J. Math. Soc. Japan
Vol. 69, No. 2 (2017) pp. 563–595
doi: 10.2969/jmsj/06920563

An example of Schwarz map of reducible Appell's hypergeometric equation E_2 in two variables

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(Received Mar. 26, 2015)
(Revised June 15, 2015)

Abstract. We study an Appell hypergeometric system E_2 of rank four which is reducible, and show that its Schwarz map admits geometric interpretations: the map can be considered as the universal Abel–Jacobi map of a 1-parameter family of curves of genus 2.

Introduction.

Schwarz maps for hypergeometric systems in single and several variables are studied by several authors (cf. [**Yo**]) for more than hundred years. These systems treated were irreducible, maybe because specialists believed that reducible systems would not give interesting Schwarz maps.

We study in this paper Appell's hypergeometric system E_2 of rank four when its parameters satisfy $a-c \in \mathbb{Z}$ or $a-c' \in \mathbb{Z}$. In this case, the system E_2 is reducible, and has a 3-dimensional subsystem isomorphic to Appell's E_1 (Proposition 1.3). If $a-c, a-c' \in \mathbb{Z}$ then E_2 has two such subsystems. By Proposition 1.5, the intersection of these subsystems is equivalent to Gauss's hypergeometric equation E. As a consequence, we have inclusions on E_2, two E_1's and E (Theorem 2.4).

We give in Theorem 3.10 the monodromy representation of the system E_2 which can be specialized to the case $a-c, a-c' \in \{0,1,2,\dots\}$. As for explicit circuit matrices with respect to a basis $\Delta_1,\dots,\Delta_4$, see Corollary 3.12.

We further specialize the parameters of the system E_2 as

$$(a,b,b',c,c') = \left(\frac{4}{3},\frac{2}{3},\frac{2}{3},\frac{4}{3},\frac{4}{3}\right)$$

in Section 4. In this case, the restriction of its monodromy group to the 2-dimensional invariant subspace is arithmetic and isomorphic to the triangle group of type $[3,\infty,\infty]$. We show in Theorem 4.1 that its Schwarz map admits geometric interpretations: the map can be considered as the universal Abel–Jacobi map of a 1-dimensional family of curves of genus 2.

The system E_2 is equivalent to the restriction of the hypergeometric system $E(3,6;a_1,\dots,a_6)$ to a two-dimensional stratum in the configuration space $X(3,6)$ of six lines in the projective plane. In Appendix A, we study a system of hypergeometric differential equations in three variables which is obtained by restricting $E(3,6;a_1,\dots,a_6)$ to

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2010 *Mathematics Subject Classification.* Primary 33C65.
Key Words and Phrases. Appell's hypergeometric function, Schwarz map.

the three-dimensional strata corresponding to configurations with only one triple point. The methods to prove Proposition 1.3 are also applicable to this system under a reducibility condition. In Appendix B, we classify families of genus 2 branched coverings of the projective line, whose period maps yield triangle groups.

In a forthcoming paper [**MT**], we study this Schwarz map using period domains for mixed Hodge structures. Moreover, we explicitly give its inverse in terms of theta functions.

1. Some generalities on Appell's systems E_2 and E_1.

We first review Gauss's hypergeometric series F, Appell's hypergeometric series F_1 and F_2 together with their integral representations and their systems of differential equations.

Gauss hypergeometric series

$$F(a,b,c;x)=\sum_{i=0}^{\infty}\frac{(a,i)(b,i)}{(c,i)\ i!}x^i,$$

where $(a,i)=a(a+1)\cdots(a+i-1)$, admits an integral representation:

$$\begin{aligned}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a,b,c;x)&=\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}dt\\&=\text{constant}\times\int_\infty^1 t^{b-c}(1-t)^{c-a-1}(x-t)^{-b}dt.\end{aligned}$$

The function $z=F$ is a solution of the hypergeometric equation

$$E(a,b,c;x):\qquad P(a,b,c;x)z=0,$$

where

$$P(a,b,c;x)=D(c-1+D)-x(a+D)(b+D),\qquad D=x\frac{d}{dx}.$$

The collection of solutions is denoted by $S(a,b,c;x)$.

Appell's hypergeometric series

$$F_1(a,b,b',c;x,y)=\sum_{i,j=0}^{\infty}\frac{(a,i+j)(b,i)(b',j)}{(c,i+j)\ i!\ j!}x^iy^j$$

admits an integral representation:

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F_1(a,b,b',c;x,y)=\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}(1-ty)^{-b'}dt.$$

The function $z=F_1$ is a solution of the hypergeometric system

$$E_1(a,b,b',c;x,y):\quad \begin{cases}(D(c-1+D+D')-x(a+D+D')(b+D))z=0,\\ (D'(c-1+D+D')-y(a+D+D')(b'+D'))z=0,\end{cases}$$

where $D = x\partial/\partial x, D' = y\partial/\partial y$. They can be written as

$$P_1(a,b,b',c;x,y)z = Q_1(a,b,b',c;x,y)z = R_1(a,b,b',c;x,y)z = 0,$$

where

$$P_1(a,b,b',c;x,y) = x(1-x)\partial_{xx} + y(1-x)\partial_{xy} + (c-(a+b+1)x)\partial_x - by\partial_y - ab,$$
$$Q_1(a,b,b',c;x,y) = y(1-y)\partial_{yy} + x(1-y)\partial_{yx} + (c-(a+b'+1)y)\partial_y - b'x\partial_x - ab',$$
$$R_1(a,b,b',c;x,y) = (x-y)\partial_{xy} - b'\partial_x + b\partial_y,$$

and $\partial_x = \partial/\partial x$, etc. The last equation $R_1(a,b,b',c;x,y)z = 0$ is derived from the integrability condition of the first two equations. The collection of solutions is denoted by $S_1(a,b,b',c;x,y)$.

Appell's hypergeometric series

$$F_2(a,b,b',c,c';x,y) = \sum_{i,j=0}^{\infty} \frac{(a,i+j)(b,i)(b',j)}{(c,i)(c',j)\ i!\ j!} x^i y^j$$

admits an integral representation:

$$\frac{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}{\Gamma(c)\Gamma(c')} F_2(a,b,b',c,c';x,y)$$
$$= \int_0^1 \int_0^1 s^{b-1} t^{b'-1} (1-s)^{c-b-1} (1-t)^{c'-b'-1} (1-sx-ty)^{-a} ds dt.$$

The function $z = F_2$ satisfies the system

$$E_2(a,b,b',c,c';x,y):\quad P_2(a,b,b',c,c';x,y)z = Q_2(a,b,b',c,c';x,y)z = 0,$$

where

$$P_2(a,b,b',c,c';x,y) = D(c-1+D) - x(a+D+D')(b+D),$$
$$Q_2(a,b,b',c,c';x,y) = D'(c'-1+D') - y(a+D+D')(b'+D').$$

The collection of solutions is denoted by $S_2(a,b,b',c,c';x,y)$.

1.1. Reducibility conditions for E_2 and E_1.

As for the reducibility of the systems E_2 and E_1, the followings are known:

FACT 1.1 ([**Bod**]). *Appell's system $E_2(a,b,b',c,c')$ is reducible if and only if at least one of*

$$a,\ b,\ b',\ c-b,\ c'-b',\ c-a,\ c'-a,\ c+c'-a$$

is an integer.

Fact 1.2 ([**MS**]). *Appell's system $E_1(a,b,b',c)$ is reducible if and only if at least one of*

$$b+b'-c,\ b,\ b',\ c-a,\ a$$

is an integer.

1.2. System $E_2(a,b,b',c,c')$ under $a=c'$.

The system $E_2(a,b,b',c,c')$ is reducible when $a=c'$ (see Fact 1.1). In fact, we see that the system $E_1(b,a-b',b',c)$ is a subsystem of $E_2(a,b,b',c,a)$; precisely, we have

Proposition 1.3.

$$(1-y)^{-b'}S_2\left(a,b,b',c,a;x,-\frac{y}{1-y}\right)\supset S_1(b,a-b',b',c;x,x(1-y)).$$

We give three "proof"s: one using power series (Subsection 1.2.1), one using integral representations (Subsection 1.2.2), and one manipulating differential equations (Subsection 1.2.3). The former two are valid only under some non-integral conditions on parameters, which we do not give explicitly. Though the last one is valid for any parameters, it would be not easy to get a geometric meaning.

1.2.1. Power series.

The following fact explains the inclusion in Proposition 1.3.

Fact 1.4 ([**B**, p. 80]).

$$(1-y)^{-b'}F_2\left(a,b,b',c,a;x,-\frac{y}{1-y}\right)=F_1(b,a-b',b',c;x,x(1-y)).$$

1.2.2. Integral representation.

We consider the integral

$$I=\iint s^{b-1}t^{b'-1}(1-s)^{c-b-1}(1-t)^{c'-b'-1}(1-sx-ty)^{-a}dsdt,$$

which is a solution of the system $E_2(a,b,b',c,c';x,y)$. We change the coordinate t into τ as

$$\tau=\frac{pt}{1-sx-yt},\quad p=1-y-sx,$$

which sends

$$t=0,\quad 1,\quad \frac{1-sx}{y}\qquad \text{to}\qquad \tau=0,\quad 1,\quad \infty.$$

The inverse map is

$$t = \frac{1-sx}{q}\tau, \qquad q = y\tau + p.$$

Since

$$1-t = \frac{(1-\tau)p}{q}, \quad 1-sx-ty = \frac{(1-sx)p}{q}, \quad dt = \frac{(1-sx)p}{q^2}d\tau + *ds,$$

we have

$$I = \iint s^{b-1}(1-s)^{c-b-1}(1-sx)^{b'-a}p^{c'-b'-a} \cdot \tau^{b'-1}(1-\tau)^{c'-b'-1} \cdot q^{a-c'} ds d\tau.$$

This implies, if $a = c'$, then the double integral above becomes the product of the Beta integral

$$\int \tau^{b'-1}(1-\tau)^{c'-b'-1} d\tau$$

and the integral

$$\begin{aligned} J &= \int s^{b-1}(1-s)^{c-b-1}(1-sx)^{b'-a}p^{c'-b'-a} ds \\ &= (1-y)^{-b'} \int s^{b-1}(1-s)^{c-b-1}(1-sx)^{b'-a}\left(1-\frac{x}{1-y}s\right)^{-b'} ds, \end{aligned}$$

which is an element of the space $(1-y)^{-b'}S_1(b, a-b', b', c; x, x/(1-y))$. This shows

$$S_2(a, b, b', c, a; x, y) \supset (1-y)^{-b'} S_1\left(b, a-b', b', c; x, \frac{x}{1-y}\right),$$

which is equivalent to

$$(1-y)^{-b'} S_2\left(a, b, b', c, a; x, -\frac{y}{1-y}\right) \supset S_1(b, a-b', b', c; x, x(1-y)).$$

