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Extension theorem for rough paths via fractional calculus

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Abstract. On the basis of fractional calculus, we introduce an integral of weakly controlled paths, which is a generalization of integrals in the context of rough path analysis. As an application, we provide an alternative proof of Lyons' extension theorem for geometric Hölder rough paths together with an explicit expression of the extension map.

1. Introduction.

Recently, several different approaches have been proposed for the study of the theory of rough paths. One of those approaches is based on fractional calculus and was introduced by Hu and Nualart [**5**] and further investigated by Besalú and Nualart [**1**] and the author [**6**]. This approach is beneficial in that the integrals along rough paths are described explicitly by ordinary Lebesgue integrals, in contrast to the rough integration of Lyons [**7**] as the limit of a type of Riemann sums. Accordingly, one can expect some applications of this approach, such as more concise proofs of fundamental results in the theory of rough paths.

In this paper, we first describe integrals along Hölder rough paths for more general integrands than those studied in the author's previous work [**6**]. In [**6**], the author introduced an integral along β-Hölder rough paths for any roughness $\beta \in (0, 1]$ by using fractional derivative operators, and proved that the integral coincides with the first level path of the rough integral along geometric β-Hölder rough paths. We show in this paper that the definition of integral introduced in [**6**] can be modified so that it is suitable for weakly controlled paths. Here, the concept of weakly controlled paths is a generalization of the usual integrands in the context of rough path analysis, which was introduced by Gubinelli [**4**] to produce a more general framework of rough integration and differential equations driven by rough paths. This generalization of the author's previous work [**6**] has an application to Lyons' extension theorem (also called the first fundamental result in the theory of rough paths) as follows. Let $X = (1, X^1, \ldots, X^N)$ be a β-Hölder rough path, that is, a multiplicative functional of degree N with finite β-Hölder estimates (see (2.2) and (2.3)). Here, N is the unique integer such that $N \leq 1/\beta < N + 1$. Lyons' extension theorem states that for any integer $k \geq N + 1$, the rough path X extends to the unique multiplicative functional of degree k that possesses β-Hölder estimates (see [**7**, Theorem 2.2.1] for the exact statement of the claim). This extension map has been constructed by a discrete approximation similar to the Riemann sums [**7**].

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By using the integration introduced in this paper, we give an expression of the extension map induced by geometric Hölder rough paths as ordinary Lebesgue integrals for fractional derivative operators (Definition 3.11). This result can also be regarded as an alternative proof of Lyons' extension theorem for geometric Hölder rough paths. The integrals defined in Definition 3.11 were not treated in the author's previous work [**6**]; the generalization of integrands in this paper makes it possible to describe the extension map explicitly. Gubinelli also proved Lyons' extension theorem in his framework (cf. [**4**, Proposition 10]), but our approach is different from his and the results are not comparable; indeed, Gubinelli's theory [**4**] is not based on fractional calculus.

The remainder of this paper is organized as follows. In Section 2, as preliminaries, we provide a brief review of concepts of rough paths, weakly controlled paths, and fractional integral and derivative operators. In Section 3, we define integrals of weakly controlled paths and introduce the main theorems along with the application to Lyons' extension theorem. The last section is devoted to the proofs of some results of Section 3.

2. Preliminaries.

In this section, we briefly review some concepts, such as rough paths [**3**], [**7**], [**8**], [**9**], weakly controlled paths [**2**], [**4**], and fractional integral and derivative operators [**10**], [**12**]. Our version of Lyons' extension theorem is also described.

2.1. Notation.

Let V and W be finite-dimensional normed spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We use $L(V,W)$ to denote the set of all linear maps from V to W. For a topological space S, let $C(S,V)$ denote the space of all V-valued continuous functions on S. For $a,b\in\mathbb{R}$ with $a<b$ and $\lambda\in(0,1]$, we denote by $C^{\lambda\text{-Höl}}([a,b],V)$ the space of all V-valued λ-Hölder continuous functions on the interval $[a,b]$. Let T denote a positive constant. This constant will be fixed throughout this paper. The simplex $\{(s,t)\in\mathbb{R}^2 : 0\le s\le t\le T\}$ is denoted by $\triangle$, which is a closed subset of $\mathbb{R}^2$. Let $\mathcal{C}_1(V)$ and $\mathcal{C}_2(V)$ denote $C([0,T],V)$ and $C(\triangle,V)$, respectively. For $f\in\mathcal{C}_1(\mathbb{C})$ and $g\in\mathcal{C}_2(\mathbb{C})$, we define $gf\in\mathcal{C}_2(\mathbb{C})$ by

$$(gf)_{s,t} := g_{s,t}f_t \quad \text{for } (s,t)\in\triangle. \tag{2.1}$$

For $g\in\mathcal{C}_2(V)$ and $\mu>0$, we set

$$\|\!|g|\!\|_\mu := \sup_{0\le s<t\le T}\frac{\|g_{s,t}\|_V}{(t-s)^\mu}.$$

Furthermore, we set $\mathcal{C}_2^\mu(V) := \{g\in\mathcal{C}_2(V) : \|\!|g|\!\|_\mu<\infty\}$ and $\mathcal{C}_1^\lambda(V) := C^{\lambda\text{-Höl}}([0,T],V)$.

Hereafter, E and F denote the Euclidean spaces $\mathbb{R}^d$ and $\mathbb{R}^e$ respectively, and $|\cdot|$ denotes the Euclidean norms of E, F, and their tensor spaces. For a positive integer k, $T^{(k)}(E)$ denotes $\bigoplus_{j=0}^k E^{\otimes j}$ and we define the norm on $T^{(k)}(E)$ as

$$\|\boldsymbol{a}\|_{T^{(k)}(E)} := \sum_{j=0}^k |a^j| \quad \text{for } \boldsymbol{a}=(a^0,a^1,\dots,a^k)\in T^{(k)}(E).$$

The set of all $X = (X^0, X^1, \dots, X^k) \in C(\triangle, T^{(k)}(E))$ such that $X^0_{s,t} = 1$ for all $(s,t) \in \triangle$ is denoted by $C_0(\triangle, T^{(k)}(E))$.

2.2. Rough paths.

Let k be a positive integer. We say that $X = (1, X^1, \dots, X^k) \in C_0(\triangle, T^{(k)}(E))$ is a multiplicative functional of degree k in E if

$$\sum_{i=0}^{j} X^i_{s,u} \otimes X^{j-i}_{u,t} = X^j_{s,t} \tag{2.2}$$

for each $j = 1, \dots, k$ and $s, t, u \in [0, T]$ with $s \le u \le t$. Let β be a real number with $0 < \beta \le 1$. We say that $X = (1, X^1, \dots, X^k) \in C_0(\triangle, T^{(k)}(E))$ has finite β-Hölder estimates if

$$\sup_{0 \le s < t \le T} \frac{|X^j_{s,t}|}{(t-s)^{j\beta}} < \infty \tag{2.3}$$

for each $j = 1, \dots, k$. We denote by $C_{0,\beta}(\triangle, T^{(k)}(E))$ the space of all $X = (1, X^1, \dots, X^k) \in C_0(\triangle, T^{(k)}(E))$ with finite β-Hölder estimates and define the distance on $C_{0,\beta}(\triangle, T^{(k)}(E))$ as

$$d_{\beta,k}(X, \tilde{X}) := \max_{1 \le j \le k} \|X^j - \tilde{X}^j\|_{j\beta} \quad \text{for } X, \tilde{X} \in C_{0,\beta}(\triangle, T^{(k)}(E)).$$

Let $x \in \mathcal{C}^1_1(E)$. We set

$$X^j_{s,t} = \int_{s<u_1<\cdots<u_j<t} dx_{u_1} \otimes \cdots \otimes dx_{u_j} \tag{2.4}$$

for each $j = 1, \dots, k$ and $(s,t) \in \triangle$. Then we see that $X = (1, X^1, \dots, X^k)$ is a multiplicative functional of degree k in E with finite 1-Hölder estimates and we call this the step-k signature of x. Let N denote the integer determined by the relation $N \le 1/\beta < N+1$. A multiplicative functional of degree N in E with finite β-Hölder estimates is called a β-Hölder rough path in E. A step-N signature is called a smooth rough path and the elements in the closure of the set of all smooth rough paths with respect to the distance $d_{\beta,N}$ are called geometric β-Hölder rough paths. The spaces of all β-Hölder rough paths, smooth rough paths, and geometric β-Hölder rough paths in E are denoted by $\Omega_\beta(E)$, $S\Omega_\beta(E)$, and $G\Omega_\beta(E)$, respectively. Let us now introduce our version of Lyons' extension theorem.

THEOREM 2.1 (cf. [**7**, Theorem 2.2.1]). *Let $X = (1, X^1, \dots, X^N) \in \Omega_\beta(E)$. For any integer $k \ge N+1$, there exists a unique extension of the rough path X to a multiplicative functional of degree k in E with finite β-Hölder estimates.*

In [**7**, Theorem 2.2.1], rough paths X of finite p-variation with $p := 1/\beta$ are treated and the exact claim includes quantitative estimates for the extension of X by using control functions ω. For Theorem 2.1 and the alternative proof of the theorem for geometric β-Hölder rough paths $X \in G\Omega_\beta(E)$ given in Section 3, we consider only a particular case where ω is given by $\omega(s,t) = C(t-s)$ for some constant C for simplicity and are not concerned with uniform estimates for the continuity of the extension map.

