J. Math. Soc. Japan
Vol. 69, No. 2 (2017) pp. 715–734
doi: 10.2969/jmsj/06920715

A note on bounded-cohomological dimension of discrete groups

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(Received May 13, 2015)
(Revised June 27, 2015)

Abstract. Bounded-cohomological dimension of groups is a relative of classical cohomological dimension, defined in terms of bounded cohomology with trivial coefficients instead of ordinary group cohomology. We will discuss constructions that lead to groups with infinite bounded-cohomological dimension, and we will provide new examples of groups with bounded-cohomological dimension equal to 0. In particular, we will prove that every group functorially embeds into an acyclic group with trivial bounded cohomology.

1. Introduction.

Bounded cohomology $H_b^*(\,\cdot\,;\mathbb{R})$ is a functional-analytic version of ordinary group cohomology, defined in terms of cocycles that are bounded with respect to the ℓ^1-norm on the bar complex [**8**], [**11**], [**19**], [**3**] (Section 2). Bounded cohomology has various applications in geometry and geometric group theory [**8**], [**14**], [**19**], [**20**]. There is a natural comparison map between bounded cohomology and ordinary group cohomology with $\mathbb{R}$-coefficients; however, this comparison map in general is neither surjective nor injective, and bounded cohomology usually is hard to calculate.

We will consider the following bounded analogue of classical cohomological dimension of groups with trivial coefficients (which should not be confused with the bounded-cohomological dimension with varying coefficients [**20**]):

Definition 1.1 (bounded-cohomological dimension [**9**]). *The bounded-cohomological dimension* of a group G is defined by

$$\operatorname{bcd}(G) := \sup\bigl\{n \in \mathbb{N} \bigm| H_b^n(G;\mathbb{R}) \not\cong 0\bigr\} \in \mathbb{N} \cup \{\infty\}.$$

In contrast with the corresponding invariant for ordinary group cohomology, not much is known about bounded-cohomological dimension. For example, bounded-cohomological dimension does *not* admit an obvious bound in terms of the geometric dimension of groups.

In this article, we will provide new examples of groups with bounded-cohomological dimension equal to 0 as well as of basic constructions that lead to groups with infinite bounded-cohomological dimension.

2010 *Mathematics Subject Classification.* Primary 55N35, 20J06, 20E99.

Key Words and Phrases. bounded cohomology, cohomological dimension, mitotic groups.

This work was supported by the CRC 1085 Higher Invariants (Universität Regensburg, funded by the DFG).

For all amenable groups G one has $\operatorname{bcd}(G) = 0$ [**8**], [**11**]. For all groups G we have $H_b^1(G;\mathbb{R}) \cong 0$ [**18**] and hence $\operatorname{bcd}(G) \neq 1$. Free groups F of rank at least 2 satisfy $\operatorname{bcd}(F) \geq 3$ [**21**], [**22**], [**24**]; however, the exact value of $\operatorname{bcd}(F)$ is unknown. If M is an oriented closed connected n-manifold with non-zero simplicial volume, then $\operatorname{bcd}\pi_1(M) \geq n$ [**8**]; this happens, for example, if M admits a metric of negative sectional curvature [**10**]. More generally, if G is a hyperbolic group, then the comparison map $H_b^*(G;\mathbb{R}) \longrightarrow H^*(G;\mathbb{R})$ is surjective in degree at least 2 [**17**], which gives lower bounds on $\operatorname{bcd} G$. Bounded cohomology in degree 2 is rather well understood in terms of quasi-morphisms/pseudo-characters [**6**]. For example, $\operatorname{bcd} G \geq 2$ whenever G is a sufficiently non-trivial amalgamated free product [**6**], [**7**], [**5**].

No examples of groups G with $\operatorname{bcd}(G) \notin \{0, \infty\}$ seem to be known.

Groups with small bounded cohomology.

Mather [**15**] showed that the (discrete) group $\operatorname{Homeo}_K(\mathbb{R}^n)$ of homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ with compact support is *acyclic* for all $n \in \mathbb{N}_{>0}$, i.e., $H_k(\operatorname{Homeo}_K(\mathbb{R}^n);\mathbb{Z}) \cong 0$ for all $k \in \mathbb{N}_{>0}$. Matsumoto and Morita [**16**] refined Mather's proof in the normed setting to obtain $\operatorname{bcd}\operatorname{Homeo}_K(\mathbb{R}^n) = 0$. This was the first example of a non-amenable group with trivial bounded-cohomological dimension.

Baumslag, Dyer, and Heller [**1**, Section 4] considered so-called mitotic groups (see Section 4.1 for the definition); mitotic groups have all the algebraic properties necessary to carry out Mather's argument and Baumslag, Dyer, Heller [**1**, Theorem 4.2] proved that all mitotic groups are acyclic.

Based on the normed refinement of Matsumoto and Morita of Mather's proof, we will adapt the argument of Baumslag, Dyer, Heller to show that mitotic groups have trivial bounded cohomology (Section 4):

THEOREM 1.2 (bounded cohomology of mitotic groups). *If G is a mitotic group, then* $\operatorname{bcd} G = 0$.

COROLLARY 1.3 (embedding groups into very acyclic groups). *There is a functor M*: Group $\longrightarrow$ Group *and a natural transformation i*: $\mathrm{id}_{\mathrm{Group}} \Longrightarrow M$ *with the following properties*:

(1) *For all groups G the group $M(G)$ is mitotic*; *in particular, $M(G)$ is acyclic and* $\operatorname{bcd} M(G) = 0$.

(2) *For all groups G, the homomorphism $i_G\colon G \longrightarrow M(G)$ is injective.*

(3) *If G is an infinite group, then $|M(G)| = |G|$, where $|\cdot|$ denotes the cardinality.*

PROOF. Baumslag, Dyer, Heller [**1**, Section 5, Theorem 4.2] constructed a functor M with these properties; Theorem 1.2 is only needed to deduce that $\operatorname{bcd} M(G) = 0$ for all groups G. □

In particular, mitotic groups in general are *not* amenable: For instance, $M(F_2)$ contains the non-amenable group F_2 as subgroup. Moreover, all algebraically closed groups are mitotic [**1**, Corollary 4.4].

EXAMPLE 1.4. Clearly, not every group G with $\operatorname{bcd} G = 0$ is acyclic: For every $n \in \mathbb{N} \cup \{\infty\}$ there is a group G that is not acyclic and satisfies

$$\operatorname{bcd} G = 0 \quad \text{and} \quad \operatorname{cd}_{\mathbb{Z}} G = n = \operatorname{cd}_{\mathbb{R}} G,$$

e.g., one can consider the amenable group $G = \mathbb{Z}^{\oplus n}$.

Groups with large bounded cohomology.

On the other hand, it is not hard to construct groups with large bounded cohomology, and hence of infinite bounded-cohomological dimension. For example, even though there does not seem to be a general Künneth theorem for bounded cohomology, we can use the interplay between bounded cohomology and ℓ^1-homology and (co)homological cross-products to propagate non-trivial classes:

PROPOSITION 1.5. *For each $n \in \mathbb{N}$ let G_n be a group with $H_b^2(G_n;\mathbb{R}) \not\cong 0$, and let $G \in \{\bigoplus_{n\in\mathbb{N}} G_n, \prod_{n\in\mathbb{N}} G_n\}$. Then*

$$\operatorname{bcd} G = \infty.$$

More precisely: There exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset H_b^2(G;\mathbb{R})$ such that for all $n \in \mathbb{N}$ we have

$$\varphi_0 \cup \cdots \cup \varphi_{n-1} \neq 0 \in H_b^{2\cdot n}(G;\mathbb{R}).$$

Here, $\bigoplus_{n\in\mathbb{N}} G_n$ denotes the subgroup of $\prod_{n\in\mathbb{N}} G_n$ of families with finite support.

The proof of Proposition 1.5 is given in Section 3.2, where we also give further classes of examples whose bounded cohomology can be easily calculated to a large extent.

