper-triangular with respect to the original partial order $<$. For example, suppose M is the image of the standard diagonal embedding $R \hookrightarrow \prod_{i \in \mathcal{I}} R$ with $1 < |\mathcal{I}| < \infty$, and take the trivial partial order on $\mathcal{I}$ in which all pairs of elements are incomparable and the trivial rank function $\rho \equiv 0$. Hence the choice of a total order $\prec$ is a crucial assumption in Proposition 2.4.

(2) On the other hand, Proposition 2.4 extends straightforwardly to partially ordered sets without a choice of total order if we assume that the vanishing submodules $\mathcal{V}^i$ together generate all of M.

The previous proposition constructed module generators. Our next task is to deal with R-linear independence. For the remainder of the manuscript, we assume that

- R is a domain, and
- for all $i \in \mathcal{I}$ the module M_i is R-torsion-free.

These assumptions imply that the submodule $M \subseteq \prod_{i \in \mathcal{I}} M_i$ is also R-torsion-free. (In fact, in our geometric applications, it is usually the case that $M_i \cong R$ for all i and that M is a free R-module.) We begin by giving conditions under which the construction in Proposition 2.4 in fact yields an R-module basis.

Proposition 2.6. *Let $R, \mathcal{I}, M_i, M$ satisfy the conditions of Proposition 2.4. Assume that R is a domain and that the R-module M_i is R-torsion-free for all $i \in \mathcal{I}$.*

(1) *If $\{x_\alpha\}$ is poset-upper-triangular, then $\{x_\alpha\}$ is R-linearly independent.*

(2) *Suppose that for all $i \in \mathcal{I}$, the sets $\{x_{i,k}\}_{k \in \mathcal{K}_i}$ constructed in Proposition 2.4 may be chosen to be R-linearly independent. Then the union $\mathcal{B} = \bigcup_{i \in \mathcal{I}} \{x_{i,k}\}_{k \in \mathcal{K}_i}$ is an R-module basis of M.*

(3) *Suppose that for all $i \in \mathcal{I}$, the index sets $\mathcal{K}_i$ constructed in Proposition 2.4 may be chosen such that either $|\mathcal{K}_i| = 0$ or $|\mathcal{K}_i| = 1$. Then the union $\mathcal{B} = \bigcup_{i \in \mathcal{I}} \{x_{i,k}\}_{k \in \mathcal{K}_i}$ is an R-module basis of M.*

(4) *Suppose that R is a principal ideal domain and each M_i is a free R-module of rank 1. Then the index sets $\mathcal{K}_i$ constructed in Proposition 2.4 may be chosen such that*

either $|\mathcal{K}_i| = 0$ *or* $|\mathcal{K}_i| = 1$ *and the union* $\mathcal{B} = \bigcup_{i \in \mathcal{I}} \{x_{i,k}\}_{k \in \mathcal{K}_i}$ *is an* R*-module basis of* M*.*

Proof. If $\{x_\alpha\}$ is poset-upper-triangular then for each $i \in \mathcal{I}$ there is at most one α with $min(x_\alpha) = i$. Using this, Part (1) follows by induction. Part (2) is by definition. Part (3) follows from Part (2). Part (4) is a special case of Part (3) since, by definition, an ideal in a PID is generated by a single element. □

In our geometric applications, we typically use S^1-equivariant cohomology, in which the ring R is the PID $\mathbb{C}[t]$. In other words we are in the case of Proposition 2.6 (4). More generally, whenever Part (3) of Proposition 2.6 holds, the module basis elements correspond to elements of the poset, so we may think of the bases $\mathcal{B}$ as being indexed by the poset $\mathcal{I}$ (or possibly a subset of $\mathcal{I}$). In this way, the combinatorics of the poset $\mathcal{I}$ interacts directly with the algebra of the module M via the basis $\mathcal{B}$. This links the combinatorial strategies of Section 3 to the algebraic problem of constructing module bases. We close the section with a partial converse to Proposition 2.4.

Lemma 2.7. *Let* $R, \mathcal{I}, M_i, M$ *satisfy the conditions of Proposition* 2.4*. Assume that* R *is a domain and* $M_i \cong R$ *for each* $i \in \mathcal{I}$*. Suppose that* $\{x_i\}_{i \in \mathcal{I}}$ *is poset-upper-triangular with respect to the partial order on* $\mathcal{I}$*, that* $\min(x_i) = i$ *for each* i*, and that* $\{x_i\}_{i \in \mathcal{I}}$ *generates* M *as an* R*-module. Then the choice of singleton set* $\{x_i\}$ *for each* $i \in \mathcal{I}$ *satisfies Conditions* (1) *and* (2) *of Proposition* 2.4*.*

Proof. The set $\{x_i\}_{i \in \mathcal{I}}$ is poset-upper-triangular with respect to $<$ so it is also poset-upper-triangular with respect to any total ordering $\prec$ compatible with $<$. This is Condition (1) of Proposition 2.4.

To prove Condition (2), we show that $x_i(i)$ generates $\mathcal{V}^i(i)$ as an R-module for each i. We first claim that each $\mathcal{V}^i$ is generated by $\{x_j\}_{i \preceq j}$. We proceed by induction. For the base case, let i_0 be the minimal element of $\mathcal{I}$ with respect to $\prec$. Then the set $\{j : i_0 \preceq j\}$ is all of $\mathcal{I}$ and the claim trivially holds.

Now suppose that for some i the submodule $\mathcal{V}^i$ is generated by $\{x_j\}_{i \preceq j}$. Let i' be minimal in $\{i'' \in \mathcal{I} : i \prec i''\}$. (A minimal i' exists because the image of the rank function is contained in $\mathbb{N}$ and the number of elements in each rank of $\mathcal{I}$ is finite by the hypotheses from Section 2.1.) Consider $x \in \mathcal{V}^{i'} \subseteq \mathcal{V}^i$. By the inductive hypothesis,

$$x = c_i x_i + \sum_{j : i \prec j} c_j x_j = c_i x_i + \sum_{j : i' \preceq j} c_j x_j.$$

The set is poset-upper-triangular so $x_j(i) = 0$ for all j with $i \prec j$. Evaluating x at i yields $x(i) = c_i x_i(i)$ which must equal 0 since $x \in \mathcal{V}^{i'}$. By assumption $x_i(i) \neq 0$ so $c_i = 0$. This means $\mathcal{V}^{i'}$ is generated by $\{x_j\}_{i' \preceq j}$ as desired.

Finally, evaluation at i yields $x(i) = c_i x_i(i)$. The element $x \in \mathcal{V}^i$ was chosen arbitrarily, so we conclude $\mathcal{V}^i(i)$ is generated by the single element $x_i(i)$ as claimed. □

2.3. Bases for submodules of products.

In this section we present the central problem of the manuscript, stated in purely algebraic and combinatorial language. Its geometric manifestation is reserved until Section 4. We also explain the core difficulty in addressing the problem, which motivates the poset pinball game introduced in Section 3.

Let $\mathcal{I}$ be a countable graded poset with a finite number of elements in each rank. Let $\mathcal{J}$ be a subset of $\mathcal{I}$ with the partial order induced from $\mathcal{I}$. Let R and R' be integral domains, M_i a torsion-free R-module for each $i \in \mathcal{I}$, and M'_j a torsion-free R'-module for each $j \in \mathcal{J}$. Let $M \subseteq \prod_{i\in\mathcal{I}} M_i$ and $M' \subseteq \prod_{i\in\mathcal{J}} M'_i$ be R- and R'-submodules, respectively, of the given products. Suppose $\gamma : R \to R'$ is a ring homomorphism and that $\phi_i : M_i \to M'_i$ are surjective additive homomorphisms for each $i \in \mathcal{J}$ satisfying

$$\phi_i(rm) = \gamma(r)\phi_i(m) \quad \text{for all } r \in R, i \in \mathcal{I}, \text{ and } m \in M. \tag{2.8}$$

We also assume the homomorphism $\prod_{i\in\mathcal{I}} \phi_i : \prod_{i\in\mathcal{I}} M_i \to \prod_{i\in\mathcal{J}} M'_i$ restricts to a surjection $\phi : M \to M'$ so that the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\phi"'] & \prod_{i\in\mathcal{I}} M_i \arrow[d, "\prod_{i\in\mathcal{J}} \phi_i"] \\ M' \arrow[r] & \prod_{i\in\mathcal{J}} M'_i \end{tikzcd} \tag{2.9}$$

commutes, where the right vertical map is understood to be 0 on the components M_i for $i \notin \mathcal{J}$.

Now suppose $\mathcal{B} = \{x_\alpha\}$ is a poset-upper-triangular basis of M. (In many examples, such bases exist because of extra geometric structure; see Section 4.) Poset-upper-triangularity implies that the elements x_α have convenient vanishing properties when viewed in the product $\prod_{i\in\mathcal{I}} M_i$. The homomorphism ϕ is surjective, so there exists a subset of the image $\phi(\mathcal{B}) \subseteq M'$ which generates M'. We may then ask the following question (see Question 4.8 for the geometric version).

QUESTION 2.8. Is it possible to obtain a poset-upper-triangular basis for $M' \subseteq \prod_{i\in\mathcal{J}} M'_i$ with respect to the induced partial order on $\mathcal{J} \subseteq \mathcal{I}$ from a subset of $\phi(\mathcal{B})$?

This turns out to be a difficult problem. The fundamental obstacle is that the maps in (2.9) do not necessarily behave well with respect to poset-flow-ups. More precisely, the intersection $\mathcal{L}_\mathcal{I}(i) \cap \mathcal{J}$ of a principal order filter $\mathcal{L}_\mathcal{I}(i)$ with the subset $\mathcal{J}$ is *not* necessarily a principal order filter of $\mathcal{J}$ when $i \notin \mathcal{J}$. As a consequence, the images $\phi(\mathcal{B}) = \{\phi(x_\alpha)\}$ need not even be poset-flow-up elements in M' in the sense of Definition 2.2. This means that poset-upper-triangularity with respect to $(\mathcal{I}, <)$ does not immediately translate via ϕ to poset-upper-triangularity with respect to $(\mathcal{J}, <)$. In Section 3, we introduce the combinatorial game of poset pinball, which was created to address these difficulties.

2.4. A criterion for module bases in the graded case.

In the above discussion, we concentrated on the problem of building a poset-upper-triangular module basis for M'. However, in some situations one may wish simply to produce a module basis. In this section we briefly recall a result which, in a graded setting, guarantees that a subset is a module basis (not necessarily poset-upper-triangular). In the applications of this result in Section 4, the ring R is the Borel-equivariant cohomology ring of a point, with coefficients in a field of characteristic zero.

Let R be a graded ring and M an R-module. Suppose M is graded compatibly with the R-module structure in the sense that $M \cong \bigoplus_{k\geq 0} M_k$ as additive groups and the R-module structure takes $R_i \times M_k$ to M_{i+k}. We assume $R_0 \cong \mathbb{F}$. Hence, since M is an R_0-module, it also has the structure of an $\mathbb{F}$-vector space, with each M_k an $\mathbb{F}$-subspace. Let $M_{\leq k} = \bigoplus_{j\leq k} M_j$ denote the subspace of M consisting of graded pieces of degree at most k. The following is [**23**, Proposition A.1].

Proposition 2.9. *Let $\mathbb{F}$ be a field. Let $R = \bigoplus_{i\geq 0} R_i$ be a graded $\mathbb{F}$-algebra such that R_k is finite-dimensional for all $k \geq 0$, and $R_0 \cong \mathbb{F}$. Let M be a free finitely-generated R-module of the form*

$$M = R \otimes_{\mathbb{F}} V$$

for a finite-dimensional graded $\mathbb{F}$-vector space V, where the R-module structure on the right hand side is given by ordinary multiplication on the first factor and the grading on M is given by

$$M_k = \bigoplus_{i+j=k} R_i \otimes_{\mathbb{F}} V_j.$$

Suppose $\{m_{\mu,k}\}$ is a subset of M satisfying

- $\deg(m_{\mu,k}) = k$,
- *the number of $m_{\mu,k}$ of degree k is precisely* $\dim_{\mathbb{F}}(V_k)$*, and*
- *the $\{m_{\mu,k}\}$ are R-linearly independent in M.*

Then the $\{m_{\mu,k}\}$ are an R-module basis of M.

3. Poset pinball: a combinatorial game on directed graphs.

3.1. The poset pinball game.

We now introduce a non-deterministic game which we call poset pinball, by analogy with pinball arcade games, which involve dropping balls on a tilted board. The game can be understood and played independently of the considerations in the previous section; however, the game was designed to address the algebraic difficulties outlined in Section 2.3 (and discussed from a geometric perspective in Section 4). We present several variants of poset pinball; which version one plays depends on the geometric, algebraic, or combinatorial context. Examples are given in Section 3.2. In Section 3.3, we discuss principal order ideals and the role they play in poset pinball.

We begin with the basic structure of the game, common to all variants. Henceforth we assume that the poset $\mathcal{I}$ is finite.

Poset pinball rules and terminology:

(1) Let $(\mathcal{I}, <)$ be a finite partially ordered set. We identify $\mathcal{I}$ with its Hasse diagram, so we think of $\mathcal{I}$ as a directed acyclic graph with vertices the elements of $\mathcal{I}$ and

with a directed edge from i to i' when i covers i' with respect to the partial order. We denote an edge from i to i' by $i \mapsto i'$.

(2) The graph $\mathcal{I}$ is the *pinball board*, or simply the *board*. The vertices are called *pinball slots*, or simply *slots*. At most one pinball can occupy a slot at any time.

(3) Let $\mathcal{J}$ be a fixed subset of $\mathcal{I}$. We call $\mathcal{J}$ an *initial subset*. Note that $\mathcal{J}$ inherits a partial order from $\mathcal{I}$.

(4) We place *pinballs* at the initial subset, i.e., for each element $j \in \mathcal{J}$, we place a pinball at the slot corresponding to j. These pinballs are markers on certain vertices, usually denoted by circling or coloring the vertices.

(5) The directed edges of the graph $\mathcal{I}$ are called *pinball slides*, or simply *slides*. When released, a pinball may *roll down* along a slide, in the direction determined by the directed edge. Specifically, if $i \mapsto i'$ is an edge, a pinball at slot i may roll down to the slot i'.

(6) During the game, we occasionally place *walls* across some slides. A wall across a slide prevents a pinball from rolling down that edge (slide). The initial board has no walls. A wall is never removed once it has been placed.

(7) Fix a total order $\prec$ on the initial subset $\mathcal{J}$ subordinate to the induced partial order on $\mathcal{J}$. We write $\mathcal{J} = \{j_1 \prec j_2 \prec \cdots \prec j_{|\mathcal{J}|}\}$ with respect to this total order.

(8) We now define the procedure for allowing a pinball to roll down (along slides). Suppose a pinball is at slot $i \in \mathcal{I}$. Consider the set of downward-pointing edges with i as top vertex. The pinball at slot i is allowed to roll down to i' as long as there is no wall across the slide $i \mapsto i'$. Hence we consider

$$\{i' \in \mathcal{I} : \text{ there exists an edge } i \mapsto i' \text{ and there is no wall across } i \mapsto i'\}. \tag{3.1}$$

Choose an arbitrary element i' in the set in (3.1) and move the pinball to slot i'. We refer to this as rolling along the slide $i \mapsto i'$. Repeat the above process using the new slot i' in the role of i above, and continue in this manner. We say that the pinball can *roll no further* if at any stage the set in (3.1) is empty, namely there are no lower available slots. When a pinball starting at slot i has rolled down successive slides until it can roll no further, the final slot at which the pinball rests is called the *rolldown of* i and denoted $roll(i)$. We refer to this process of associating to i its rolldown $roll(i)$ as rolling (or dropping) the pinball. This procedure is *not* deterministic because of the choices made when rolling along each slide (just like real-life pinball!). Note also that the rolldown $roll(j)$ of a pinball which was originally at a slot $j \in \mathcal{J}$ might *not* be an element of $\mathcal{J}$.

(9) We drop pinballs successively according to the total order $\prec$ on $\mathcal{J}$. Hence we first drop the pinball from slot j_1 as described above. For every $k = 1, 2, \ldots, |\mathcal{J}|$, after rolling the k-th pinball, we may place more walls along the slides of the board. Each version of pinball has a separate set of rules for placing walls; the details for

each variant are given below. Once the first $k-1$ pinballs are dropped, we drop the pinball at j_k and continue until all $|\mathcal{J}|$ pinballs are dropped.