The bi-rational coordinate change $(s,t) \to (s,\tau)$ is so made that the lines defining the integrand of the integral $\iint \cdots ds dt$ may become the union of vertical lines and horizontal lines in the (s,τ)-space. Actual blow-up and blow-down process is as follows (see Figure 1). Name the six lines in the st-projective plane as:

$$\ell_1 : s = 0, \quad \ell_2 : t = 0, \quad \ell_3 : s = 1, \quad \ell_4 : t = 1, \quad \ell_5 : 1-sx-ty = 0, \quad \ell_6 : \infty.$$

Blow up at the 4 points (shown by $\bullet$)

$$\ell_2 \cap \ell_5, \quad \ell_4 \cap \ell_5, \quad \ell_1 \cap \ell_3 \cap \ell_6 = 0:1:0, \quad \ell_2 \cap \ell_4 \cap \ell_6 = 1:0:0,$$

and blow-down along the proper transforms of the line ℓ_6 and two lines:

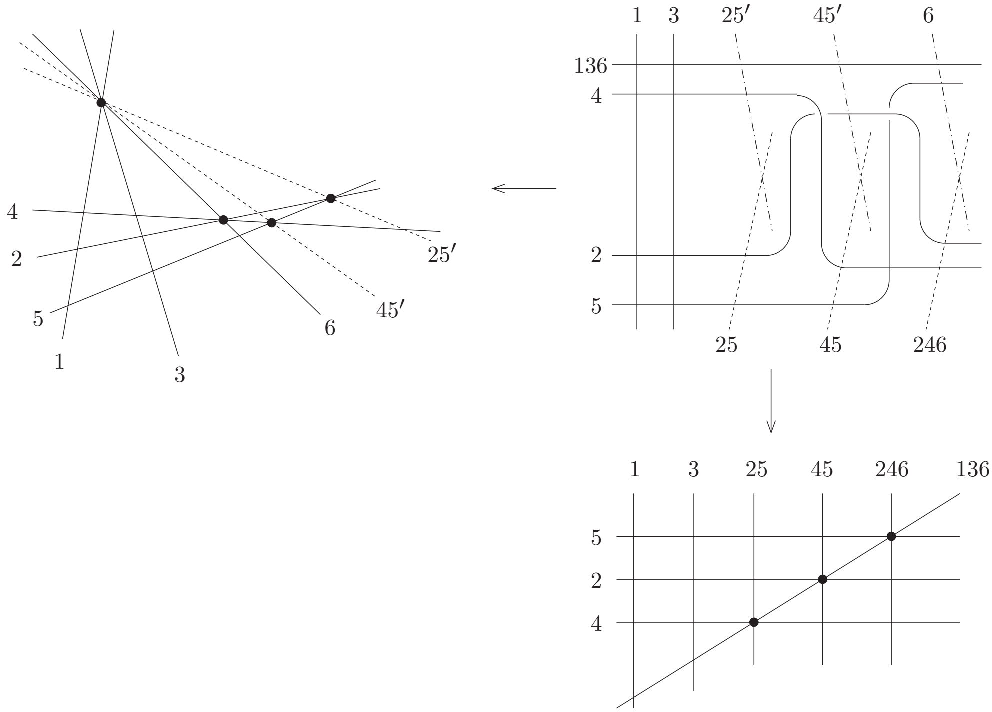

Figure 1. Birational map $(s,t) \to (s,\tau)$.

$$25' : \ 1 - sx = 0 \quad \text{and} \quad 45' : \ p = 1 - y - sx;$$

these three lines are dotted. This takes the st-projective plane to $\mathbb{P}^1(s) \times \mathbb{P}^1(\tau)$. In the figure, lines labeled $1, 2, \ldots$ stand for $\ell_1, \ell_2, \ldots$, and the lines labeled $25, 45, 246$ on the right are the blow-ups of the intersection points $\ell_2 \cap \ell_5, \ell_4 \cap \ell_5, \ell_2 \cap \ell_4 \cap \ell_6$, respectively. The line obtained by blowing up the point $\ell_1 \cap \ell_3 \cap \ell_6 = 0 : 1 : 0$ is the line defined by $q = y\tau + 1 - y - sx$, which should be labeled by 136.

1.2.3. System of differential equations.

A proof of the inclusion in Proposition 1.3 that is valid for any parameters is done as follows. Let z be a solution of the system $E_1(a,b,b',c;x,y)$. Then, the system E_1 : $P_1 z = Q_1 z = R_1 z = 0$ yields $\mathbb{C}(x,y)$-linear expressions of z_{xx}, z_{xy} and z_{yy} in terms of z_x, z_y and z. Substitute these expressions into the system $E_2 : P_2 z = Q_2 z = 0$. Then, we get two linear forms in z_x, z_y and z. We now have only to see their coefficients vanish for the given parameters after a change of coordinates and a change of the unknown by multiplying a simple factor. We do not here present the actual computation, because if we put $x^3 = 0$ in the proof of Proposition A.1 in Subsection A.3, manipulating differential equations, it gives essentially a proof of Proposition 1.3.

1.3. System $E_2(a,b,b',c,c')$ under $a = c = c'$.

When $a = c = c'$, applying Proposition 1.3 also for $a = c$, we see that the system $E_2(a,b,b',a,a)$ has two subsystems isomorphic to E_1. The intersection of the two E_1's would be the Gauss's hypergeometric equation. In fact, we have the following proposition.

PROPOSITION 1.5.

$$S_2(a,b,b',a,a;x,y)\supset(1-x)^{-b}(1-y)^{-b'}S\left(b,b',a;\frac{xy}{(1-x)(1-y)}\right).$$

Similar to the argument of the previous Subsection, we can give three "proof"s: one using power series, one using integral representations, and one manipulating differential equations. We give a sketch of them in the following.

1.3.1. Power series.
The following identity explains the inclusion above.

FACT 1.6 ([**B**, p. 81]).

$$F_2(a,b,b',a,a;x,y)=(1-x)^{-b}(1-y)^{-b'}F\left(b,b',a;\frac{xy}{(1-x)(1-y)}\right).$$

1.3.2. Integral representation.
We continue the argument in Section 1.2.2. In the integral J in Section 1.2.2 above, change the coordinate from s to σ as

$$s=\frac{\sigma}{M},\quad \sigma=\frac{(1-x)s}{1-xs},\qquad M=x\sigma+1-x$$

sending

$$s=0,\ 1,\ \frac{1}{x}\quad\text{to}\quad\sigma=0,\ 1,\ \infty.$$

Since

$$1-s=\frac{(1-x)(1-\sigma)}{M},\quad 1-sx=\frac{1-x}{M},\quad 1-\frac{xs}{1-y}=\frac{(1-x)(1-y)-xy\sigma}{(1-y)M},$$

we have

$$J=(1-x)^{c-b-a}(1-y)^{-b'}\int\sigma^{b-1}(1-\sigma)^{c-b-1}\left(1-\frac{xy\sigma}{(1-x)(1-y)}\right)^{-b'}M^{a-c}d\sigma.$$

This implies, if $a=c$, then

$$J=(1-x)^{-b}(1-y)^{-b'}\int\sigma^{b-1}(1-\sigma)^{a-b-1}\left(1-\frac{xy\sigma}{(1-x)(1-y)}\right)^{-b'}d\sigma.$$

This shows

$$(1-y)^{-b'}S_1\left(b,a-b',b',a;x,\frac{x}{1-y}\right)\supset(1-x)^{-b}(1-y)^{-b'}S\left(b,b',a;\frac{xy}{(1-x)(1-y)}\right),$$

which of course implies the inclusion relation in Proposition 1.5 by combination with that in Proposition 1.3.

1.3.3. System of differential equations.
Put

$$z = (1-x)^{-b}(1-y)^{-b'}u(t), \quad t = \frac{xy}{(1-x)(1-y)}.$$

We have

$$z_x = (1-x)^{-b}(1-y)^{-b'}u't_x + \cdots, \quad z_{xx} = (1-x)^{-b}(1-y)^{-b'}u''(t_x)^2 + \cdots,$$

and so on. Assume that $u(t) \in S(b, b', a; t)$. The equation $P(b, b', a; t)u(t) = 0$ gives a linear expression of u'' in u' and u. Substitute these expressions into

$$P_2(a, b, b', a, a; x, y)z = x(1-x)z_{xx} - xyz_{xy} + \cdots,$$

then we get the product of $(1-x)^{-b}(1-y)^{-b'}$ and a $\mathbb{C}(x, y)$-linear combination of u and u'. The coefficients vanish if $a = c$. If we do the same for $Q_2(a, b, b', a, a; x, y)z$, then we find that it vanishes when $a = c'$.

2. Solutions expressed as indefinite integrals.

We show that some indefinite integrals solve the system $E_2(a, b, b', a, a)$. We begin with some well-known facts.

Lemma 2.1.

$$P(a, b, c; x)\Phi = bx\frac{\partial}{\partial s}\left(\frac{s(1-s)}{x-s}\Phi\right),$$

where

$$P(a, b, c; x) = D(D+c-1) - x(D+a)(D+b), \quad \Phi = s^{b-c}(1-s)^{c-a-1}(x-s)^{-b}.$$

Proof. Note that

$$\left(D + D_s + \frac{c-a-1}{1-s}\right)\Phi = (-a-1)\Phi, \quad D = x\frac{\partial}{\partial x}, \quad D_s = s\frac{\partial}{\partial s}.$$

This implies

$$(D+a)\Phi = -\left(D_s + \frac{c-a-1}{1-s} + 1\right)\Phi,$$
$$(D+c-1)\Phi = -\left(D_s + \frac{(c-a-1)s}{1-s} + 1\right)\Phi.$$

Since

$$D\Phi = \frac{-bx}{x-s}\Phi, \quad (D+b)\Phi = -\frac{bs}{x-s}\Phi, \quad D_s + 1 = \frac{\partial}{\partial s}s,$$

we have

$$\begin{aligned} P\Phi &= \left\{ \left(D_s + \frac{(c-a-1)s}{1-s} + 1 \right) \frac{bx}{x-s} - x \left(D_s + \frac{c-a-1}{1-s} + 1 \right) \frac{bs}{x-s} \right\} \Phi \\ &= bx(D_s+1)\frac{1-s}{x-s}\Phi \\ &= bx\frac{\partial}{\partial s}\left(\frac{s(1-s)}{x-s}\Phi \right). \end{aligned}$$

□

Lemma 2.2. *The indefinite integral*

$$u = \int_p^s \Phi ds, \quad \Phi = s^{b-c}(1-s)^{c-a-1}(t-s)^{-b}, \quad p \in \{0, 1, t, \infty\}$$

solves $E_1(a,0,b,c;s,t)$. In particular, $S_1(a,0,b,c;s,t) \supset S(a,b,c;t)$.