EXAMPLE 1.6. Let $G := \bigoplus_{\mathbb{N}} F_2$. Then G clearly is not acyclic and because of $H_b^2(F_2;\mathbb{R}) \not\cong 0$ and $H^1(F_2;\mathbb{R}) \not\cong 0$ we obtain

$$\operatorname{bcd} G = \infty \quad \text{and} \quad \operatorname{cd}_{\mathbb{Z}} G = \infty = \operatorname{cd}_{\mathbb{R}} G.$$

EXAMPLE 1.7. There are acyclic groups with infinite bounded-cohomological dimension: For example, we can consider Higman's group

$$H := \langle a,b,c,d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle;$$

it is well known that H is acyclic and that H can be decomposed as a non-trivial amalgamated free product [**1**, Section 3]. Hence, $H_b^2(H;\mathbb{R}) \not\cong 0$ [**6**], [**5**]. Therefore,

$$\operatorname{bcd}\Bigl(\bigoplus_{\mathbb{N}} H\Bigr) = \infty.$$

On the other hand, acyclicity of H, the Künneth theorem, and the compatiblity of homology with colimits shows that $\bigoplus_{\mathbb{N}} H$ is acyclic.

However, so far, no examples of *finitely generated* non-amenable groups G seem to be known where $\operatorname{bcd} G$ can be computed explicitly.

Question 1.8. What can be said about the bounded-cohomological dimension of $(\bigoplus_{\mathbb{Z}} F_2) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\bigoplus_{\mathbb{Z}} F_2$ by shifting the summands?

Organisation of this article.

In Section 2, we briefly recall the definition of bounded cohomology and ℓ^1-homology of discrete groups, as well as some basic properties and constructions. In Section 3, we will give simple examples of groups with large bounded cohomology; in particular, we will prove Proposition 1.5. Finally, in Section 4, we will compute the bounded cohomology of mitotic groups, which proves Theorem 1.2.

2. Bounded cohomology and ℓ^1-homology.

We briefly review the definitions and basic properties of bounded cohomology and ℓ^1-homology of (discrete) groups with constant coefficients:

2.1. Bounded cohomology and ℓ^1-homology.

Bounded cohomology and ℓ^1-homology are normed refinements of classical group (co)homology: We will use the following concrete description:

Definition 2.1 (ℓ^1-norm, bounded cohomology, ℓ^1-homology). Let G be a group. We denote the standard chain complex by $C_*(G;\mathbb{R})$; more precisely, for $k \in \mathbb{N}$ we write $C_k(G;\mathbb{R}) := \bigoplus_{g\in G^k} \mathbb{R}\cdot g$ and

$$\begin{aligned}
\partial_k \colon C_k(G;\mathbb{R}) &\longrightarrow C_{k-1}(G;\mathbb{R}) \\
G^k \ni (g_1,\dots,g_k) &\longmapsto (g_2,\dots,g_k) \\
&\qquad + \sum_{j=1}^{k-1} (-1)^j \cdot (g_1,\dots,g_j\cdot g_{j+1},\dots,g_k) \\
&\qquad + (-1)^k \cdot (g_1,\dots,g_{k-1}).
\end{aligned}$$

We denote the ℓ^1-norm on $C_k(G;\mathbb{R})$ associated with the basis G^k by $\|\cdot\|_1$. Notice that $\|\partial_k\| \leq k+1$ with respect to the ℓ^1-norms.

- The completion of $C_*(G;\mathbb{R})$ with respect to the ℓ^1-norm is denoted by $C_*^{\ell^1}(G;\mathbb{R})$, the *$\ell^1$-chain complex of G*.
- The topological dual of $C_*(G;\mathbb{R})$ with respect to the ℓ^1-norm is denoted by $C_b^*(G;\mathbb{R})$, the *bounded cochain complex of G*.
- The homology $H_*^{\ell^1}(G;\mathbb{R})$ of $C_*^{\ell^1}(G;\mathbb{R})$ is called *ℓ^1-homology of G*. The reduced homology $\overline{H}_*^{\ell^1}(G;\mathbb{R})$ (i.e., kernel modulo closure of the image of the boundary operator) of $C_*^{\ell^1}(G;\mathbb{R})$ is called *reduced ℓ^1-homology of G*.
- The cohomology $H_b^*(G;\mathbb{R})$ of $C_b^*(G;\mathbb{R})$ is *bounded cohomology of G*. The reduced cohomology $\overline{H}_b^*(G;\mathbb{R})$ of $C_b^*(G;\mathbb{R})$ is called *reduced bounded cohomology of G*.

Clearly, all these constructions are functorial with respect to group homomorphisms and the inclusion $C_b^*(\,\cdot\,;\mathbb{R}) \hookrightarrow C^*(\,\cdot\,;\mathbb{R})$ induces a natural transformation between bounded cohomology and ordinary group cohomology, the so-called *comparison map.*

The ℓ^1-norm and its dual norm induce semi-norms on ℓ^1-homology and bounded cohomology, respectively. By definition, these semi-norms are norms on reduced ℓ^1-homology and reduced bounded cohomology, which then consist of Banach spaces.

More background on (co)homology of normed (co)chain complexes and on descriptions of bounded cohomology and ℓ^1-homology in terms of homological algebra can be found in the literature [**8**], [**11**], [**19**], [**16**], [**12**], [**3**].

2.2. Evaluation and duality.

Evaluation gives rise to a weak form of duality between bounded cohomology and ℓ^1-homology. If G is a group and $k \in \mathbb{N}$, then the evaluation map

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle\colon C_b^k(G;\mathbb{R}) \otimes_{\mathbb{R}} C_k(G;\mathbb{R}) &\longrightarrow \mathbb{R} \\ (f,c) &\longmapsto f(c) \end{aligned}$$

is compatible with the (co)boundary operators and it is continuous with respect to the (dual) ℓ^1-norm and hence induces a well-defined natural *Kronecker product*

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon \overline{H}_b^k(G;\mathbb{R}) \otimes_{\mathbb{R}} \overline{H}_k^{\ell^1}(G;\mathbb{R}) \longrightarrow \mathbb{R}.$$

Proposition 2.2 (weak duality principle [**16**], [**12**]). *Let G be a group and let $k \in \mathbb{N}$. Then the map*

$$\overline{H}_b^k(G;\mathbb{R}) \longrightarrow \big(\overline{H}_k^{\ell^1}(G;\mathbb{R})\big)'$$

induced by the Kronecker product is surjective.

Proposition 2.3 (bounded cohomology and ℓ^1-acyclicity [**16**]). *Let G be a group. Then* $\operatorname{bcd} G = 0$ *if and only if G is* ℓ^1-acyclic, *i.e., $H_k^{\ell^1}(G;\mathbb{R}) \cong 0$ for all $k \in \mathbb{N}_{>0}$.*

2.3. The cross-product in bounded cohomology and ℓ^1-homology.

The explicit descriptions of the (co)homological cross-products are continuous with respect to the (dual) ℓ^1-norm and lead to well-defined cross-products in bounded cohomology and ℓ^1-homology:

For groups G, H the homological cross-product is induced from the maps

$$\begin{aligned} \cdot\times\cdot\colon C_p(G;\mathbb{R}) \otimes_{\mathbb{R}} C_q(H;\mathbb{R}) &\longrightarrow C_{p+q}(G\times H;\mathbb{R}) \\ (g_1,\dots,g_p)\otimes(h_1,\dots,h_q) &\longmapsto \sum_{\sigma\in S_{p,q}} (-1)^{|\sigma|}\cdot \big(g_{\sigma_1(j)}, h_{\sigma_2(j)}\big)_{j\in\{1,\dots,p+q\}}. \end{aligned}$$

Here, $S_{p,q}$ is the set of all (p,q)-shuffles $\sigma = (\sigma_1,\sigma_2)$ [**4**], and $|\sigma|$ denotes the sign of shuffles $\sigma \in S_{p+q}$.

This cross-product is bounded in every degree with respect to the norms induced from the ℓ^1-norm. Because the compatibility with the boundary operators carries over to the completed chain complexes, we obtain a corresponding well-defined natural cross-product on (reduced) ℓ^1-homology.

Dually, for groups G and H the cohomological cross-product is induced from the maps

$$\begin{aligned}\cdot \times \cdot : C^p(G;\mathbb{R}) \otimes_{\mathbb{R}} C^q(H;\mathbb{R}) &\longrightarrow C^{p+q}(G \times H;\mathbb{R}) \\ \varphi \otimes \psi &\longmapsto (-1)^{p\cdot q} \cdot \bigl(((g_1,h_1),\dots,(g_{p+q},h_{p+q})) \\ &\qquad \mapsto \varphi(g_1,\dots,g_p)\cdot\psi(h_{p+1},\dots,h_{p+q})\bigr),\end{aligned}$$

as suggested by the Alexander–Whitney map. These maps preserve boundedness and are continuous and thus induce a well-defined natural cross-product on (reduced) bouneded cohomology.