2.3. Weakly controlled paths.

Let β be a real number with $0 < \beta \leq 1$, k a positive integer, and $X \in C_{0,\beta}(\triangle, T^{(k)}(E))$. We say that a k-tuple $Y = (Y^{(0)}, Y^{(1)}, \ldots, Y^{(k-1)})$ is a path weakly controlled by X with values in F if Y satisfies the following two properties:

(1) for each $l = 0, \ldots, k-1$, $Y^{(l)} \in \mathcal{C}_1^{\beta}(L(E^{\otimes l}, F))$;

(2) for each $l = 0, \ldots, k-1$, $R_l^{k-1-l}(X,Y) \in \mathcal{C}_2^{(k-l)\beta}(L(E^{\otimes l}, F))$, where

$$R_l^{k-1-l}(X,Y)_{s,t} := Y_t^{(l)} - \sum_{i=0}^{k-1-l} Y_s^{(l+i)} X_{s,t}^i \quad \text{for } (s,t) \in \triangle. \tag{2.5}$$

It is sometimes referred to as a weakly controlled path for $X \in C_{0,\beta}(\triangle, T^{(k)}(E))$. The space of all paths weakly controlled by $X \in C_{0,\beta}(\triangle, T^{(k)}(E))$ with values in F is denoted by $\mathcal{Q}_X^{\beta,k}(F)$, which is a normed space under the norm $Y \mapsto \sum_{l=0}^{k-1} |Y_0^{(l)}| + \|Y\|_{X,\beta,k}$. Here, $\|Y\|_{X,\beta,k}$ is defined by

$$\|Y\|_{X,\beta,k} := \sum_{l=0}^{k-1} \| R_l^{k-1-l}(X,Y) \|_{(k-l)\beta} \quad \text{for } Y \in \mathcal{Q}_X^{\beta,k}(F). \tag{2.6}$$

Although the highest level path X^k is not necessary for our definition of paths weakly controlled by $X \in C_{0,\beta}(\triangle, T^{(k)}(E))$, we need it in applications. The multiplicative property (2.2) is not assumed for $X \in C_{0,\beta}(\triangle, T^{(k)}(E))$ in the definition of paths weakly controlled by X. In the following examples, however, such properties play an essential role in confirming property (2) in the definition above.

Example 2.2. Let $X \in G\Omega_\beta(E)$. For $\xi \in E$, we define $X^{1,\xi} \in \mathcal{C}_1^{\beta}(E)$ as

$$X_t^{1,\xi} := \xi + X_{0,t}^1 \quad \text{for } t \in [0,T].$$

Let φ be an $L(E,F)$-valued $(N-1)$-times continuously Fréchet differentiable function on E whose $(N-1)$th derivative $\nabla^{N-1}\varphi$ is Lipschitz continuous on E. For each $l = 0, \ldots, N-1$, we set $Y^{(l)} \in \mathcal{C}_1^{\beta}(L(E^{\otimes l}, L(E,F)))$ as

$$Y_t^{(l)} := \nabla^l \varphi(X_t^{1,\xi}) \quad \text{for } t \in [0,T]. \tag{2.7}$$

We note that, for each $j = 1, \ldots, N$ and $(s,t) \in \triangle$, the symmetric part of $X_{s,t}^j$ is equal to $(X_{s,t}^1)^{\otimes j}/j!$. By using this property of $X \in G\Omega_\beta(E)$, the symmetry of the derivatives of φ, and Taylor's theorem, we can show that, for each $l = 0, \ldots, N-1$ and $(s,t) \in \triangle$,

$$|R_l^{N-1-l}(X,Y)_{s,t}| \leq \|\nabla^{N-1}\varphi\|_{Lip} (\| X^1 \|_\beta^{N-l}/(N-l)!)(t-s)^{(N-l)\beta},$$

where

$$\|\nabla^{N-1}\varphi\|_{Lip} := \sup_{x,y\in E,\ x\neq y} \frac{|\nabla^{N-1}\varphi(x) - \nabla^{N-1}\varphi(y)|}{|x-y|}.$$

Thus, $Y = (Y^{(0)}, Y^{(1)}, \ldots, Y^{(N-1)})$ belongs to $\mathcal{Q}_X^{\beta,N}(L(E,F))$. In addition, if $1/3 < \beta \leq 1$, then Y belongs to $\mathcal{Q}_X^{\beta,N}(L(E,F))$ for every $X \in \Omega_\beta(E)$.

Example 2.3 (cf. [**4**, Proposition 4]). Let $1/3 < \beta \leq 1/2$, $X \in C_{0,\beta}(\triangle, T^{(2)}(E))$, and $Y = (Y^{(0)}, Y^{(1)}) \in \mathcal{Q}_X^{\beta,2}(F)$. Let ψ be an $L(E,F)$-valued continuously Fréchet differentiable function on F whose derivative $\nabla\psi$ is Lipschitz continuous and bounded on F. We set $Z^{(0)} \in \mathcal{C}_1^\beta(L(E,F))$ and $Z^{(1)} \in \mathcal{C}_1^\beta(L(E,L(E,F)))$ as

$$Z_t^{(0)} := \psi(Y_t^{(0)}) \quad \text{and} \quad Z_t^{(1)} := \nabla\psi(Y_t^{(0)})Y_t^{(1)} \quad \text{for } t \in [0,T]. \tag{2.8}$$

Then, $Z = (Z^{(0)}, Z^{(1)})$ belongs to $\mathcal{Q}_X^{\beta,2}(L(E,F))$.

Example 2.4. Let X be a multiplicative functional of degree k in E with finite β-Hölder estimates. For each $l = 0, \ldots, k-1$, we set $Y^{(l)} \in \mathcal{C}_1^\beta(L(E^{\otimes l}, L(E, E^{\otimes(k+1)})))$ as

$$(Y_t^{(l)}(\eta))(\xi) := (X_{0,t}^{k-l} \otimes \eta) \otimes \xi \quad \text{for } t \in [0,T], \tag{2.9}$$

where $\eta \in E^{\otimes l}$ and $\xi \in E$. From (2.2), for each $l = 0, \ldots, k-1$ and $(s,t) \in \triangle$,

$$R_l^{k-1-l}(X,Y)_{s,t} = X_{0,t}^{k-l} - \sum_{i=0}^{k-1-l} X_{0,s}^{k-l-i} \otimes X_{s,t}^i = X_{s,t}^{k-l}. \tag{2.10}$$

Then, from (2.3), $Y = (Y^{(0)}, Y^{(1)}, \ldots, Y^{(k-1)})$ belongs to $\mathcal{Q}_X^{\beta,k}(L(E, E^{\otimes(k+1)}))$.

The weakly controlled path in Example 2.4 is used in the proof of Lyons' extension theorem (Theorem 3.14).

2.4. Fractional integrals and derivatives.

Let a and b be real numbers with $a < b$. For $p \in [1,\infty)$, $L^p(a,b)$ denotes the real L^p-space on the interval $[a,b]$ with respect to the Lebesgue measure. Let $f \in L^1(a,b)$ and $\alpha \in (0,\infty)$. The left- and right-sided Riemann–Liouville fractional integrals of f of order α are defined for almost all $t \in (a,b)$ by

$$I_{a+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I_{b-}^\alpha f(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where $(-1)^{-\alpha} := e^{-i\pi\alpha}$ and $\Gamma(\alpha)$ denotes the gamma function, namely $\Gamma(\alpha) := \int_0^\infty r^{\alpha-1}e^{-r}\, dr$. We use $I^\alpha_{\substack{a+ \\ (b-)}}(L^p)$ to denote the image of $L^p(a,b)$ by the operator $I^{\alpha}_{\substack{a+\\(b-)}}$. Here, we note a simple criterion for functions to belong to $I^{\alpha}_{\substack{a+\\(b-)}}(L^p)$. This criterion is used frequently in Section 4 without being explicitly noted: if $f \in C^{\lambda\text{-Höl}}([a,b],\mathbb{R})$ with $0 < \lambda \leq 1$, then $f \in I^{\alpha}_{a+}(L^p) \cap I^{\alpha}_{b-}(L^p)$ for any $1 \leq p < \infty$ and $0 < \alpha < \lambda$. Let $f \in I^{\alpha}_{\substack{a+\\(b-)}}(L^1)$ with $0 < \alpha < 1$. The left- and right-sided Weyl–Marchaud fractional derivatives of f of order α are defined for almost all $t \in (a,b)$ by

$$D^{\alpha}_{a+}f(t) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha\int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}}\,ds\right) \tag{2.11}$$

and

$$D^{\alpha}_{b-}f(t) := \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha\int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}}\,ds\right), \tag{2.12}$$

respectively. The integrals above are well-defined for almost all $t \in (a,b)$.