Proof. Since $u_s = \Phi$, we have

$$u_{ss} = \left(\frac{b-c}{s} - \frac{c-a-1}{1-s} - \frac{-b}{t-s} \right) u_s, \quad u_{st} = \frac{-b}{t-s}u_s.$$

Lemma 2.1 leads to

$$P(a,b,c;t)u = bt\frac{s(1-s)}{t-s}u_s.$$

Let P_1, Q_1 and R_1 be the operators generating the system $E_1(a,0,b,c;s,t)$:

$$\begin{aligned} P_1(a,0,b,c;s,t) &= s(1-s)\partial_{ss} + t(1-s)\partial_{st} + (c-(a+1)s)\partial_s, \\ Q_1(a,0,b,c;s,t) &= t^{-1}P(a,b,c;t) + s(1-t)\partial_{st} - bs\partial_s, \\ R_1(a,0,b,c;s,t) &= (s-t)\partial_{st} - b\partial_s; \end{aligned}$$

refer to Section 1. Note that $t^{-1}P(a,b,c;t) = t(1-t)\partial_{tt} + \cdots$. By using the above identities, we have

$$R_1 u = (s-t)u_{st} - bu_s = 0,$$

and

$$P_1 u = s(1-s)\left(\frac{b-c}{s} - \frac{c-a-1}{1-s} - \frac{-b}{t-s} \right) u_s + t(1-s)\frac{bu_s}{s-t} + (c-(a+1)s)u_s = 0,$$

$$Q_1 u = bt\frac{s(1-s)}{t-s}u_s + \left(\frac{s(1-t)}{s-t} - bs \right) u_s = 0.$$

Furthermore, for $z \in S(a,b,c;t)$, the forms of operators above imply that z lies in $S_1(a,0,b,c;s,t)$. □

We now use the following fact:

Fact 2.3 ([**B**, p. 78]).

$$F_1(a,b,b',c;x,y) = (1-x)^{-a} F_1\left(a, c-b-b', b', c; \frac{-x}{1-x}, \frac{y-x}{1-x}\right).$$

From this fact we get, when $c = b + b'$,

$$S_1(a,b,b',b+b';x,y) = (1-x)^{-a} S_1\left(a, 0, b', b+b'; \frac{-x}{1-x}, \frac{y-x}{1-x}\right).$$

If we put

$$y = \frac{x}{1-\eta},$$

then

$$\frac{y-x}{1-x} = \frac{x\eta}{(1-x)(1-\eta)}.$$

Thus we have

$$\begin{aligned} S_2(a,b,b',a,a;x,y) &\supset (1-y)^{-b'} S_1\left(b, a-b', b', a; x, \frac{x}{1-y}\right) \\ &= (1-y)^{-b'}(1-x)^{-b} S_1\left(b, 0, b', a; \frac{-x}{1-x}, \frac{xy}{(1-x)(1-y)}\right) \\ &\supset (1-y)^{-b'}(1-x)^{-b} S\left(b, b', a; \frac{xy}{(1-x)(1-y)}\right). \end{aligned}$$

This agrees with the inclusion in Proposition 1.5. In particular, by the inclusion

$$(1-y)^{b'}(1-x)^{b} S_2(a,b,b',a,a;x,y) \supset S_1\left(b, 0, b', a; \frac{-x}{1-x}, \frac{xy}{(1-x)(1-y)}\right)$$

and Lemma 2.2 we get a solution of $E_2(a,b,b',a,a;x,y)(1-y)^{b'}(1-x)^{b}$ represented by the indefinite integral:

$$f_1 := \int_0^{\boldsymbol{x}} s^{b'-a}(1-s)^{a-b-1}(\boldsymbol{x}\boldsymbol{y}-s)^{-b'}\,ds, \qquad \boldsymbol{x} = \frac{-x}{1-x},\ \boldsymbol{y} = \frac{-y}{1-y}.$$

The starting point of the path of integration can be any point $p \in \{0, 1, t, \infty\}$, so we choose $p = 0$, just for simplicity. By exchanging the role of x and y, we get an inclusion

$$(1-y)^{b'}(1-x)^{b} S_2(a,b,b',a,a;x,y) \supset S_1\left(b', b, 0, a; \frac{xy}{(1-x)(1-y)}, \frac{-y}{1-y}\right)$$

and another solution of $(1-y)^{b'}(1-x)^{b} E_2(a,b,b',a,a;x,y)$ represented by the indefinite integral:

$$\int_0^{\boldsymbol{y}} s^{b-a}(1-s)^{a-b'-1}(\boldsymbol{xy}-s)^{-b}ds.$$

After the change $s \to \boldsymbol{xy}/s$, it can be also expressed as

$$f_2 := (\boldsymbol{xy})^{-a+1}\int_0^{\boldsymbol{x}} s^{b'-1}(1-s)^{-b}(\boldsymbol{xy}-s)^{a-b'-1}ds.$$

Thus, we have:

THEOREM 2.4. *We have the following inclusions of the spaces of solutions*:

$$\begin{array}{ccc} (1-y)^{b'}(1-x)^b S_2(a,b,b',a,a;x,y) & \supset & S_1\left(b,0,b',a;\dfrac{-x}{1-x},\dfrac{xy}{(1-x)(1-y)}\right) \\ \cup & & \cup \\ S_1\left(b',b,0,a;\dfrac{xy}{(1-x)(1-y)},\dfrac{-y}{1-y}\right) & \supset & S\left(b,b',a;\dfrac{xy}{(1-x)(1-y)}\right). \end{array}$$

Moreover, the collection of solutions $(1-y)^{b'}(1-x)^b S_2(a,b,b',a,a;x,y)$ *is spanned by* f_1, f_2 *and* $S\left(b,b',a;\dfrac{xy}{(1-x)(1-y)}\right)$.

This will play a key role to understand the Schwarz map of a system E_2 with specific parameters, which will be introduced in Section 4.1.

3. Monodromy representation of E_2.

In this section, we study the monodromy representations of local systems associated with $E_2 = E_2(a,b,b',c,c')$. Though it is assumed in [**MY**] that the parameters satisfy the irreducibility condition in Fact 1.1:

$$a,\ b,\ b',\ c-b,\ c'-b',\ a-c-c', \quad a-c,\ a-c' \notin \mathbb{Z},$$

in this section, we only assume the weaker condition

$$a,\ b,\ b',\ c-b,\ c'-b',\ a-c-c' \notin \mathbb{Z}. \tag{3.1}$$

We modify Theorem 7.1 in [**MY**] so that the statements remain valid for these parameters. We will apply the result of this section in Section 4.4.

3.1. Twisted homology groups and the intersection form.

Let

$$X = \{(x,y) \in \mathbb{C}^2 \mid x(x-1)y(y-1)(x+y-1) \neq 0\}$$

be the complement of the singular locus of E_2. For each $(x,y) \in X$, we consider a multi-valued function

$$\psi = t_1^{b-1}(1-t_1)^{c-b-1}t_2^{b'-1}(1-t_2)^{c'-b'-1}(1-t_1x-t_2y)^{-a}$$

on

$$T_{x,y} = \{(t_1, t_2) \in \mathbb{C}^2 \mid t_1(1-t_1)t_2(1-t_2)(1-t_1x-t_2y) \neq 0\}.$$

As in Section 1.6 and Section 1.7 of Chapter 2 in [**AK**], we define the twisted homology group $H_2(T_{x,y}, \psi)$ associated with ψ and locally finite one $H_2^{\mathrm{lf}}(T_{x,y}, \psi)$. Under some genericity condition, the integral of ψ over a twisted cycle gives a solution of E_2.

If $a-c,\ a-c' \notin \mathbb{Z}$, then the natural map $\imath : H_2(T_{x,y}, \psi) \to H_2^{\mathrm{lf}}(T_{x,y}, \psi)$ is bijective, and the inverse map reg : $H_2^{\mathrm{lf}}(T_{x,y}, \psi) \to H_2(T_{x,y}, \psi)$ is called the regularization. In general, the map $\imath : H_2(T_{x,y}, \psi) \to H_2^{\mathrm{lf}}(T_{x,y}, \psi)$ is neither injective nor surjective, however we still have the isomorphism

$$\mathrm{reg} : \mathrm{Im}(\imath) \to H_2(T_{x,y}, \psi)/\ker(\imath).$$

Under the condition (3.1), thanks to the vanishing theorem of cohomology groups in [**C**], the ranks of $H_2^{\mathrm{lf}}(T_{x,y}, \psi)$ and $H_2(T_{x,y}, \psi)$ are equal to the Euler characteristic $\chi(T_{x,y}) = 4$ of $T_{x,y}$, and the bilinear form—*the intersection form*—

$$\mathfrak{I} : H_2^{\mathrm{lf}}(T_{x,y}, \psi) \times H_2(T_{x,y}, \psi^{-1}) \to \mathbb{C}$$

is non-degenerate.

3.2. Monodromy representation.

Under the condition (3.1), we consider the monodromy representations of the local systems

$$\mathcal{H}_2^{\mathrm{lf}}(\psi) = \bigcup_{(x,y)\in X} H_2^{\mathrm{lf}}(T_{x,y}, \psi), \quad \mathcal{H}_2(\psi^{-1}) = \bigcup_{(x,y)\in X} H_2(T_{x,y}, \psi^{-1})$$

over X. By adding the condition

$$a-c, a-c' \notin \{-1, -2, -3, \dots\}, \tag{3.2}$$

we can identify the local solution space to E_2 on a small simply connected domain $U \subset X$ with the trivial vector bundle $\bigcup_{(x,y)\in U} H_2^{\mathrm{lf}}(T_{x,y}, \psi)$ via the Euler type integral representation of solutions to E_2. In this case, the monodromy representation of E_2 is equivalent to that of the local system $\mathcal{H}_2^{\mathrm{lf}}(\psi)$. However we do not assume (3.2) in this section.