Definition 2.4 (cross-product on bounded cohomology/ℓ^1-homology). Let G and H be groups and let $p,q \in \mathbb{N}$. Then the *cross-product* on reduced ℓ^1-homology and reduced bounded cohomology are defined via:

$$\begin{aligned}\cdot \times \cdot \, \overline{H}^{\ell^1}_p(G;\mathbb{R}) \otimes_{\mathbb{R}} \overline{H}^{\ell^1}_q(H;\mathbb{R}) &\longrightarrow \overline{H}^{\ell^1}_{p+q}(G \times H;\mathbb{R}) \\ [c]\otimes[d] &\longmapsto [c \times d], \\ \cdot \times \cdot \, \overline{H}^p_b(G;\mathbb{R}) \otimes_{\mathbb{R}} \overline{H}^q_b(H;\mathbb{R}) &\longmapsto \overline{H}^{p+q}_b(G \times H;\mathbb{R}) \\ [f]\otimes[g] &\longmapsto [f \times g].\end{aligned}$$

As in the case of ordinary group (co)homology these cross-products are compatible in the following sense:

Proposition 2.5 (compatibility of cross-products). *Let G and H be groups, let $p,q \in \mathbb{N}$ and let $\varphi \in \overline{H}^p_b(G;\mathbb{R})$, $\psi \in \overline{H}^q_b(H;\mathbb{R})$ as well as $\alpha \in \overline{H}^{\ell^1}_p(G;\mathbb{R})$, $\beta \in \overline{H}^{\ell^1}_q(H;\mathbb{R})$. Then*

$$\langle \varphi \times \psi, \alpha \times \beta\rangle = (-1)^{p\cdot q} \cdot \langle \varphi,\alpha\rangle \cdot \langle \psi,\beta\rangle.$$

Proof. For classical group (co)homology this can be deduced from the above explicit descriptions of the cross-products on the (co)chain level and the fact that the Alexander–Whitney map A satisfies $A \circ (\,\cdot \times \cdot\,) \simeq \mathrm{id}$ on the (co)chain level (Lemma 2.6 below).

Because this natural chain homotopy Ω can be chosen to be bounded in each degree (Lemma 2.6), the corresponding arguments carry over to the ℓ^1-chain complex and the bounded cochain complex:

Let $f \in C^p_b(G;\mathbb{R})$, $g \in C^q_b(H;\mathbb{R})$, $c \in C^{\ell^1}_p(G;\mathbb{R})$, $d \in C^{\ell^1}_q(H;\mathbb{R})$ be (co)cycles representing $\varphi,\psi,\alpha,\beta$, respectively. Let $\overline{C}_*$ be the completion of $C^{\ell^1}_*(G;\mathbb{R}) \otimes_{\mathbb{R}} C^{\ell^1}_*(H;\mathbb{R})$ with respect to the norm induced by the ℓ^1-norms. Then A extends to a chain map $\overline{A}\colon C^{\ell^1}_*(G \times H;\mathbb{R}) \longrightarrow \overline{C}_*$ that is bounded in each degree, and also Ω extends to $\overline{\Omega}$ satisfying

$$\overline{A} \circ (\,\cdot \times \cdot\,) - \mathrm{id} = \partial\overline{\Omega} + \overline{\Omega} \circ \partial.$$

Moreover, $f \otimes g$ also can be evaluated on elements of $\overline{C}_{p+q}$ because f and g are bounded. Therefore,

$$\begin{aligned}
(-1)^{p\cdot q} \cdot (f \times g)(c \times d) &= (f \otimes g)\big(A \circ (\,\cdot\, \times \,\cdot\,)(c \otimes d)\big) \\
&= (f \otimes g)\big(\overline{A} \circ (\,\cdot\, \times \,\cdot\,)(c \otimes d)\big) \\
&= (f \otimes g)(c \otimes d) \\
&\quad - (f \otimes g)\big(\partial \circ \overline{\Omega}(c \otimes d)\big) - (f \otimes g)\big(\overline{\Omega} \circ \partial(c \otimes d)\big) \\
&= (f \otimes g)(c \otimes d) \\
&= f(c) \cdot g(d),
\end{aligned}$$

as desired. $\square$

Lemma 2.6. *Let G be a group. Then the cross-product*

$$\cdot \times \cdot : C_*(G;\mathbb{R}) \otimes_{\mathbb{R}} C_*(G;\mathbb{R}) \longmapsto C_*(G \times G;\mathbb{R})$$

and the Alexander–Whitney map given by

$$\begin{aligned}
A\colon C_q(G \times G) &\longrightarrow \big(C_*(G) \otimes_{\mathbb{R}} C_*(G)\big)_q \\
(G \times G)^q \ni \big((g_1,h_1),\dots,(g_q,h_q)\big) &\longmapsto \sum_{j=0}^{q} (g_1,\dots,g_j) \otimes (h_{j+1},\dots,h_q)
\end{aligned}$$

are natural chain maps that are mutually chain homotopy inverses of each other. More precisely, there exist natural chain homotopies

$$\begin{aligned}
\Xi&\colon (\,\cdot\, \times \,\cdot\,) \circ A \simeq \mathrm{id}, \\
\Omega&\colon A \circ (\,\cdot\, \times \,\cdot\,) \simeq \mathrm{id}
\end{aligned}$$

that are bounded in each degree (*with respect to the norms induced from the respective ℓ^1-norms*), *where the bounds in every degree q depend only on q and* not *on the group G.*

Proof. This is a consequence of the classic proof via the acyclic model theorem [**4**]. $\square$

A more systematic study of acyclic models in the context of ℓ^1-homology was carried out by Bouarich [**2**]. Moreover, for sufficiently well-behaved products the spectral sequence of Monod applies [**19**].

Furthermore, (reduced) bounded cohomology carries a natural ring structure via the cup-product:

Definition 2.7 (cup-product on bounded cohomology). Let G be a group, and let $p, q \in \mathbb{N}$. Then the *cup-product on $\overline{H}_b^*(G;\mathbb{R})$* is given by

$$\begin{aligned}
\cdot \cup \cdot : \overline{H}_b^p(G;\mathbb{R}) \otimes_{\mathbb{R}} \overline{H}_b^q(G;\mathbb{R}) &\longrightarrow \overline{H}_b^{p+q}(G;\mathbb{R}) \\
\varphi \otimes \psi &\longmapsto \overline{H}_b^{p+q}(\Delta_G;\mathbb{R})(\varphi \times \psi),
\end{aligned}$$

where $\Delta_G\colon G \longrightarrow G \times G$ is the diagonal map.

As in classical group cohomology, also the relation

$$\varphi \times \psi = \overline{H}_b^p(p_G;\mathbb{R})(\varphi) \cup \overline{H}_b^q(p_H;\mathbb{R})(\psi) \in \overline{H}_b^{p+q}(G \times H;\mathbb{R})$$

holds for all $\varphi \in \overline{H}_b^p(G;\mathbb{R})$, $\psi \in \overline{H}_b^q(H;\mathbb{R})$, where $p_G\colon G \times H \longrightarrow G$ and $p_H\colon G \times H \longrightarrow H$ are the projections onto the factors.

3. Groups with large bounded cohomology.

We will now construct groups with large bounded cohomology by taking (free) products and exploiting the relation with ℓ^1-homology. In particular, we will prove Proposition 1.5 and related results.

3.1. ℓ^1-Betti numbers.

We introduce (reduced) ℓ^1-Betti numbers of groups and discuss their basic properties as well as their influence on bounded cohomology.

DEFINITION 3.1 (ℓ^1-Betti numbers). Let G be a group and let $k \in \mathbb{N}$. Then the *k-th ℓ^1-Betti number* $\overline{b}_k^{\ell^1}(G)$ is defined as the cardinality of an $\mathbb{R}$-basis of $\overline{H}_k^{\ell^1}(G;\mathbb{R})$; we also write $\overline{b}_k^{\ell^1}(G) = \infty$ if this cardinality is infinite.

For example, ℓ^1-Betti numbers satisfy the following simple inheritance properties:

PROPOSITION 3.2. *Let G be a group and let $k \in \mathbb{N}_{>0}$.*

(1) *We have* $\dim_{\mathbb{R}} H_b^k(G;\mathbb{R}) \geq \overline{b}_k^{\ell^1}(G)$. *In particular: If* $\overline{b}_k^{\ell^1}(G) \neq 0$, *then we have* $\operatorname{bcd} G \geq k$.

(2) *Conversely, if* $H_b^2(G;\mathbb{R}) \not\cong 0$, *then* $\overline{b}_2^{\ell^1}(G) \neq 0$.