The following three formulas are important in this paper. The first is the composition formula:

$$D^{\alpha}_{\substack{a+\\(b-)}}(D^{\beta}_{\substack{a+\\(b-)}}f) = D^{\alpha+\beta}_{\substack{a+\\(b-)}}f \tag{2.13}$$

for $f \in I^{\alpha+\beta}_{\substack{a+\\(b-)}}(L^1)$, $0 < \alpha < 1$, and $0 < \beta < 1$, with $\alpha + \beta < 1$. The second is the basic integration by parts formula of order α:

$$(-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f(t)g(t)\,dt = \int_a^b f(t)D^{\alpha}_{b-}g(t)\,dt \tag{2.14}$$

for $f \in I^{\alpha}_{a+}(L^p)$, $g \in I^{\alpha}_{b-}(L^q)$, $0 < \alpha < 1$, $1 \leq p < \infty$, and $1 \leq q < \infty$, with $1/p + 1/q \leq 1 + \alpha$. The third is also regarded as an integration by parts formula of oreder α. Let $f \in C^{\lambda\text{-Höl}}([a,b],\mathbb{R})$ and $g \in C^{\mu\text{-Höl}}([a,b],\mathbb{R})$ with $\lambda + \mu > 1$. Then, the Riemann–Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists [**11**] and is expressed as follows: for each $\alpha \in (1-\mu, \lambda)$,

$$\int_a^b f(t)\,dg(t) = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f_{a+}(t)D^{1-\alpha}_{b-}g_{b-}(t)\,dt + f(a)(g(b)-g(a)) \tag{2.15}$$

$$= (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f(t)D^{1-\alpha}_{b-}g_{b-}(t)\,dt, \tag{2.16}$$

where $f_{a+}(t) := f(t) - f(a)$ and $g_{b-}(t) := g(t) - g(b)$. For proofs of (2.15) and (2.16), see [**12**, Theorem 4.2.1 and Proposition 2.2].

3. Main theorems.

In the remainder of this paper, we will assume the following: (a,b) is an element of $\triangle$ with $a < b$, β is a real number with $0 < \beta \leq 1$, N is the unique integer such that $N \leq 1/\beta < N+1$, k is a positive integer, and γ is a real number with $0 < \gamma < \min\{1/k, \beta\}$.

3.1. Some fractional operators and their properties.

In this subsection, we introduce some variants of the fractional derivatives and integral operators for later use. Let $\mu > 0$ and $\Psi \in \mathcal{C}_2^{\mu}(V)$. For $\alpha \in (0, \min\{\mu, 1\})$, we define $\mathcal{D}_{a+}^{\alpha}\Psi$ and $\mathcal{D}_{b-}^{\alpha}\Psi$ as $\mathcal{D}_{a+}^{\alpha}\Psi(a) := 0$,

$$\mathcal{D}_{a+}^{\alpha}\Psi(u) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{a,u}}{(u-a)^{\alpha}} + \alpha\int_a^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}}\,dv\right) \quad \text{for } u \in (a,T]$$

and $\mathcal{D}_{b-}^{\alpha}\Psi(b) := 0$,

$$\mathcal{D}_{b-}^{\alpha}\Psi(r) := \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\left(\frac{\Psi_{r,b}}{(b-r)^{\alpha}} + \alpha\int_r^b \frac{\Psi_{r,v}}{(v-r)^{\alpha+1}}\,dv\right) \quad \text{for } r \in [0,b).$$

It is straightforward to show that, for each $u \in [a,T]$ and $r \in [0,b]$,

$$\|\mathcal{D}_{a+}^{\alpha}\Psi(u)\|_V \le \frac{1}{\Gamma(1-\alpha)}\frac{\mu}{\mu-\alpha}\|\Psi\|_{\mu}(u-a)^{\mu-\alpha} \tag{3.1}$$

and

$$\|\mathcal{D}_{b-}^{\alpha}\Psi(r)\|_V \le \frac{1}{\Gamma(1-\alpha)}\frac{\mu}{\mu-\alpha}\|\Psi\|_{\mu}(b-r)^{\mu-\alpha}. \tag{3.2}$$

If $\Psi \in \mathcal{C}_2^{\lambda}(V)$ is of the form $\Psi_{s,t} = \psi(t) - \psi(s)$ for some $\psi \in \mathcal{C}_1^{\lambda}(V)$ with $0 < \lambda \le 1$, then the identities $\mathcal{D}_{a+}^{\alpha}\Psi = D_{a+}^{\alpha}\psi_{a+}$ and $\mathcal{D}_{b-}^{\alpha}\Psi = D_{b-}^{\alpha}\psi_{b-}$ hold, by definition, for any $\alpha \in (0,\lambda)$. Using these operators, we further introduce the following.

Definition 3.1. For $X = (1, X^1, \ldots, X^k) \in C_{0,\beta}(\triangle, T^{(k)}(E))$ and $j = 1, \ldots, k$, we define a function $\mathcal{R}_{b-}^{(j,\gamma)}X$ on $[0,b]$ as follows: for each $r \in [0,b]$,

$$\mathcal{R}_{b-}^{(1,\gamma)}X(r) := \mathcal{D}_{b-}^{\gamma}X^1(r)$$

and

$$\mathcal{R}_{b-}^{(j,\gamma)}X(r) := \mathcal{D}_{b-}^{j\gamma}X^j(r) - \sum_{i=1}^{j-1}\mathcal{D}_{b-}^{(j-i)\gamma}(X^{j-i} \otimes \mathcal{R}_{b-}^{(i,\gamma)}X)(r)$$

for $j = 2, \ldots, k$, inductively.

We note that $\mathcal{R}_{b-}^{(j,\gamma)}X$ is well-defined by the assumption that $0 < \gamma < \min\{1/k, \beta\}$. With regard to the second term of $\mathcal{R}_{b-}^{(j,\gamma)}X(r)$,

$$\mathcal{D}_{b-}^{(j-i)\gamma}(X^{j-i} \otimes \mathcal{R}_{b-}^{(i,\gamma)}X)(r) = \frac{(-1)^{1+(j-i)\gamma}(j-i)\gamma}{\Gamma(1-(j-i)\gamma)}\int_r^b \frac{X_{r,v}^{j-i} \otimes \mathcal{R}_{b-}^{(i,\gamma)}X(v)}{(v-r)^{(j-i)\gamma+1}}\,dv \tag{3.3}$$

holds for each $i = 1, \ldots, j-1$ from (2.1) and $\mathcal{R}_{b-}^{(i,\gamma)}X(b) = 0$. Furthermore, for each $j = 1, \ldots, k$, there exists a constant $C_{j,\beta,\gamma}$ such that, for each $r \in [0,b]$,

$$|\mathcal{R}^{(j,\gamma)}_{b-}X(r)| \leq C_{j,\beta,\gamma}\left(1+\max_{1\leq i\leq j-1}\|X^i\|_{i\beta}\right)^{j-1}\max_{1\leq i\leq j}\|X^i\|_{i\beta}(b-r)^{j(\beta-\gamma)}. \tag{3.4}$$

We will prove (3.4) in Section 4.

Definition 3.2. Let $X=(1,X^1,\dots,X^k)\in C_{0,\beta}(\triangle,T^{(k)}(E))$, $j=1,\dots,k$, μ a real number with $\mu>1-j\gamma$, and Ψ a function in $\mathcal{C}^{\mu}_2(L(E^{\otimes(j-1)},L(E,F)))$. An F-valued function $\mathcal{I}^{j,\gamma}_X(\Psi)$ on $\triangle$ is defined as

$$\mathcal{I}^{j,\gamma}_X(\Psi)_{s,t} := (-1)^{1-j\gamma}\int_s^t \mathcal{D}^{1-j\gamma}_{s+}\Psi(u)\mathcal{R}^{(j,\gamma)}_{t-}X(u)\,du \quad \text{for } (s,t)\in\triangle. \tag{3.5}$$

From (3.1) and (3.4), for each $(s,t)\in\triangle$,

$$|\mathcal{I}^{j,\gamma}_X(\Psi)_{s,t}| \leq C_{j,\beta,\gamma,\mu}\|\Psi\|_{\mu}\left(1+\max_{1\leq i\leq j-1}\|X^i\|_{i\beta}\right)^{j-1}\max_{1\leq i\leq j}\|X^i\|_{i\beta}(t-s)^{\mu+j\beta}. \tag{3.6}$$

It is also straightforward to show that $\mathcal{I}^{j,\gamma}_X(\Psi)$ belongs to $\mathcal{C}_2(F)$. Thus, $\mathcal{I}^{j,\gamma}_X(\Psi)$ belongs to $\mathcal{C}^{\mu+j\beta}_2(F)$. Furthermore, from (3.5) and (3.6), we obtain the following proposition.