(10) Fix the board $\mathcal{I}$, the initial subset $\mathcal{J}$, the choice of total order $\prec$, and a particular outcome of poset pinball, which we write as $\{(j, roll(j)) : j \in \mathcal{J}\}$. Then we denote by $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ the set of slots in $\mathcal{I}$ occupied by the rolldown elements, i.e.

$$\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec) := \{roll(j) : j \in \mathcal{J}\} \subseteq \mathcal{I}.$$

We call $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ the *rolldown set* for the given outcome of pinball. We also denote by $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ the set of rolldown elements for the first k pinballs, i.e.

$$\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec) := \{roll(j_\ell) : j_\ell \in \mathcal{J}, 1 \leq \ell \leq k\} = \{roll(j_1), roll(j_2), \ldots, roll(j_k)\} \subseteq \mathcal{I}.$$

We refer to $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ as the *rolldown set up to step k*. We emphasize that since pinball is *not* deterministic, the sets $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ and $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ might *not* be uniquely determined by $\mathcal{I}, \mathcal{J}$ and $\prec$.

The different versions of poset pinball are distinguished by how the walls are placed after each pinball in the initial subset rolls down. We now describe these variants of poset pinball.

Basic pinball. In this version, the walls are placed as follows. Let $k = 1, 2, \ldots, |\mathcal{J}|$. Suppose the k-th pinball j_k has been dropped. We then place a wall across every edge of the form $i \mapsto roll(j_k)$. Hence the walls in basic pinball simply enforce the rule that at most one pinball may occupy a given slot at any time.

Upper-triangular pinball. This version takes into account the partial order structure on $\mathcal{I}$. Let $k = 1, 2, \ldots, |\mathcal{J}|$. Suppose the k-th pinball j_k has been dropped. We then place a wall across:

- every edge of the form $i \mapsto roll(j_k)$ for $i \in \mathcal{I}$, and
- every edge of the form $i \mapsto i'$ for $i \in \mathcal{I}, i' \in \mathcal{L}_{\mathcal{I}}(j_k)$.

The rules of upper-triangular pinball ensure that for each $j \in \mathcal{J}$, the element j has a unique maximal rolldown $roll(j)$ in its principal order ideal.

Betti pinball. This version of pinball is motivated by the geometric applications discussed later, in which we allow the Betti numbers of an underlying topological space to impose additional constraints on the pinball game. Here we assume that $\mathcal{I}$ is a finite graded poset with rank function $\rho : \mathcal{I} \to \mathbb{N}$. We also assume $\boldsymbol{b} = (b_0, b_1, \ldots, b_n)$ is a sequence of nonnegative integers. (In geometric applications, these b_j are in fact the Betti numbers of a topological space, so we refer to $\boldsymbol{b}$ as the *target Betti numbers*; in particular, Betti pinball can be successful only if $b_0 + b_1 + \cdots + b_n = |\mathcal{I}|$.) Let $k = 1, 2, \ldots, |\mathcal{J}|$. Suppose the k-th pinball j_k has been dropped. Then the walls are placed as follows:

- Place a wall across any edge of the form $i \mapsto roll(j_k)$ for $i \in \mathcal{I}$.

- Let $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ denote the rolldown set up to step k. Let $j \in \{0, 1, \ldots, n\}$ and suppose that

$$b_j = |\{i \in \mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec) : \rho(i) = j\}|, \tag{3.2}$$

i.e. there are exactly b_j rolldown elements of rank j at step k. Then place walls across every edge of the form $i \mapsto i'$ where $\deg(i') = j$.

These rules ensure that the number of rolldown elements of rank j do not exceed the given target b_j.

We say that a game of Betti pinball is *successful* if, after all the pinballs in the initial subset $\mathcal{J}$ are dropped, there are precisely b_j rolldowns of rank j for each j. In other words, after a successful game of Betti pinball, the ranks of the elements of the rolldown set $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ are given by the target Betti numbers $\boldsymbol{b}$.

Upper-triangular Betti pinball. This version adds both the walls for upper-triangular pinball and those for Betti pinball at each pinball step. We leave it to the reader to write the rules. As in Betti pinball, we assume we are given a graded poset $\mathcal{I}$ and the data of target Betti numbers $\boldsymbol{b} = (b_0, b_1, b_2, \ldots, b_n)$. Also as in Betti pinball, we say that a game of upper-triangular Betti pinball is *successful* if the ranks of the rolldowns are given by the target Betti numbers.

3.2. Playing pinball: examples.

Here we illustrate our poset pinball game with two concrete examples. In both cases, the ambient graded poset $\mathcal{I}$ is the symmetric group S_4 equipped with the usual Bruhat order. We take the rank of a permutation w to be the standard *length* of w with respect to Bruhat order. For simplicity, we do not draw the entire graph of $\mathcal{I}$ in the figures below, but only those vertices and edges relevant in the game. We label each vertex by its corresponding permutation, factored into simple transpositions. The graph is drawn so that all directed edges point towards the bottom of the page, so the minimal permutation e is at the bottom of the figure. The initial subset $\mathcal{J}$ is indicated by the circled vertices, and the final rolldown set is indicated by the vertices with squares around them. Each example is accompanied by a table recording each step of the poset pinball game as it was played.

EXAMPLE 3.1. In our first example, the rolldown set is in fact unique for basic, upper-triangular, or Betti pinball with $\boldsymbol{b} = (1, 3)$. In particular, the partial order induced on $\mathcal{J}$ by $\mathcal{I}$ is already a total order, so there is a unique total order with respect to which to play pinball. Notice that the rolldown set $\mathcal{R}(\mathcal{J}, \mathcal{I}, \prec)$ in this example is a union of principal order ideals; we explore this phenomenon further in Section 3.3.

The initial subset is $\mathcal{J} = \{e, s_3, s_3s_2, s_3s_2s_1\}$. The final drop-down set is $\mathcal{R}(\mathcal{I}, \mathcal{J}) = \{e, s_3, s_2, s_1\}$. (The reader may wish to explore how the game changes if the Betti numbers are $\boldsymbol{b} = (1, 2, 1)$.)

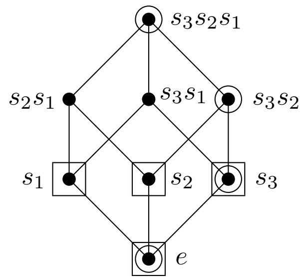

Figure 3.1. Example of basic pinball.

pinball step	w_k	v_k
1	$w_1 = e$	$v_1 = e$
2	$w_2 = s_3$	$v_2 = s_3$
3	$w_3 = s_3s_2$	$v_3 = s_2$
4	$w_4 = s_3s_2s_1$	$v_4 = s_1$

(3.3)

EXAMPLE 3.2. In this example, we play Betti pinball with target Betti numbers $\boldsymbol{b} = (1, 3, 4, 3, 1)$. The final drop-down set $\mathcal{R}(\mathcal{I}, \mathcal{J})$ is indicated by the squared vertices. Dotted lines indicate a path in the partial order, with some intermediate vertices omitted for visual simplicity. For similar reasons, not all edges are drawn in the figure below for elements of rank greater than 2. In this example, the drop-down set $\mathcal{R}(\mathcal{J}, \mathcal{I}, \prec)$ is *not* a union of principal order ideals because of the constraints imposed by the target Betti numbers.

The rolldown set in Figure 3.2 could also be the outcome of a game of upper-triangular Betti pinball. However, if we instead chose the rolldown $roll(s_1s_2s_3s_1s_2s_1) = s_1s_2s_3s_2$ then we would obtain a successful outcome of Betti pinball that is not a successful outcome of upper-triangular Betti pinball.

Note that there are other vertices whose rolldowns are not unique. In particular

- w_8 could roll down to either s_2s_3 or s_3s_2
- w_{10} could roll down to any one of $s_2s_1s_3$, $s_3s_2s_3$, or $s_3s_2s_1$
- w_{11} could roll down to any one of $s_1s_2s_3$, $s_1s_3s_1$, $s_2s_1s_3$, or $s_3s_2s_3$ that w_{10} left unoccupied
- w_{12} could roll down to any one of $s_1s_2s_3s_1$, $s_1s_2s_3s_2$, $s_2s_3s_1s_2$, $s_2s_3s_2s_1$, or $s_3s_1s_2s_1$

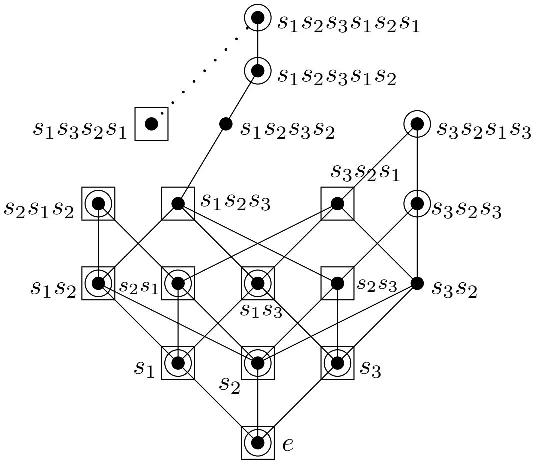

Figure 3.2. An example of Betti pinball.

pinball step	w_k	v_k
1	$w_1 = e = [1,2,3,4]$	$v_1 = e = [1,2,3,4]$
2	$w_2 = s_3 = [1,2,4,3]$	$v_2 = s_3 = [1,2,4,3]$
3	$w_3 = s_2 = [1,3,2,4]$	$v_3 = s_2 = [1,3,2,4]$
4	$w_4 = s_1 = [2,1,3,4]$	$v_4 = s_1 = [2,1,3,4]$
5	$w_5 = s_1s_3 = s_3s_1 = [2,1,4,3]$	$v_5 = s_1s_3 = s_3s_1 = [2,1,4,3]$
6	$w_6 = s_1s_2 = [2,3,1,4]$	$v_6 = s_1s_2 = [2,3,1,4]$
7	$w_7 = s_2s_1 = [3,1,2,4]$	$v_7 = s_2s_1 = [3,1,2,4]$
8	$w_8 = s_3s_2s_3 = [1,4,3,2]$	$v_8 = s_2s_3 = [1,3,4,2]$
9	$w_9 = s_2s_1s_2 = [3,2,1,4]$	$v_9 = s_2s_1s_2 = [3,2,1,4]$
10	$w_{10} = s_3s_2s_1s_3 = [4,1,3,2]$	$v_{10} = s_3s_2s_1 = [4,1,2,3]$
11	$w_{11} = s_1s_2s_3s_1s_2 = [3,4,2,1]$	$v_{11} = s_1s_2s_3 = [2,3,4,1]$
12	$w_{12} = s_1s_2s_3s_1s_2s_1 = [4,3,2,1]$	$v_{12} = s_1s_3s_2s_1 = [4,2,1,3]$

(3.4)

3.3. Principal order ideals and poset pinball.

In this section, we briefly explore the role played by principal order ideals in poset pinball. We are motivated by our geometric applications, in which principal order ideals can correspond naturally to subvarieties in an ambient variety (cf. Remark 3.5).

We begin with a simple statement about basic pinball.

PROPOSITION 3.3. *Let $(\mathcal{I}, <)$ be a finite poset and let $\mathcal{J} \subseteq \mathcal{I}$ be a subset. Let $\prec$ be a total ordering on $\mathcal{J}$ compatible with the partial order $<$ induced from $\mathcal{I}$. Suppose $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec) \subseteq \mathcal{I}$ is a rolldown set from a game of basic pinball played with board $\mathcal{I}$, initial set $\mathcal{J}$, and total order $\prec$. Then $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ is a union of principal order ideals of $\mathcal{I}$.*

Proof. A subset $\mathcal{K} \subseteq \mathcal{I}$ is a union of principal order ideals if and only if for all $i \in \mathcal{K}$, the principal order ideal $\mathcal{L}_{\mathcal{I}}(i)$ is entirely contained in $\mathcal{K}$. Let $\mathcal{J} = \{j_1, j_2, \ldots, j_{|\mathcal{J}|}\}$ be the totally-ordered initial subset. We will induct on $k = 1, 2, \ldots, |\mathcal{J}|$ to show that each $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ is a union of principal order ideals. In the base case, the rolldown $roll(j_1)$ must be the minimal element in $\mathcal{I}$ by definition of basic pinball. A minimal element $\{roll(j_1)\}$ is a principal order ideal, so the claim holds for $k = 1$. Now assume that $\mathcal{R}_{k-1}(\mathcal{I}, \mathcal{J}, \prec)$ is a union of principal order ideals. Let $roll(j_k) \leq j_k$ be a rolldown of j_k. If $i < roll(j_k)$ then i must be in the rolldown set $\mathcal{R}_{k-1}(\mathcal{I}, \mathcal{J}, \prec)$ by the rules of basic pinball. Hence $\mathcal{L}_{\mathcal{I}}(i)$ is contained in

$$\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec) = \mathcal{R}_{k-1}(\mathcal{I}, \mathcal{J}, \prec) \cup \{roll(j_k)\}.$$

The rolldown set $\mathcal{R}_{k-1}(\mathcal{I}, \mathcal{J}, \prec)$ is itself a union of principal order ideals by the inductive hypothesis. Hence $\mathcal{R}_k(\mathcal{I}, \mathcal{J}, \prec)$ is also a union of principal order ideals. The case $k = |\mathcal{J}|$ proves the proposition. $\square$

The previous proposition only applies to basic pinball. In upper-triangular or Betti pinball, the additional walls placed during the game imply that the resulting rolldown set $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ may not be a union of principal order ideals. Indeed, Example 3.2 is an instance of Betti pinball in which the associated rolldown set $\mathcal{R}(\mathcal{I}, \mathcal{J}, \prec)$ is not a union of principal order ideals. On the other hand, in many examples (like the example in the Introduction or Example 6.13), Betti pinball does produce rolldown sets which are unions of principal order ideals. This leads us to ask the following.

Question 3.4. What are combinatorial conditions on $\mathcal{I}, \mathcal{J}, \prec$, and (in the case of Betti pinball) the target Betti numbers $(b_0, \ldots, b_n)$ which guarantee that outcomes of upper-triangular or Betti pinball are unions of principal order ideals?

A concrete answer to this question would yield new perspectives on the geometric problems that motivate our pinball game, such as computing cohomology rings or Betti numbers.

Remark 3.5. There is one easy answer to Question 3.4. If the initial subset $\mathcal{J}$ is itself a principal order ideal $\mathcal{L}_{\mathcal{I}}(i)$ for some $i \in \mathcal{I}$ then *any* version of pinball results in a union of principal order ideals since the rolldown set equals the original set $\mathcal{J}$. This situation arises naturally in geometric contexts, as we show in the following two examples.

(1) The poset $\mathcal{I}$ is the Weyl group W of a complex reductive algebraic group $\mathcal{G}$, identified with the T-fixed points of its flag variety $\mathcal{G}/\mathcal{B}$, where T is a maximal torus in the Borel subgroup $\mathcal{B} \subseteq \mathcal{G}$. In this case, a principal order ideal $\mathcal{L}_W(w)$ of $w \in W$ corresponds naturally to the T-fixed points in a Schubert subvariety of $\mathcal{G}/\mathcal{B}$.

(2) Let T be a torus and let X be a complex projective algebraic variety (possibly singular) equipped with a T-action that has isolated T-fixed points X^T. Choose a one-parameter subgroup $S : \mathbb{G}_m \to T$ with $X^S = X^T$. Then X is partitioned into locally closed subsets X_p defined by

$$X_p := \{x \in X : \lim_{z \mapsto 0} S(z) \cdot x = p\}.$$

The disjoint union $\bigsqcup X_p$ is called a Bialynicki-Birula decomposition of X [**6**]. Moreover Knutson states that X^T can be given a poset structure by taking the transitive closure of the rule that $p \leq q$ when $p \in \overline{X^q}$ [**26**]. The principal order ideal $\mathcal{L}_{X^T}(p)$ of $p \in X^T$ then corresponds naturally to the T-fixed points in $\overline{X_p}$.

4. Poset pinball for GKM-compatible subspaces.

4.1. Background: GKM theory in equivariant cohomology.

The algebraic questions discussed in the previous sections arise naturally in equivariant algebraic topology. Suppose G is a topological group and X is a topological space with a continuous G-action. Let pt denote the topological space consisting of one point, equipped with the trivial G-action, and let E_G^* denote a generalized equivariant cohomology theory with a commutative cup product. (Examples include Borel-equivariant cohomology $H_G^*(-;\mathfrak{r})$ for various coefficient rings $\mathfrak{r}$, topological equivariant K-theory in the sense of Atiyah and Segal, and equivariant cobordism; cf. [**33**, Chapter XIII].) Then $E_G^*(\mathrm{pt})$ is a commutative ring, and $E_G^*(X)$ is naturally an $E_G^*(\mathrm{pt})$-module for any G-space X via the map induced on cohomology by the G-equivariant map $X \to \mathrm{pt}$.

We work in a situation in which $E_G^*(X)$ has a well-studied combinatorial description, often called *GKM theory* due to an influential manuscript of Goresky–Kottwitz–MacPherson [**18**]. We present one of many variations and generalizations of GKM theory in the literature (see [**21**] and references therein).

In this manuscript we say *the GKM package* holds for a G-space X when the following statements hold.