We fix a small positive real number ε, and let $(\varepsilon, \varepsilon)$ be a base point in X. Denote the germs at this point of the local systems $\mathcal{H}_2^{\mathrm{lf}}(\psi)$ and $\mathcal{H}_2(\psi^{-1})$ by

$$H_2^{\mathrm{lf}}(T, \psi), \quad H_2(T, \psi^{-1}), \qquad T = T_{\varepsilon,\varepsilon},$$

respectively. Let

$$\mathcal{M}^{\mu} : \pi_1(X, (\varepsilon, \varepsilon)) \ni \rho \mapsto \mathcal{M}^{\mu}_{\rho} \in GL(H_2^{\mathrm{lf}}(T, \psi)),$$
$$\widetilde{\mathcal{M}}^{-\mu} : \pi_1(X, (\varepsilon, \varepsilon)) \ni \rho \mapsto \widetilde{\mathcal{M}}^{-\mu}_{\rho} \in GL(H_2(T, \psi^{-1}))$$

be the monodromy representations of $\mathcal{H}_2^{\mathrm{lf}}(\psi)$ and $\mathcal{H}_2(\psi^{-1})$ with respect to $(\varepsilon, \varepsilon)$. $\mathcal{M}_\rho^\mu$ and $\widetilde{\mathcal{M}}_\rho^{-\mu}$ are called the circuit transformations along ρ.

PROPOSITION 3.1. (1) *The image* $\mathrm{Im}(\imath)$ *of the natural map* $\imath : H_2(T,\psi) \to H_2^{\mathrm{lf}}(T,\psi)$ *is invariant under the monodromy representation* $\mathcal{M}^\mu$.

(2) *The kernel* $\ker(\imath')$ *of the natural map* $\imath' : H_2(T,\psi^{-1}) \to H_2^{\mathrm{lf}}(T,\psi^{-1})$ *is invariant under the monodromy representation* $\widetilde{\mathcal{M}}^{-\mu}$.

PROOF. (1) It is clear that the space $H_2(T,\psi)$ is invariant under the monodromy representation $\widetilde{\mathcal{M}}^\mu$, which is the monodromy representation on $H_2(T,\psi)$ (replacing ψ^{-1} as ψ). Since the map $\imath$ commutes with small deformations, the space $\imath(H_2(T,\psi))$ is invariant under $\mathcal{M}^\mu$.
(2) Since the map $\imath'$ commutes with small deformations, $\ker(\imath')$ is invariant under $\widetilde{\mathcal{M}}^{-\mu}$. □

REMARK 3.2. We will see that if $a-c \in \mathbb{Z}$ or $a-c' \in \mathbb{Z}$, then both of $\mathrm{Im}(\imath)$ and $\ker(\imath')$ are proper subspaces. Thus monodromy representations $\mathcal{M}^\mu$ and $\widetilde{\mathcal{M}}^{-\mu}$ are reducible in this case.

LEMMA 3.3. (1) *Let* Δ *and* $\widetilde{\Delta}$ *be elements of* $H_2^{\mathrm{lf}}(T,\psi)$ *and* $H_2(T,\psi^{-1})$, *respectively. Then we have*

$$\mathfrak{I}(\mathcal{M}_\rho^\mu(\Delta), \widetilde{\mathcal{M}}_\rho^{-\mu}(\widetilde{\Delta})) = \mathfrak{I}(\Delta, \widetilde{\Delta}).$$

(2) *Suppose that* $W = H_2^{\mathrm{lf}}(T,\psi)$ *is decomposed into the direct sum of the eigenspaces* $W_1, \dots, W_r$ *of* $\mathcal{M}_\rho^\mu$ *of eigenvalues* $\lambda_1, \dots, \lambda_r$. *Then* $\widetilde{W} = H_2(T,\psi^{-1})$ *is decomposed into the direct sum of the eigenspaces* $\widetilde{W}_1, \dots, \widetilde{W}_r$ *of* $\widetilde{\mathcal{M}}_\rho^{-\mu}$ *of eigenvalues* $\lambda_1^{-1}, \dots, \lambda_r^{-1}$. *The eigenspace* $\widetilde{W}_i$ *is characterized as*

$$\widetilde{W}_i = \bigcap_{1\le j\le r}^{j\neq i} W_j^\perp, \quad W_j^\perp = \{\widetilde{w} \in \widetilde{W} \mid \mathfrak{I}(w_j, \widetilde{w}) = 0 \ \ \textit{for any } w_j \in W_j\}.$$

PROOF. (1) The intersection number $\mathfrak{I}(\Delta, \widetilde{\Delta})$ is stable under small deformations of Δ and $\widetilde{\Delta}$.

(2) Take a basis $w_1, \dots w_n$ of W consisting of eigenvectors of $\mathcal{M}_\rho^\mu$. Since the intersection form $\mathfrak{I}$ is non degenerate, there exist $\widetilde{w}_1, \dots, \widetilde{w}_n \in \widetilde{W}$ such that $\mathfrak{I}(w_i, \widetilde{w}_j) = \delta_{ij}$ (Kronecker's symbol). By (1), $\widetilde{w}_i$ is an eigenvector of the reciprocal of the eigenvalue of w_i. These bases of W and $\widetilde{W}$ imply the assertion (2) in this lemma. □

3.3. Twisted cycles.

Let $\square_j$ $(i = 1, \dots, 6)$ be locally finite chains shown in Figure 2. We specify a branch of ψ on each chain by the assignment of $\arg(f_j)$ on it as in Table 1, where

$$f_1 = t_1, \quad f_2 = 1 - t_1, \quad f_3 = t_2, \quad f_4 = 1 - t_2, \quad f_5 = 1 - t_1 x - t_2 y,$$

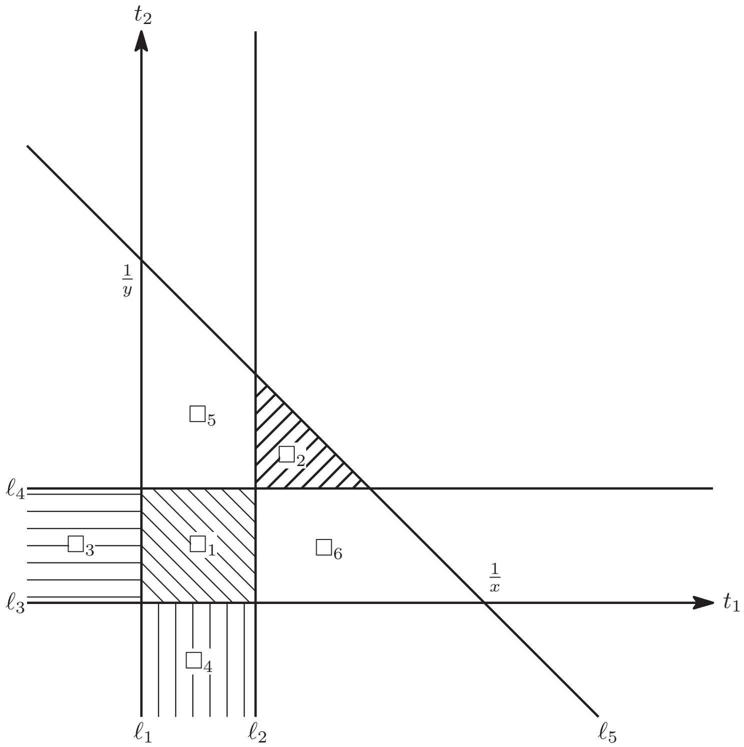

Figure 2. 2-cycles.

Table 1. List of $\arg(f_j)$ on $\square_i$.

	$f_1 = t_1$	$f_2 = 1 - t_1$	$f_3 = t_2$	$f_4 = 1 - t_2$	$f_5 = 1 - t_1x - t_2y$
$\square_1$	0	0	0	0	0
$\square_2$	0	π	0	π	0
$\square_3$	$-\pi$	0	0	0	0
$\square_4$	0	0	$-\pi$	0	0
$\square_5$	0	0	0	π	0
$\square_6$	0	π	0	0	0

and load ψ to get the locally finite twisted cycles $\square_i^{\psi} \in H_2^{\mathrm{lf}}(T, \psi)$. It will be shown that $\Delta_1 = \square_1^{\psi}, \dots, \Delta_4 = \square_4^{\psi}$ form a basis of $H_2^{\mathrm{lf}}(T, \psi)$ in Corollary 3.6.

We choose elements in $H_2(T, \psi^{-1})$ as

$$\widetilde{\Delta}_1 = \mathrm{reg}(\square_1^{\psi^{-1}}), \quad \widetilde{\Delta}_2 = \mathrm{reg}(\square_2^{\psi^{-1}}),$$
$$\widetilde{\Delta}_3 = (\mu_{125} - 1)\mathrm{reg}(\square_3^{\psi^{-1}}), \quad \widetilde{\Delta}_4 = (\mu_{345} - 1)\mathrm{reg}(\square_4^{\psi^{-1}}),$$

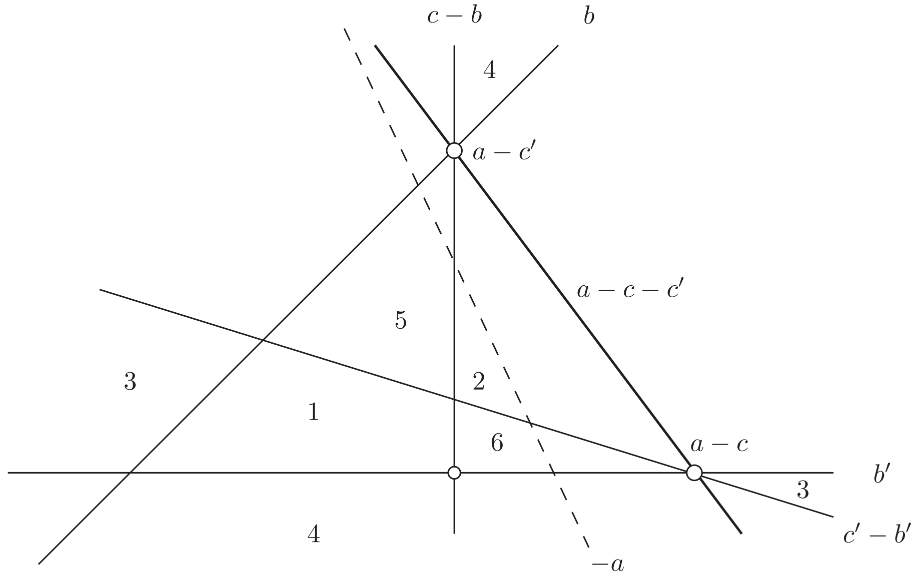

Figure 3. Exponents around the line at infinity.