Proposition 3.3. *In the setting of Definition* 3.2, *the map* $\Psi\mapsto\mathcal{I}^{j,\gamma}_X(\Psi)$ *is bounded linear from* $\mathcal{C}^{\mu}_2(L(E^{\otimes(j-1)},L(E,F)))$ *to* $\mathcal{C}^{\mu+j\beta}_2(F)$; *in particular, it is Lipschitz continuous.*

Let $X,\tilde{X}\in C_{0,\beta}(\triangle,T^{(k)}(E))$. For each $j=1,\dots,k$, there exists a constant $C_{j,\beta,\gamma}$ such that, for each $r\in[0,b]$,

$$\begin{aligned}&|\mathcal{R}^{(j,\gamma)}_{b-}X(r)-\mathcal{R}^{(j,\gamma)}_{b-}\tilde{X}(r)|\\&\quad\leq C_{j,\beta,\gamma}\left(1+\max_{1\leq i\leq j-1}\|X^i\|_{i\beta}+\max_{1\leq i\leq j-1}\|\tilde{X}^i\|_{i\beta}\right)^{j-1}\max_{1\leq i\leq j}\|X^i-\tilde{X}^i\|_{i\beta}(b-r)^{j(\beta-\gamma)}.\end{aligned} \tag{3.7}$$

We will prove (3.7) in Section 4. From (3.1) and (3.7),

$$\begin{aligned}&\|\mathcal{I}^{j,\gamma}_X(\Psi)-\mathcal{I}^{j,\gamma}_{\tilde{X}}(\Psi)\|_{\mu+j\beta}\\&\quad\leq C_{j,\beta,\gamma,\mu}\|\Psi\|_{\mu}\left(1+\max_{1\leq i\leq j-1}\|X^i\|_{i\beta}+\max_{1\leq i\leq j-1}\|\tilde{X}^i\|_{i\beta}\right)^{j-1}\max_{1\leq i\leq j}\|X^i-\tilde{X}^i\|_{i\beta}.\end{aligned} \tag{3.8}$$

This yields the following proposition.

Proposition 3.4. *In the setting of Definition* 3.2, *the map* $(1,X^1,\dots,X^j)\mapsto\mathcal{I}^{j,\gamma}_X(\Psi)$ *is locally Lipschitz continuous from* $C_{0,\beta}(\triangle,T^{(j)}(E))$ *to* $\mathcal{C}^{\mu+j\beta}_2(F)$.

3.2. Integration of weakly controlled paths via fractional calculus.

Throughout this subsection, γ will be a real number with $(1-\beta)/N<\gamma<\beta$. Before defining integrals of weakly controlled paths, we introduce several function spaces that are required for the discussion in this subsection. We set

$$M_\beta(E,F) := \{(X,Y) : X \in C_{0,\beta}(\triangle, T^{(N)}(E)), Y \in \mathcal{Q}_X^{\beta,N}(F)\}$$

equipped with a distance

$$m_\beta((X,Y),(\tilde{X},\tilde{Y})) := d_{\beta,N}(X,\tilde{X}) + \sum_{j=1}^{N} |Y_0^{(j-1)} - \tilde{Y}_0^{(j-1)}| + d_{X,\tilde{X},\beta}(Y,\tilde{Y}) \tag{3.9}$$

for $(X,Y),(\tilde{X},\tilde{Y}) \in M_\beta(E,F)$. Here,

$$d_{X,\tilde{X},\beta}(Y,\tilde{Y}) := \sum_{j=1}^{N} \| R_{j-1}^{N-j}(X,Y) - R_{j-1}^{N-j}(\tilde{X},\tilde{Y}) \|_{(N-j+1)\beta}. \tag{3.10}$$

We define the subset $S_\beta(E,F)$ of $M_\beta(E,F)$ by $S_\beta(E,F) := \{(X,Y) : X \in S\Omega_\beta(E), Y \in \mathcal{Q}_X^{1,N}(F)\}$ and let $\overline{S_\beta}(E,F)$ denote the closure of $S_\beta(E,F)$ with respect to the distance m_β. In Example 2.2, if φ is sufficiently smooth and all derivatives are bounded on E, then the pair (X,Y) belongs to $\overline{S_\beta}(E,L(E,F))$. In Example 2.3, if ψ is sufficiently smooth and all derivatives are bounded on F and the pair (X,Y) is in $\overline{S_\beta}(E,F)$, then the pair (X,Z) belongs to $\overline{S_\beta}(E,L(E,F))$. These can be proved by straightforward calculation. The following is our definition of the integral of weakly controlled paths along rough paths.

Definition 3.5. For $(X,Y) \in M_\beta(E,L(E,F))$, an F-valued function $I^\gamma(X,Y)$ on $\triangle$ is defined by

$$I^\gamma(X,Y)_{s,t} := \sum_{n=1}^{N} Y_s^{(n-1)} X_{s,t}^n + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t} \quad \text{for } (s,t) \in \triangle.$$

We note that the inequality $1 - n\gamma < (N-n+1)\beta$ follows from the assumption that $(1-\beta)/N < \gamma < \beta$. Therefore, $\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}$ is well-defined and so is $I^\gamma(X,Y)_{s,t}$. The following theorem justifies treating $I^\gamma(X,Y)$ as the integral of Y along X.

Theorem 3.6. *Let $(X,Y) \in S_\beta(E,L(E,F))$. Then, for each $(s,t) \in \triangle$, $I^\gamma(X,Y)_{s,t}$ coincides with the Riemann–Stieltjes integral $\int_s^t Y_u^{(0)} dX_{0,u}^1$.*

We will prove Theorem 3.6 in Section 4. The integral $I^\gamma(\cdot,\cdot)$ can be regarded as a continuous map.

Theorem 3.7. *The map $(X,Y) \mapsto I^\gamma(X,Y)$ is locally Lipschitz continuous from $M_\beta(E,L(E,F))$ to $\mathcal{C}_2^\beta(F)$.*

Proof. From Proposition 3.3, $I^\gamma(X,Y)$ belongs to $\mathcal{C}_2^\beta(F)$. Set $(s,t) \in \triangle$ with $s < t$. For $(X,Y),(\tilde{X},\tilde{Y}) \in M_\beta(E,L(E,F))$,

$$
\begin{aligned}
|I^\gamma(X,Y)_{s,t} - I^\gamma(\tilde{X},\tilde{Y})_{s,t}| \le \sum_{n=1}^{N} \Big\{ & |Y_s^{(n-1)} - \tilde{Y}_s^{(n-1)}||X_{s,t}^n| + |\tilde{Y}_s^{(n-1)}||X_{s,t}^n - \tilde{X}_{s,t}^n| \\
& + |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}| \\
& + |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t} - \mathcal{I}_{\tilde{X}}^{n,\gamma}(R_{n-1}^{N-n}(\tilde{X},\tilde{Y}))_{s,t}| \Big\}.
\end{aligned} \tag{3.11}
$$

By the definition of weakly controlled paths, we have

$$
\begin{aligned}
& |Y_s^{(n-1)} - \tilde{Y}_s^{(n-1)}| \\
& \quad \le |Y_0^{(n-1)} - \tilde{Y}_0^{(n-1)}| + |R_{n-1}^{N-n}(X,Y)_{0,s} - R_{n-1}^{N-n}(\tilde{X},\tilde{Y})_{0,s}| \\
& \qquad + \sum_{i=1}^{N-n} |Y_0^{(n-1+i)} X_{0,s}^i - \tilde{Y}_0^{(n-1+i)} \tilde{X}_{0,s}^i| \\
& \quad \le |Y_0^{(n-1)} - \tilde{Y}_0^{(n-1)}| + \|R_{n-1}^{N-n}(X,Y) - R_{n-1}^{N-n}(\tilde{X},\tilde{Y})\|_{(N-n+1)\beta} T^{(N-n+1)\beta} \\
& \qquad + \sum_{i=1}^{N-n} \Big\{ |Y_0^{(n-1+i)} - \tilde{Y}_0^{(n-1+i)}||X_{0,s}^i| + |\tilde{Y}_0^{(n-1+i)}||X_{0,s}^i - \tilde{X}_{0,s}^i| \Big\} \\
& \quad \le \left(1 + T^{(N-n+1)\beta} + \sum_{i=1}^{N-n} \left(\|X^i\|_{i\beta} + |\tilde{Y}_0^{(n-1+i)}| \right) T^{i\beta} \right) m_\beta((X,Y),(\tilde{X},\tilde{Y}))
\end{aligned} \tag{3.12}
$$

for each $n = 1, \ldots, N$. Then, from (3.6), (3.8), (3.9), (3.10), (3.11), and (3.12), we obtain the statement of this theorem immediately. □

Corollary 3.8. *Let $X \in C_{0,\beta}(\triangle, T^{(N)}(E))$. Then, the map $Y \mapsto I^\gamma(X,Y)$ is locally Lipschitz continuous from $\mathcal{Q}_X^{\beta,N}(L(E,F))$ to $\mathcal{C}_2^\beta(F)$.*

Proof. Apply (3.6) and (2.6) to (3.11) and (3.12) with $X = \tilde{X}$. □

From Theorems 3.6 and 3.7, we see that, for each $s, t, u \in [0,T]$ with $s \le u \le t$, the identity

$$
I^\gamma(X,Y)_{s,u} + I^\gamma(X,Y)_{u,t} = I^\gamma(X,Y)_{s,t} \tag{3.13}
$$

holds for $(X,Y) \in \overline{S_\beta}(E, L(E,F))$. Using this identity, we obtain the following proposition.