(3) *If H is a group that is a retract of G, i.e., there are group homomorphisms $i\colon H \longrightarrow G$ and $r\colon G \longrightarrow H$ with $r \circ i = \mathrm{id}_H$, then*

$$\overline{b}_k^{\ell^1}(G) \geq \overline{b}_k^{\ell^1}(H).$$

(4) *If H is a group, then*

$$\overline{b}_k^{\ell^1}(G * H) \geq \overline{b}_k^{\ell^1}(G) + \overline{b}_k^{\ell^1}(H).$$

In particular: If $\overline{b}_k^{\ell^1}(G) \neq 0$, *then* $\overline{b}_k^{\ell^1}\Big(\bigstar_{\mathbb{N}} G\Big) = \infty$.

(5) *If G is countable and* $\overline{b}_k^{\ell^1}(G) = \infty$, *then*

$$\dim_{\mathbb{R}} H_k^{\ell^1}(G;\mathbb{R}) = \dim_{\mathbb{R}} \overline{H}_k^{\ell^1}(G;\mathbb{R}) = |\mathbb{R}|,$$
$$\dim_{\mathbb{R}} H_b^k(G;\mathbb{R}) = \dim_{\mathbb{R}} \overline{H}_b^k(G;\mathbb{R}) = |\mathbb{R}|.$$

PROOF. The first part follows from Proposition 2.2. The second part follows from an observation of Matsumoto and Morita [**16**, Corollary 2.7 and Theorem 2.3]. The third part is a direct consequence of functoriality of reduced ℓ^1-homology.

The fourth part can be shown as follows: From Proposition 2.2 we deduce that there exist families $(\varphi_i)_{i\in I} \subset \overline{H}_b^k(G;\mathbb{R})$, $(\alpha_i)_{i\in I} \subset \overline{H}_k^{\ell^1}(G;\mathbb{R})$ and $(\psi_j)_{j\in J} \subset \overline{H}_b^k(H;\mathbb{R})$, $(\beta_j)_{j\in J} \subset \overline{H}_k^{\ell^1}(H;\mathbb{R})$ with $|I| = \overline{b}_k^{\ell^1}(G)$ and $|J| = \overline{b}_k^{\ell^1}(H)$ that satisfy

$$\langle \varphi_i, \alpha_{i'}\rangle = \delta_{i,i'} \quad \text{and} \quad \langle \psi_j, \beta_{j'}\rangle = \delta_{j,j'}$$

for all $i, i' \in I$ and all $j, j' \in J$. Let $i_G \colon G \longrightarrow G*H$, $i_H \colon H \longrightarrow G*H$, $p_G \colon G*H \longrightarrow G$, $p_H \colon G*H \longrightarrow H$ be the canonical inclusions and projections associated with the free factors. Then

$$\begin{aligned}
\bigl\langle \overline{H}_b^k(p_G;\mathbb{R})(\varphi_i), \overline{H}_k^{\ell^1}(i_G;\mathbb{R})(\alpha_{i'})\bigr\rangle &= \langle \varphi_i, \alpha_{i,i'}\rangle = \delta_{i,i'},\\
\bigl\langle \overline{H}_b^k(p_G;\mathbb{R})(\varphi_i), \overline{H}_k^{\ell^1}(i_H;\mathbb{R})(\beta_j)\bigr\rangle &= \bigl\langle \varphi_i, \overline{H}_k^{\ell^1}(1;\mathbb{R})(\beta_j)\bigr\rangle = \langle \varphi_i, 0\rangle = 0
\end{aligned}$$

etc. for all $i, i' \in I$, $j, j' \in J$. Hence, $\overline{b}_k^{\ell^1}(G*H) \geq |I| + |J| = \overline{b}_k^{\ell^1}(G) + \overline{b}_k^{\ell^1}(H)$.

We now prove the last part: By definition, $\overline{H}_k^{\ell^1}(G;\mathbb{R})$ and $\overline{H}_b^k(G;\mathbb{R})$ are Banach spaces, and Banach spaces of infinite dimension have dimension at least $|\mathbb{R}|$. On the other hand, countability of G implies that we have both $\dim_{\mathbb{R}} C_k^{\ell^1}(G;\mathbb{R}) \leq |\mathbb{R}|$ and $\dim_{\mathbb{R}} C_b^k(G;\mathbb{R}) \leq |\mathbb{R}|$. Hence,

$$\begin{aligned}
|\mathbb{R}| \leq \dim_{\mathbb{R}} \overline{H}_k^{\ell^1}(G;\mathbb{R}) \leq \dim_{\mathbb{R}} H_k^{\ell^1}(G;\mathbb{R}) \leq \dim_{\mathbb{R}} C_k^{\ell^1}(G;\mathbb{R}) \leq |\mathbb{R}|,\\
|\mathbb{R}| \leq \dim_{\mathbb{R}} \overline{H}_b^k(G;\mathbb{R}) \leq \dim_{\mathbb{R}} H_b^k(G;\mathbb{R}) \leq \dim_{\mathbb{R}} C_b^k(G;\mathbb{R}) \leq |\mathbb{R}|. \qquad \square
\end{aligned}$$

While it is not clear whether ℓ^1-homology or bounded cohomology satisfy a simple Künneth theorem, we at least have the following weak version:

PROPOSITION 3.3. *Let G and H be groups and let $p, q \in \mathbb{N}$. Then*

$$\overline{b}_{p+q}^{\ell^1}(G \times H) \geq \overline{b}_p^{\ell^1}(G) \cdot \overline{b}_q^{\ell^1}(H).$$

In particular: If $\overline{b}_p^{\ell^1}(G) \neq 0$ and $\overline{b}_q^{\ell^1}(H) \neq 0$, then $\operatorname{bcd}(G \times H) \geq p + q$.

PROOF. From Proposition 2.2 we deduce that there exist families $(\varphi_i)_{i\in I} \subset \overline{H}_b^p(G;\mathbb{R})$, $(\alpha_i)_{i\in I} \subset \overline{H}_p^{\ell^1}(G;\mathbb{R})$ and $(\psi_j)_{j\in J} \subset \overline{H}_b^k(H;\mathbb{R})$, $(\beta_j)_{j\in J} \subset \overline{H}_k^{\ell^1}(H;\mathbb{R})$ with $|I| = \overline{b}_p^{\ell^1}(G)$ and $|J| = \overline{b}_q^{\ell^1}(H)$ that satisfy

$$\langle \varphi_i, \alpha_{i'}\rangle = \delta_{i,i'} \quad \text{and} \quad \langle \psi_j, \beta_{j'}\rangle = \delta_{j,j'}$$

for all $i, i' \in I$ and all $j, j' \in J$. Hence, the compatibility of the cross-products (Proposition 2.5) yields

$$(-1)^{p\cdot q} \cdot \langle \varphi_i \times \psi_j, \alpha_{i'} \times \beta_{j'}\rangle = \langle \varphi_i, \alpha_{i'}\rangle \cdot \langle \psi_j, \beta_{j'}\rangle = \delta_{i,i'} \cdot \delta_{j,j'} = \delta_{(i,j),(i',j')}$$

for all $(i,j),(i',j') \in I \times J$; thus, $\overline{b}^{\ell^1}_{p+q}(G \times H) \geq |I| \cdot |J| \geq \overline{b}^{\ell^1}_p(G) \cdot \overline{b}^{\ell^1}_q(H)$.

The second part follows then with help of Proposition 3.2. □

3.2. Examples.

The observations from Section 3.1 are now applied to concrete examples:

Definition 3.4 (infinite chains of cup-products in bounded cohomology). Let G be a group. Then G *admits infinite chains of cup-products in bounded cohomology* if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \overline{H}^*_b(G;\mathbb{R})$ of non-zero degree such that for all $n \in \mathbb{N}$ we have

$$\varphi_0 \cup \cdots \cup \varphi_{n-1} \neq 0 \in \overline{H}^*_b(G;\mathbb{R}).$$

Proposition 3.5. *Let G_0 be a group with $\overline{b}^{\ell^1}_3(G_0) = \infty$ and for each $n \in \mathbb{N}_{>0}$ let G_n be a group with $\overline{b}^{\ell^1}_2(G_n) = \infty$. Let G be $\bigoplus_{n\in\mathbb{N}} G_n$ or $\prod_{n\in\mathbb{N}} G_n$. Then*

$$\overline{b}^{\ell^1}_k(G) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, \\ \infty & \text{if } k \in \mathbb{N}_{\geq 2}, \end{cases} \quad \text{and} \quad \dim_{\mathbb{R}} \overline{H}^k_b(G;\mathbb{R}) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, \\ \infty & \text{if } k \in \mathbb{N}_{\geq 2}, \end{cases}$$

for all $k \in \mathbb{N}$, and thus $\operatorname{bcd}(G) = \infty$. *Moreover, G admits infinite chains of cup-products in bounded cohomology and for all $k \in \mathbb{N}_{\geq 4}$ there exist non-trivial classes in $H^k_b(G;\mathbb{R})$ that decompose as cup-products of classes in degree* 2 *and* 3.