where $\mu_{ij\cdots} = \mu_i\mu_j\cdots$ and

$$\mu_1 = e^{2\pi\sqrt{-1}b},\ \mu_2 = e^{2\pi\sqrt{-1}(c-b)},\ \mu_3 = e^{2\pi\sqrt{-1}b'},\ \mu_4 = e^{2\pi\sqrt{-1}(c'-b')},\ \mu_5 = e^{-2\pi\sqrt{-1}a}.$$

Note that in terms of μ_j's the irreducibility condition in Fact 1.1 is

$$\mu_i (i = 1, \dots, 5),\ \mu_{12345},\quad \mu_{125},\ \mu_{345} \neq 1,$$

the condition (3.1) is

$$\mu_i (i = 1, \dots, 5),\ \mu_{12345} \neq 1,$$

and (see Figure 3)

$$c - a \in \mathbb{Z} \Leftrightarrow \mu_{125} = 1, \quad c' - a \in \mathbb{Z} \Leftrightarrow \mu_{345} = 1,$$

and that $(a, b, b', c, c') = (4/3, 2/3, 2/3, 4/3, 4/3)$ satisfies (3.1) and $\mu_{ijk} = 1$ for any i, j, k. Explicitly, $\widetilde{\Delta}_3$ and $\widetilde{\Delta}_4$ can be written as:

$$(\mu_{125}-1)\left[\left(\frac{\circlearrowleft_\infty^+}{\mu_{125}-1} + \left[\frac{-1}{\delta}, -\delta\right] - \frac{\circlearrowleft_0^-}{\mu_1^{-1}-1}\right) \times \left(\frac{\circlearrowleft_0^+}{\mu_3^{-1}-1} + [\delta, 1-\delta] - \frac{\circlearrowleft_1^-}{\mu_4^{-1}-1}\right)\right]^{\psi^{-1}},$$

$$(\mu_{345}-1)\left[\left(\frac{\circlearrowleft_0^+}{\mu_1^{-1}-1} + [\delta, 1-\delta] - \frac{\circlearrowleft_1^-}{\mu_2^{-1}-1}\right) \times \left(\frac{\circlearrowleft_\infty^+}{\mu_{345}-1} + \left[\frac{-1}{\delta}, -\delta\right] - \frac{\circlearrowleft_0^-}{\mu_3^{-1}-1}\right)\right]^{\psi^{-1}},$$

where δ is a small positive real number, $[-1/\delta, -\delta]$ and $[\delta, 1-\delta]$ are closed intervals, $\circlearrowleft_\infty^+$ is the negatively oriented circle of which radius, center and terminal are $1/\delta$, 0 and $-1/\delta$,

$\circlearrowleft_q^{\pm}$ $(q = 0, 1)$ is the positively oriented circle of which radius, center and terminal are δ, q and $q \pm \delta$.

Notice that the definition of the twisted cycles $\widetilde{\Delta}_i$ $(i = 1, \dots, 4)$ makes sense even in the case $a - c \in \mathbb{Z}$ or $a - c' \in \mathbb{Z}$. Indeed, this specialization gives no harm to $\widetilde{\Delta}_i$ $(i = 1, 2)$, and thanks to the above expression, when $\mu_{125} = 1$ and $\mu_{345} = 1$, we have

$$\widetilde{\Delta}_3 = \left[\circlearrowleft_\infty^+ \times \left(\frac{\circlearrowleft_0^+}{\mu_3^{-1} - 1} + [\delta, 1 - \delta] - \frac{\circlearrowleft_1^-}{\mu_4^{-1} - 1}\right)\right]^{\psi^{-1}},$$

$$\widetilde{\Delta}_4 = \left[\left(\frac{\circlearrowleft_0^+}{\mu_1^{-1} - 1} + [\delta, 1 - \delta] - \frac{\circlearrowleft_1^-}{\mu_2^{-1} - 1}\right) \times \circlearrowleft_\infty^+\right]^{\psi^{-1}},$$

respectively.

Remark 3.4. Suppose that $a - c$, $a - c' \in \mathbb{Z}$. Then the twisted cycles $\imath'(\widetilde{\Delta}_3)$ and $\imath'(\widetilde{\Delta}_4)$ are homologous to 0 in $H_2^{\mathrm{lf}}(T, \psi^{-1})$, since they are the boundary of locally finite 3-chains given by the replacement $\circlearrowleft_\infty^+ \to \odot_\infty^+$ in their expressions, where $\odot_\infty^+$ is the annulus $\{t \in \mathbb{C} \mid 1/\delta \le |t|\}$. They belong to $\ker(\imath')$. By Proposition 3.5, it turns out that $\mathrm{Im}(\imath)$ is spanned by Δ_1 and Δ_2.

3.4. Intersection matrices.

Proposition 3.5. *The intersection matrix* $H^\mu = \left(\mathfrak{I}(\Delta_i, \widetilde{\Delta}_j)\right)_{1 \le i,j, \le 4}$ *for* $\Delta_1, \dots, \Delta_4 \in H_2^{\mathrm{lf}}(T, \psi)$ *and* $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_4 \in H_2(T, \psi^{-1})$ *is given by*

$$\begin{pmatrix} \frac{(\mu_{12}-1)(\mu_{34}-1)}{(\mu_1-1)(\mu_2-1)(\mu_3-1)(\mu_4-1)} & \frac{1}{(\mu_2-1)(\mu_4-1)} & -\frac{\mu_1(\mu_{34}-1)(\mu_{125}-1)}{(\mu_1-1)(\mu_3-1)(\mu_4-1)} & -\frac{\mu_3(\mu_{12}-1)(\mu_{345}-1)}{(\mu_1-1)(\mu_2-1)(\mu_3-1)} \\ \frac{\mu_{24}}{(\mu_2-1)(\mu_4-1)} & \frac{\mu_{245}-1}{(\mu_2-1)(\mu_4-1)(\mu_5-1)} & 0 & 0 \\ -\frac{\mu_{34}-1}{(\mu_1-1)(\mu_3-1)(\mu_4-1)} & 0 & \frac{\mu_1(\mu_{34}-1)(\mu_{25}-1)}{(\mu_1-1)(\mu_3-1)(\mu_4-1)} & \frac{\mu_3(\mu_{345}-1)}{(\mu_1-1)(\mu_3-1)} \\ -\frac{\mu_{12}-1}{(\mu_1-1)(\mu_2-1)(\mu_3-1)} & 0 & \frac{\mu_1(\mu_{125}-1)}{(\mu_1-1)(\mu_3-1)} & \frac{\mu_3(\mu_{12}-1)(\mu_{45}-1)}{(\mu_1-1)(\mu_2-1)(\mu_3-1)} \end{pmatrix}.$$

Its determinant is

$$\frac{\mu_{1234}(\mu_5 - 1)(\mu_{12345} - 1)}{(\mu_1 - 1)^2(\mu_2 - 1)^2(\mu_3 - 1)^2(\mu_4 - 1)^2},$$

which does not vanish under the assumption (3.1).

Proof. Follow Section 3 of Chapter VIII in [**Yo**] for the computation of the intersection numbers. By a straightforward calculation, we have its determinant. □

Proposition 3.5 yields the following corollary.

Corollary 3.6. *The twisted cycles* $\Delta_1, \dots, \Delta_4$ *and* $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_4$ *form a basis of* $H_2^{\mathrm{lf}}(T, \psi)$ *and that of* $H_2(T, \psi^{-1})$, *respectively.*

We express the twisted cycles $\square_5^\psi$ and $\square_6^\psi$ as linear combinations of Δ_i.

LEMMA 3.7. *We have*

$$\square_5^\psi = -\frac{\mu_4\mu_5 - 1}{\mu_5 - 1}\Delta_1 - \frac{\mu_{345} - 1}{\mu_5 - 1}\Delta_4,$$
$$\square_6^\psi = -\frac{\mu_2\mu_5 - 1}{\mu_5 - 1}\Delta_1 - \frac{\mu_{125} - 1}{\mu_5 - 1}\Delta_3.$$

PROOF. Set

$$\square_5^\psi = \sum_{i=1}^{4} \gamma_i \Delta_i = (\gamma_1, \dots, \gamma_4)\ {}^t(\Delta_1, \dots, \Delta_4),$$

and compute the intersection numbers $\mathfrak{I}(\square_5^\psi, \widetilde{\Delta}_i)$. Then we have

$$\begin{aligned}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)H^\mu &= (\mathfrak{I}(\square_5^\psi, \widetilde{\Delta}_1), \dots, \mathfrak{I}(\square_5^\psi, \widetilde{\Delta}_4)) \\ &= \Big(\frac{-\mu_4(\mu_1\mu_2 - 1)}{(\mu_1 - 1)(\mu_2 - 1)(\mu_4 - 1)}, \frac{-(\mu_4\mu_5 - 1)}{(\mu_2 - 1)(\mu_4 - 1)(\mu_5 - 1)}, \frac{\mu_1\mu_4(\mu_{125} - 1)}{(\mu_1 - 1)(\mu_4 - 1)}, 0\Big).\end{aligned}$$

Thus the vector $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by the right multiplication of the inverse of H^μ to the last term.

Since

$$\begin{aligned}&(\mathfrak{I}(\square_6^\psi, \widetilde{\Delta}_1), \dots, \mathfrak{I}(\square_6^\psi, \widetilde{\Delta}_4)) \\ &= \Big(\frac{-\mu_2(\mu_3\mu_4 - 1)}{(\mu_2 - 1)(\mu_3 - 1)(\mu_4 - 1)}, \frac{-(\mu_2\mu_5 - 1)}{(\mu_2 - 1)(\mu_4 - 1)(\mu_5 - 1)}, 0, \frac{\mu_2\mu_3(\mu_{345} - 1)}{(\mu_2 - 1)(\mu_3 - 1)}\Big),\end{aligned}$$

we have the expression of $\square_6^\psi$. □

REMARK 3.8. In [**MY**], we take a basis Δ_1, Δ_2, $\square_5^\psi$ and $\square_6^\psi$ of $H_2^{\mathrm{lf}}(T, \psi)$. If $a - c, a - c' \in \mathbb{Z}$, then each of $\square_5^\psi$ and $\square_6^\psi$ is a scalar multiple of Δ_1 by Lemma 3.7.