Proposition 3.9. *Let $(X,Y) \in \overline{S_\beta}(E, L(E,F))$. Then, for each $(s,t) \in \triangle$,*

$$
I^\gamma(X,Y)_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{m-1} \sum_{n=1}^{N} Y_{t_i}^{(n-1)} X_{t_i,t_{i+1}}^n,
$$

where the limit is taken over all finite partitions $\mathcal{P} = \{t_0, t_1, \ldots, t_m\}$ of the interval $[s,t]$

such that $s = t_0 \le t_1 \le \cdots \le t_m = t$ *and* $|\mathcal{P}| := \max_{0\le i\le m-1}|t_{i+1}-t_i|$.

PROOF. From (3.13) and Definition 3.5, for any partition $\mathcal{P} = \{t_0, t_1, \dots, t_m\}$,

$$\begin{aligned} I^\gamma(X,Y)_{s,t} &= \sum_{i=0}^{m-1} I^\gamma(X,Y)_{t_i,t_{i+1}} \\ &= \sum_{i=0}^{m-1}\left\{\sum_{n=1}^{N} Y_{t_i}^{(n-1)} X^n_{t_i,t_{i+1}} + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{t_i,t_{i+1}}\right\}. \end{aligned}$$

It then suffices to show that, for each $n = 1, \dots, N$,

$$\lim_{|\mathcal{P}|\to 0}\sum_{i=0}^{m-1} |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{t_i,t_{i+1}}| = 0. \tag{3.14}$$

From (3.6), we have

$$\begin{aligned} |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{t_i,t_{i+1}}| &\le C_{n,\beta,\gamma} \|\!|R_{n-1}^{N-n}(X,Y)|\!\|_{(N-n+1)\beta} \\ &\quad \times \left(1 + \max_{1\le j\le n-1}\|\!|X^j|\!\|_{j\beta}\right)^{n-1} \max_{1\le j\le n}\|\!|X^j|\!\|_{j\beta}(t_{i+1}-t_i)^{(N+1)\beta}. \end{aligned}$$

Thus, from the relation $(N+1)\beta > 1$,

$$\sum_{i=0}^{m-1} |\mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{t_i,t_{i+1}}| \le C\sum_{i=0}^{m-1}(t_{i+1}-t_i)^{(N+1)\beta} \le C|\mathcal{P}|^{(N+1)\beta-1}(t-s) \to 0$$

as $|\mathcal{P}| \to 0$. Here, C is a positive constant that does not depend on $\mathcal{P}$. Therefore, (3.14) holds. Thus we obtain the claim of the proposition. □

REMARK 3.10. Let us make a few comments about our integration.

1. Take $X \in G\Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,N}(L(E,F))$ as in Example 2.2. Then, $I^\gamma(X,Y)$ is the same as the integral introduced in the author's previous work [**6**, Definition 2.3]. Thus we see from [**6**, Theorem 2.6] that $I^\gamma(X,Y)$ coincides with the first level path of the rough integral along $X \in G\Omega_\beta(E)$.

2. The relation to the integration of weakly controlled paths introduced by Gubinelli [**4**] is stated as follows. Let $1/3 < \beta \le 1/2$. Based on Proposition 3.9 and [**4**, Corollaries 2 and 3], we see that $I^\gamma(X,Y)$ coincides with the integral introduced in [**4**, Corollary 3] on $\triangle$ if $I^\gamma(X,Y)$ satisfies (3.13). Therefore, for $(X,Y) \in \overline{S_\beta}(E, L(E,F))$, $I^\gamma(X,Y)$ is consistent with the integral introduced in [**4**, Corollary 3] on $\triangle$. However, it is unknown whether (3.13) is true for every $X \in \Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,2}(L(E,F))$.

3. If $N = 1$, then $I^\gamma(X,Y)_{a,b}$ coincides with the Riemann–Stieltjes integral $\int_a^b Y_t^{(0)} dX^1_{0,t}$ for $X \in \Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,1}(L(E,F))$. This follows from (2.15) with $f = Y^{(0)}$, $g = X^1_{0,\cdot}$, and $\alpha = 1-\gamma$. In particular, $I^\gamma(X,Y)_{a,b}$ is independent

of the choice of γ. If $N \geq 2$, then this value is independent of the choice of γ for $(X, Y) \in \overline{S_\beta}(E, L(E, F))$ from Proposition 3.9. However, it is unknown whether such a property holds for every $X \in \Omega_\beta(E)$ and $Y \in \mathcal{Q}_X^{\beta,N}(L(E, F))$.

3.3. Lyons' extension theorem.

Throughout this subsection, we assume the following: j is an integer with $j \geq N$ and γ_j is a real number with $(1-\beta)/j < \gamma_j < \min\{1/j, \beta\}$. To construct the extension map we first define the following functional.

Definition 3.11. For $X = (1, X^1, \ldots, X^j) \in C_{0,\beta}(\triangle, T^{(j)}(E))$, an $E^{\otimes(j+1)}$-valued function $\hat{X}^{j+1}$ on $\triangle$ is defined by

$$\hat{X}^{j+1}_{s,t} := \sum_{n=1}^{j} (-1)^{1-n\gamma_j} \int_s^t \mathcal{D}^{1-n\gamma_j}_{s+} X^{j+1-n}(u) \otimes \mathcal{R}^{(n,\gamma_j)}_{t-} X(u)\, du \quad \text{for } (s,t) \in \triangle.$$

We note that the inequalities $0 < 1 - n\gamma_j < (j+1-n)\beta$ follow from the assumption that $(1-\beta)/j < \gamma_j < \min\{1/j, \beta\}$. Thus, $\hat{X}^{j+1}$ is well-defined and

$$\begin{aligned} &|||\hat{X}^{j+1}|||_{(j+1)\beta} \\ &\leq C_{j,\beta,\gamma_j} \left(\max_{1\leq i\leq j} |||X^i|||_{i\beta} \right)^2 \left(\left(1 + \max_{1\leq i\leq j-1} |||X^i|||_{i\beta} \right)^j - 1 \right) \left(\max_{1\leq i\leq j-1} |||X^i|||_{i\beta} \right)^{-1} \end{aligned}$$

from (2.3), (3.1), (3.5), and (3.6). Furthermore, from Propositions 3.3 and 3.4, we obtain the following proposition.

Proposition 3.12. *For $X = (1, X^1, \ldots, X^j) \in C_{0,\beta}(\triangle, T^{(j)}(E))$, the map $X \mapsto \hat{X}^{j+1}$ is locally Lipschitz continuous from $C_{0,\beta}(\triangle, T^{(j)}(E))$ to $\mathcal{C}_2^{(j+1)\beta}(E^{\otimes(j+1)})$.*

The following is a key proposition for the proof of Theorem 3.14 below.

Proposition 3.13. *Let $X = (1, X^1, \ldots, X^j)$ be a step-j signature in E. Then, $(1, X^1, \ldots, X^j, \hat{X}^{j+1})$ is the step-$(j+1)$ signature, that is, for each $(s,t) \in \triangle$, $\hat{X}^{j+1}_{s,t}$ coincides with the Riemann–Stieltjes integral $\int_s^t X^j_{s,u} \otimes dX^1_{0,u}$.*

We will prove Proposition 3.13 in Section 4. From Propositions 3.12 and 3.13, for geometric β-Hölder rough paths $X \in G\Omega_\beta(E)$, we can see that the definition of $\hat{X}^{j+1}$ is independent of the choice of γ_j. The following is our version of Lyons' extension theorem for $X \in G\Omega_\beta(E)$.

Theorem 3.14. *Let $X = (1, X^1, \ldots, X^N) \in G\Omega_\beta(E)$. For any integer $k \geq N+1$, there exists an extension of the rough path X to a multiplicative functional of degree k in E with finite β-Hölder estimates.*

Proof. We take an arbitrary γ_N such that $(1-\beta)/N < \gamma_N < \min\{1/N, \beta\} = \beta$ and define $\hat{X}^{N+1}$ as in Definition 3.11. We set $\hat{X}^{(N+1)} := (1, X^1, \ldots, X^N, \hat{X}^{N+1})$. From Proposition 3.12, $\hat{X}^{(N+1)}$ belongs to $C_{0,\beta}(\triangle, T^{(N+1)}(E))$. By the definition of

$X \in G\Omega_\beta(E)$, there exists a sequence of smooth rough paths $X(m)$ which converges to X with respect to the distance $d_{\beta,N}$. Hence, from Propositions 3.12 and 3.13, $\lim_{m\to\infty} d_{\beta,N+1}(X(m)^{(N+1)}, \hat{X}^{(N+1)}) = 0$, where $X(m)^{(N+1)}$ is the step-$(N+1)$ signature of $X(m)^1_{0,\cdot} \in \mathcal{C}^1_1(E)$. Thus, $\hat{X}^{(N+1)}$ is a multiplicative functional of degree $(N+1)$ in E. This implies the statement of the theorem for $k = N+1$. By repeating this argument with $\gamma_{N+1}, \ldots, \gamma_{k-1}$, the desired statement is proven for any $k \geq N+1$. $\square$

We remark that [**7**, Theorem 2.2.1] implies the uniqueness of extensions even for $X \in \Omega_\beta(E)$. In particular, for $X \in G\Omega_\beta(E)$, the extension by Theorem 3.14 coincides with those introduced by Lyons [**7**, Theorem 2.2.1] and by Gubinelli [**4**, Proposition 10]. However, it is unknown for non-geometric Hölder rough paths $X \in \Omega_\beta(E)$ whether $\hat{X}^{(k)}$ defined as in Theorem 3.14 is a multiplicative functional of degree k in E. Also, one would expect that $\hat{X}^k$ should possess a uniform estimate as in [**7**, Theorem 2.2.1] and [**4**, Proposition 10], but it is doubtful whether $\hat{X}^k$ could provide sharper estimates than those provided in the previous studies; there is room for argument on this point.