- The G-fixed set X^G consists of countably many isolated points, i.e. $X^G \cong \bigcup_{i \in \mathcal{I}} F_i$ with $F_i \cong \mathrm{pt}$ for all $i \in \mathcal{I}$.
- The indexing set $\mathcal{I}$ for the fixed points $\bigcup_{i \in \mathcal{I}} F_i$ may be equipped with a graded partial order such that there are only finitely many elements of each rank. We denote the rank function by $\rho_X : \mathcal{I} \to \mathbb{N}$.
- The space X is a stratified G-space $X = \bigcup_{i \in \mathcal{I}} X_i$ with $F_i \in X_i$ for each $i \in \mathcal{I}$ and the cohomology $E_G^*(X) = \varprojlim E_G^*(X_i)$.
- The restriction map
$$\iota^* : E_G^*(X) \to \prod_{i \in \mathcal{I}} E_G^*(F_i) \cong \prod_{i \in \mathcal{I}} E_G^*(\mathrm{pt}) \tag{4.1}$$
is injective.
- There exist nonzero equivariant cohomology classes $e_{ij} \in E_G^*(\mathrm{pt})$ satisfying the following conditions.
  - If $e_{ij} \neq 1$ then i and j are comparable in $\mathcal{I}$, and
  - if there is a covering relation between $i > j$ in $\mathcal{I}$ then $e_{ij} \neq 1$,

and the image of ι^* in (4.1) is precisely

$$\text{image}(\iota^*) = \left\{ x \in \prod_{i \in \mathcal{I}} E_G^*(\text{pt}) \,\middle|\, e_{ij} \,\big|\, (x(i) - x(j)) \ \text{ for all } j < i \text{ in the partial order} \right\}. \tag{4.2}$$

- There exists a module basis $\{x_i\}_{i \in \mathcal{I}}$ for $E_G^*(X)$ that is indexed by the (isolated) fixed points F_i, where the $x_i \in E_G^*(X)$ are equivariant cohomology classes satisfying

$$x_i(i') = 0 \ \text{ for } i' \not\geq i \tag{4.3}$$

(where $x_i(i')$ notates the i'-th component of the image of x_i in $\prod_{i \in \mathcal{I}} E_G^*(\text{pt})$) and

$$x_i(i) \text{ generates the ideal } e_i E_G^*(\text{pt}) \tag{4.4}$$

where $e_i := \prod_{j<i} e_{ij}$. (In this case $\{x_i\}_{i \in \mathcal{I}}$ is a free $E_G^*(\text{pt})$-module basis of $E_G^*(X)$ as proven in, e.g., [**21**, Proposition 4.1].)

The vanishing condition in Equation (4.3) says exactly that the x_i are poset-flow-ups, so the $\{x_i\}_{i \in \mathcal{I}}$ are in fact a poset-upper-triangular module basis by Proposition 2.6.

For convenience, we often collect the information needed to determine $E_G^*(X)$ using Equation (4.2) in a directed, labeled graph called the *GKM graph of X*. The vertices of the GKM graph are the fixed points F_i, or equivalently the elements of the poset $\mathcal{I}$. There is an edge between F_i and F_j exactly when $e_{ij} \neq 1$. If it exists, the edge between F_i and F_j is labeled e_{ij} and is directed from F_i to F_j exactly when $i > j$. Note that the GKM graph of X contains the Hasse diagram of the poset $\mathcal{I}$, but possibly includes edges which are not poset covering relations.

We now recall some situations in which the GKM package holds.

Remark 4.1. (1) Let $G = T$ be a compact torus and let $E_G^* = H_T^*(-;\mathbb{F})$ be Borel-equivariant cohomology with coefficients in a field $\mathbb{F}$ of characteristic zero. Let (M, ω, Φ) be a compact Hamiltonian T-manifold with moment map $\Phi : M \to \mathfrak{t}^*$. Suppose that M has finitely many (isolated) fixed points, and for every codimension 1 subtorus $K \subseteq T$, each connected component of the fixed submanifold M^K has (real) dimension less than or equal to 2. Assume in addition that there are finitely many one-dimensional T-orbits in M and that there exists a T-invariant Palais–Smale metric. Let $\Psi := \Phi^\xi$ denote a generic component of the moment map and consider its negative gradient flow with respect to the given Palais–Smale metric. Denote by $\lambda(p)$ the Morse index of Ψ at a critical point $p \in M^T$. It is known [**17**, Remark 4.3] that Ψ is index-increasing, i.e. if $p, q \in M^T$ and $\Psi(p) < \Psi(q)$, then $\lambda(p) < \lambda(q)$. We give the fixed point set M^T a partial order by defining $p < q$ precisely if $\Psi(p) < \Psi(q)$ and there exists a one-dimensional T-orbit O whose closure contains p and q. In particular $p < q$ implies $\lambda(p) < \lambda(q)$. (The Hasse diagram of this poset M^T has an edge between p and q exactly when $p < q$ and $\lambda(q) = \lambda(p)+2$, i.e. the Morse index increases from p to q by precisely 2.) By standard equivariant symplectic geometry arguments, the Morse index is always even. Thus the function $\rho : M^T \to \mathbb{Z}_{\geq 0}$ defined by $\rho(p) := \lambda(p)/2$ is a well-defined rank function, and the fact that Ψ is index-increasing implies that ρ gives M^T the structure of a graded poset.

Moreover, in this situation it is also known that the map (4.1) is injective and that the description of the image of ι^* given in (4.2) is valid [**18**, Theorem 14.1 (9)]. Finally, in this situation, Goldin and Tolman prove that there exists a collection $\{x_p\}_{p \in M^T}$ of canonical classes [**17**, Definition 1.1] in $H_T^*(M; \mathbb{F})$ indexed by the T-fixed points M^T. These canonical classes give a module basis for $H_T^*(M; \mathbb{F})$ satisfying Equations (4.3) and (4.4) with respect to the partial order on M^T defined above [**17**, Proposition 4.4]. In particular, they give a poset-upper-triangular module basis with respect to the partial order on M^T. Moreover the x_p are homogeneous classes of degree $2\rho(p)$. (In fact, their canonical classes form a module basis for $H_T^*(M; \mathbb{Z})$ with integer coefficients, but we will not use that here.)

(2) Suppose $\mathcal{G}$ is a Kac–Moody group and $\mathcal{P}$ a parabolic subgroup with corresponding flag variety $X = \mathcal{G}/\mathcal{P}$. For $G = T_{\mathcal{G}}/Z(\mathcal{G})$ where $T_{\mathcal{G}}$ is the maximal torus of $\mathcal{G}$ and $Z(\mathcal{G})$ is its center, Equations (4.1) and (4.2) hold for many cases of E_G^* (see e.g. [**21**]). The set of T-fixed points of $\mathcal{G}/\mathcal{P}$ may be identified with the quotient $W_{\mathcal{G}}/W_{\mathcal{P}}$ where $W_{\mathcal{G}}, W_{\mathcal{P}}$ are the Weyl groups of $\mathcal{G}, \mathcal{P}$ respectively. The index set $\mathcal{I}$ may be identified with this countable quotient and given a poset structure and rank function induced by the Bruhat order and length on $W_{\mathcal{G}}$ respectively. In the cases of Borel-equivariant cohomology $E_T^* = H_T^*(-)$ or equivariant K-theory $E_T^* = K_T^*$, the equivariant Schubert classes $\{\sigma_w\}_{w \in W_{\mathcal{G}}/W_{\mathcal{P}}}$ corresponding to the Schubert varieties in $\mathcal{G}/\mathcal{P}$ form a module basis satisfying Equations (4.3) and (4.4) (see e.g. [**31**] and also [**42**], [**43**]). Unlike the previous example, this one includes cases of infinite posets $\mathcal{I}$. Harada–Henriques–Holm give some explicit computations for $\Omega SU(2)$, an infinite-dimensional affine Grassmannian [**21**].

We close the section with a brief discussion of a more general class of spaces for which poset-upper-triangular bases satisfying conditions (4.3) and (4.4) exist in Borel-equivariant cohomology.

THEOREM 4.2. *Let $S \cong S^1$ or $S \cong \mathbb{C}^*$ be a rank-one torus and let X be an S-space satisfying the following conditions*:

(1) *the set of S-fixed points X^S is isolated*;

(2) *the set $(X^S, <)$ can be equipped with a poset structure satisfying the conditions of Section* 2.1;

(3) *the S-equivariant cohomology $H_S^*(X; \mathbb{F})$ is a free $H_S^*(\mathrm{pt}; \mathbb{F})$-module*; *and*

(4) *the ring map $\iota_X^* : H_S^*(X; \mathbb{F}) \to H_S^*(X^S; \mathbb{F})$ induced by the inclusion $\iota_X : X^S \hookrightarrow X$ is injective.*

Then the module $H_S^*(X; \mathbb{F})$ has a poset-upper-triangular basis with respect to any choice of total order compatible with the partial order on X^S.

PROOF. The space X has isolated S-fixed points and $H_S^*(X;\mathbb{F})$ is a free $H_S^*(\mathrm{pt};\mathbb{F})$-module isomorphic to a submodule of $\bigoplus_{i\in X^S} H_S^*(\mathrm{pt};\mathbb{F})$. The ring $H_S^*(\mathrm{pt};\mathbb{F})$ is isomorphic to the polynomial ring $\mathbb{F}[t]$ in one variable, which is a PID. The claim now follows from part (4) of Proposition 2.6. □

REMARK 4.3. The poset structure in Condition (2) of Theorem 4.2 is often induced from an S-action for S a subgroup of T, as described in Remark 3.5 for complex projective varieties equipped with a T-action and in Remark 4.1 (1) for Hamiltonian T-manifolds. The existence of a paving by complex affine cells, as is sometimes given by a Bialynicki-Birula decomposition, implies Conditions (3) and (4).

4.2. Subspaces of GKM spaces.

GKM theory is a powerful combinatorial tool that can provide natural and computationally convenient bases for $E_G^*(X)$. However, there are many G-spaces for which the GKM package does not hold. This brings us to the central geometric problem of this manuscript. More specifically, we will describe a geometric framework within which we propose to exploit the GKM theory on an ambient space X in order to analyze the equivariant geometry of a subspace $Y \subseteq X$. We carried out this program for a special case in a previous paper [**23**]; one of the main goals of the current manuscript is to both generalize and formalize the techniques therein.

Let G be a topological group and suppose that X is a G-space. Throughout this section we assume that the G-action on X is such that the GKM package holds for X as described in Section 4.1. Let $X^G = \bigcup_{i\in\mathcal{I}} F_i$ denote the set of (isolated) fixed points and let $\{x_i\}_{i\in\mathcal{I}}$ denote a choice of poset-upper-triangular basis for $E_G^*(X)$.

We wish to analyze subspaces Y of X using GKM theory on X. For this to be feasible, we need to place certain conditions on Y. The following definition is motivated in part by results discussed above, e.g. Theorem 4.2.

DEFINITION 4.4. Let G be a topological group, X a G-space, and E_G^* an equivariant cohomology theory. Let $Y \subseteq X$ be a subspace of X and $G' \subseteq G$ a topological subgroup of G preserving Y. We call the pair (Y, G') *GKM-compatible* with the pair (X, G) with respect to E_G^* if the following conditions hold.

(1) The G'-fixed set of Y is the intersection of Y with X^G, i.e.

$$Y^{G'} = Y \cap X^G. \tag{4.5}$$

(2) The G'-equivariant cohomology $E_{G'}^*(Y)$ is a free $E_{G'}^*(\mathrm{pt})$-module.

(3) The ring map $\iota_Y^* : E_{G'}^*(Y) \to E_{G'}^*(Y^{G'})$ induced by the inclusion $\iota_Y : Y^{G'} \hookrightarrow Y$ is injective.

When the groups are clear from context, we may simply say that Y is a GKM-compatible subspace of X. Similarly, when there is no ambiguity we often neglect to mention the choice of cohomology theory E_G^*.

Remark 4.5. In certain cases, Conditions (2) and (3) are related. For instance, if $G = T$ is a compact torus and $E_T^* = H_T^*(-, \mathbb{F})$ is Borel-equivariant cohomology with coefficients in a field $\mathbb{F}$ of characteristic zero, then Condition (2) implies Condition (3).

Suppose now that (Y, G') is GKM-compatible with (X, G) with respect to E_G^*. Since we assume that $X^G = \bigcup_{i \in \mathcal{I}} F_i$ consists of isolated fixed points, the intersection $Y \cap X^G$ is indexed by some subset $\mathcal{J}$ of $\mathcal{I}$. In this setting, the relationship between $E_G^*(X)$ to $E_{G'}^*(Y)$ fits into the algebraic framework discussed in Section 2.3. Indeed, consider first the sequence of ring homomorphisms

$$E_G^*(X) \longrightarrow E_{G'}^*(X) \longrightarrow E_{G'}^*(Y) \tag{4.6}$$

where the first map is the forgetful map associated to the inclusion of groups $G' \hookrightarrow G$ and the second is induced from the inclusion of spaces $Y \hookrightarrow X$. (For $X = Y = \mathrm{pt}$, the composition (4.6) specializes to the usual forgetful map $E_G^*(\mathrm{pt}) \to E_{G'}^*(\mathrm{pt})$.) Moreover, Condition (1) in the definition of GKM-compatibility implies that the map (4.6) fits into a commutative diagram

$$\begin{tikzcd} E_G^*(X) \arrow[r, "\iota^*"] \arrow[d] & E_G^*(X^G) \cong \prod_{i \in \mathcal{I}} E_G^*(\mathrm{pt}) \arrow[d, "\pi"] \\ E_{G'}^*(Y) \arrow[r, "\iota_Y^*"'] & E_{G'}^*(Y^{G'} = Y \cap X^G) \cong \prod_{i \in \mathcal{J}} E_{G'}^*(\mathrm{pt}) \end{tikzcd} \tag{4.7}$$

where the right arrow π is 0 on the components $i \notin \mathcal{J}$ and is the forgetful map $E_G^*(\mathrm{pt}) \to E_{G'}^*(\mathrm{pt})$ on the components $i \in \mathcal{J}$. By assumption, the GKM package holds for X, so the restriction map ι^* is injective. Finally, Condition (3) of the definition of GKM-compatibility assures us that ι_Y^* is an injection, and Condition (2) ensures that we can find a module basis for $E_{G'}^*(Y)$. The commutative diagram (4.7) is thus an instance of the diagram (2.9) in Section 2.3, with $R = E_G^*(\mathrm{pt}), R' = E_{G'}^*(\mathrm{pt}), M = E_G^*(X)$, and $M' = E_{G'}^*(Y)$. The forgetful map $E_G^*(\mathrm{pt}) \to E_{G'}^*(\mathrm{pt})$ satisfies the required property (2.8) for $\gamma : R \to R'$ by naturality. In contrast to Section 2.3, we do *not* assume here that the map $E_G^*(X) \to E_{G'}^*(Y)$ is surjective. We discuss this further below.

We first give a rich class of examples of GKM-compatible subspaces of GKM spaces.

Remark 4.6. Let $X = \mathcal{G}/\mathcal{B}$ be the flag variety of a complex reductive algebraic group. As observed in Remark 4.1, the GKM package holds for X with respect to the standard action of the maximal torus T of $\mathcal{G}$. In Section 5 we define a family of subvarieties $Y \subseteq X$ called Hessenberg varieties. In many cases, there is a natural S^1 subtorus of T which preserves Y and makes (Y, S^1) GKM-compatible with the pair (X, T) with respect to Borel-equivariant cohomology with $\mathbb{Q}$ coefficients. Such examples include the regular nilpotent Hessenberg varieties in classical Lie type and the Springer varieties in Lie type A.

Hessenberg varieties are our primary examples, but the following question arises naturally.

Question 4.7. What are other classes of GKM-compatible subspaces of GKM spaces?

We now phrase our main question in these geometric terms (see Question 2.8 for the algebraic version). Let $\{x_i\}_{i\in\mathcal{I}}$ be a poset-upper-triangular module basis of $E_G^*(X)$ and let $\overline{x}_i$ denote the image of x_i under the composition $E_G^*(X) \to E_{G'}^*(Y)$, given in (4.6).

Question 4.8. Suppose the GKM package holds for (X, G) and suppose (Y, G') is GKM-compatible with (X, G). Under what circumstances can we exploit the GKM theory on X to explicitly construct a computationally convenient module basis for $E_{G'}^*(Y)$ using the images $\{\overline{x}_i\}_{i\in\mathcal{I}}$ in $E_{G'}^*(Y)$?