Similar to Lemma 3.7, we have the following.

LEMMA 3.9. *We have*

$$\mathrm{reg}(\square_5^{\psi^{-1}}) = -\frac{\mu_4\mu_5 - 1}{\mu_4(\mu_5 - 1)}\widetilde{\Delta}_1 - \frac{1}{\mu_3\mu_4(\mu_5 - 1)}\widetilde{\Delta}_4,$$
$$\mathrm{reg}(\square_6^{\psi^{-1}}) = -\frac{\mu_2\mu_5 - 1}{\mu_2(\mu_5 - 1)}\widetilde{\Delta}_1 - \frac{1}{\mu_1\mu_2(\mu_5 - 1)}\widetilde{\Delta}_3.$$

3.5. Circuit transformations.

3.5.1. Generators of the fundamental group.

We give generators of the fundamental group $\pi_1(X, (\varepsilon, \varepsilon))$. Let ρ_1 (and ρ_3, ρ_5) be a loop in

$$L_x = \{(x, \varepsilon) \in \mathbb{C}^2 \mid x \neq 0, 1 - \varepsilon, 1\}$$

starting from $(x,y)=(\varepsilon,\varepsilon)$, approaching to the point $(x,y)=(0,\varepsilon)$,(and $(x,y)=(1-\varepsilon,\varepsilon)$, $(1,\varepsilon)$) with $\mathrm{Im}(x)>0$, turning once around the point positively, and tracing back to $(\varepsilon,\varepsilon)$. Let ρ_2 (and ρ_4) be a loop in

$$L_y=\{(\varepsilon,y)\in\mathbb{C}^2 \mid y\neq 0,1-\varepsilon,1\}$$

starting from $(x,y)=(\varepsilon,\varepsilon)$, approaching to the point $(x,y)=(\varepsilon,0)$, (and $(x,y)=(\varepsilon,1)$) with $\mathrm{Im}(y)>0$, turning once around the point positively, and tracing back to $(\varepsilon,\varepsilon)$. By van Kampen's theorem, it is seen that the loops $\rho_1,\dots,\rho_5$ generate $\pi_1(X,(\varepsilon,\varepsilon))$. The circuit matrices of $\mathcal{M}^\mu$ and $\widetilde{\mathcal{M}}^\mu$ along ρ_i will be denoted as

$$\mathcal{M}_i^\mu=\mathcal{M}_{\rho_i}^\mu,\quad \widetilde{\mathcal{M}}_i^{-\mu}=\widetilde{\mathcal{M}}_{\rho_i}^{-\mu}\quad (i=1,\dots,5).$$

3.5.2. Expressions of the circuit transformations.

Theorem 3.10. *Under the assumption* (3.1), *the circuit transformations* $\mathcal{M}_i^\mu\in GL(H_2^{\mathrm{lf}}(T,\psi))$ $(i=1,\dots,5)$ *are given as*

$$\mathcal{M}_1^\mu(\Delta)=\mu_{12}^{-1}\Delta-(\mu_{12}^{-1}-1)\Big(\mathfrak{I}(\Delta,\widetilde{\Delta}_1),\mathfrak{I}(\Delta,\widetilde{\Delta}_4)\Big){H_{14}^{\mu}}^{-1}\begin{pmatrix}\Delta_1\\ \Delta_4\end{pmatrix}, \tag{3.3}$$

$$\mathcal{M}_2^\mu(\Delta)=\mu_{34}^{-1}\Delta-(\mu_{34}^{-1}-1)\Big(\mathfrak{I}(\Delta,\widetilde{\Delta}_1),\mathfrak{I}(\Delta,\widetilde{\Delta}_3)\Big){H_{13}^{\mu}}^{-1}\begin{pmatrix}\Delta_1\\ \Delta_3\end{pmatrix},$$

$$\mathcal{M}_3^\mu(\Delta)=\Delta-\frac{1-\mu_{245}}{\mathfrak{I}(\Delta_2,\widetilde{\Delta}_2)}\mathfrak{I}(\Delta,\widetilde{\Delta}_2)\Delta_2,$$

$$\mathcal{M}_4^\mu(\Delta)=\Delta-\frac{1-\mu_{145}}{\mathfrak{I}(\Delta_{145},\widetilde{\Delta}_{145})}\mathfrak{I}(\Delta,\widetilde{\Delta}_{145})\Delta_{145},$$

$$\mathcal{M}_5^\mu(\Delta)=\Delta-\frac{1-\mu_{235}}{\mathfrak{I}(\Delta_{235},\widetilde{\Delta}_{235})}\mathfrak{I}(\Delta,\widetilde{\Delta}_{235})\Delta_{235},$$

where Δ *is any element of* $H_2^{\mathrm{lf}}(T,\psi)$, H_{1j}^μ $(j=3,4)$ *is the submatrix of* H^μ *consisting of the* $(1,1)$, $(1,j)$, $(j,1)$ *and* (j,j) *entries of* H^μ, *and*

$$\Delta_{145}=\Delta_2+\Box_5^\psi,\qquad \Delta_{235}=\Delta_2+\Box_6^\psi,$$
$$\widetilde{\Delta}_{145}=\widetilde{\Delta}_2+\mathrm{reg}(\Box_5^{\psi^{-1}}),\qquad \widetilde{\Delta}_{235}=\widetilde{\Delta}_2+\mathrm{reg}(\Box_6^{\psi^{-1}}).$$

Proof. Under the condition $c,c'\notin\mathbb{Z}$, the linear transformation $\mathcal{M}_1^\mu$ satisfies the assumption of Lemma 3.3 (2). In fact, a fundamental system of solutions can be given by the hypergeometric series F_2 multiplied by the power functions

$$1,\quad x^{1-c},\quad y^{1-c'},\quad x^{1-c}y^{1-c'}.$$

Thus the eigenvalues of $\mathcal{M}_1^\mu$ are 1 and $e^{-2\pi\sqrt{-1}c}=\mu_{12}^{-1}$, and each of the eigenspaces is two dimensional. It is easy to see that the locally finite chains $\Box_1$ and $\Box_4$ are invariant under the deformation along ρ_1. Hence the twisted cycles Δ_1 and Δ_4 span the eigenspace of $\mathcal{M}_1^\mu$ of eigenvalue 1. Similarly, we can show that the twisted cycles $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_4$ span the eigenspace of $\widetilde{\mathcal{M}}_1^{-\mu}$ of eigenvalue 1. By Lemma 3.3 (2), the eigenspace of $\mathcal{M}_1^{\mu}$ of eigenvalue μ_{12}^{-1} is

$$\langle \widetilde{\Delta}_1, \widetilde{\Delta}_4 \rangle^{\perp} = \{\Delta \in H_2^{\mathrm{lf}}(T,\psi) \mid \mathfrak{I}(\Delta, \widetilde{\Delta}_1) = \mathfrak{I}(\Delta, \widetilde{\Delta}_4) = 0\}.$$

It is easy to see that the right hand side $\mathcal{M}_1^{\mu}$ of (3.3) satisfies

$$\mathcal{M}_1^{\mu}(\Delta_j) = \Delta_j, \quad j = 1, 4, \qquad \mathcal{M}_1^{\mu}(\Delta) = \mu_{12}^{-1}\Delta, \quad \Delta \in \langle \widetilde{\Delta}_1, \widetilde{\Delta}_4 \rangle^{\perp}.$$

By Proposition 3.5, $\Delta_1, \ldots, \Delta_4$ form a basis even in the case $c \in \mathbb{Z}$ or $c' \in \mathbb{Z}$. Thus the representation matrix M_1^{μ} of $\mathcal{M}_1^{\mu}$ with respect to this basis is continuous on the parameters a, b, b', c, c'. On the other hand, the expression $\mathcal{M}_1^{\mu}$ is also continuous since the factor $\mu_{12} - 1$ in the denominator of

$$(H_{14}^{\mu})^{-1} = \frac{(\mu_1 - 1)(\mu_2 - 1)}{\mu_{12} - 1} \begin{pmatrix} \dfrac{(\mu_4 - 1)(\mu_{45} - 1)}{\mu_4(\mu_5 - 1)} & \dfrac{(\mu_4 - 1)(\mu_{345} - 1)}{\mu_4(\mu_5 - 1)} \\ \dfrac{\mu_4 - 1}{\mu_{34}(\mu_5 - 1)} & \dfrac{\mu_{34} - 1}{\mu_{34}(\mu_5 - 1)} \end{pmatrix}.$$

cancels with the factor $\mu_{12}^{-1} - 1$ in the righthand side of (3.3). Similarly, we have the expression of $\mathcal{M}_2^{\mu}$.

To study $\mathcal{M}_5^{\mu}$, we work temporarily under the condition $\mu_{235} \neq 1$. We decompose ρ_5 into $\bar{\rho}_5 \cdot \rho_5^{\circ} \cdot \bar{\rho}_5^{-1}$, where $\bar{\rho}_5$ is the approach to $x = 1$ and ρ_5° is the turning path. We trace the deformation of the triangle $\square_{235}$ made by $\square_2$ and $\square_6$ along $\bar{\rho}_5$. After the deformation, this becomes a small triangle near the point $(t_1, t_2) = (1, 0)$. We see the argument of $f_4 = 1 - t_2$ on this triangle. Since $y = \varepsilon$ and $\mathrm{Im}(x) > 0$ in $\bar{\rho}_5$, $t_1 > 1$ in this triangle, and

$$1 - t_2 = (\varepsilon - 1)t_2 + xt_1$$

on the line $\ell_5 : f_5 = 0$, $f_4 = 1 - t_2$ varies negative to positive via the upper-half space, i.e, $\arg(f_4)$ decreases by π along $\bar{\rho}_5$. Note that this change is compatible with our assignment of $\arg(f_4)$ on $\square_2$ and $\square_6$ in Table 1. Thus the twisted cycle $\Delta_{235} = \Delta_2 + \square_6^{\psi}$ plays the role of a vanishing cycle as the line ℓ_5 approaches the point $(1, 0)$. Since ρ_5° corresponds to the move of f_5 turning around the point $(1, 0)$, the cycle Δ_{235} is an eigenvector of $\mathcal{M}_5^{\mu}$ of eigenvalue μ_{235}. We can similarly show that $\widetilde{\Delta}_{235}$ is an eigenvector of $\widetilde{\mathcal{M}}_5^{-\mu}$ of eigenvalue μ_{235}^{-1}. On the other hand, we can find three chambers not affected by the move of the line ℓ_5 along ρ_5°. For example, $\square_3$, $\{(t_1, t_2) \in \mathbb{R}_2 \mid t_1 < 0, t_2 < 0\}$ and $\{(t_1, t_2) \in \mathbb{R}_2 \mid t_1 > 1, t_2 > 1\}$. Hence $\mathcal{M}_5^{\mu}$ has a three-dimensional eigenspace of eigenvalue 1. Lemma 3.3 (2) yields that this eigenspace is expressed as

$$\langle \widetilde{\Delta}_{235} \rangle^{\perp} = \{\Delta \in H_2^{\mathrm{lf}}(T,\psi) \mid \mathfrak{I}(\Delta, \widetilde{\Delta}_{235}) = 0\}.$$