Proof. We only need to consider the case $k \geq 2$. Every $k \in \mathbb{N}_{\geq 2}$ can be written in the form $k = 2 \cdot r + 3 \cdot s$ with $r \in \mathbb{N}$ and $s \in \{0,1\}$. Because

$$G_1 \times \cdots \times G_r \quad \text{and} \quad G_0 \times G_1 \times \cdots \times G_r$$

are retracts of G, the calculation of the dimensions follows from Proposition 3.2 and Proposition 3.3. The assertion on the cup-products follows from the argumentation via iterated cross-products and the relation between the cohomological cross-product and the cup-product on bounded cohomology. □

Remark 3.6 (exact cardinality). If $G_0, G_1, \ldots$ are countable groups that satisfy the assumptions of Proposition 3.5 and $G := \bigoplus_{n\in\mathbb{N}} G_n$, then

$$\dim_{\mathbb{R}} H^k_b(G;\mathbb{R}) = \dim_{\mathbb{R}} \overline{H}^k_b(G;\mathbb{R}) = |\mathbb{R}|$$

for all $k \in \mathbb{N}_{\geq 2}$ by Proposition 3.5 and Proposition 3.2.

Furthermore, by taking the infinite free product with the examples by Soma [**22**], we can also enforce that the difference between reduced and non-reduced bounded cohomology is infinite-dimensional in degree $3, 5, 6, \ldots$.

Proposition 3.7. *For each $n \in \mathbb{N}$ let G_n be a group such that there exists a degree $k_n \in \mathbb{N}_{>1}$ with $\overline{b}^{\ell^1}_{k_n}(G_n) \neq 0$. Then*

$$\operatorname{bcd}\Bigl(\bigoplus_{n\in\mathbb{N}} G_n\Bigr) = \infty \quad \textit{and} \quad \operatorname{bcd}\Bigl(\prod_{n\in\mathbb{N}} G_n\Bigr) = \infty,$$

and $\bigoplus_{n\in\mathbb{N}} G_n$ *and* $\prod_{n\in\mathbb{N}} G_n$ *admit infinite chains of cup-products in bounded cohomology.*

Proof. Similarly to the proof of Proposition 3.5 this follows inductively from Proposition 3.2 and Proposition 3.3. □

We can now also easily deduce a proof for Proposition 1.5:

Proof of Proposition 1.5. Because $H_b^2(G_n;\mathbb{R}) \not\cong 0$, we know that $\overline{b}_2^{\ell^1}(G) \neq 0$ (Proposition 3.2). Therefore, Proposition 3.7 provides the desired conclusion. □

Some concrete groups with large ℓ^1-Betti numbers or large bounded-cohomological dimension are:

Example 3.8. Let $n \in \mathbb{N}_{\geq 2}$ and let $(M_k)_{k\in\mathbb{N}}$ be a sequence of oriented closed connected n-manifolds with positive simplicial volume, e.g., hyperbolic manifolds [**8**], [**23**]. Then $\overline{b}_n^{\ell^1}(\pi_1(M_k)) \neq 0$ [**8**], and so Proposition 3.2 shows that

$$\overline{b}_n^{\ell^1}\Bigl(\mathop{\bigstar}_{k\in\mathbb{N}} \pi_1(M_k)\Bigr) = \infty.$$

Moreover, Proposition 3.7 tells us that

$$\operatorname{bcd}\Bigl(\bigoplus_{k\in\mathbb{N}} \pi_1(M_k)\Bigr) = \infty \quad \text{and} \quad \operatorname{bcd}\Bigl(\prod_{k\in\mathbb{N}} \pi_1(M_k)\Bigr) = \infty.$$

Example 3.9. It is well known that $\overline{b}_2^{\ell^1}(F_2) = \infty$ [**18**]. If M is an oriented closed connected hyperbolic 3-manifold, then $\overline{b}_3^{\ell^1}(\pi_1(M)) \neq 0$ (as in the previous example). Let $H := \bigstar_{\mathbb{N}} \pi_1(M)$. Hence, we have $\overline{b}_3^{\ell^1}(H) = \infty$ by Proposition 3.2. So, Proposition 3.5 allows us to compute the size of (reduced) bounded cohomology and (reduced) ℓ^1-homology of $H \times \bigoplus_{\mathbb{N}} F_2$ in all degrees. In particular,

$$\operatorname{bcd}\Bigl(H \times \bigoplus_{\mathbb{N}} F_2\Bigr) = \infty.$$

Furthermore, by Proposition 3.7,

$$\operatorname{bcd}\Bigl(\bigoplus_{\mathbb{N}} F_2\Bigr) = \infty \quad \text{and} \quad \operatorname{bcd}\Bigl(\prod_{\mathbb{N}} F_2\Bigr) = \infty,$$

but because the exact structure of $H_b^*(F_2;\mathbb{R})$ is unknown, it is currently out of reach to calculate the bounded cohomology ring of $\bigoplus_{\mathbb{N}} F_2$ or $\prod_{\mathbb{N}} F_2$ completely.

If we are not interested in having many non-trivial cup-products, then we can also take large free products:

EXAMPLE 3.10. For $n \in \mathbb{N}$ let G_n be a group with $\overline{b}^{\ell^1}_{k_n}(G_n) \neq 0$ for some $k_n \geq n$; e.g., we could take the fundamental group of an oriented closed connected hyperbolic n-manifold of dimension at least n. Then Proposition 3.2 shows that

$$\operatorname{bcd}\Bigl(\mathop{\bigstar}_{n\in\mathbb{N}} G_n\Bigr) = \infty.$$

4. Groups with small bounded cohomology.

We will first recall the notion of mitotic groups and their basic properties (Section 4.1). We will prove Theorem 1.2 in Section 4.3, i.e., that mitotic groups have bounded-cohomological dimension equal to 0. As a preparation for this proof, we recall the uniform boundary condition in Section 4.2.

4.1. Mitotic groups.

We recall the notion of mitotic groups, due to Baumslag, Dyer, Heller [**1**, Section 4]. Roughly speaking, a mitosis of a group G is an ambient group that allows to divide G into two copies of itself by means of conjugation (Figure 1). For group elements g, h we use the conjugation notation $g^h := h \cdot g \cdot h^{-1}$.

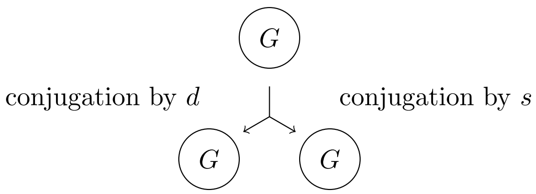

Figure 1. A mitosis of a group, schematically; the original group G commutes with G^s inside M and g commutes with g^d for all $g \in G$.

DEFINITION 4.1 (mitotic group). Let G be a subgroup of a group M. Then M is a *mitosis of* G if there exist $s, d \in M$ with the following properties:

(1) The group M is generated by $G \cup \{s, d\}$.

(2) For all $g \in G$ we have $g^d = g \cdot g^s$.

(3) For all $g, g' \in G$ we have $[g', g^s] = 1$.

We then also call the inclusion $G \hookrightarrow M$ a mitosis and the elements d, s as above are *witnesses* for this mitosis. A group M is *mitotic*, if for every finitely generated subgroup $G \subset M$ there exists a subgroup $M' \subset M$ such that $G \subset M'$ is a mitosis of G.

If M is a mitosis of a group G, witnessed by $s, d \in M$, then

$$\begin{aligned} \mu\colon G \times G &\longrightarrow M \\ (g', g) &\longmapsto g' \cdot g^s \end{aligned}$$

is a well-defined group homomorphism; if $\Delta_G\colon G \longrightarrow G \times G$ denotes the diagonal, then $\mu \circ \Delta_G$ is nothing but conjugation with d. Using the Künneth theorem, the fact that conjugations act trivially on homology, and an induction argument, Baumslag, Dyer, Heller [**1**, Theorem 4.2] established that mitotic groups are acyclic:

Theorem 4.2. *All mitotic groups are acyclic.*

In view of the universal coefficient theorem, we obtain that also group cohomology with $\mathbb{R}$-coefficients is trivial for mitotic groups.