Ideally we would like the module basis for $E_{G'}^*(Y)$ to be a linear combination of elements in $\{\overline{x}_i\}_{i\in\mathcal{I}}$ or even to be a subset of $\{\overline{x}_i\}_{i\in\mathcal{I}}$. For this to be possible, we need that

$$\text{the ring homomorphism } E_G^*(X) \to E_{G'}^*(Y) \text{ is a surjection.} \tag{4.8}$$

Condition (4.8) is not included in the definition of GKM-compatibility because in many applications, we can use other topological data, together with poset pinball, to deduce surjectivity. For instance, when $G = T$ is a compact torus and $E_T^* = H_T^*(-;\mathbb{F})$ is Borel-equivariant cohomology with coefficients a field of characteristic zero, the ring surjection in Condition (4.8) follows from a successful game of Betti pinball; and to play Betti pinball all we need is prior knowledge of the target Betti numbers. We give concrete examples of such arguments in the cases of Peterson varieties in classical Lie type in Section 5, and Springer varieties in type A in Section 6. They were also part of our arguments in previous work [**23**]. We discuss this in more detail in Section 4.3.

Remark 4.9. In some situations, Condition (4.8) may be seen to hold directly without using Betti-number arguments. For instance, suppose the spaces X and Y in the discussion above are complex algebraic varieties. If there is a G-invariant affine paving of X and a subset of those affine cells yields an affine paving of Y then Condition (4.8) holds for any subgroup $G' \subseteq G$.

4.3. Borel-equivariant cohomology with field coefficients.

We now specialize to the case where $G = T$ is a compact torus and $E_T^* = H_T^*(-;\mathbb{F})$ is Borel-equivariant cohomology with coefficients in a field $\mathbb{F}$ of characteristic zero. Let X denote an ambient T-space satisfying the GKM package and let (Y, S) be a GKM-compatible subspace for a subtorus S of T. Moreover, let $\{x_i\}$ be a poset-upper-triangular basis for $H_T^*(X;\mathbb{F})$ indexed by the set of isolated fixed points $\mathcal{I} = X^T$ and let $\rho_X : \mathcal{I} \to \mathbb{N}$ be the rank function on the poset $\mathcal{I}$. Borel-equivariant cohomology is a graded theory, so we may speak of the degree of a class x_i. For most of the discussion we assume that

$$\text{the ordinary cohomology of the spaces } X \text{ and } Y \text{ vanish in odd degrees.} \tag{4.9}$$

Remark 4.10. It is sometimes possible to apply the theory of GKM spaces and GKM-compatible subspaces without Assumption (4.9) on the ordinary cohomology of

X and Y. For instance, mod-2 GKM theory (e.g. [**22**] and references therein) can be applied to *real* projective spaces $\mathbb{RP}^n$ which have non-vanishing cohomology in odd degree (working with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$). It would be possible to rephrase our theorems and the game of poset pinball to account for these mod-2 GKM spaces and Borel-equivariant cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, but we have chosen to simplify exposition by assuming (4.9).

Assumption (4.9) implies Conditions (2) and (3) in the definition of GKM-compatibility, as the following remark elaborates.

Remark 4.11. Suppose X is as above and suppose Y is a T'-invariant subspace satisfying the vanishing assumption (4.9). It follows that the Leray–Serre spectral sequence for Borel-equivariant cohomology of Y collapses, so $H^*_S(Y;\mathbb{F})$ is a free $H^*_S(\mathrm{pt};\mathbb{F})$-module. By the localization theorem in Borel-equivariant cohomology (e.g. [**3**, Theorem (3.5)], [**20**, Theorem 11.4.4]) the inclusion $\iota : Y^S \hookrightarrow Y$ induces an injection

$$\iota^* : H^*_S(Y;\mathbb{F}) \hookrightarrow H^*_S(Y^S;\mathbb{F}),$$

so Conditions (2) and (3) of GKM-compatibility are automatically satisfied for such Y.

We need a homogeneity condition on the classes x_i in the module basis, so we define the following.

Definition 4.12. Let X be as above. The poset-upper-triangular basis $\{x_i\}$ of $H^*_T(X;\mathbb{F})$ is *rank-homogeneous* if each x_i has homogenous degree with respect to the standard $\mathbb{Z}$-grading on Borel-equivariant cohomology, and if

$$\deg(x_i) = 2\rho_X(i) \text{ for all } i \in \mathcal{I}.$$

For instance, the bases in Remark 4.1 for Borel-equivariant cohomology are all rank-homogeneous. We now tackle Question 4.8 in this setting, namely we build a module basis for $H^*_S(Y;\mathbb{F})$ from the basis $\{x_i\}_{i\in\mathcal{I}}$ for $H^*_T(X;\mathbb{F})$. As was discussed at the end of Section 4.2, we often know the Betti numbers $b_j := \dim_{\mathbb{F}} H^{2j}(Y;\mathbb{F})$ of Y in advance but do not know that the ring map $H^*_T(X;\mathbb{F}) \to H^*_S(Y;\mathbb{F})$ is surjective. Poset pinball can be useful in this situation: a successful game of Betti pinball using the target Betti numbers $b_j = \dim_{\mathbb{F}} H^{2j}(Y;\mathbb{F})$ may yield a module basis for $H^*_S(Y;\mathbb{F})$, from which we may deduce surjectivity. We explain this in the next two propositions. Recall we denote by $\overline{x}$ the image in $H^*_S(Y;\mathbb{F})$ of a class x in $H^*_T(X;\mathbb{F})$.

Proposition 4.13. *Let T be a compact torus and X a T-space for which the GKM package holds. Let $S \subseteq T$ be a subtorus and suppose (Y, S) is GKM-compatible with (X, T). We assume $H^*(Y;\mathbb{F})$ is finite-dimensional. Suppose $\{x_i\}_{i\in\mathcal{I}}$ is a rank-homogeneous poset-upper-triangular $H^*_T(\mathrm{pt},\mathbb{F})$-module basis of $H^*_T(X,\mathbb{F})$. Suppose the ordinary cohomology of Y vanishes in odd degrees and let $b_j := \dim_{\mathbb{F}} H^{2j}(Y;\mathbb{F})$ be the even Betti numbers of Y. Suppose there exists a subset $\mathcal{K} \subseteq \mathcal{I}$ such that the images $\mathcal{B}' := \{\overline{x}_k\}_{k\in\mathcal{K}}$ in $H^*_S(Y;\mathbb{F})$ under the ring map $H^*_T(X;\mathbb{F}) \to H^*_S(Y;\mathbb{F})$ in (4.6)*

(1) *are $H^*_S(\mathrm{pt},\mathbb{F})$-linearly independent in $H^*_S(Y;\mathbb{F})$, and*

(2) *for each* $j \in \mathbb{Z}_{\geq 0}$ *there exist precisely* b_j *elements in* $\mathcal{B}'$ *of homogeneous degree* $2j$.

Then $\mathcal{B}'$ *is a* $H^*_S(\mathrm{pt}, \mathbb{F})$*-module basis for* $H^*_S(Y)$.

PROOF. We apply Proposition 2.9 using $R = H^*_S(\mathrm{pt}; \mathbb{F}), M = H^*_S(Y; \mathbb{F})$, and $V = H^*(Y; \mathbb{F})$, where Conditions (1) and (2) above are the hypotheses of Proposition 2.9. □

REMARK 4.14. Note that we cannot apply Proposition 2.6 (2) in the proof above since we are not assuming that the subset $\mathcal{K}$ is obtained using the algorithm described in Proposition 2.4. For instance we make no assumptions about whether the elements $\overline{x_k}$ span the appropriate vanishing submodules.

If a subset $\mathcal{K}$ in the above proposition exists, then the $\{\overline{x}_k\}_{k \in \mathcal{K}}$ form a module basis for $H^*_S(Y; \mathbb{F})$, and so the ring map $H^*_T(X; \mathbb{F}) \to H^*_S(Y; \mathbb{F})$ is surjective. In other words, we may deduce surjectivity from the Betti numbers—though it may be challenging to find a subset $\mathcal{K}$. The following proposition shows that poset pinball can sometimes accomplish this.

PROPOSITION 4.15. *Suppose we have* $T, X, S, Y, \{x_i\}_{i \in \mathcal{I}}, b_j := \dim_{\mathbb{F}} H^{2j}(Y; \mathbb{F})$ *as in Proposition* 4.13. *Let* $\mathcal{J}$ *be the* S*-fixed points of* Y *and* $\mathcal{I}$ *be the* T*-fixed points of* X. *Suppose that* either *of the following holds for poset pinball played with ambient poset* $\mathcal{I}$ *and initial subset* $\mathcal{J}$:

(1) *An instance of upper-triangular Betti pinball with target Betti numbers* $\boldsymbol{b} = (b_0, b_1, b_2, \ldots)$ *is successful.*

(2) *An instance of Betti pinball with target Betti numbers* $\boldsymbol{b} = (b_0, b_1, b_2, \ldots)$ *is successful, and the classes corresponding to the rolldown set* $\{\overline{x}_{roll(j)}\}_{j \in \mathcal{J}}$ *are* $H^*_S(\mathrm{pt}; \mathbb{F})$*-linearly independent.*

Then the classes corresponding to the rolldown set $\{\overline{x}_{roll(j)}\}_{j \in \mathcal{J}}$ *are an* $H^*_S(Y; \mathbb{F})$*-module basis for* $H^*_S(Y; \mathbb{F})$.

PROOF. If successful, Betti pinball yields the correct nonzero Betti numbers $(b_0, b_1, b_2, \ldots)$ by construction. Hence in either case, Condition (2) of Proposition 4.13 is satisfied by the classes corresponding to the rolldowns $roll(j)$ for $j \in \mathcal{J}$. It remains to verify Condition (1). In the first case, the classes corresponding to the rolldowns are poset-upper-triangular by the rules of upper-triangular pinball; by Proposition 2.6 they are linearly independent. In the second, linear independence is assumed. The result follows. □

We refer to any basis found using these methods as a *poset pinball basis.*

REMARK 4.16. Alternatively, if we know in advance that the map $H^*_T(X; \mathbb{F}) \to H^*_T(Y; \mathbb{F})$ is surjective, then it would be possible to deduce the Betti numbers of Y from a successful game of upper-triangular pinball together with an argument that the classes corresponding to the rolldowns satisfy the conditions of Proposition 2.4.

REMARK 4.17. Betti pinball often yields a set of elements in $H_S^*(Y;\mathbb{F})$ which are *not* upper-triangular with respect to the original partial order $<$ on $\mathcal{I}$. In fact, Example 6.13 shows a case in which *no* successful game of Betti pinball yields a poset-upper-triangular set of elements in $H_S^*(Y;\mathbb{F})$. However we can frequently find a total order $\prec$ compatible with the original partial order, with respect to which the classes associated to the rolldown set *are* upper-triangular. Applying Proposition 2.6 with respect to the total order $\prec$ we conclude that these classes are linearly independent. If Betti pinball is successful in this case, Proposition 4.15 guarantees that the set forms a module basis.

We close this section by addressing Question 4.8 via matchings with respect to different integer functions on the underlying posets. This is a complementary approach to pinball that is easier to use in some contexts. Betti pinball does not determine the degree $\rho_X(roll(j))$ of the rolldown of the vertex $j \in \mathcal{J}$. However, in some cases, geometric considerations on the subspace Y naturally give rise to a function $\deg_Y : \mathcal{J} \to \mathbb{Z}_{\geq 0}$ with the property that

$$b_k = \dim H^{2k}(Y;\mathbb{F}) = |\{j \in \mathcal{J} : \deg_Y(j) = k\}|.$$

For example many varieties Y have a paving by complex affine cells, as mentioned in Remark 4.9 and described in special cases in Sections 5 and 6. In this situation, it is natural to define $\deg_Y(j)$ to be the complex dimension of the affine cell of Y containing the S-fixed point of Y associated to $j \in \mathcal{J}$. Given $\deg_Y$ we could define a new version of poset pinball in which we require that rolldowns satisfy $\rho_X(roll(j)) = \deg_Y(j)$. Instead of taking this approach, we construct module bases for $H_S^*(Y;\mathbb{F})$ directly, using matchings compatible with rank functions (and defined below).

THEOREM 4.18. *Let T be a compact torus and X a T-space for which the GKM package holds. Let $S \subseteq T$ be a subtorus and suppose (Y,S) is GKM-compatible with (X,T). We assume $H^*(Y;\mathbb{F})$ is finite-dimensional. Suppose $\{x_i\}_{i\in\mathcal{I}}$ is a rank-homogeneous poset-upper-triangular $H_T^*(\mathrm{pt},\mathbb{F})$-module basis of $H_T^*(X,\mathbb{F})$. Suppose the ordinary cohomology of Y vanishes in odd degrees and let $b_k := \dim_{\mathbb{F}} H^{2k}(Y;\mathbb{F})$ be the even Betti numbers of Y. Suppose $\deg_Y : \mathcal{J} \to \mathbb{Z}_{\geq 0}$ is a function with*

$$b_k = |\{j \in \mathcal{J} : \deg_Y(j) = k\}|.$$

Suppose there is an injection $f : \mathcal{J} \to \mathcal{I}$ with $\deg_Y(j) = \rho_X(f(j))$ and a total ordering $\prec$ compatible with the partial order $<$ with respect to which $\{\overline{x}_{f(j)}\}$ is upper-triangular. Then $\{\overline{x}_{f(j)}\}$ is a module basis for $H_S^*(Y;\mathbb{F})$.

If it exists, we call the map $f : \mathcal{J} \to \mathcal{I}$ a *matching*.

PROOF. By hypothesis, the set $\{\overline{x}_{f(j)}\}_{j\in\mathcal{J}}$ is upper-triangular with respect to the total ordering $\prec$ compatible with $<$. Proposition 2.6 then implies that $\{\overline{x}_{f(j)}\}_{j\in\mathcal{J}}$ is linearly independent. The matching condition implies that for each $k \in \mathbb{Z}_{\geq 0}$ there exist precisely b_k elements in $\{\overline{x}_{f(j)}\}_{j\in\mathcal{J}}$ of homogeneous degree $2k$. Thus the set $\{\overline{x}_{f(j)}\}_{j\in\mathcal{J}}$ satisfies the hypotheses of Proposition 2.9, and the result follows. $\square$

Remark 4.19. Module bases constructed from a matching may be very different from bases obtained by poset pinball if the degree function $\deg_Y$ is not compatible with the partial order on $\mathcal{I}$. On the other hand, if the degree function $\deg_Y : \mathcal{J} \to \mathbb{Z}_{\geq 0}$ also satisfies

$$\deg_Y(j) \leq \rho_X(j) \text{ for all } j \in \mathcal{J}, \tag{4.10}$$

namely $2\deg_Y(j)$ is bounded by the cohomology degree of x_j in $H^*(X)$, then a matching basis could arise as a poset pinball basis.

5. Example: Peterson varieties and other regular nilpotent Hessenberg varieties.

Regular nilpotent Hessenberg varieties are a family of subvarieties of the flag varieties $\mathcal{G}/\mathcal{B}$ which fit into the geometric framework of Section 4.2, so we can analyze their equivariant cohomology using poset pinball. We discuss facts about regular nilpotent Hessenberg varieties in different Lie types in Section 5.1. Then in Section 5.2 we explicitly calculate the Borel-equivariant cohomology of Peterson varieties, a collection of regular nilpotent Hessenberg varieties. To do this, we find a module basis, and show the basis may be obtained either via poset pinball (as in Proposition 4.15) or by a matching (as in Theorem 4.18). We studied the case of Lie type A previously [**23**]; the results here generalize that earlier work to all classical Lie types. In this section we always work in Borel-equivariant cohomology with coefficients in a field $\mathbb{F}$ of characteristic zero.

5.1. Background on regular nilpotent Hessenberg varieties.

Let $\mathcal{G}$ be a complex reductive linear algebraic group, and let $\mathcal{B}$ and $T \subseteq \mathcal{B}$ denote choices of a Borel subgroup and a maximal torus of $\mathcal{G}$, respectively. We denote by $\mathfrak{g}$ and $\mathfrak{b}$ the Lie algebras of $\mathcal{G}$ and $\mathcal{B}$. The homogeneous space $\mathcal{G}/\mathcal{B}$ is a generalized flag variety. A linear subspace $H \subseteq \mathfrak{g}$ is called a *Hessenberg space* if

- H contains the Lie algebra $\mathfrak{b}$, and
- H is closed under Lie bracket with $\mathfrak{b}$, i.e. $[H, \mathfrak{b}] \subseteq H$.

Let $N \in \mathfrak{g}$. The *Hessenberg variety* $\mathcal{H}(N, H)$ associated to N and H is the subvariety of $\mathcal{G}/\mathcal{B}$ defined by

$$\mathcal{H}(N, H) := \{g\mathcal{B} : \mathrm{Ad}(g^{-1})(N) \in H\} \subseteq \mathcal{G}/\mathcal{B}. \tag{5.1}$$

When N is regular (also called principal) nilpotent, then the Hessenberg variety $\mathcal{H}(N, H)$ is called a *regular nilpotent Hessenberg variety.*

Let Φ denote the set of roots of $\mathfrak{g}$ and $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to $\mathfrak{b}$. Denote by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots in Φ^+. If $\alpha \in \Phi$ is a root, let $\mathfrak{g}_\alpha$ be its corresponding root space. Fix a basis element E_α for each $\mathfrak{g}_\alpha$. Let W denote the Weyl group associated to $\mathcal{G}$. We use the natural action of the maximal torus T on $\mathcal{G}/\mathcal{B}$ given by left multiplication on cosets. The fixed point set $(\mathcal{G}/\mathcal{B})^T$ may be naturally identified with the Weyl group W.