So $\mathcal{M}_5^{\mu}$ has desired eigenvalues and eigenspaces. Since the factor

$$\frac{1 - \mu_{235}}{\mathfrak{I}(\Delta_{235}, \widetilde{\Delta}_{235})} = (1 - \mu_{235}) \Big/ \left(\frac{\mu_{235} - 1}{(\mu_2 - 1)(\mu_3 - 1)(\mu_5 - 1)} \right) = (\mu_2 - 1)(\mu_3 - 1)(\mu_5 - 1)$$

is continuous on μ_{235} at 1, the expression of $\mathcal{M}_5^\mu$ is valid even in the case $\mu_{235}=1$. Similarly, we have the expressions of $\mathcal{M}_3^\mu$ and $\mathcal{M}_4^\mu$. $\square$

COROLLARY 3.11. *Under the assumption* (3.1), *the circuit transformations* $\widetilde{\mathcal{M}}_i^{-\mu}\in GL(H_2(T,\psi^{-1}))$ $(i=1,\dots,5)$ *are given as*

$$\widetilde{\mathcal{M}}_1^{-\mu}(\widetilde{\Delta})=\mu_{12}\widetilde{\Delta}-(\mu_{12}-1)\left(\widetilde{\Delta}_1,\widetilde{\Delta}_4\right)\begin{pmatrix}\mathfrak{I}(\Delta_1,\widetilde{\Delta})\\ \mathfrak{I}(\Delta_4,\widetilde{\Delta})\end{pmatrix}H_{14}^{\mu\ -1},$$

$$\widetilde{\mathcal{M}}_2^{-\mu}(\widetilde{\Delta})=\mu_{34}\widetilde{\Delta}-(\mu_{34}-1)\left(\widetilde{\Delta}_1,\widetilde{\Delta}_3\right)\begin{pmatrix}\mathfrak{I}(\Delta_1,\widetilde{\Delta})\\ \mathfrak{I}(\Delta_3,\widetilde{\Delta})\end{pmatrix}H_{13}^{\mu\ -1},$$

$$\widetilde{\mathcal{M}}_3^{-\mu}(\widetilde{\Delta})=\widetilde{\Delta}-\frac{1-\mu_{245}^{-1}}{\mathfrak{I}(\Delta_2,\widetilde{\Delta}_2)}\mathfrak{I}(\Delta_2,\widetilde{\Delta})\widetilde{\Delta}_2,$$

$$\widetilde{\mathcal{M}}_4^{-\mu}(\widetilde{\Delta})=\widetilde{\Delta}-\frac{1-\mu_{145}^{-1}}{\mathfrak{I}(\Delta_{145},\widetilde{\Delta}_{145})}\mathfrak{I}(\Delta_{145},\widetilde{\Delta})\widetilde{\Delta}_{145},$$

$$\widetilde{\mathcal{M}}_5^{-\mu}(\widetilde{\Delta})=\widetilde{\Delta}-\frac{1-\mu_{235}^{-1}}{\mathfrak{I}(\Delta_{235},\widetilde{\Delta}_{235})}\mathfrak{I}(\Delta_{235},\widetilde{\Delta})\widetilde{\Delta}_{235},$$

where $\widetilde{\Delta}$ *is any element of* $H_2(T,\psi^{-1})$.

3.5.3. Circuit matrices.

Let M_i^μ and $\widetilde{M}_i^{-\mu}$ $(i=1,\dots,5)$ be the circuit matrices along the loop ρ_i with respect to the basis ${}^t(\Delta_1,\dots,\Delta_4)$ of $H_2^{\mathrm{lf}}(T,\psi)$, and to $(\widetilde{\Delta}_1,\dots,\widetilde{\Delta}_4)$ of $H_2(T,\psi^{-1})$, respectively. That is, we have transformations

$$\begin{pmatrix}\Delta_1\\ \vdots\\ \Delta_4\end{pmatrix}\mapsto M_i^\mu\begin{pmatrix}\Delta_1\\ \vdots\\ \Delta_4\end{pmatrix},\quad (\widetilde{\Delta}_1,\dots,\widetilde{\Delta}_4)\mapsto(\widetilde{\Delta}_1,\dots,\widetilde{\Delta}_4)\widetilde{M}_i^{-\mu}$$

by the continuation along ρ_i.

COROLLARY 3.12. *The circuit matrices are expressed as*

$$M_i^\mu=\lambda_iI_4-(\lambda_i-1)H^\mu\widetilde{R}_i(R_iH^\mu\widetilde{R}_i)^{-1}R_i\quad(i=1,2),$$
$$M_j^\mu=I_4-(1-\lambda_j)H^\mu\widetilde{r}_j(r_jH^\mu\widetilde{r}_j)^{-1}r_j\quad(j=3,4,5),$$

$$\widetilde{M}_i^{-\mu}=\frac{1}{\lambda_i}I_4-\left(\frac{1}{\lambda_i}-1\right)\widetilde{R}_i(R_iH^{\mu\ t}\widetilde{R}_i)^{-1}R_iH^\mu\quad(i=1,2),$$
$$\widetilde{M}_j^{-\mu}=I_4-\left(1-\frac{1}{\lambda_j}\right)\widetilde{r}_j(r_jH^\mu\widetilde{r}_j)^{-1}r_jH^\mu\quad(j=3,4,5),$$

where

$$\lambda_1=\mu_{12}^{-1},\quad\lambda_2=\mu_{34}^{-1},\quad\lambda_3=\mu_2\mu_4\mu_5,\quad\lambda_4=\mu_1\mu_4\mu_5,\quad\lambda_5=\mu_2\mu_3\mu_5,$$

$$R_1 = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix}, \quad r_3 = (0,1,0,0),$$

$$r_4 = \left(-\frac{\mu_{45}-1}{\mu_5-1}, 1, 0, -\frac{\mu_{345}-1}{\mu_5-1}\right), \quad r_5 = \left(-\frac{\mu_{25}-1}{\mu_5-1}, 1, -\frac{\mu_{125}-1}{\mu_5-1}, 0\right),$$

$$\widetilde{R}_1 = {}^tR_1, \quad \widetilde{R}_2 = {}^tR_2, \quad \widetilde{r}_3 = {}^tr_3, \quad \widetilde{r}_4 = \begin{pmatrix} \dfrac{-(\mu_{45}-1)}{\mu_4(\mu_5-1)} \\ 1 \\ 0 \\ \dfrac{-1}{\mu_{34}(\mu_5-1)} \end{pmatrix}, \quad \widetilde{r}_5 = \begin{pmatrix} \dfrac{-(\mu_{25}-1)}{\mu_2(\mu_5-1)} \\ 1 \\ \dfrac{-1}{\mu_{12}(\mu_5-1)} \\ 0 \end{pmatrix}.$$

Their explicit forms are

$$M_1^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{(\mu_1-1)(\mu_4\mu_5-1)}{\mu_1(\mu_5-1)} & \dfrac{1}{\mu_1\mu_2} & 0 & \dfrac{(\mu_1-1)(\mu_{345}-1)}{\mu_1(\mu_5-1)} \\ \dfrac{-(\mu_2-1)}{\mu_1\mu_2} & 0 & \dfrac{1}{\mu_1\mu_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_2^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{(\mu_3-1)(\mu_2\mu_5-1)}{\mu_3(\mu_5-1)} & \dfrac{1}{\mu_3\mu_4} & \dfrac{(\mu_3-1)(\mu_{125}-1)}{\mu_3(\mu_5-1)} & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{-(\mu_4-1)}{\mu_3\mu_4} & 0 & 0 & \dfrac{1}{\mu_3\mu_4} \end{pmatrix},$$

$$M_3^\mu = \begin{pmatrix} 1 & \mu_5-1 & 0 & 0 \\ 0 & \mu_{245} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_4^\mu = \begin{pmatrix} 1-\mu_1+\mu_{145} & -\mu_1(\mu_5-1) & 0 & \mu_1(\mu_{345}-1) \\ \dfrac{-(\mu_1-1)(\mu_4\mu_5-1)}{\mu_5-1} & \mu_1 & 0 & \dfrac{-(\mu_1-1)(\mu_{345}-1)}{\mu_5-1} \\ -(\mu_4\mu_5-1) & \mu_5-1 & 1 & -(\mu_{345}-1) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_5^\mu = \begin{pmatrix} 1-\mu_3+\mu_{235} & -\mu_3(\mu_5-1) & \mu_3(\mu_{125}-1) & 0 \\ \dfrac{-(\mu_3-1)(\mu_2\mu_5-1)}{\mu_5-1} & \mu_3 & \dfrac{-(\mu_3-1)(\mu_{125}-1)}{\mu_5-1} & 0 \\ 0 & 0 & 1 & 0 \\ -(\mu_2\mu_5-1) & \mu_5-1 & -(\mu_{125}-1) & 1 \end{pmatrix},$$

$$\widetilde{M_1}^{-\mu}=\begin{pmatrix}1 & \dfrac{-(\mu_1-1)(\mu_4\mu_5-1)}{\mu_4(\mu_5-1)} & \mu_1(\mu_2-1)(\mu_{125}-1) & 0\\ 0 & \mu_1\mu_2 & 0 & 0\\ 0 & 0 & \mu_1\mu_2 & 0\\ 0 & \dfrac{-(\mu_1-1)}{\mu_3\mu_4(\mu_5-1)} & 0 & 1\end{pmatrix},$$