4.2. The uniform boundary condition.

We we will now review the uniform boundary condition, as studied by Matsumoto and Morita [**16**].

Definition 4.3 (uniform boundary condition). Let $q \in \mathbb{N}$ and let $\kappa \in \mathbb{R}_{>0}$. A group G satisfies the (q,κ)-*uniform boundary condition* ((q,κ)-*UBC*) if the following holds: for all $z \in \operatorname{im} \partial_{q+1} \subset C_q(G;\mathbb{R})$ there is a chain $c \in C_{q+1}(G;\mathbb{R})$ with

$$\partial_{q+1}(c) = z \quad \text{and} \quad \|c\|_1 \leq \kappa \cdot \|z\|_1.$$

A group G satisfies q-UBC if there exists a $\kappa \in \mathbb{R}_{>0}$ such that G satisfies (q,κ)-UBC.

For example, the uniform boundary condition allows to upgrade acyclicity of a group to vanishing of bounded cohomology [**16**, Theorem 2.8]:

Theorem 4.4. *Let G be a group and let $q \in \mathbb{N}$. Then the following are equivalent*:

(1) *The group G satisfies q-UBC.*

(2) *The comparison map $H_b^{q+1}(G;\mathbb{R}) \longrightarrow H^{q+1}(G;\mathbb{R})$ is injective.*

In particular: *If G is acyclic and G satisfies q-UBC, then $H_b^{q+1}(G;\mathbb{R}) \cong 0$.*

More geometrically, the uniform boundary condition also has applications in the context of simplicial volume of non-compact manifolds [**13**].

We introduce the following version of the uniform boundary condition:

Definition 4.5 (uniform boundary condition). Let $q \in \mathbb{N}$, $\kappa \in \mathbb{R}_{>0}$. A group homomorphism $\varphi\colon H \longrightarrow K$ satisfies the (q,κ)-*uniform boundary condition* ((q,κ)-*UBC*) if there exists a linear map

$$S\colon \partial_{q+1}\bigl(C_{q+1}(H;\mathbb{R})\bigr) \longrightarrow C_{q+1}(K;\mathbb{R})$$

with

$$\partial_{q+1} \circ S = C_q(\varphi;\mathbb{R})|_{\operatorname{im} \partial_{q+1}} \quad \text{and} \quad \|S\| \leq \kappa.$$

Here, $\|S\|$ denotes the norm of S with respect to the restricition of the ℓ^1-norm to $\partial_{q+1}(C_{q+1}(H;\mathbb{R}))$ and the ℓ^1-norm on $C_{q+1}(K;\mathbb{R})$.

Clearly, every group homomorphism satisfies $(0,0)$-UBC.

4.3. Bounded cohomology of mitotic groups.

We will now prove Theorem 1.2, i.e., that mitotic groups have trivial bounded cohomology. The proofs of Baumslag, Dyer, Heller and the normed refinement of Matsumoto and Morita of Mather's argument for $\mathrm{Homeo}_K(\mathbb{R}^n)$ serve as a blueprint.

In view of Theorem 4.4 and Theorem 4.2 we only need to show that mitotic groups satisfy the uniform boundary condition in each positive degree. To this end, we first prove that mitoses allow to increase the degree in which the uniform boundary condition is satisfied. More precisely, following the arguments of Matsumoto and Morita [**16**] step by step, one obtains the following (a detailed proof is given in Appendix A):

Proposition 4.6. *Let $q \in \mathbb{N}$, $\kappa \in \mathbb{R}_{>0}$. Then there is a constant $c_{q,\kappa} \in \mathbb{R}_{>0}$ such that: let*

$$H \xrightarrow{\varphi} H' \xrightarrow{\varphi'} K \xrightarrow{\psi} G \xrightarrow{i} M$$

be a chain of group homomorphisms with the following properties:

- *The homomorphism $i\colon G \hookrightarrow M$ is a mitosis.*
- *For all $k \in \{1, \dots, q-1\}$ we have $H_k(\varphi'; \mathbb{R}) = 0$.*
- *For all $k \in \{0, \dots, q-1\}$ the group homomorphisms $\varphi\colon H \longrightarrow H'$ and $\psi\colon K \longrightarrow G$ satisfy (k, κ)-UBC.*

Then for all $k \in \{1, \dots, q\}$ we obtain

$$H_k(i \circ \psi \circ \varphi' \circ \varphi; \mathbb{R}) = 0$$

and the composition $i \circ \psi \circ \varphi' \circ \varphi$ satisfies $(k, c_{q,\kappa})$-UBC for all $k \in \{0, \dots, q\}$.

We can then easily complete the proof of Theorem 1.2 by induction:

Proof of Theorem 1.2. Let M be a mitotic group, let $q \in \mathbb{N}_{>0}$, and let $z \in C_q(M; \mathbb{R})$ be a boundary, say $z = \partial_{q+1}(c)$ for some $c \in C_{q+1}(M; \mathbb{R})$. Because z and c are finite linear combinations of tuples of M, there exists a finitely generated subgroup G_0 such that $z \in C_q(G_0; \mathbb{R})$ and $c \in C_{q+1}(G_0; \mathbb{R})$; i.e., z is a boundary in G_0.

As M is mitotic, we can extend G_0 to a sequence $G_0 \subset G_1 \subset G_2 \subset \cdots \subset M$ of finitely generated subgroups of M such that each step $G_j \hookrightarrow G_{j+1}$ is a mitosis. We now proceed by induction over q: If $q = 1$, then the sequence

$$G_0 \xrightarrow{i_0} G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} G_4$$

of mitoses satisfies the assumptions of Proposition 4.6 in degree 1, and hence the composition $i_3 \circ i_2 \circ i_1 \circ i_0$ satisfies $(1, c_{1,0})$-UBC.

For the induction step, let $q \in \mathbb{N}_{>1}$, let

$$n_q := \sum_{j=0}^{q} 3^j$$

and suppose that compositions of n_{q-1} mitoses in M satisfy $(q-1, \kappa_{q-1})$-UBC, where κ_{q-1} depends only on q, but not on the groups involved.

Then the chain

$$G_0 \longrightarrow G_{n_{q-1}} \longrightarrow G_{2\cdot n_{q-1}} \longrightarrow G_{3\cdot n_{q-1}=n_q-1} \xrightarrow{i_{n_q-1}} G_{n_q}$$

of inclusions satisfies the assumptions of Proposition 4.6 in degree q, and hence the inclusion $G_0 \hookrightarrow G_{n_q}$ satisfies $(q, c_{q,\kappa_{q-1}})$-UBC where $c_{q,\kappa_{q-1}}$ depends only on q, but *not* on z or the chain $G_0 \subset G_1 \subset \cdots$.

In particular, there is a chain $c' \in C_{q+1}(M;\mathbb{R})$ with

$$\partial_{q+1} c' = z \in C_q(M;\mathbb{R}) \quad \text{and} \quad \|c'\|_1 \leq \kappa_q \cdot \|z\|_1.$$

Hence, M satisfies (q, κ_q)-UBC. Because M is acyclic by Theorem 4.2, we obtain $H_b^q(M;\mathbb{R}) \cong 0$ from Theorem 4.4. □

Appendix A. Detailed proof of Proposition 4.6.