4. Some proofs.

In this section, we prove (3.4), (3.7), Theorem 3.6, and Proposition 3.13. Let us recall the following assumptions: (a, b) is an element of $\triangle$ with $a < b$, β is a real number with $0 < \beta \leq 1$, k is a positive integer, and γ is a real number with $0 < \gamma < \min\{1/k, \beta\}$.

4.1. Proof of (3.4) and (3.7).

In this subsection, we prove (3.4) and (3.7). Let $X, \tilde{X} \in C_{0,\beta}(\triangle, T^{(k)}(E))$. For each $j = 1, \ldots, k$, we set $K_j := \max_{1\leq i\leq j} \|X^i\|_{i\beta}$ and $\tilde{K}_j := \max_{1\leq i\leq j} \|\tilde{X}^i\|_{i\beta}$.

LEMMA 4.1. *Under the above notation and assumptions, for each $j = 2, \ldots, k$ and $r \in [0, b]$,*

$$|\mathcal{R}^{(j,\gamma)}_{b-} X(r) - \mathcal{R}^{(j,\gamma)}_{b-} \tilde{X}(r)| \leq C(1 + C(K_{j-1} + \tilde{K}_{j-1}))^{j-1} \max_{1\leq i\leq j} \|X^i - \tilde{X}^i\|_{i\beta}(b-r)^{j(\beta-\gamma)}, \tag{4.1}$$

where $C = (\beta/(\beta-\gamma))\Gamma(1-\gamma)^{-1}$. If $\tilde{X} = (1, 0, \ldots, 0)$, then, for each $j = 2, \ldots, k$ and $r \in [0, b]$,

$$|\mathcal{R}^{(j,\gamma)}_{b-} X(r)| \leq C(1 + CK_{j-1})^{j-1} K_j (b-r)^{j(\beta-\gamma)}. \tag{4.2}$$

PROOF OF (3.4) AND (3.7). From (4.1) and (4.2), and the relation $C \leq \beta/(\beta-\gamma)$, we obtain (3.4) and (3.7) with $C_{j,\beta,\gamma} = (\beta/(\beta-\gamma))^j$. $\square$

PROOF OF LEMMA 4.1. We prove (4.1) by induction on j. We set $r \in [0, b]$ with $0 \leq r < b$ since $\mathcal{R}^{(j,\gamma)}_{b-} X(b) = \mathcal{R}^{(j,\gamma)}_{b-} \tilde{X}(b) = 0$ holds from the definition. From (3.3),

$$\begin{aligned}|\mathcal{R}^{(2,\gamma)}_{b-} X(r) - \mathcal{R}^{(2,\gamma)}_{b-} \tilde{X}(r)| &\leq |\mathcal{D}^{2\gamma}_{b-} X^2(r) - \mathcal{D}^{2\gamma}_{b-} \tilde{X}^2(r)| \\ &\quad + \frac{\gamma}{\Gamma(1-\gamma)} \int_r^b \frac{|X^1_{r,v} - \tilde{X}^1_{r,v}||\mathcal{R}^{(1,\gamma)}_{b-} X(v)|}{(v-r)^{\gamma+1}}\, dv\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma}{\Gamma(1-\gamma)}\int_r^b \frac{|\tilde{X}^1_{r,v}||\mathcal{R}^{(1,\gamma)}_{b-}X(v)-\mathcal{R}^{(1,\gamma)}_{b-}\tilde{X}(v)|}{(v-r)^{\gamma+1}}\,dv\\
&=: A_1+A_2+A_3.
\end{aligned}
$$

From (3.2), we have

$$
A_1 \le \frac{1}{\Gamma(1-2\gamma)}\frac{2\beta}{2\beta-2\gamma}\|X^2-\tilde{X}^2\|_{2\beta}(b-r)^{2(\beta-\gamma)} \le C\|X^2-\tilde{X}^2\|_{2\beta}(b-r)^{2(\beta-\gamma)}
$$

and

$$
\begin{aligned}
A_2 &\le \frac{\gamma}{\Gamma(1-\gamma)}\int_r^b (v-r)^{\beta-\gamma-1}\,dv\|X^1-\tilde{X}^1\|_\beta \frac{1}{\Gamma(1-\gamma)}\frac{\beta}{\beta-\gamma}\|X^1\|_\beta(b-r)^{\beta-\gamma}\\
&= \frac{\gamma}{\Gamma(1-\gamma)}\frac{(b-r)^{\beta-\gamma}}{\beta-\gamma}\|X^1-\tilde{X}^1\|_\beta C\|X^1\|_\beta(b-r)^{\beta-\gamma}\\
&\le C^2\|X^1\|_\beta\|X^1-\tilde{X}^1\|_\beta(b-r)^{2(\beta-\gamma)}.
\end{aligned}
$$

In a similar way, we get

$$
A_3 \le C^2\|\tilde{X}^1\|_\beta\|X^1-\tilde{X}^1\|_\beta(b-r)^{2(\beta-\gamma)}.
$$

By combining these estimates, we obtain

$$
A_1+A_2+A_3 \le C(1+C(\|X^1\|_\beta+\|\tilde{X}^1\|_\beta))\max_{1\le l\le 2}\|X^l-\tilde{X}^l\|_{l\beta}(b-r)^{2(\beta-\gamma)}.
$$

Hence, (4.1) holds for $j=2$. Suppose that (4.1) holds for each $j=2,\dots,J$ with $J\le k-1$. By using the induction hypothesis and calculations similar to those shown above, we have

$$
\begin{aligned}
&|\mathcal{R}^{(J+1,\gamma)}_{b-}X(r)-\mathcal{R}^{(J+1,\gamma)}_{b-}\tilde{X}(r)|\\
&\quad=|\mathcal{D}^{(J+1)\gamma}_{b-}X^{J+1}(r)-\mathcal{D}^{(J+1)\gamma}_{b-}\tilde{X}^{J+1}(r)|\\
&\qquad+\sum_{i=1}^{J}\Bigg\{\frac{(J+1-i)\gamma}{\Gamma(1-(J+1-i)\gamma)}\int_r^b\frac{|X^{J+1-i}_{r,v}-\tilde{X}^{J+1-i}_{r,v}||\mathcal{R}^{(i,\gamma)}_{b-}X(v)|}{(v-r)^{(J+1-i)\gamma+1}}\,dv\\
&\qquad\qquad+\frac{(J+1-i)\gamma}{\Gamma(1-(J+1-i)\gamma)}\int_r^b\frac{|\tilde{X}^{J+1-i}_{r,v}||\mathcal{R}^{(i,\gamma)}_{b-}X(v)-\mathcal{R}^{(i,\gamma)}_{b-}\tilde{X}(v)|}{(v-r)^{(J+1-i)\gamma+1}}\,dv\Bigg\}\\
&\quad\le C|\!|\!|X^{J+1}-\tilde{X}^{J+1}|\!|\!|_{(J+1)\beta}(b-r)^{(J+1)(\beta-\gamma)}\\
&\qquad+\sum_{i=1}^{J}\Bigg\{C|\!|\!|X^{J+1-i}-\tilde{X}^{J+1-i}|\!|\!|_{(J+1-i)\beta}(b-r)^{(J+1-i)(\beta-\gamma)}\\
&\qquad\qquad\times C(1+CK_{i-1})^{i-1}K_i(b-r)^{i(\beta-\gamma)}\\
&\qquad\qquad+C|\!|\!|\tilde{X}^{J+1-i}|\!|\!|_{(J+1-i)\beta}(b-r)^{(J+1-i)(\beta-\gamma)}\\
&\qquad\qquad\times C(1+C(K_{i-1}+\tilde{K}_{i-1}))^{i-1}\max_{1\le l\le i}|\!|\!|X^l-\tilde{X}^l|\!|\!|_{l\beta}(b-r)^{i(\beta-\gamma)}\Bigg\}
\end{aligned}
$$