We begin with some useful facts.

LEMMA 5.1. (1) *Any regular nilpotent Lie algebra element $N \in \mathfrak{g}$ is $\mathcal{G}$-conjugate to the regular nilpotent element of the form*

$$N_0 := \sum_{\alpha_i \in \Delta} E_{\alpha_i}. \tag{5.2}$$

(2) *Suppose $H \subseteq \mathfrak{g}$ is a Hessenberg space and $N_1, N_2 \in \mathfrak{g}$ are $\mathcal{G}$-conjugate. The corresponding varieties $\mathcal{H}(N_1, H)$ and $\mathcal{H}(N_2, H)$ are isomorphic, with explicit isomorphism given by $\mathcal{H}(N_1, H) \to \mathcal{H}(N_2, H)$; $g\mathcal{B} \mapsto g_2 g\mathcal{B}$, where $g_2 \in \mathcal{G}$ satisfies $N_1 = \mathrm{Ad}(g_2^{-1})(N_2)$.*

(3) *If $N \in \mathfrak{g}$ is a sum of simple root vectors, there exists a circle subgroup S^1 of the maximal torus T such that the restriction of the natural T-action on $\mathcal{G}/\mathcal{B}$ to the S^1-subgroup preserves $\mathcal{H}(N, H)$. Moreover, the points in $\mathcal{H}(N, H)$ that are fixed by this S^1-action satisfy*

$$(\mathcal{H}(N, H))^{S^1} = \mathcal{H}(N, H) \cap (\mathcal{G}/\mathcal{B})^T. \tag{5.3}$$

PROOF. Part (1) is a standard result (see e.g. Collingswood–McGovern [**12**, Theorem 4.1.6]).

Part (2) is a straightforward consequence of Definition (5.1).

To prove (3), we explicitly construct the required subgroup S^1. By definition $\mathfrak{g}_\alpha$ is an eigenspace for the action of Ad T with eigenfunction $\alpha : T \to \mathbb{C}^*$. This means that Ad $t(x) = \alpha(t)x$ for all $x \in \mathfrak{g}_\alpha, t \in T$, and that α is a character of T, which we think of as an element of $\mathfrak{t}^*$. (See also [**24**, 16.4].) The characters $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a maximal $\mathbb{Z}$-linearly independent set in $\mathfrak{t}^*$ by definition of simple roots, so the map $\phi : T \to (\mathbb{C}^*)^n$ given by $\phi(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$ is an isomorphism of linear algebraic groups.

In particular, the preimage of the diagonal subgroup $\{(c, c, \ldots, c) | c \in \mathbb{C}^*\}$ is a rank-one subtorus $S \cong \mathbb{C}^*$ of T whose elements t_c are parametrized by c. The elements of S also satisfy

$$(\mathrm{Ad}\ t_c)\left(\sum E_{\alpha_i}\right) = \sum cE_{\alpha_i} = c\left(\sum E_{\alpha_i}\right)$$

for all $c \in \mathbb{C}^*$ and any sum of simple root vectors, since $E_\alpha \in \mathfrak{g}_\alpha$. In particular Ad $(g^{-1})\left(\mathrm{Ad}\ (t_c^{-1})N\right) = c\mathrm{Ad}\ (g^{-1})N$ for a nonzero scalar c. Since H is a vector space we have

$$t_c g\mathcal{B} \in \mathcal{H}(N, H) \Longleftrightarrow g\mathcal{B} \in \mathcal{H}(N, H).$$

We now confirm that $(\mathcal{G}/\mathcal{B})^S = (\mathcal{G}/\mathcal{B})^T$. We saw that the composition of the maps ϕ^{-1} and α_i send $c \mapsto t_c \mapsto \alpha_i(t_c) = c$ so the composition has degree one for each simple root α_i. Under the natural pairing of characters and one-parameter subgroups [**24**, 16.1], we have

$$\langle S, \alpha_i \rangle = 1 \text{ for all simple roots } \alpha_i. \tag{5.4}$$

This implies that S is a regular subgroup [**24**, 24.4], from which $(\mathcal{G}/\mathcal{B})^S = (\mathcal{G}/\mathcal{B})^T$ follows

[**24**, Section 24, Exercise 6]. This in turn implies Equation (5.3). Finally, to obtain the real rank-one torus S^1, we may restrict to the unit-length elements in $\mathbb{C}^*$. $\square$

For the rest of this section, we assume that $N = N_0$, which by Lemma 5.1 results in no loss of generality. We will also assume that the S^1-action on $\mathcal{H}(N_0, H)$ is that constructed in Lemma 5.1.

Our next goal is to explicitly describe the S^1-fixed points in $\mathcal{H}(N_0, H)$. By Equation (5.3), this is equivalent to identifying the T-fixed points in $\mathcal{G}/\mathcal{B}$ that lie in $\mathcal{H}(N_0, H)$. The next proposition does this in arbitrary Lie type. We need the following notation. Given a Hessenberg space H, let $\mathcal{M}_H$ denote the set of roots defined by the condition

$$H = \mathfrak{b} \oplus \bigoplus_{\alpha \in \mathcal{M}_H} \mathfrak{g}_\alpha. \tag{5.5}$$

For each $w \in W = N(T)/T$, choose a representative $\tilde{w} \in N(T)$. The coset $\tilde{w}\mathcal{B}$ is independent of the choice of representative $\tilde{w}$ since $T \subseteq \mathcal{B}$, so we denote it $w\mathcal{B}$. Recall the T-fixed points in $\mathcal{G}/\mathcal{B}$ are the flags $\{w\mathcal{B} : w \in W\}$.

PROPOSITION 5.2. *Let $\mathfrak{g}$ be of arbitrary Lie type. Let $H \subseteq \mathfrak{g}$ be a Hessenberg space and $\mathcal{H}(N_0, H)$ be the regular nilpotent Hessenberg variety corresponding to H and N_0. The flag $w\mathcal{B} \in (\mathcal{G}/\mathcal{B})^T$ is in the regular nilpotent Hessenberg variety $\mathcal{H}(N_0, H)$ if and only if $w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+$.*

PROOF. The element $w\mathcal{B}$ is in $\mathcal{H}(N_0, H)$ if and only if $\mathrm{Ad}(\tilde{w}^{-1})(N_0) \in H$ for any representative $\tilde{w} \in N(T)$ of $w \in W$. Since $\mathrm{Ad}(\tilde{w}^{-1})(N_0) = \sum_{\alpha_i \in \Delta} E_{w^{-1}\alpha_i}$ we have $w\mathcal{B} \in \mathcal{H}(N_0, H)$ if and only if $w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+$. $\square$

We next recall a result which allows us to deduce the Betti numbers of $\mathcal{H}(N_0, H)$. The original and stronger result, restated below, is that certain nilpotent Hessenberg varieties are paved by (complex) affines.

LEMMA 5.3 ([**39**, Theorem 6.1] and [**40**, Theorem 4.3]). *Assume either that*

- *the Lie algebra $\mathfrak{g}$ is of classical Lie type and $N = N_0$ is the regular nilpotent element $N_0 = \sum_{\alpha_i \in \Delta} E_{\alpha_i}$ or*
- *the Lie algebra $\mathfrak{g}$ is of Lie type A and N is a nilpotent linear operator in Jordan canonical form.*

Let $H \subseteq \mathfrak{g}$ be a Hessenberg space and let $\mathcal{H}(N, H)$ denote the Hessenberg variety corresponding to H and N. Then $\mathcal{H}(N, H)$ has a paving by complex affines obtained by intersecting with an appropriate Bruhat decomposition $\bigcup \mathcal{C}_w$ of $\mathcal{G}/\mathcal{B}$. The homology classes corresponding to the subspaces $\overline{\mathcal{C}_w \cap \mathcal{H}(N, H)}$ generate $H_*(\mathcal{H}(N, H))$.

Moreover, in the case when $\mathfrak{g}$ is of classical Lie type and $N = N_0$ is the regular nilpotent element, the intersection $\mathcal{C}_w \cap \mathcal{H}(N_0, H)$ is nonempty exactly when $w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+$. The degree of the homology class corresponding to w is

$$2|\{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \mathcal{M}_H\}|.$$

In particular, the homology of $\mathcal{H}(N_0, H)$ is $\mathbb{Z}$-torsion-free, and nonzero only in even degree. It follows that the $2j$-th Betti number $b_{2j} = \dim_{\mathbb{F}} H^{2j}(\mathcal{H}(N_0, H); \mathbb{F})$ is

$$b_{2j} = \left|\left\{w \in W : w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+, \quad |\Phi^+ \cap w(\mathcal{M}_H)| = j\right\}\right|. \tag{5.6}$$

We saw in Remark 4.1 that the T-space $\mathcal{G}/\mathcal{B}$ satisfies the GKM package of Section 4.1 and that the cohomology ring $H_T^*(\mathcal{G}/\mathcal{B}; \mathbb{F})$ has a well-known set of poset-upper-triangular generators with respect to the Bruhat order on $(\mathcal{G}/\mathcal{B})^T \cong W$: the equivariant Schubert classes $\{\sigma_v\}_{v \in W}$. Our goal is to construct computationally convenient module bases for $H_{S^1}^*(\mathcal{H}(N_0, H); \mathbb{F})$ using the equivariant Schubert classes, using the methods laid out in previous sections. For this we need the following preliminary observation, which we state in more generality than we use here.

THEOREM 5.4. *Suppose N is the regular nilpotent operator $\sum_{\alpha_i \in \Delta} E_{\alpha_i}$ in classical Lie type, or a nilpotent linear operator in Jordan canonical form in Lie type A. The pair $(\mathcal{H}(N, H), S^1)$ is GKM-compatible with the pair $(\mathcal{G}/\mathcal{B}, T)$ with respect to Borel-equivariant cohomology $H^*(-; \mathbb{F})$ with coefficients in a field $\mathbb{F}$ of characteristic zero.*

PROOF. A matrix in Jordan canonical form in Lie type A is by definition a sum of simple root vectors such as the E_{α_i} above. Equation (5.3) of Lemma 5.1 Part (3) gives Condition (1) of GKM compatibility. The complex paving by affines of $\mathcal{H}(N, H)$ described in Lemma 5.3 implies that the ordinary cohomology of $\mathcal{H}(N, H)$ is zero in odd degrees. Conditions (2) and (3) now follow from the argument in Remark 4.11. $\square$

In order to effectively compute a module basis for $H_{S^1}^*(\mathcal{H}(N_0, H); \mathbb{F})$, we need more information about the components $\sigma_v(w)$ of the equivariant Schubert classes in the direct sum

$$H_T^*(\mathcal{G}/\mathcal{B}; \mathbb{F}) \hookrightarrow H_T^*((\mathcal{G}/\mathcal{B})^T; \mathbb{F}) \cong \bigoplus_{w \in W} H_T^*(\mathrm{pt}; \mathbb{F}) \cong \bigoplus_{w \in W} \mathrm{Sym}_{\mathbb{F}}(\mathfrak{t}^*), \tag{5.7}$$

where $\mathrm{Sym}_{\mathbb{F}}(\mathfrak{t}^*)$ denotes the ring of polynomials with coefficients in the field $\mathbb{F}$ on the Lie algebra $\mathfrak{t}$. Billey gave a complete description of the polynomial $\sigma_v(w)$ for any $v, w \in W$ in arbitrary Lie type [**8**, Theorem 4]. We will only need the following consequences of her formula.

PROPOSITION 5.5 (Corollaries of Billey's formula [**8**, Theorem 4]). *Let $v, w \in W$, and $\sigma_v \in H_T^*(\mathcal{G}/\mathcal{B}; \mathbb{F})$ be the equivariant Schubert class corresponding to v. Then*:

(1) *Given a reduced word decomposition of w, the component $\sigma_v(w)$ is a sum of terms, with one summand for each reduced subword of w that equals v. In particular, $\sigma_v(w) = 0$ if $w \not\geq v$ in Bruhat order.*

(2) *Suppose $w > v$ in Bruhat order. Each summand in $\sigma_v(w)$ is a monomial in the positive roots Φ^+ with a positive integer coefficient.*

In this context, the commutative diagram (4.7) becomes

$$\begin{array}{ccc} H_T^*(\mathcal{G}/\mathcal{B};\mathbb{F}) & \hookrightarrow & H_T^*((\mathcal{G}/\mathcal{B})^T;\mathbb{F}) \cong \bigoplus_{w\in W} \mathrm{Sym}_{\mathbb{F}}(\mathfrak{t}^*) \\ \downarrow{\scriptstyle \pi_{\mathcal{G}/\mathcal{B}}} & & \downarrow{\scriptstyle \pi} \\ H_{S^1}^*(\mathcal{H}(N_0,H);\mathbb{F}) & \hookrightarrow & H_{S^1}^*((\mathcal{H}(N_0,H))^{S^1};\mathbb{F}) \cong \bigoplus_{\substack{w\in W \text{ such that} \\ w^{-1}\Delta\subseteq\mathcal{M}_H\cup\Phi^+}} \mathrm{Sym}_{\mathbb{F}}(\mathrm{Lie}(S^1)^*) \end{array} \tag{5.8}$$

where the left vertical arrow $\pi_{\mathcal{G}/\mathcal{B}}$ is the composition of the natural maps $H_T^*(\mathcal{G}/\mathcal{B};\mathbb{F}) \to H_{S^1}^*(\mathcal{G}/\mathcal{B};\mathbb{F})$ and $H_{S^1}^*(\mathcal{G}/\mathcal{B};\mathbb{F}) \to H_{S^1}^*(\mathcal{H}(N_0,H);\mathbb{F})$ as in Section 4.2. For $v \in W$ let

$$p_v := \pi_{\mathcal{G}/\mathcal{B}}(\sigma_v) \in H_{S^1}^*(\mathcal{H}(N_0,H);\mathbb{F})$$

denote the image of a Schubert class σ_v under $\pi_{\mathcal{G}/\mathcal{B}}$. Given this setup, the following proposition—which holds in arbitrary Lie type—is a straightforward consequence of Proposition 5.5.

Proposition 5.6. *Let $H \subseteq \mathfrak{g}$ be a Hessenberg space and $\mathcal{H}(N_0,H)$ denote the regular nilpotent Hessenberg variety corresponding to H and N_0. Let $w \in W$ satisfy $w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+$. For $v \in W$,*

(1) *$p_v(w) = 0$ if $w \not\geq v$ in Bruhat order, and*

(2) *$p_v(w) \neq 0$ if $w > v$ in Bruhat order.*

Proof. Since S^1 is a torus of rank one, the symmetric algebra $\mathrm{Sym}_{\mathbb{F}}(\mathrm{Lie}(S^1)^*)$ may be identified with a polynomial ring in one variable. Denote this variable t. Given $S^1 \subseteq T$ constructed in Lemma 5.1 Part (3), consider the natural map $H_T^*(\mathrm{pt};\mathbb{F}) \to H_{S^1}^*(\mathrm{pt};\mathbb{F})$. The induced map $\mathrm{Sym}_{\mathbb{F}}(\mathfrak{t}^*) \to \mathrm{Sym}_{\mathbb{F}}(\mathrm{Lie}(S^1)^*)$ sends each simple root $\alpha_i \in \mathfrak{t}^* \subseteq \mathrm{Sym}_{\mathbb{F}}(\mathfrak{t}^*)$ to $t \in \mathrm{Sym}_{\mathbb{F}}(\mathrm{Lie}(S^1)^*)$ by Equation (5.4). Moreover, the arrow labeled π in Equation (5.8) is defined by restricting the class $(p(w))_{w\in W} \in H_T^*((\mathcal{G}/\mathcal{B})^T;\mathbb{F})$ to the components indexed by Weyl group elements with $w^{-1}\Delta \subseteq \mathcal{M}_H \cup \Phi^+$. In other words π sends $(p(w))_{w\in W}$ to $(p(w))_{w\in W : w^{-1}\Delta\subseteq\mathcal{M}_H\cup\Phi^+}$. Proposition 5.5 now implies that $p_v(w)$ is either zero or a polynomial in t with positive integer coefficients, so the claim follows. □

5.2. The S^1-equivariant cohomology of the Peterson variety in classical Lie types.

In this section, we explicitly build module bases for the S^1-equivariant cohomology of Peterson varieties in classical Lie type. These are special cases of the regular nilpotent Hessenberg varieties in Section 5.1, for which the Hessenberg space is chosen to be

$$H_\Delta := \mathfrak{b} \oplus \bigoplus_{\alpha\in-\Delta} \mathfrak{g}_\alpha. \tag{5.9}$$

In other words $\mathcal{M}_H = -\Delta$. For notational simplicity, we fix $\mathfrak{g}$ and denote the corresponding Peterson variety by

$$\mathcal{Y} := \mathcal{H}(N_0, H_\Delta).$$

The set of S^1-fixed points $\mathcal{Y}^{S^1}$ is a key ingredient in the combinatorial constructions from Section 3, since it corresponds to the initial subset $\mathcal{J} \subseteq \mathcal{I}$ with $\mathcal{I} = W \cong (\mathcal{G}/\mathcal{B})^T$. Proposition 5.2 characterizes the Weyl group elements $w \in W$ whose flags $w\mathcal{B}$ are in an arbitrary regular nilpotent Hessenberg variety; Proposition 5.8 refines this characterization for Peterson varieties. We need the following lemma, which we give for convenience, though it is probably familiar to experts. We follow the notation of Section 5.1. Also, given a subset $J \subseteq \Delta$ we denote by W_J the Weyl group generated by the simple reflections corresponding to the simple roots in J, and by Φ_J (respectively Φ_J^+ or Φ_J^-) the corresponding root system (respectively positive or negative roots). Recall that the roots Φ are partially ordered by the condition that $\alpha \leq \beta$ if $\beta - \alpha$ is a sum of positive roots.