For the convenience of the reader, we present a detailed proof of Proposition 4.6, following the arguments of Matsumoto and Morita [**16**]:

Proof of Proposition 4.6. It suffices to prove the claims in degree q. We abbreviate $f := \psi \circ \varphi' \circ \varphi$. The fact that $H_q(i \circ f;\mathbb{R}) = 0$ was proved by Baumslag, Dyer, Heller [**1**, Proposition 4.1]. However, in order to make the normed refinement more transparent, we repeat the argument:

Let $d, s \in M$ be witnesses for the mitosis $i\colon G \hookrightarrow M$. Then

$$\begin{aligned} \mu\colon G \times G &\longrightarrow M \\ (g', g) &\longmapsto g' \cdot g^s \end{aligned}$$

is a group homomorphism. Denoting the diagonal maps by Δ_H, Δ_G and the conjugations on M by $\gamma_d = \ \cdot\ ^d$, $\gamma_s = \ \cdot\ ^s$, we obtain

$$\gamma_d \circ i \circ f = \mu \circ \Delta_G \circ f = \mu \circ (f \times f) \circ \Delta_H.$$

On the other hand, the Künneth theorem (and its naturality) and the homological assumption on $H_*(f;\mathbb{R})$ shows that the diagram

$$\begin{tikzcd}
H_q(H \times H;\mathbb{R}) \arrow[r, "H_q(p_1;\mathbb{R}) \oplus H_q(p_2;\mathbb{R})"] \arrow[d, "H_q(f\times f;\mathbb{R})"'] & H_q(H;\mathbb{R}) \oplus H_q(H;\mathbb{R}) \arrow[d, "H_q(f;\mathbb{R})\oplus H_q(f;\mathbb{R})"] \\
H_q(G \times G;\mathbb{R}) \arrow[d, "H_q(\mu;\mathbb{R})"'] & H_q(G;\mathbb{R}) \oplus H_q(G;\mathbb{R}) \arrow[l, "H_q(i_1;\mathbb{R}) + H_q(i_2;\mathbb{R})"] \arrow[d, "H_q(i;\mathbb{R})\oplus H_q(\gamma_s\circ i;\mathbb{R})"] \\
H_q(M;\mathbb{R}) & H_q(M;\mathbb{R}) \oplus H_q(M;\mathbb{R}) \arrow[l, "\mathrm{id} + \mathrm{id}"]
\end{tikzcd}$$

is commutative; here, i_1, i_2, p_1, p_2 denote the corresponding inclusions and projections of the factors. Hence, we obtain

$$\begin{aligned} H_q(\gamma_d;\mathbb{R}) \circ H_q(i \circ f) &= H_q(\mu;\mathbb{R}) \circ H_q(f \times f;\mathbb{R}) \circ H_q(\Delta_H;\mathbb{R}) \\ &= H_q(i \circ f;\mathbb{R}) + H_q(\gamma_s;\mathbb{R}) \circ H_q(i \circ f;\mathbb{R}). \end{aligned}$$

As conjugations act trivially on homology, $H_q(\gamma_d;\mathbb{R}) = \mathrm{id} = H_q(\gamma_s;\mathbb{R})$, and so $H_q(i \circ f;\mathbb{R}) = 0$.

We will now refine this argument and prove that $i \circ f$ satisfies a strong uniform boundary condition in degree q:

Let $S_0, \dots, S_{q-1}$ and $T_0, \dots, T_{q-1}$ be sections that witness that φ and ψ satisfy $(0,\kappa)$-UBC, …, $(q-1,\kappa)$-UBC; for simplicity, we omit the indices and denote all these maps by S or T respectively. Let $z \in B_q(H) := \partial_{q+1}(C_{q+1}(H;\mathbb{R}))$. We construct an explicit ∂_{q+1}-primitive for $C_q(i \circ f;\mathbb{R})$ in two steps: We first deal with the Künneth argument, and then we will take care of the conjugations.

Normed refinement of the Künneth argument. We first study the intermediate degree part of z, viewed in $C_*(H;\mathbb{R}) \otimes_{\mathbb{R}} C_*(H;\mathbb{R})$, i.e., the chain

$$D(z) := A \circ \Delta(z) - z \otimes 1 - 1 \otimes z,$$

where, $A\colon C_*(H \times H;\mathbb{R}) \longrightarrow C_*(H;\mathbb{R}) \otimes_{\mathbb{R}} C_*(H;\mathbb{R})$ is the Alexander–Whitney map (Lemma 2.6) and $\Delta := C_*(\Delta_H;\mathbb{R})$. Moreover, we write $\varphi_* := C_*(\varphi;\mathbb{R})$ etc.

Similar to Matsumoto and Morita [**16**, p. 544] we define the map

$$\begin{aligned} E := \;& (\psi_* \otimes_{\mathbb{R}} \psi_*) \circ (\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) \circ (S \otimes_{\mathbb{R}} S) \circ (\mathrm{id} \otimes_{\mathbb{R}} \partial) \\ +\;& \bigl(T \otimes_{\mathbb{R}} (\psi_* - T \circ \partial)\bigr) \circ (\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) \circ \bigl(\varphi_* \otimes_{\mathbb{R}} \varphi_* - \partial \circ (S \otimes_{\mathbb{R}} S) \circ (\mathrm{id} \otimes_{\mathbb{R}} \partial)\bigr) \\ :\;& B_q \longrightarrow \bigl(C_*(G;\mathbb{R}) \otimes_{\mathbb{R}} C_*(G;\mathbb{R})\bigr)_{q+1} \end{aligned}$$

on $B_q := \mathrm{im}\bigl(\partial_{q+1, C_*(H;\mathbb{R}) \otimes_{\mathbb{R}} C_*(H;\mathbb{R})}\bigr) \cap \bigoplus_{j=1}^{q-1} C_j(H;\mathbb{R}) \otimes_{\mathbb{R}} C_{q-j}(H;\mathbb{R})$.

LEMMA A.1 (explicit primitives for $D(z)$). *This map E has the following properties*:

(1) *The map E is well-defined.*

(2) *We have $D(z) \in B_q$ and the map E produces explicit primitives, i.e.,*

$$(f_* \otimes_{\mathbb{R}} f_*) D(z) = \partial_{q+1} E\bigl(D(z)\bigr).$$

(3) *Moreover,*

$$\|E\| \leq \kappa + 2 \cdot (q+1) \cdot \kappa^2 \cdot \bigl(1 + (q+1) \cdot \kappa + (q+1)^2 \cdot \kappa^2\bigr)$$

with respect to the norms induced by the respective ℓ^1-norms. Notice that this bound does only depend on q and κ, but not *on the groups or homomorphisms that are involved.*

The proof of this lemma is given below. We now continue with the proof of Proposition 4.6: In view of the naturality of the cross-product map $B\colon C_*(\,\cdot\,;\mathbb{R})\otimes_{\mathbb{R}} C_*(\,\cdot\,;\mathbb{R}) \longrightarrow C_*(\,\cdot\,\times\,\cdot\,;\mathbb{R})$ and Lemma 2.6 we obtain

$$\begin{aligned}(f\times f)_*\circ\Delta(z) &= (f\times f)_*\circ B\circ A\circ\Delta(z) + (f\times f)_*(\partial\circ\Xi+\Xi\circ\partial)\circ\Delta(z)\\ &= B\circ(f_*\otimes_{\mathbb{R}} f_*)\circ A\circ\Delta(z) + (f\times f)_*\circ\partial\circ\Xi\circ\Delta(z).\end{aligned}$$

The construction of $D(z)$ and the explicit primitives from Lemma A.1 now lead to

$$(f\times f)_*\circ\Delta(z) = (f\times f)_*\circ B(z\otimes 1) + (f\times f)_*\circ B(1\otimes z) + \partial E'(z),$$

where

$$E' := B\circ E\circ D + (f\times f)_*\circ\Xi\circ\Delta;$$

notice that E' is bounded and that $\|E'\|$ admits a bound that only depends on q and κ, but not on the specific groups or homomorphisms. By definition of the cross-product, we have $B(z\otimes 1) = j_{1*}(z)$ and $B(1\otimes z) = j_{2*}(z)$, where $j_1, j_2\colon H \longrightarrow H\times H$ are the inclusions of the factors. Therefore,

$$\begin{aligned}(f\times f)_*\circ\Delta(z) &= (f\times f)_*\circ j_{1*}(z) + (f\times f)_*\circ j_{2*}(z) + \partial\circ E'(z)\\ &= i_{1*}\circ f_*(z) + i_{2*}\circ f_*(z) + \partial\circ E'(z).\end{aligned}$$

Normed refinement of the conjugation argument. Applying μ_* to this equation and using the chain homotopy Θ from Lemma A.2 below associated with the conjugation by $k := s\cdot d^{-1}$ on M leads then to

$$\begin{aligned}(i\circ f)_*(z) &= \bigl(\mu\circ(f\times f)\circ j_1\bigr)_*(z)\\ &= \bigl(\mu\circ(f\times f)\circ\Delta_H\bigr)_*(z) - \bigl(\mu\circ(f\times f)\circ j_2\bigr)_*(z) - \mu_*\circ\partial\circ E'(z)\\ &= \gamma_{d*}\circ(i\circ f)_*(z) - \gamma_{s*}\circ(i\circ f)_*(z) - \partial\circ\mu_*\circ E'(z)\\ &= \gamma_{d*}\circ(i\circ f)_*(z) - \gamma_{k*}\circ\gamma_{d*}\circ(i\circ f)_*(z) - \partial\circ\mu_*\circ E'(z)\\ &= (\partial\circ\Theta+\Theta\circ\partial)\circ(i\circ f)_*(z) - \partial\circ\mu_*\circ E'(z)\\ &= \partial\bigl(\Theta\circ(i\circ f)_*(z) - \mu_*\circ E'(z)\bigr).\end{aligned}$$

Because $\|\Theta\circ(i\circ f)_* - \mu_*\circ E'\|$ admits a bound $c_{q,\kappa}$ on $B_q(H)$ that only depends on q and κ (as the same holds for E' and Θ) we see that $i\circ f$ satisfies $(q, c_{q,\kappa})$-UBC, as desired. □

Proof of Lemma A.1. We mainly follow the corresponding arguments by Matsumoto and Morita.