(from the induction hypothesis)

$$\begin{aligned}
&\le C \max_{1\le l\le J+1} |||X^l-\tilde{X}^l|||_{l\beta}(b-r)^{(J+1)(\beta-\gamma)}\\
&\quad\times\left(1+C\sum_{i=1}^{J}\left(K_i+|||\tilde{X}^{J+1-i}|||_{(J+1-i)\beta}\right)(1+C(K_{i-1}+\tilde{K}_{i-1}))^{i-1}\right)\\
&\le C \max_{1\le l\le J+1} |||X^l-\tilde{X}^l|||_{l\beta}(b-r)^{(J+1)(\beta-\gamma)}\left(1+C(K_J+\tilde{K}_J)\sum_{i=1}^{J}(1+C(K_J+\tilde{K}_J))^{i-1}\right)\\
&= C(1+C(K_J+\tilde{K}_J))^J \max_{1\le l\le J+1} |||X^l-\tilde{X}^l|||_{l\beta}(b-r)^{(J+1)(\beta-\gamma)},
\end{aligned}$$

as desired. Therefore, (4.1) holds for $j=J+1$. □

4.2. Proofs of Theorem 3.6 and Proposition 3.13.

Using Proposition 4.2 stated below, we prove Theorem 3.6 and Proposition 3.13. Let $X\in C_{0,\beta}(\triangle, T^{(k)}(E))$ and $Y\in \mathcal{Q}_X^{\beta,k}(L(E,F))$. For each $l=0,\dots,k-1$, $m=0,\dots,k-1-l$, and $(s,t)\in\triangle$, we set

$$R_l^m(X,Y)_{s,t} := Y_t^{(l)} - \sum_{i=0}^{m} Y_s^{(l+i)} X_{s,t}^i. \tag{4.3}$$

PROPOSITION 4.2. *Let X be a step-k signature in E and $Y\in\mathcal{Q}_X^{1,k}(L(E,F))$. Take $\gamma\in(0,1/k)$. Then, for each $l=0,\dots,k-1$, $m=0,\dots,k-1-l$, and $(s,t)\in\triangle$,*

$$\int_s^t R_l^m(X,Y)_{s,u}\,dX_{0,u}^1 = \sum_{n=1}^{m+1} \mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{m-n+1}(X,Y))_{s,t}, \tag{4.4}$$

where the left-hand side is the Riemann–Stieltjes integral of $R_l^m(X,Y)_{s,\cdot}$ along $X_{0,\cdot}^1$.

For the proof of this proposition, we need the following two lemmas.

LEMMA 4.3. *Let X be a multiplicative functional of degree k in E with finite β-Hölder estimates. Then, for each $j=1,\dots,k$, $\mathcal{R}_{b-}^{(j,\gamma)}X$ is $\min\{\beta-\gamma,1-j\gamma\}$-Hölder continuous on the interval $[0,b]$.*

LEMMA 4.4. *Let X be a multiplicative functional of degree k in E with finite 1-Hölder estimates. Take $\gamma\in(0,1/k)$. Then, for each $j=2,\dots,k$ and $r\in(a,b)$,*

$$\mathcal{R}_{b-}^{(j,\gamma)}X(r) = D_{b-}^{j\gamma}(X_{a,\cdot}^j - X_{a,b}^j)(r) - \sum_{i=1}^{j-1} D_{b-}^{(j-i)\gamma}(X_{a,\cdot}^{j-i}\otimes\mathcal{R}_{b-}^{(i,\gamma)}X)(r). \tag{4.5}$$

The right-hand side of (4.5) is well-defined from Lemma 4.3. We omit the details of the proofs of the lemmas since these are proved by rewriting the proofs of [**6**, Lemma 3.2 and Proposition 3.3] in the obvious way. In [**6**, Lemma 3.2 and Proposition 3.3], we consider only the case where k is equal to $N=\lfloor 1/\beta\rfloor$. It is straightforward to prove that [**6**, Lemma 3.2 and Proposition 3.3] is generalized for any integer k and that Lemmas 4.3 and 4.4 follow from the generalizations with the parameters β_n and γ_n in [**6**, Lemma 3.2 and Proposition 3.3] chosen as $\beta_n=\min\{n\beta,1\}$ and $\gamma_n=\gamma$, respectively, for $n=1,\dots,k$;

we leave the details to the reader. Let us introduce one more notation for the proof of Proposition 4.2. Let X be a multiplicative functional of degree k in E. For $j = 1, \dots, k$, we set $\mathcal{T}(X)^j \in \mathcal{C}_2(E^{\otimes j})$ as follows: for each $(s,t) \in \triangle$, $\mathcal{T}(X)^1_{s,t} := X^1_{s,t}$ and

$$\mathcal{T}(X)^j_{s,t} := X^j_{s,t} - \sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,t} \otimes X^{j-i}_{s,t} \tag{4.6}$$

for $j = 2, \dots, k$, inductively. Then, for each $j = 2, \dots, k$ and $(s,t) \in \triangle$, the identity

$$\sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,t} \otimes X^{j-i}_{s,t} = \sum_{i=1}^{j-i} X^i_{s,t} \otimes \mathcal{T}(X)^{j-i}_{s,t} \tag{4.7}$$

holds. This is proved by simple calculation and induction on j. By using (4.7) and induction on j, we can show that, for each $s, u, t \in [0,T]$ with $s \le u \le t$, the identity

$$X^j_{u,t} = X^j_{s,t} - X^j_{s,u} - \sum_{i=1}^{j-1} \mathcal{T}(X)^i_{s,u} \otimes (X^{j-i}_{s,t} - X^{j-i}_{s,u}) \tag{4.8}$$

holds for $j = 2, \dots, k$. (4.8) is used in the proof of Proposition 4.2. Furthermore, we remark the following identity for later use. Let $f, g \in \mathcal{C}^\lambda_1(\mathbb{R})$ with $0 < \lambda \le 1$. From (2.11), for each $\alpha \in (0, \lambda)$ and $t \in (a,b)$,

$$D^\alpha_{a+}(fg)(t) - D^\alpha_{a+} f(t) g(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{f(s)(g(t)-g(s))}{(t-s)^{\alpha+1}}\, ds. \tag{4.9}$$

We now have all the tools to prove Proposition 4.2.

Proof of Proposition 4.2. Fix l with $0 \le l \le k-1$. We prove (4.4) by induction on m. Using (4.3) and (2.15), we have

$$\int_s^t R^0_l(X,Y)_{s,u}\, dX^1_{0,u} = \mathcal{I}^{1,\gamma}_X(R^0_l(X,Y))_{s,t}.$$

Hence, (4.4) holds for $m = 0$. Suppose that (4.4) holds for $m = M$ with $0 \le M \le k-2-l$. Using (4.3), (2.4), and the induction hypothesis, we have

$$\begin{aligned}\int_s^t R^{M+1}_l(X,Y)_{s,u}\, dX^1_{0,u} &= \int_s^t R^M_l(X,Y)_{s,u}\, dX^1_{0,u} - Y^{(l+M+1)}_s \int_s^t X^{M+1}_{s,u} \otimes dX^1_{0,u} \\ &= \sum_{n=1}^{M+1} \mathcal{I}^{n,\gamma}_X(R^{M-n+1}_{l+n-1}(X,Y))_{s,t} - Y^{(l+M+1)}_s X^{M+2}_{s,t}.\end{aligned}$$

For the proof of (4.4) for $m = M+1$, it then suffices to show the following identity:

$$\begin{aligned}&\sum_{n=1}^{M+1} \mathcal{I}^{n,\gamma}_X(R^{M-n+1}_{l+n-1}(X,Y) - R^{M+1-n+1}_{l+n-1}(X,Y))_{s,t} \\ &\quad = \mathcal{I}^{M+2,\gamma}_X(R^0_{l+M+1}(X,Y))_{s,t} + Y^{(l+M+1)}_s X^{M+2}_{s,t}.\end{aligned}$$

By the definition of $\mathcal{I}_X^{n,\gamma}$ (see (3.5)), for each $n = 1, \dots, M+1$, we have

$$\begin{aligned}
&\mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))_{s,t} \\
&\quad = (-1)^{1-n\gamma} \int_s^t \mathcal{D}_{s+}^{1-n\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))(u) \mathcal{R}_{t-}^{(n,\gamma)} X(u)\, du.
\end{aligned}$$