LEMMA 5.7. *Let Φ be a finite root system of arbitrary Lie type with Weyl group W. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the simple roots in a choice of positive roots Φ^+ of Φ. Let $w \in W$ and define*

$$J := \{\alpha_i : w^{-1}(\alpha_i) < 0\}. \tag{5.10}$$

Then

$$w^{-1}\Delta \subseteq -\Delta \cup \Phi^+ \tag{5.11}$$

if and only if w is the maximal element of the Weyl group W_J.

PROOF. Bourbaki proves that if w_J is the (unique) maximal element of W_J then $\ell(w_J) = \Phi_J^+$ and w_J^{-1} sends Δ_J to $-\Delta_J$ [**9**, Corollary 3 in VI.1.1.6]. Every element of Φ_J^+ is a linear combination of the simple roots Δ_J with nonnegative coefficients, so $w_J^{-1}(\Phi_J^+) = \Phi_J^-$ by linearity. The length of w_J is the number of positive roots that w_J^{-1} sends to negative roots [**9**, Corollary 2 in VI.1.1.6] so $w_J^{-1}(\Phi^+ - \Phi_J^+) \subseteq \Phi^+$. We conclude $w_J^{-1}\Delta \subseteq -\Delta \cup \Phi^+$ as desired.

Conversely, suppose $w \in W$ and $w^{-1}\Delta \subseteq -\Delta \cup \Phi^+$. We first show that $w^{-1}(\Phi^+ - \Phi_J^+) \subseteq \Phi^+$. Choose $\alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i$ to be an arbitrary positive root with $w^{-1}(\alpha) < 0$. (In particular, each coefficient c_i is non-negative.) Since w^{-1} is linear on Φ we obtain

$$w^{-1}(\alpha) = \sum_{\alpha_i \in \Delta} c_i w^{-1}(\alpha_i) = \sum_{\alpha_k \notin J} c_k w^{-1}(\alpha_k) + \sum_{\alpha_j \in J} c_j w^{-1}(\alpha_j) < 0. \tag{5.12}$$

Now let $\alpha_i \in \Delta$ with $\alpha_i \notin J$, so $w^{-1}(\alpha_i) > 0$. If the coefficient c_i of α_i is strictly positive then

$$0 < c_i w^{-1}(\alpha_i) \leq \sum_{\alpha_k \notin J} c_k w^{-1}(\alpha_k) < \sum_{\alpha_j \in J} -c_j w^{-1}(\alpha_j), \tag{5.13}$$

where the last inequality follows from Equation (5.12). Each summand in Equation (5.13) is non-negative by definition of J and the fact that the coefficients c_j, c_k are non-negative. For each $j \in J$ the root $-w^{-1}(\alpha_j)$ is in Δ by the hypothesis that $w^{-1}\Delta \subseteq -\Delta \cup \Phi^+$.

Since each $-w^{-1}(\alpha_j)$ is simple and simple roots form a base for all roots Φ, the rightmost inequality in Equation (5.13) implies that $w^{-1}(\alpha_i)$ is a linear combination of the roots $\{-w^{-1}(\alpha_j)\}_{\alpha_j \in J}$. This contradicts the fact that $w^{-1}(\Delta)$ is a linearly independent set of roots. We conclude that $c_i = 0$ for all $i \notin J$, from which it follows that $w^{-1}(\Phi^+ - \Phi_J^+) \subseteq \Phi^+$. The definition of w forces $w^{-1}(\Phi_J^+) \subseteq \Phi^-$ so in fact Φ_J^+ are precisely the positive roots that w sends to negative roots. We showed that w_J sends exactly the same set of positive roots to negative roots. But by a result of Kostant, each Weyl group element is determined by the set of positive roots that it sends to negative roots [**28**]. We conclude that $w = w_J$ as desired. □

Combining the previous lemma and Proposition 5.2 immediately gives the following.

Proposition 5.8. *Let $\mathfrak{g}$ be of arbitrary Lie type. Fix the Hessenberg space $H_\Delta := \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha$ and let $\mathcal{Y}$ be the Peterson variety corresponding to H_Δ and N_0. The S^1-fixed points $\mathcal{Y}^{S^1}$ of $\mathcal{Y}$ are in one-to-one correspondence with subsets J of Δ, with explicit bijection given by*

$$J \subseteq \Delta \longleftrightarrow \text{ the maximal element } w_J \text{ in } W_J \longleftrightarrow w_J \mathcal{B} \in \mathcal{Y}^{S^1}. \tag{5.14}$$

Now assume $\mathfrak{g}$ has classical Lie type. In the next result, we combine the concrete characterization of $\mathcal{Y}^{S^1}$ with the Betti numbers from Lemma 5.3 to construct a rolldown $roll(w) \in W$ for each element w in the initial subset $\mathcal{J} = \mathcal{Y}^{S^1}$. We show that these rolldowns arise from a successful game of upper-triangular Betti pinball, so the set $\{p_{roll(w)}\}_{w \in \mathcal{Y}^{S^1}}$ forms an $H^*_{S^1}(\mathrm{pt}; \mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$.

Denote by $s_i \in W$ the simple reflection corresponding to the simple root α_i in Δ.

Theorem 5.9. *Let $\mathfrak{g}$ be of classical Lie type. Fix the Hessenberg space $H_\Delta := \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha$ and let $\mathcal{Y}$ be the Peterson variety corresponding to H_Δ and N_0. For each subset $J \subseteq \Delta$ let w_J be maximal Weyl group element of W_J. Suppose $J = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_s} : i_1 < i_2 < \cdots < i_s\}$. Let*

$$v_J := s_{i_1} s_{i_2} \cdots s_{i_s} \in W. \tag{5.15}$$

Then the association $w_J \mapsto v_J$ for $J \subseteq \Delta$ is a possible outcome of a successful game of upper-triangular Betti pinball, where $v_J = roll(w_J)$. In particular, the equivariant cohomology classes $\{p_{v_J}\} \subseteq H^*_{S^1}(\mathcal{Y})$ form a $H^*_{S^1}(\mathrm{pt}; \mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$.

Proof. We prove the claim by playing upper-triangular Betti pinball. The board $\mathcal{I}$ is the Weyl group W, identified with the set of T-fixed points on $\mathcal{G}/\mathcal{B}$ and equipped with Bruhat order. We use the rank function on $\mathcal{I} = W$ given by $\rho(w) = \ell(w)$, the Bruhat length. The initial subset is $\mathcal{J} = \{w_J : J \subseteq \Delta\} \cong \mathcal{Y}^{S^1}$. The inclusion $J' \subseteq J$ implies that $w_{J'} \in W_J$. This in turn implies $w_{J'} \leq w_J$ since w_J is maximal in W_J. In other words, Bruhat order induces the partial order on the initial subset $\mathcal{J}$ given by

$$w_{J'} < w_J \Leftrightarrow J' \subset J. \tag{5.16}$$

Fix any total order $\prec$ of $\mathcal{J}$ subordinate to this partial order.

Specializing Formula (5.6) in Lemma 5.3 to Peterson varieties, we see that the nonzero Betti numbers of $\mathcal{Y}$ are $b_{2j} = \binom{|\Delta|}{j}$, namely the number of subsets $J \subseteq \Delta$ with $|J| = j$. These b_{2j} are our target Betti numbers.

We now play upper-triangular Betti pinball. We will show that for each w_J our choice $v_J := roll(w_J)$ satisfies all the rules for basic pinball, upper-triangular pinball, and Betti pinball. Indeed, for all J we have $v_J < w_J$ in Bruhat order by construction, so v_J is a possible basic pinball rolldown for w_J. Second we prove that at each step of upper-triangular pinball, no wall on the board prevents w_J from rolling down to v_J. It suffices to show that if $w_{J'} \prec w_J$ then the element $v_J \not< w_{J'}$ in Bruhat order. The reflection $s_{\alpha_i} < v_J$ precisely when $\alpha_i \in J$ by construction of the element v_J. Hence $v_J < w_{J'}$ if and only if $J \subseteq J'$, from which it follows that if $w_{J'} \prec w_J$ then $v_J \not< w_{J'}$. Third, we saw above that $\rho(v_J) = \ell(v_J) = |J|$ and there are precisely $\binom{|\Delta|}{j}$ subsets J with degree $|J|$, so the v_J are also rolldowns in Betti pinball.

Finally, since the equivariant Schubert classes $\{\sigma_w\}_{w \in W}$ are a rank-homogeneous poset-upper-triangular basis with respect to Bruhat order, we conclude from Proposition 4.15 that the classes $\{p_{v_J} : J \subseteq \Delta\}$ form a $H^*_{S^1}(\mathrm{pt}; \mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$, as desired. □

REMARK 5.10. A subset of the images $\{p_w\}_{w \in W}$ of the equivariant Schubert classes generate the ring $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$, so the ring map

$$\pi_{\mathcal{G}/\mathcal{B}} : H^*_T(\mathcal{G}/\mathcal{B}; \mathbb{F}) \to H^*_{S^1}(\mathcal{Y}; \mathbb{F})$$

is surjective when $\mathcal{Y}$ is the Peterson variety corresponding to $\mathcal{G}$ of classical Lie type. (This is the map from Diagram (5.8), which is Diagram (4.6) for the special case of regular nilpotent Hessenberg varieties.) We note that Carrell and Kaveh have shown that surjectivity of $\pi_{\mathcal{G}/\mathcal{B}}$ is equivalent to the statement that $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$ is generated by the Chern classes of $\mathcal{B}$-equivariant vector bundles [**11**].

We can also construct the module basis $\{p_{v_J}\}_{J \subseteq \Delta}$ in the above theorem from a matching compatible with degrees, as discussed in Section 4.3. The additional ingredient which enables this construction is the geometric data of the dimensions of the affine cells that pave $\mathcal{Y}$, as recorded in Lemma 5.3.

THEOREM 5.11. *Let $\mathfrak{g}, \mathcal{Y}, \{w_J : J \subseteq \Delta\}, \{v_J : J \subseteq \Delta\}$ be as in Theorem* 5.9. *Then the $H^*_{S^1}(\mathrm{pt}; \mathbb{F})$-module basis $\{p_{v_J} : J \subseteq \Delta\}$ of $H^*_{S^1}(\mathcal{Y}; \mathbb{F})$ can be obtained via a matching compatible with degrees in the sense of Theorem* 4.18.

PROOF. Define a degree function $\deg_{\mathcal{Y}} : \mathcal{J} \to \mathbb{Z}_{\geq 0}$ by

$$\deg_{\mathcal{Y}}(w_J) = \dim_{\mathbb{C}}(\mathcal{C}_{w_J} \cap \mathcal{Y}),$$

the complex dimension of the affine cell $\mathcal{C}_{w_J} \cap \mathcal{Y}$ associated to w_J in Lemma 5.3. Lemmas 5.3 and 5.7 together show that $\deg_{\mathcal{Y}}(w_J) = |J|$. Take the rank function $\rho : \mathcal{I} \to \mathbb{Z}_{\geq 0}$ to be the usual Bruhat length, namely $\rho(w) = \ell(w)$. (Bruhat length of w equals half the cohomology degree of the Schubert class σ_w.) This means that $\deg_{\mathcal{Y}}(w_J) = \rho(v_J)$. In particular the map $f : \mathcal{J} \to \mathcal{I}$ given by $f(w_J) = v_J$ satisfies $\deg_{\mathcal{Y}}(w_J) = \rho(v_J) = |J|$.

The proof of Theorem 5.9 showed that the $\{p_{v_J}\}$ are a rank-homogeneous poset-upper-triangular basis with respect to any total ordering compatible with $<$. The result now follows from Theorem 4.18. $\square$

In previous work [**23**], the authors constructed an $H^*_{S^1}(\mathrm{pt};\mathbb{F})$-module basis for $H^*_{S^1}(\mathcal{Y};\mathbb{F})$ without reference to poset pinball, in the case when $\mathfrak{g}$ has Lie type A. In fact, the formula for v_J given in Equation (5.15) generalizes to arbitrary Lie type the explicit formulas for what was called $v_{\mathcal{A}}$ in earlier work [**23**, Equation (2.7) and Definition 4.1]. We deduce that the basis discussed in [**23**] in fact arises from poset pinball.

Moreover, our previous paper [**23**] used the poset pinball basis $\{p_{v_J}\}$ in Lie type A to explicitly analyze the structure constants of $H^*_{S^1}(\mathcal{Y})$ via a kind of Monk's formula in equivariant cohomology. We conclude this section with a question for future work. Recent work of DeWitt–Harada [**13**] and Drellich [**14**] give steps towards answering this question.

Question 5.12. What is an explicit combinatorial formula for the structure constants of $H^*_{S^1}(\mathcal{Y})$ with respect to the basis $\{p_{v_J}\}$ in each classical Lie type?

6. Example: Springer varieties in type A.

In this section we analyze a special class of nilpotent Hessenberg varieties in Lie type A: Springer varieties, and in particular the subregular Springer varieties. The flag variety $GL_n(\mathbb{C})/\mathcal{B}$ can be identified with

$$\mathcal{F}\ell ags(\mathbb{C}^n) = \{V_\bullet : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n \text{ such that } \dim_{\mathbb{C}}(V_i) = i\}.$$

Suppose $N : \mathbb{C}^n \to \mathbb{C}^n$ is a nilpotent linear operator and $\mathfrak{b}$ is the standard Borel subalgebra of upper-triangular matrices in $\mathfrak{g}$. The *Springer variety* $\mathcal{S}_N$ is the Hessenberg variety associated to N and the Hessenberg space $H = \mathfrak{b}$, namely

$$\mathcal{S}_N := \mathcal{H}(N, \mathfrak{b}). \tag{6.1}$$

In Lie type A, this can be expressed as

$$\mathcal{S}_N := \{V_\bullet : NV_i \subseteq V_i \text{ for all } 1 \leq i \leq n\} \subseteq \mathcal{F}\ell ags(\mathbb{C}^n).$$

Springer discovered that the symmetric group S_n acts on the ordinary cohomology $H^*(\mathcal{S}_N;\mathbb{C})$ for any Springer variety [**37**]. This representation is graded by the degree of the cohomology classes. Springer also showed that the top-dimensional cohomology group is an irreducible representation, and that any irreducible representation of S_n arises in this way. Indeed, the irreducible representation corresponding to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ arises from the top-dimensional cohomology of the Springer variety $\mathcal{S}_N$ for N with Jordan canonical form given by Jordan blocks of size $\lambda_1, \lambda_2, \ldots, \lambda_s$.

In this section, we use poset pinball to construct an explicit module basis for the S^1-equivariant cohomology with complex coefficients of subregular Springer varieties of Lie type A. Moreover, we construct an S_n-representation on this explicit module basis and obtain an equivariant Springer representation. Goresky and MacPherson give a related construction for a different torus action [**19**, Section 7].

6.1. An S^1-action and the S^1-fixed points of Springer varieties.

In this section, we describe an S^1-action on arbitrary Springer varieties of Lie type A and make some initial observations. For instance, we will see that the fixed points $\mathcal{S}_N^{S^1}$ may be identified with the set of permutations whose descents are in positions given by the partition of n determined by the Jordan canonical form of N. By contrast, Carrell obtains a similar result in general Lie type using a different torus action [**10**]. In Section 6.3, we will specialize to a particular class of nilpotent operators called the subregular operators.

Lemma 5.1 shows that for any $g \in GL_n(\mathbb{C})$ the Springer variety $\mathcal{S}_N$ is homeomorphic to $\mathcal{S}_{g^{-1}Ng}$. We may therefore assume without loss of generality that N is in Jordan canonical form, with Jordan blocks weakly decreasing in size. We denote by λ_N both the partition of n and the Young diagram corresponding to this decomposition of N into Jordan blocks.