Ad 1. Showing that E is well-defined is the most delicate point of the whole proof of Theorem 1.2. Let $x\in B_q$. Because x is a boundary, a straightforward calculation shows that

$$(\mathrm{id} \otimes_{\mathbb{R}} \partial)(x) \in \bigoplus_{j=1}^{q-1} B_j(H) \otimes_{\mathbb{R}} B_{q-1-j}(H);$$

here, one should also note that $B_0(H) = 0$ by definition of the chain complex $C_*(H; \mathbb{R})$. In particular, $(S \otimes_{\mathbb{R}} S)$ indeed can be applied to $(\mathrm{id} \otimes_{\mathbb{R}} \partial)(x)$. This takes care of the first summand of E and the last part of the second summand of E.

For the remaining terms, we consider the element

$$U(x) := \big(\varphi_* \otimes \varphi_* - \partial \circ (S \otimes_{\mathbb{R}} S) \circ (\mathrm{id} \otimes_{\mathbb{R}} \partial)\big)(x).$$

Using the fact that the maps of type S are sections of φ_* on boundaries, one readily computes $(\mathrm{id} \otimes_{\mathbb{R}} \partial) \circ U(x) = 0$. Exactness of the tensor product over $\mathbb{R}$ then implies that

$$U(x) \in \bigoplus_{j=1}^{q-1} C_j(H; \mathbb{R}) \otimes_{\mathbb{R}} Z_{q-j}(H),$$

where $Z_*(H)$ denotes the cycles in $C_*(H; \mathbb{R})$. On the other hand, we clearly also have $\partial U(x) = 0$, and so $(\partial \otimes_{\mathbb{R}} \mathrm{id}) \circ U(x) = 0$ and (again by exactness of the tensor product over $\mathbb{R}$) it follows that

$$U(x) \in \bigoplus_{j=1}^{q-1} Z_j(H) \otimes_{\mathbb{R}} Z_{q-j}(H).$$

By assumption, $H_k(\varphi'; \mathbb{R}) = 0$ for all $k \in \{1, \dots, q-1\}$; thus,

$$(\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) \circ U(x) \in \bigoplus_{j=1}^{q-1} B_j(H) \otimes_{\mathbb{R}} B_{q-j}(H).$$

In particular, we indeed can apply the maps of type T to all components of $(\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) U(x)$ and of $(\mathrm{id} \otimes_{\mathbb{R}} \partial) \circ (\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) \circ U(x)$. Therefore, E is well-defined.

Ad 2. Because z is a boundary, a straightforward calculation shows that also $D(z)$ is a boundary. Moreover, by construction of $D(z)$, all summands of $D(z)$ are of intermediate degree. Hence, $D(z) \in B_q$, and so E can indeed be applied to $D(z)$.

Because $(\mathrm{id} \otimes_{\mathbb{R}} \partial) \circ U(D(z)) = 0$, a calculation shows that

$$\begin{aligned} \partial_{q+1} E\big(D(z)\big) &= (\psi_* \otimes_{\mathbb{R}} \psi_*) \circ (\varphi'_* \otimes_{\mathbb{R}} \varphi'_*) \circ (\varphi_* \otimes_{\mathbb{R}} \varphi_*)\big(D(z)\big) \\ &= (f_* \otimes_{\mathbb{R}} f_*) D(z). \end{aligned}$$

Ad 3. The bound on $\|E\|$ follows directly from the explicit definition of E and corresponding bounds on the building blocks of E: Chain maps induced by group homomorphisms have norm 1, the maps of type S and T have norms bounded by κ (by assumption), and the boundary operator on $C_q(\,\cdot\,; \mathbb{R})$ has norm bounded by $q+1$. □

Lemma A.2. *Let G be a group and let $k \in G$. Then*

$$\begin{aligned}\Theta_q\colon C_q(G;\mathbb{R}) &\longrightarrow C_{q+1}(G;\mathbb{R})\\ G^q \ni (g_1,\dots,g_q) &\longmapsto \sum_{j=1}^{q+1} (-1)^j \cdot (g_1,\dots,g_{j-1},k,k^{-1}\cdot g_j\cdot k,\dots,k^{-1}\cdot g_q\cdot k)\end{aligned}$$

defines a chain homotopy between the identity and $C_*(\gamma_k;\mathbb{R})$, where γ_k denotes the conjugation on G by k. Moreover, for all $q\in\mathbb{N}$ we have

$$\|\Theta_q\| \le q+1.$$

Proof. This is a straightforward computation. □

References

[1] G. Baumslag, E. Dyer and A. Heller, The topology of discrete groups, J. Pure Appl. Algebra, **16** (1980), 1–47.

[2] A. Bouarich, Théorèmes de Zilber-Eilemberg et de Brown en homologie ℓ^1, Proyecciones, **23** (2004), 151–186.

[3] T. Bühler, On the algebraic foundations of bounded cohomology, Mem. Amer. Math. Soc., **214** (2011), no. 1006.

[4] A. Dold, Lectures on Algebraic Topology, reprint of the 1972 edition, Classics in Mathematics, Springer, 1995.

[5] K. Fujiwara, The second bounded cohomology of an amalgamated free product of groups, Trans. Amer. Math. Soc., **352** (2000), 1113–1129.

[6] R. I. Grigorchuk, Some results on bounded cohomology, Combinatorial and geometric group theory (Edinburgh, 1993), London Math. Soc. Lecture Note Ser., **204**, Cambridge University Press, 1995, 111–163.

[7] R. I. Grigorchuk, Bounded cohomology of group constructions, Mat. Zametki, **59** (1996), 546–550; translation in Math. Notes, **59** (1996), 392–394.

[8] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math., **56** (1983), 5–99.

[9] T. Hartnick and A. Ott, Bounded cohomology via partial differential equations, I, preprint, arXiv:1310.4806 [math.GR], 2013.

[10] H. Inoue and K. Yano, The Gromov invariant of negatively curved manifolds, Topology, **21** (1981), 83–89.

[11] N. V. Ivanov, Foundations of the theory of bounded cohomology, J. Soviet Math., **37** (1987), 1090–1114.

[12] C. Löh, Isomorphisms in ℓ^1-homology, Münster J. of Math., **1** (2008), 237–266.

[13] C. Löh and R. Sauer, Simplicial volume of Hilbert modular varieties, Comment. Math. Helv., **84** (2009), 457–470.

[14] C. Löh, Simplicial Volume, Bull. Man. Atl., 2011, 7–18.

[15] J. N. Mather, The vanishing of the homology of certain groups of homeomorphisms, Topology, **10** (1971), 297–298.

[16] S. Matsumoto and S. Morita, Bounded cohomology of certain groups of homeomorphisms, Proc. Amer. Math. Soc., **94** (1985), 539–544.

[17] I. Mineyev, Straightening and bounded cohomology of hyperbolic groups, Geom. Funct. Anal., **11** (2001), 807–839.

[18] Y. Mitsumatsu, Bounded cohomology and l^1-homology of surfaces, Topology, **23** (1984), 465–471.

[19] N. Monod, Continuous Bounded Cohomology of Locally Compact Groups, Lecture Notes in Math., **1758**, Springer, 2001.

[20] N. Monod, An invitation to bounded cohomology, International Congress of Mathematicians, **II**, Eur. Math. Soc., 2006., 1183–1211.

[21] T. Soma, Bounded cohomology of closed surfaces, Topology, **36** (1997), 1221–1246.

[22] T. Soma, Existence of non-Banach bounded cohomology, Topology, **37** (1998), 179–193.

[23] W. P. Thurston, The geometry and topology of 3-manifolds, mimeographed notes, 1979.

[24] T. Yoshida, On 3-dimensional bounded cohomology of surfaces, Homotopy theory and related topics (Kyoto, 1984), Adv. Stud. Pure Math., **9**, North-Holland, 1987, 173–176.

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