We calculate the integrand as follows: for each $u \in (s,t)$,

$$\begin{aligned}
&\mathcal{D}_{s+}^{1-n\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))(u) \\
&\quad = \frac{1}{\Gamma(n\gamma)}\left(\frac{Y_s^{(l+M+1)} X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}} + (1-n\gamma)\int_s^u \frac{Y_v^{(l+M+1)} X_{v,u}^{M+1-n+1}}{(u-v)^{(1-n\gamma)+1}}\, dv\right) \\
&\qquad \text{(from (4.3))} \\
&\quad = \frac{1}{\Gamma(n\gamma)} \frac{Y_s^{(l+M+1)} X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}} + \frac{1-n\gamma}{\Gamma(n\gamma)} \int_s^u \frac{Y_v^{(l+M+1)}(X_{s,u}^{M+1-n+1} - X_{s,v}^{M+1-n+1})}{(u-v)^{(1-n\gamma)+1}}\, dv \\
&\qquad - \sum_{i=1}^{M+1-n} \frac{1-n\gamma}{\Gamma(n\gamma)} \int_s^u \frac{Y_v^{(l+M+1)}(\mathcal{T}(X)_{s,v}^i \otimes (X_{s,u}^{M+1-n+1-i} - X_{s,v}^{M+1-n+1-i}))}{(u-v)^{(1-n\gamma)+1}}\, dv \\
&\qquad \text{(from (4.8))} \\
&\quad = \frac{1}{\Gamma(n\gamma)} \frac{Y_s^{(l+M+1)} X_{s,u}^{M+1-n+1}}{(u-s)^{1-n\gamma}} \\
&\qquad + D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} X_{s,\cdot}^{M+1-n+1})(u) - D_{s+}^{1-n\gamma} Y_\cdot^{(l+M+1)}(u) X_{s,u}^{M+1-n+1} \\
&\qquad - \sum_{i=1}^{M+1-n} \Bigl\{ D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i X_{s,\cdot}^{M+1-n+1-i})(u) \\
&\qquad\qquad - D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u) X_{s,u}^{M+1-n+1-i} \Bigr\} \quad \text{(from (4.9))} \\
&\quad = -D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u) X_{s,u}^{M+1-n+1} \\
&\qquad + D_{s+}^{1-n\gamma}\left(Y_\cdot^{(l+M+1)}\left(X_{s,\cdot}^{M+1-n+1} - \sum_{i=1}^{M+1-n} \mathcal{T}(X)_{s,\cdot}^i \otimes X_{s,\cdot}^{M+1-n+1-i}\right)\right)(u) \\
&\qquad + \sum_{i=1}^{M+1-n} D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u) X_{s,u}^{M+1-n+1-i} \\
&\quad = -D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u) X_{s,u}^{M+1-n+1} \\
&\qquad + D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^{M+1-n+1})(u) \quad \text{(from (4.6))} \\
&\qquad + \sum_{i=1}^{M+1-n} D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u) X_{s,u}^{M+1-n+1-i}.
\end{aligned}$$

Therefore, for each $n = 1, \dots, M+1$, we obtain

$$\mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))_{s,t}$$

$$= -(-1)^{1-(M+2)\gamma} \int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)$$
$$\times D_{t-}^{(M+1-n+1)\gamma}(X_{s,\cdot}^{M+1-n+1} \otimes \mathcal{R}_{t-}^{(n,\gamma)} X)(u)\, du$$
(from (2.13), Lemma 4.3, and (2.14) with $\alpha = (M+1-n+1)\gamma$)
$$+ A_n^{M+1-n+1}$$
$$+ \sum_{i=1}^{M+1-n} (-1)^{1-(M+2-i)\gamma} \int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u)$$
$$\times D_{t-}^{(M+1-n+1-i)\gamma}(X_{s,\cdot}^{M+1-n+1-i} \otimes \mathcal{R}_{t-}^{(n,\gamma)} X)(u)\, du.$$
(from (2.13), Lemma 4.3, and (2.14) with $\alpha = (M+1-n+1-i)\gamma$) (4.10)

Here, $A_n^{M+1-n+1}$ is defined by

$$A_n^j := (-1)^{1-n\gamma} \int_s^t D_{s+}^{1-n\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^j)(u) \mathcal{R}_{t-}^{(n,\gamma)} X(u)\, du$$

for each $n = 1, \ldots, M+1$ and $j = 1, \ldots, M+1$. Also, we have

$$A_1^{M+1} = \int_s^t Y_u^{(l+M+1)} \mathcal{T}(X)_{s,u}^{M+1}\, dX_{0,u}^1 \quad \text{(from (2.16))}$$
$$= \int_s^t Y_u^{(l+M+1)}\, dX_{s,u}^{M+2} - \sum_{i=1}^{M} \int_s^t Y_u^{(l+M+1)} \mathcal{T}(X)_{s,u}^i\, dX_{s,u}^{M+2-i}$$
(from (4.6) and (2.2))
$$= (-1)^{1-(M+2)\gamma} \int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)$$
$$\times D_{t-}^{(M+2)\gamma}(X_{s,\cdot}^{M+2} - X_{s,t}^{M+2})(u)\, du$$
$$+ Y_s^{(l+M+1)}(X_{s,t}^{M+2} - X_{s,s}^{M+2})$$
$$- \sum_{i=1}^{M} (-1)^{1-(M+2-i)\gamma} \int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u)$$
$$\times D_{t-}^{(M+2-i)\gamma}(X_{s,\cdot}^{M+2-i} - X_{s,t}^{M+2-i})(u)\, du. \quad \text{(from (2.15) and (2.16))} \quad (4.11)$$

Hence, by combining (4.10) and (4.11), we have

$$\sum_{n=1}^{M+1} \mathcal{I}_X^{n,\gamma}(R_{l+n-1}^{M-n+1}(X,Y) - R_{l+n-1}^{M+1-n+1}(X,Y))_{s,t}$$
$$= (-1)^{1-(M+2)\gamma} \int_s^t D_{s+}^{1-(M+2)\gamma}(Y_\cdot^{(l+M+1)} - Y_s^{(l+M+1)})(u)$$
$$\times \Bigg(D_{t-}^{(M+2)\gamma}(X_{s,\cdot}^{M+2} - X_{s,t}^{M+2})(u)$$

$$
\begin{aligned}
&\quad - \sum_{n=1}^{M+1} D_{t-}^{(M+1-n+1)\gamma}(X_{s,\cdot}^{M+1-n+1} \otimes \mathcal{R}_{t-}^{(n,\gamma)}X)(u)\Big)\,du \\
&+ Y_s^{(l+M+1)} X_{s,t}^{M+2} \\
&+ \sum_{n=2}^{M+1} A_n^{M+1-n+1} \\
&- \sum_{i=1}^{M} (-1)^{1-(M+2-i)\gamma} \int_s^t D_{s+}^{1-(M+2-i)\gamma}(Y_\cdot^{(l+M+1)} \mathcal{T}(X)_{s,\cdot}^i)(u) \\
&\times \Big(D_{t-}^{(M+2-i)\gamma}(X_{s,\cdot}^{M+2-i} - X_{s,t}^{M+2-i})(u) \\
&\quad - \sum_{n=1}^{M+1-i} D_{t-}^{(M+1-n+1-i)\gamma}(X_{s,\cdot}^{M+1-n+1-i} \otimes \mathcal{R}_{t-}^{(n,\gamma)}X)(u)\Big)\,du \\
= {}& \mathcal{I}_X^{M+2,\gamma}(R_{l+M+1}^0(X,Y))_{s,t} + Y_s^{(l+M+1)} X_{s,t}^{M+2} + \sum_{n=2}^{M+1} A_n^{M+1-n+1} - \sum_{i=1}^{M} A_{M+2-i}^i \\
&\text{(from (4.5))} \\
= {}& \mathcal{I}_X^{M+2,\gamma}(R_{l+M+1}^0(X,Y))_{s,t} + Y_s^{(l+M+1)} X_{s,t}^{M+2},
\end{aligned}
$$

as desired. Therefore, (4.4) holds for $m = M+1$ and thus the claim of the proposition holds by induction. □

Proof of Theorem 3.6. From (2.5) for $l = 0$ and $k = N$ and Proposition 4.2 for $l = 0$ and $m = N-1$,

$$
\begin{aligned}
\int_s^t Y_u^{(0)} dX_{0,u}^1 &= \sum_{i=0}^{N-1} \left\{ Y_s^{(i)} \int_s^t X_{s,u}^i \otimes dX_{0,u}^1 \right\} + \int_s^t R_0^{N-1}(X,Y)_{s,u}\, dX_{0,u}^1 \\
&= \sum_{n=1}^{N} Y_s^{(n-1)} X_{s,t}^n + \sum_{n=1}^{N} \mathcal{I}_X^{n,\gamma}(R_{n-1}^{N-n}(X,Y))_{s,t}.
\end{aligned}
$$

This is the claim of the theorem. □

Proof of Proposition 3.13. Under the assumptions of Proposition 3.13, we can take $Y = (Y^{(0)}, Y^{(1)}, \dots, Y^{(k-1)}) \in \mathcal{Q}_X^{1,k}(L(E, E^{\otimes(k+1)}))$ as in Example 2.4. Then, from (2.10) for $l = n-1$ and $k = j$ and Proposition 4.2 for $l = 0$ and $m = j-1$,

$$
\hat{X}_{s,t}^{j+1} = \sum_{n=1}^{j} \mathcal{I}_X^{n,\gamma_j}(R_{n-1}^{j-n}(X,Y))_{s,t} = \int_s^t R_0^{j-1}(X,Y)_{s,u}\, dX_{0,u}^1.
$$

From (2.10) for $l = 0$ and $k = j$, we obtain the claim of the proposition. □

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