In Lemma 5.1 Part (3) we defined a circle subgroup of the standard maximal torus T^n of diagonal matrices in $U(n,\mathbb{C})$. It can be described very explicitly in this type A setting as

$$S^1 := \left\{ \left. \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \right| t \in \mathbb{C}, \ \|t\| = 1 \right\} \subseteq T^n \subseteq U(n,\mathbb{C}). \tag{6.2}$$

The maximal torus T^n acts canonically on $GL(n,\mathbb{C})/\mathcal{B} \cong \mathcal{F}\ell ags(\mathbb{C}^n)$ so $S^1 \subseteq T^n$ also acts naturally. In this case Lemma 5.1 states

$$\mathcal{F}\ell ags(\mathbb{C}^n)^{S^1} = \mathcal{F}\ell ags(\mathbb{C}^n)^{T^n}.$$

Lemma 5.1 Part (3) showed that when N is in Jordan canonical form, the subgroup S^1 in Equation (6.2) preserves the Springer variety $\mathcal{S}_N$. (The reader can do the same with an explicit matrix calculation.)

The following proposition is a summary of results in the literature, phrased in our language.

PROPOSITION 6.1. *For each nilpotent operator $N : \mathbb{C}^n \to \mathbb{C}^n$ the Springer variety $\mathcal{S}_N$ has no odd-dimensional cohomology. If N is in Jordan canonical form and S^1 is as in Equation* (6.2) *then the pair $(\mathcal{S}_N, S^1)$ is GKM-compatible with $(GL_n(\mathbb{C})/\mathcal{B}, T)$ with respect to Borel-equivariant cohomology $H^*(-;\mathbb{F})$.*

PROOF. Spaltenstein proved that the ordinary cohomology of Springer varieties is zero in odd degrees [**36**]. The result follows from the argument in Remark 4.11. □

We now compute the fixed points of the Springer variety $\mathcal{S}_N$ with respect to this S^1-action. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ of n the permutation $w \in S_n$ has *descents* in the positions given by λ if

$$w(i) > w(i+1) \Leftrightarrow i \in \{\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{s-1}\}.$$

For example, the permutation $w = (24581736)$ has descents in the positions given by $\lambda = (4, 2, 2)$.

THEOREM 6.2. *Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator in Jordan canonical form whose Jordan blocks weakly decrease in size. Let λ_N be the corresponding partition of n. The S^1-fixed points of $\mathcal{S}_N$ are given by the set*

$$\{w \in S_n : w^{-1} \text{ has descents in the positions given by } \lambda_N\}.$$

The bijection sends the permutation w to the fixed point $w\mathcal{B}$, where w also denotes the permutation matrix whose i^{th} column is the standard basis vector $e_{w(i)}$ for all i.

PROOF. Since $\mathcal{F}\ell ags(\mathbb{C}^n)^{T^n} = \mathcal{F}\ell ags(\mathbb{C}^n)^{S^1}$ it suffices to find the intersection

$$\mathcal{F}\ell ags(\mathbb{C}^n)^{T^n} \cap \mathcal{S}_N.$$

The T^n-fixed points of $\mathcal{F}\ell ags(\mathbb{C}^n)$ consist precisely of the permutation flags $\{w\mathcal{B} : w \in S_n\}$ where w is a permutation matrix in $GL(n, \mathbb{C})$ whose i^{th} column has the standard basis vector $e_{w(i)}$. The definition of Springer varieties in Equation (6.1) says that $w\mathcal{B}$ is in $\mathcal{S}_N$ exactly when $w^{-1}Nw$ is upper-triangular. Let $E_{i,j}$ denote the $n \times n$ matrix with 1 in the (i, j)-th entry and 0 in all other entries. The matrix N is in Jordan canonical form, so $N = \sum_{i \notin A} E_{i,i+1}$ where

$$A = \{(\lambda_N)_1, (\lambda_N)_1 + (\lambda_N)_2, \ldots, (\lambda_N)_1 + (\lambda_N)_2 + (\lambda_N)_3 + \cdots + (\lambda_N)_s\}.$$

(In other words, the sum is over pairs $i, i+1$ in the same part of the partition λ_N.) This means $w\mathcal{B}$ is in $\mathcal{S}_N$ if and only if

$$w^{-1} \sum_{i \notin A} E_{i,i+1} w = \sum_{i \notin A} E_{w^{-1}(i), w^{-1}(i+1)} \in \mathfrak{b},$$

or equivalently

$$i \notin A \Rightarrow w^{-1}(i) < w^{-1}(i+1).$$

As desired, this implies w^{-1} has descents in the positions given by the partition λ_N. $\square$

EXAMPLE 6.3. Let $n = 4$ and take N to be the matrix with 2 Jordan blocks each of size 2, so $N = E_{12} + E_{34}$. By Theorem 6.2, the S^1-fixed points of $\mathcal{S}_N$ are the inverses of the following permutations, written in one-line notation:

$$1234, 1324, 1423, 2314, 2413, 3412.$$

Informally, the matrix $w^{-1}Nw$ is the sum of all $E_{w^{-1}(i), w^{-1}(i+1)}$ over i such that $i, i+1$ are in the same part of the partition λ_N. For example, the matrix corresponding to the fixed point $w^{-1} = 2314$ is $E_{23} + E_{14}$ while the matrix corresponding to $w^{-1} = 3412$ is $E_{34} + E_{12}$.

Theorem 6.2 describes the fixed points w of the Springer varieties in terms of the descents of the permutation w^{-1}. In general the inverse w^{-1} is easier to characterize in straightforward combinatorial terms than w. However poset pinball uses the permutations w. The subregular N described in the next section are operators for which we can easily recover a word for w from information about its inverse. This is not true for most operators, which is why the argument in the next section does not extend immediately to all Springer fibers.

6.2. The subregular Springer representation.

A nilpotent linear operator $N : \mathbb{C}^n \to \mathbb{C}^n$ is *subregular* if the partition associated to its Jordan canonical form is $(n-1, 1)$, namely it has one Jordan block of size $n-1$ and one of size 1. If N is subregular, the Springer variety $\mathcal{S}_N$ is called a *subregular Springer variety* and Springer's representation on $H^*(\mathcal{S}_N; \mathbb{C})$ is called the *subregular Springer representation.* In this section, we recall some results about the subregular Springer variety and representation.

Theorem 6.2 implies that the S^1-fixed points of the subregular Springer variety are in bijective correspondence with permutations whose descents are given by the partition $(n-1, 1)$. The one-line notation of such a permutation w increases in the first $n-1$ entries. Since the last entry can be any integer between 1 and n, the fixed points in one-line notation are precisely

$$\mathcal{S}_N^{S^1} = \{w_i := 1\ 2\ 3 \cdots\ i-1\ \hat{i}\ i+1 \cdots n\ i \text{ for each } i \text{ with } 1 \leq i \leq n\},$$

where $\hat{i}$ indicates that the integer i is skipped. Note that w_n is the identity element in S_n.

Garsia–Procesi gave a more general construction of the Springer representation using a classical description of irreducible representations of the symmetric group [**16**] which we describe in the special case of subregular Springer varieties. A filling of any Young diagram with n boxes with the numbers $1, 2, \ldots, n$ without repetition is *row-strict* if each row increases left-to-right. (Hence a row-strict subregular filling is either a standard Young tableau or has $2, 3, \ldots, n-1, n$ in the top row and 1 in the bottom row.)

Let $M^{(n-1,1)}$ denote the complex vector space whose basis is the set of row-strict fillings with shape $(n-1, 1)$. Define an S_n-action on $M^{(n-1,1)}$ as follows. Given $w \in S_n$ and a row-strict filling T of shape $(n-1, 1)$, define the filling $w(T)$ by the following rules.

- For each $i = 1, 2, 3, \ldots, n$, place $w(i)$ in the box where T had entry i.
- Reorder each row so it increases left-to-right.

By construction $w(T)$ is row-strict of shape $(n-1, 1)$. This is a well-defined action of S_n on the set of row-strict fillings of shape $(n-1, 1)$, and extends by $\mathbb{C}$-linearity to a representation of S_n on $M^{(n-1,1)}$. (This representation generalizes to arbitrary partitions λ of n, see [**15**, Section 7.2].)

Combining several results (see Garsia–Procesi's summary [**16**, page 84] and e.g. Fulton's text for background [**15**, Section 7.2]) yields the following.

Proposition 6.4 (Garsia–Procesi). *The subregular Springer representation on $H^*(\mathcal{S}_{(n-1,1)};\mathbb{C})$ is isomorphic, as an ungraded representation, to the S_n-representation $M^{(n-1,1)}$ defined above. Moreover, suppose $M^{(n-1,1)}$ is given the grading inherited from the cohomology ring $H^*(\mathcal{S}_{(n-1,1)};\mathbb{C})$. Then the set of $n-1$ vectors*

$$v_j := \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline j \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 2 & 3 & \cdots & n \\ \hline 1 \\ \cline{1-1}\end{array}$$

for $j = 2,\ldots,n$ form a $\mathbb{C}$-basis $\{v_2, v_3, \ldots, v_n\}$ for the top-degree graded piece of $M^{(n-1,1)}$. A basis for the zero-degree graded piece is given by the vector

$$v_0 := \sum_{j=1}^{n} \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline j \\ \cline{1-1}\end{array}.$$

We can compute the character of the subregular Springer representation from this description. The representation preserves degrees, so we analyze the characters on each degree separately. Denote the character of the Springer representation on $H^{2i}(\mathcal{S}_N;\mathbb{C})$ by $\chi^i : S_n \to \mathbb{Z}$. Recall that by definition if $w \in S_n$ the integer $\chi^1(w)$ is the trace of the linear operator on $H^2(\mathcal{S}_N;\mathbb{C})$ corresponding to w.

Corollary 6.5. *Let S_n act on $H^*(\mathcal{S}_N;\mathbb{C}) \cong M^{(n-1,1)}$ via the subregular Springer representation. Then*

$$\chi^1(w) = \#\{\text{ fixed points of } w\} - 1 = \#\{j \in \{1,2,\ldots,n\} : w(j) = j\} - 1$$

for all $w \in S_n$. Also for all $w \in S_n$

$$\chi^0(w) = 1.$$

In particular the subregular Springer representation in the zero-degree piece is the trivial representation.

Proof. The first part of the claim is an exercise from Sagan [**34**, Exercise 2.12.4], and a nice exercise for the reader, given the explicit basis for $H^2(\mathcal{S}_N;\mathbb{C})$ described in Proposition 6.4.

The second part can be seen by inspection. By definition $w \cdot v_0 = v_0$ for all $w \in S_n$. This means $\chi^0(w) = 1$ for all w, so the zero-degree piece is the trivial representation, as desired. □

6.3. Lifting the Springer action to the S^1-equivariant cohomology of subregular Springer varieties.

We now lift the Springer representation on the ordinary cohomology $H^*(\mathcal{S}_N;\mathbb{C})$ to an action of S_n on the S^1-equivariant cohomology $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ in the case when N is subregular. To define the lift, we first build a convenient $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module basis of $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ using upper-triangular Betti pinball.

In this context, the commutative diagram (4.7) becomes:

$$\begin{tikzcd} H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C}) \arrow[r, hook] \arrow[d] & H_T^*((GL(n,\mathbb{C})/\mathcal{B})^T;\mathbb{C}) \cong \prod_{w\in S_n} H_T^*(\mathrm{pt};\mathbb{C}) \cong \prod_{w\in S_n} \mathrm{Sym}(\mathfrak{t}^*) \arrow[d] \\ H_{S^1}^*(\mathcal{S}_N;\mathbb{C}) \arrow[r, hook] & H_{S^1}^*(\mathcal{S}_N^{S^1};\mathbb{C}) \cong \prod_{w\in \mathcal{S}_N^{S^1}} H_{S^1}^*(\mathrm{pt};\mathbb{C}) \cong \prod_{w\in \mathcal{S}_N^{S^1}} \mathrm{Sym}(\mathrm{Lie}(S^1)^*) \end{tikzcd} \tag{6.3}$$

where the left vertical arrow is the composition

$$H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C}) \longrightarrow H_{S^1}^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C}) \longrightarrow H_{S^1}^*(\mathcal{S}_N;\mathbb{C})$$

and the right vertical arrow is zero on each component corresponding to $w \notin \mathcal{S}_N^{S^1}$. As before, we denote the equivariant Schubert class corresponding to w in $H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ by σ_w for each $w \in S_n$.

For each $i = 1, 2, \ldots, n-1$, let s_i be the permutation on $\{1, 2, \ldots, n\}$ that exchanges i and $i+1$ and leaves the other numbers fixed. (In general Lie type s_i is the reflection s_{α_i} for a choice of simple roots.) The proof of the next theorem is similar to that of Theorem 5.9.

THEOREM 6.6. *Let N be a subregular nilpotent linear operator $N : \mathbb{C}^n \to \mathbb{C}^n$ and let $\mathcal{S}_N$ denote its associated subregular Springer variety. Then the association*

$$w_i \mapsto v_i := \begin{cases} e & \text{if } i = n \\ s_i & \text{if } 1 \le i \le n-1 \end{cases}$$

is an outcome of a successful game of upper-triangular Betti pinball, where $v_i = roll(w_i)$. In particular, the classes $\{p_{v_i}\}_{i=1}^n = \{p_e, p_{s_1}, p_{s_2}, \ldots, p_{s_{n-1}}\}$ form an $H_{S^1}^*(\mathrm{pt};\mathbb{C})$-module basis for $H_{S^1}^*(\mathcal{S}_N;\mathbb{C})$.

PROOF. Recall that w_n is the identity $e \in S_n$. For each $i = 1, 2, \ldots, n-1$ note that

$$w_i = s_{n-1}s_{n-2}\cdots s_{i+1}s_i \tag{6.4}$$

is a reduced word decomposition of w_i. Thus for each $i = 1, 2, \ldots, n-1$, we have $w_i > s_i$ and $w_i \not> s_j$ for any $j < i$. Moreover $w_n < w_{n-1} < \cdots < w_1$ so Bruhat order totally orders the S^1-fixed points. In particular the choice of total order required to play pinball is uniquely determined in this case.

Now we play upper-triangular Betti pinball with board $S_n = (GL(n,\mathbb{C})/\mathcal{B})^{S^1}$ and initial subset $(\mathcal{S}_N)^{S^1}$. We saw that $v_i < w_i$ for all i. In upper-triangular pinball, walls are never placed between w_i and v_i because if $w_j < w_i$ then $j > i$ and so $v_i \not< w_j$. Finally $\ell(v_n) = 0$ and $\ell(v_i) = 1$ for $i = 1, 2, \ldots, n-1$. Comparing with the Betti numbers of $\mathcal{S}_N$ in Proposition 6.4, we conclude that the $\{v_i\}_{i=1}^n$ are a successful outcome of upper-triangular Betti pinball. Applying Proposition 4.15 gives the claim. $\square$

With a basis for $H_{S^1}^*(\mathcal{S}_N;\mathbb{C})$ in hand, we may now discuss our construction of an S_n-representation, for which we depend on a previous construction by Kostant and Kumar of a W-representation on $H_T^*(\mathcal{G}/\mathcal{B};\mathbb{C})$ when $\mathcal{G}$ is any Kac–Moody group [**29**]. We work in Lie type A, for which $W = S_n$ and $\mathcal{G}/\mathcal{B} \cong \mathcal{F}\ell ags(\mathbb{C}^n)$.

We now define Kostant and Kumar's action in the case of S_n acting on $H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$. Recall that we identify $H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ with its image under the injection at the top of Equation (6.3) so each class $\sigma \in H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ is a tuple of elements in $H_T^*(\mathrm{pt};\mathbb{C})$ indexed by fixed points $u \in W$. As before, denote the u-th coordinate of a class $\sigma \in H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ by $\sigma(u)$. Let $w, u \in W$ and let σ be any element of $H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$. The element $w \cdot \sigma$ is defined by the equation

$$(w \cdot \sigma)(u) := \sigma(uw). \tag{6.5}$$

This Kostant–Kumar action is defined componentwise, so it commutes with the natural maps induced on $H_T^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ and $H_T^*((GL(n,\mathbb{C})/\mathcal{B})^T;\mathbb{C})$ by $S^1 \hookrightarrow T$. Therefore Kostant–Kumar's action descends to an action on $H_{S^1}^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$. (We warn the reader that not all S_n-actions on $H_T^*(\mathcal{G}/\mathcal{B};\mathbb{C})$ descend to $H_{S^1}^*(\mathcal{G}/\mathcal{B};\mathbb{C})$; the second author analyzes another natural S_n-action that does *not* [**41**].)

The main result of this section is that Kostant–Kumar's action gives rise to an S_n-action on $H_{S^1}^*(\mathcal{S}_N;\mathbb{C})$, and that this action lifts the Springer representation to $H_{S^1}^*(\mathcal{S}_N;\mathbb{C})$. The first step is to show that Kostant–Kumar's action on $H_{S^1}^*(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ preserves the $H_{S^1}^*(\mathrm{pt};\mathbb{C})$-submodule spanned by the elements corresponding to the rolldowns from Theorem 6.6.

The following proposition gives a special case of a more general formula of Kostant–Kumar [**29**, Proposition 4.24.g]. The interested reader may also prove it using the following special cases of Billey's formula for the classes σ_{s_j}:

$$\sigma_{s_j}(w) = \alpha_j$$
$$\text{for } w \in \{s_j\} \cup \{s_j s_i : i \neq j, i = 1, \ldots, n-1\} \cup \{s_i s_j : i \neq j \pm 1, i \neq j, i = 1, \ldots, n-1\}$$

and

$$\sigma_{s_j}(s_{j-1}s_j) = \alpha_{j-1} + \alpha_j \text{ and } \sigma_{s_j}(s_{j+1}s_j) = \alpha_j + \alpha_{j+1}.$$

PROPOSITION 6.7 (Kostant–Kumar). *For each i, j with $1 \leq i, j \leq n-1$ we have*

(1) *if $i \neq j$ then*

$$s_i \cdot \sigma_{s_j} = \sigma_{s_j},$$

(2) *if $i = j$ then*

$$s_j \cdot \sigma_{s_j} = \begin{cases} \alpha_j \sigma_e - \sigma_{s_j} + \sigma_{s_{j+1}} & \text{if } j = 1, \\ \alpha_j \sigma_e - \sigma_{s_j} + \sigma_{s_{j-1}} + \sigma_{s_{j+1}} & \text{if } j = 2, 3, \ldots, n-2, \\ \alpha_j \sigma_e - \sigma_{s_j} + \sigma_{s_{j-1}} & \text{if } j = n-1, \end{cases}$$

(3) *and for all $w \in S_n$*

$$w\sigma_e = \sigma_e.$$

Our choice of S^1 from (6.2) induces the linear projection $\mathfrak{t}^* \to \mathrm{Lie}(S^1)^*$ which sends the simple roots α_i to t, where t denotes the polynomial variable in $\mathrm{Sym}(\mathrm{Lie}(S^1)^*)$. The following corollary is immediate from this observation together with the formulae in Proposition 6.7.

COROLLARY 6.8. *The Kostant–Kumar action of S_n on $H^*_{S^1}(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C})$ preserves the $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-submodule that is spanned by the images of the classes $\{\sigma_e, \sigma_{s_1}, \sigma_{s_2}, \dots, \sigma_{s_{n-1}}\}$.*

PROOF. By definition, Kostant–Kumar's action of each $w \in S_n$ is an $H^*_T(\mathrm{pt};\mathbb{C})$-module homomorphism, in the sense that if $f \in \mathrm{Sym}(\mathfrak{t}^*)$ and $\sigma \in H^*_T(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C})$ then $w \cdot (f\sigma) = (f)(w \cdot \sigma)$. Proposition 6.7 thus implies that the $H^*_T(\mathrm{pt};\mathbb{C})$-span of $\{\sigma_e, \sigma_{s_1}, \sigma_{s_2}, \dots, \sigma_{s_{n-1}}\}$ is an S_n-subrepresentation of $H^*_T(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C})$. The ring homomorphism $H^*_T(\mathrm{pt};\mathbb{C}) \to H^*_{S^1}(\mathrm{pt};\mathbb{C})$ is a surjection and the additive homomorphism $H^*_T(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C}) \to H^*_{S^1}(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C})$ respects multiplication in the sense of Equation (2.8). Hence the images of the classes $\{\sigma_e, \sigma_{s_1}, \sigma_{s_2}, \dots, \sigma_{s_{n-1}}\}$ span an $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-submodule of $H^*_{S^1}(GL_n(\mathbb{C})/\mathcal{B};\mathbb{C})$ that is preserved by the action of S_n. $\square$

As a consequence the formulae in the following corollary induce well-defined actions on the ordinary and equivariant cohomology rings of the subregular Springer varieties. For each $w \in S_n$ we denote by p_w the image in $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ of σ_w under the left vertical arrow of (6.3).

COROLLARY 6.9. *Let N be a subregular nilpotent linear operator $N : \mathbb{C}^n \to \mathbb{C}^n$ and let $\mathcal{S}_N$ denote its associated subregular Springer variety. Kostant–Kumar's S_n-action on $H^*_T(\mathcal{G}/\mathcal{B};\mathbb{C})$, described in Proposition 6.7, naturally induces an S_n-representation on $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ as follows. For each i, j with $1 \le i, j \le n-1$ define*:

(1) *if $i \neq j$ then*

$$s_i \cdot p_{s_j} = p_{s_j},$$

(2) *if $i = j$ then*

$$s_j \cdot p_{s_j} = \begin{cases} tp_e - p_{s_j} + p_{s_{j+1}} & \text{if } j = 1, \\ tp_e - p_{s_j} + p_{s_{j-1}} + p_{s_{j+1}} & \text{if } j = 2, 3, \dots, n-2, \\ tp_e - p_{s_j} + p_{s_{j-1}} & \text{if } j = n-1, \end{cases}$$

(3) *for all $w \in S_n$*

$$w \cdot p_e = p_e.$$

Then this is a well-defined S_n-action on $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$. Moreover, this action induces a well-defined S_n-representation on the ordinary cohomology

$$H^*(\mathcal{S}_N;\mathbb{C}) \cong H^*_{S^1}(\mathcal{S}_N;\mathbb{C})/(t)H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$$

by setting $t=0$ in the previous formulae.

Proof. From Theorem 6.6 we know that $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ is a free $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module with basis $\{p_e, p_{s_1}, \ldots, p_{s_{n-1}}\}$. In addition $H^*_T(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ is a free $H^*_T(\mathrm{pt};\mathbb{C})$-module with module basis given by the equivariant Schubert classes $\{\sigma_w\}_{w\in S_n}$. By Proposition 6.7, Kostant–Kumar's action preserves the $H^*_T(\mathrm{pt};\mathbb{C})$-submodule of $H^*_T(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C})$ generated by the degree-0 and degree-2 classes $\{\sigma_e, \sigma_{s_1}, \ldots, \sigma_{s_{n-1}}\}$. By definition of the classes $\{p_w\}$, this submodule maps isomorphically onto $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ under the natural map

$$H^*_T(GL(n,\mathbb{C})/\mathcal{B};\mathbb{C}) \to H^*_{S^1}(\mathcal{S}_N;\mathbb{C}).$$

The action of S_n on $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ is defined via this isomorphism. The explicit formulas to be proven follow immediately from Proposition 6.7 and the definition of the classes p_w.

By Proposition 6.1 the S^1-equivariant cohomology of $\mathcal{S}_N$ is a free $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module. Let $M = H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ denote the S^1-equivariant cohomology of the Springer variety $\mathcal{S}_N$ considered as an $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module and let t be the degree 2 polynomial variable in $H^*_{S^1}(\mathrm{pt};\mathbb{C}) \cong \mathbb{C}[t]$. The ordinary cohomology of $\mathcal{S}_N$ is isomorphic to the quotient $M/(t)M$ [**18**, Equation 1.2.4]. The $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module structure on the quotient factors through $\mathbb{C}$ via the ring homomorphism $H^*_{S^1}(\mathrm{pt};\mathbb{C}) \cong \mathbb{C}[t] \to \mathbb{C}$ taking t to 0. In particular the images of the $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module basis $\{p_e, p_{s_1}, \ldots, p_{s_{n-1}}\}$ in $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ form a $\mathbb{C}$-module basis for $H^*(\mathcal{S}_N;\mathbb{C})$. The S_n-action defined on the free $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ is $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-linear and thus, via the quotient map taking t to 0, yields a well-defined action on the free $\mathbb{C}$-module $H^*(\mathcal{S}_N;\mathbb{C})$ as desired. □

We refer to the S_n-actions on both $H^*_{S^1}(\mathcal{S}_N;\mathbb{C})$ and $H^*(\mathcal{S}_N;\mathbb{C})$ given in the above corollary as *Kostant–Kumar representations*. We now compute the character of the Kostant–Kumar representation on the complex vector spaces $H^{2i}(\mathcal{S}_N;\mathbb{C})$, denoted by ψ^i. We then compare the Kostant–Kumar representation with the Springer representation.

Proposition 6.10. *Let N be a subregular nilpotent linear operator $N:\mathbb{C}^n \to \mathbb{C}^n$ with Springer variety $\mathcal{S}_N$. Let $\psi^i : W \to \mathbb{Z}$ denote the character of the Kostant–Kumar representation on $H^{2i}(\mathcal{S}_N;\mathbb{C})$. Then for each $w \in S_n$*

$$\psi^1(w) = \#\{\text{ fixed points of } w\} - 1 = \#\{j \in \{1,2,\ldots,n\} : w(j) = j\} - 1.$$

The S_n-representation on $H^0(\mathcal{S}_N;\mathbb{C})$ is the trivial representation, hence for each $w \in S_n$

$$\psi^0(w) = 1.$$

Proof. For the purposes of this proof we use cycle notation for permutations, so e.g. $(1,2,3,4)$ sends 1 to 2, 2 to 3, 3 to 4, and 4 to 1. Each element of S_n may be written as a product of disjoint cycles, where the product is denoted by concatenation. The character is a class function, so it suffices to compute $\psi^1(w)$ on a representative of each conjugacy class. Thus we may assume without loss of generality that w has the form

$$w = (1, 2, \ldots, \mu_1)(\mu_1 + 1, \mu_1 + 2, \ldots, \mu_2) \cdots (\mu_{j-1} + 1, \mu_{j-1} + 2, \ldots, \mu_j = n), \qquad (6.6)$$

for some $\mu_1 < \mu_2 < \cdots < \mu_j$ where cycles may have length 1.

Choose a, b with $1 \leq a < b \leq n$. A cycle $(a, a+1, \ldots, b-1, b)$ of length at least 2 has reduced word decomposition $s_a s_{a+1} \cdots s_{b-1}$. Using this word and the formula in Corollary 6.9 we easily check that

$$\begin{aligned} &(a, a+1, \ldots, b-1, b) \cdot p_k \\ &\quad = \begin{cases} -p_k + (\,\mathbb{Z}\text{-linear combination of } \{p_j\}_{j \neq k}), & \text{if } k = a, \\ \ \mathbb{Z}\text{-linear combination of } \{p_j\}_{j \neq k}, & \text{if } a+1 \leq k \leq b-1, \\ p_k + (\,\mathbb{Z}\text{-linear combination of } \{p_j\}_{j \neq k}), & \text{else.} \end{cases} \end{aligned} \qquad (6.7)$$

For any $a = 1, 2, \ldots, n$, a cycle (a) of length 1 corresponds to a fixed point of w. The cycle (a) also denotes the identity element in S_n so

$$(a) \cdot p_k = p_k \text{ for all } k = 1, 2, \ldots, n-1. \qquad (6.8)$$

For each $k = 1, 2, \ldots, n-1$ consider the basis element p_k. The index k appears in precisely one of the cycles in Equation 6.6. From Equations (6.7) and (6.8) we conclude that, as desired,

$$\psi^1(w) = \text{ (number of cycles of length 1) } - 1 = \#\{j \in \{1, 2, \ldots, n\} : w(j) = j\} - 1.$$

Finally, the class p_e generates $H^0(\mathcal{S}_N; \mathbb{C})$ and $wp_e = p_e$ for all $w \in S_n$. So $H^0(\mathcal{S}_N; \mathbb{C})$ is the trivial 1-dimensional representation, and $\psi^0(w) = 1$ for all $w \in S_n$. □

Finally, we observe that the Kostant–Kumar S_n-representation on $H^*(\mathcal{S}_N; \mathbb{C})$ agrees with the Garsia–Procesi description of the Springer representation [**16**]. In fact, since S_n-representations are uniquely determined by their characters, the following is immediate from Corollary 6.5 and Proposition 6.10.

Corollary 6.11. *Let N be a subregular nilpotent linear operator $N : \mathbb{C}^n \to \mathbb{C}^n$ and let $\mathcal{S}_N$ denote its associated subregular Springer variety. The S_n-representation on $H^*_{S^1}(\mathcal{S}_N; \mathbb{C})$ constructed in Corollary 6.9 lifts the classical subregular Springer representation on $H^*(\mathcal{S}_N; \mathbb{C})$ to $H^*_{S^1}(\mathcal{S}_N; \mathbb{C})$ via the homomorphism $H^*_{S^1}(\mathcal{S}_N; \mathbb{C}) \to H^*(\mathcal{S}_N; \mathbb{C})$.*

Remark 6.12. The extent to which the constructions in Section 6.3 can be generalized to other classes of Springer varieties is an open question. We have preliminary experimental evidence that suggests that module bases constructed via poset pinball for other Springer varieties are not in general poset-upper-triangular with respect to Bruhat order. This may make computations using them more difficult. We leave further exploration to future work.

We conclude the paper with a concrete example of Betti pinball in the Springer case which does yield a module basis, albeit not a poset-upper-triangular one.

EXAMPLE 6.13. In our last example, we play Betti pinball with target Betti numbers $\boldsymbol{b} = (1, 3, 5, 3)$. For visual simplicity, not all edges in the poset are drawn above rank 2. The reader can verify that the rolldown set $\mathcal{R}(\mathcal{I}, \mathcal{J})$ in this case is a union of principal order ideals. The double lines indicate covering relations in the partial order which cause the failure of poset-upper-triangularity of the rolldown set. Specifically, poset-upper-triangularity fails in this example because of the drop-downs labeled v_8, v_{10}, v_{11}, and v_{12} in the table.

This example is a pinball game arising from the Springer variety associated to the nilpotent matrix for the partition $(2, 1, 1)$ of $n = 4$. In this case we can check by direct computation that the rolldown set $\mathcal{R}(\mathcal{I}, \mathcal{J})$ yields a module basis for the submodule corresponding to $\mathcal{J}$, even though it is not poset-upper-triangular.

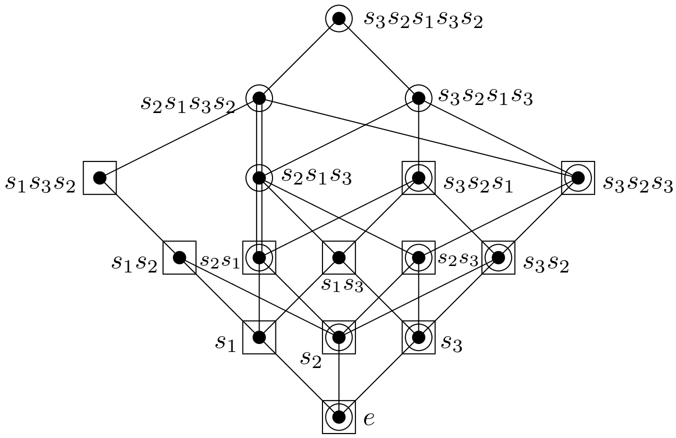

Figure 6.1. An example of Betti pinball for which the rolldown set is not poset-upper-triangular, but is a union of principal order ideals.

pinball step	w_k	v_k
1	$w_1 = e = [1,2,3,4]$	$v_1 = e = [1,2,3,4]$
2	$w_2 = s_3 = [1,2,4,3]$	$v_2 = s_3 = [1,2,4,3]$
3	$w_3 = s_2 = [1,3,2,4]$	$v_3 = s_2 = [1,3,2,4]$
4	$w_4 = s_2 s_3 = [1,3,4,2]$	$v_4 = s_2 s_3 = [1,3,4,2]$
5	$w_5 = s_3 s_2 = [1,4,2,3]$	$v_5 = s_3 s_2 = [1,4,2,3]$
6	$w_6 = s_2 s_1 = [3,1,2,4]$	$v_6 = s_1 = [2,1,3,4]$
7	$w_7 = s_3 s_2 s_3 = [1,4,3,2]$	$v_7 = s_3 s_2 s_3 = [1,4,3,2]$
8	$w_8 = s_2 s_1 s_3 = [3,1,4,2]$	$v_8 = s_1 s_3 = s_3 s_1 = [2,1,4,3]$
9	$w_9 = s_3 s_2 s_1 = [4,1,2,3]$	$v_9 = s_2 s_1 = [3,1,2,4]$
10	$w_{10} = s_2 s_1 s_3 s_2 = [3,4,1,2]$	$v_{10} = s_1 s_2 = [2,3,1,4]$
11	$w_{11} = s_3 s_2 s_1 s_3 = [4,1,3,2]$	$v_{11} = s_3 s_2 s_1 = [4,1,2,3]$
12	$w_{12} = s_3 s_2 s_1 s_3 s_2 = [4,3,1,2]$	$v_{12} = s_1 s_3 s_2 = [2,4,1,3]$

(6.9)

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