$$\widetilde{M_2}^{-\mu}=\begin{pmatrix}1 & \dfrac{-(\mu_3-1)(\mu_2\mu_5-1)}{\mu_2(\mu_5-1)} & 0 & \mu_3(\mu_4-1)(\mu_{345}-1)\\ 0 & \mu_3\mu_4 & 0 & 0\\ 0 & \dfrac{-(\mu_3-1)}{\mu_1\mu_2(\mu_5-1)} & 1 & 0\\ 0 & 0 & 0 & \mu_3\mu_4\end{pmatrix},$$

$$\widetilde{M_3}^{-\mu}=\begin{pmatrix}1 & 0 & 0 & 0\\ \dfrac{-(\mu_5-1)}{\mu_5} & \dfrac{1}{\mu_{245}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$\widetilde{M_4}^{-\mu}=\begin{pmatrix}\dfrac{1-\mu_4\mu_5+\mu_{145}}{\mu_{145}} & \dfrac{(\mu_1-1)(\mu_4\mu_5-1)}{\mu_1\mu_4(\mu_5-1)} & \dfrac{(\mu_4\mu_5-1)(\mu_{125}-1)}{\mu_4\mu_5} & 0\\ \dfrac{\mu_5-1}{\mu_1\mu_5} & \dfrac{1}{\mu_1} & \dfrac{-(\mu_5-1)(\mu_{125}-1)}{\mu_5} & 0\\ 0 & 0 & 1 & 0\\ \dfrac{-1}{\mu_{134}\mu_5} & \dfrac{\mu_1-1}{\mu_{134}(\mu_5-1)} & \dfrac{\mu_{125}-1}{\mu_{345}} & 1\end{pmatrix},$$

$$\widetilde{M_5}^{-\mu}=\begin{pmatrix}\dfrac{1-\mu_2\mu_5+\mu_{235}}{\mu_{235}} & \dfrac{(\mu_3-1)(\mu_2\mu_5-1)}{\mu_2\mu_3(\mu_5-1)} & 0 & \dfrac{(\mu_2\mu_5-1)(\mu_{345}-1)}{\mu_2\mu_5}\\ \dfrac{\mu_5-1}{\mu_3\mu_5} & \dfrac{1}{\mu_3} & 0 & \dfrac{-(\mu_5-1)(\mu_{345}-1)}{\mu_5}\\ \dfrac{-1}{\mu_{123}\mu_5} & \dfrac{\mu_3-1}{\mu_{123}(\mu_5-1)} & 1 & \dfrac{\mu_{345}-1}{\mu_{125}}\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

They satisfy

$$M_i^{\mu}H^{\mu}\widetilde{M_i}^{-\mu}=H^{\mu}\quad (i=1,\dots,5).$$

Under the additional condition (3.2), $M_1^{\mu},\dots,M_5^{\mu}$ *are regarded as circuit matrices of* $E_2(a,b,b'c,c')$.

Proof. We identify

$$\Delta=\sum_{i=1}^{4}z_i\Delta_i\in H_2^{\mathrm{lf}}(T,\psi),\quad \widetilde{\Delta}=\sum_{i=1}^{4}\widetilde{z}_i\widetilde{\Delta}_i\in H_2(T,\psi^{-1})$$

with the row and column vectors

$$z = (z_1, \dots, z_4), \quad \widetilde{z} = \begin{pmatrix} \widetilde{z}_1 \\ \vdots \\ \widetilde{z}_4 \end{pmatrix},$$

respectively. Note that

$$\mathfrak{I}(\Delta, \widetilde{\Delta}) = zH^{\mu}\widetilde{z}.$$

Theorem 3.10 yields these expressions. We have

$$M_i^{\mu} H^{\mu} \widetilde{M}_i^{-\mu} = H^{\mu} \quad (i = 1, \dots, 5)$$

by Lemma 3.3 (1). □

REMARK 3.13. As a result, M_i^{μ} $(i = 1, \dots, 5)$ coincides with the circuit matrix with respect to the basis $\Delta_1, \dots, \Delta_4$ by Theorem 7.1 in [**MY**].

4. Schwarz maps for $\mathcal{E}$ as the universal Abel–Jacobi maps.

In this section, we introduce a system $\mathcal{E}$, and describe its Schwarz map, which is the main result of this paper.

4.1. The system $\mathcal{E}$: a restriction of the system $E(3, 6; 1/3)$.

We introduce in this subsection a system $\mathcal{E}$, which is a system E_2 with specific parameters, and mention a reason why this system is of special interest.

Let $X(3, 6)$ be the configuration space of six lines $\ell_1, \dots, \ell_6$ in general position in the projective plane $\mathbb{P}^2 = \{p : q : r\}$. We identify the space $X(3, 6)$ with

$$\left\{ \begin{pmatrix} 1 & 0 & 1 & x^1 & x^2 & 0 \\ 0 & 1 & 1 & x^3 & x^4 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \;\middle|\; \text{no } 3 \times 3 \text{ - minors vanish} \right\},$$

where ℓ_6 is the line at infinity in the pq-plane given by $r = 0$. The system

$$E(3, 6; a), \quad a = (a_1, \dots, a_6), \quad a_1 + \dots + a_6 = 3$$

is generated by the linear differential equations which annihilate functions on $X(3, 6)$ defined by the integral

$$\iint_{\text{a cycle}} \prod_{j=1}^{5} f_j(x; p, q)^{a_j - 1} dp \wedge dq, \qquad \begin{pmatrix} f_1 = p, \quad f_2 = q, \quad f_3 = p + q + 1, \\ f_4 = x^1 p + x^3 q + 1, \quad f_5 = x^2 p + x^4 q + 1 \end{pmatrix}.$$

The Schwarz map of the system $E(3, 6; a)$ is studied (cf. [**MSY**], [**MSTY**]) in two cases a_j=1/2 and $a_j \equiv 1/6 \mod \mathbb{Z}$. We have been interested in the case $a_j \equiv 1/3 \mod \mathbb{Z}$.

On the other hand, let X_2 be the 2-dimensional stratum defined by $x^2 = x^3 = 0$, which is the space of six lines such that the three lines $\{\ell_1, \ell_3, \ell_6\}$ meet at a point, the three lines $\{\ell_1, \ell_2, \ell_5\}$ meet at another point, and nothing further special occurs. It is known ([**MSY**]) that the restriction of $E(3,6;a)$ onto X_2 is the Appell's hypergeometric system E_3, which is projectively equivalent (multiplying a function to the unknown) to

$$E_2(a_1, 1-a_5, 1-a_6, 2-a_2-a_5, 2-a_3-a_6; x, y), \quad x = 1/x^2,\ y = 1/x^4.$$

Setting $a = (4/3, 1/3, 1/3, 1/3, 1/3, 1/3)$, we define

$$\mathcal{E} := E_2\left(\frac{4}{3}, \frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}}; x, y\right), \quad \text{where} \quad \frac{\mathbf{2}}{\mathbf{3}} = \left(\frac{2}{3}, \frac{2}{3}\right), \ \frac{\mathbf{4}}{\mathbf{3}} = \left(\frac{4}{3}, \frac{4}{3}\right).$$

We believe that the study of the system $\mathcal{E}$ will be the first step of understanding the system $E(3,6;a)$, $a_j \equiv 1/3 \bmod \mathbb{Z}$.

The Schwarz map of a system is defined by the ratio of linearly independent solutions. The main object of this paper is the Schwarz map of the hypergeometric system $\mathcal{E}$. The system $\mathcal{E}$ admits solutions stated in Proposition 2.4. The next subsection gives a geometric background of understanding these solutions.

4.2. A family of curves of genus 2.

Consider a family of curves of genus 2 given as triple covers of $\mathbb{P}^1$:

$$C_t : S^3 = s^2(1-s)(t-s)^2, \quad t: \text{ parameter},$$

branching at four points $\{0, 1, t, \infty\}$. We choose two linearly independent holomorphic 1-forms:

$$\omega_1 = \omega_1(t) = s^{-2/3}(s-1)^{-1/3}(s-t)^{-2/3}ds = \frac{ds}{S},$$
$$\omega_2 = \omega_2(t) = s^{-1/3}(s-1)^{-2/3}(s-t)^{-1/3}ds = \frac{s(s-t)ds}{S^2},$$

and put

$$\varphi_1(s,t) = \int_0^s \omega_1(t), \quad \varphi_2(s,t) = \int_0^s \omega_2(t).$$

For a fixed t, the *Abel–Jacobi map* for the curve C_t is a multi-valued map

$$C_t \ni s \longmapsto (\varphi_1(s,t),\ \varphi_2(s,t)) \in \mathbb{C}^2.$$

It is a single-valued map to its Jacobian $\mathbb{C}^2/L$, where L is a lattice generated by its periods: integrals over possible loops with base $s = 0$:

$$(\varphi_1(0,t),\ \varphi_2(0,t)).$$

4.3. The Schwarz map of $\mathcal{E}$.

Proposition 2.4 for $a = c = c' = 4/3, b = b' = 2/3$ implies that after the coordinate change

$$\boldsymbol{x} = \frac{-x}{1-x}, \quad \boldsymbol{z} = \boldsymbol{x}\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{y} = \frac{-y}{1-y},$$

and the change of unknown: $u \to (1-y)^{2/3}(1-x)^{2/3}u$, two linearly independent solutions of

$$E\left(\frac{\mathbf{2}}{\mathbf{3}}, \frac{4}{3}; \boldsymbol{z}\right)$$

and the two indefinite integrals

$$\int_0^{\boldsymbol{x}} s^{-2/3}(1-s)^{-1/3}(\boldsymbol{z}-s)^{-2/3}ds, \quad \boldsymbol{z}^{-1/3}\int_0^{\boldsymbol{x}} s^{-1/3}(1-s)^{-2/3}(\boldsymbol{z}-s)^{-1/3}ds$$

form a set of fundamental solutions of $\mathcal{E}$.

On the other hand, the integral representation of the Gauss hypergeometric equation given in Section 1 asserts that the integral above along any *closed* path gives a solution of $E(\mathbf{2/3}, 4/3)$. Thus we find that the Schwarz map of $\mathcal{E}$ is the totality of the Abel–Jacobi map of the family $\{C_t\}$ after a slight modification (multiplying $t^{-1/3}$ to the second coordinate).

Thus we get

THEOREM 4.1. *If we change the coordinates (x, y) of $\mathcal{E} = E_2(4/3, \mathbf{2/3}, \mathbf{4/3}; x, y)$ as*

$$s(=\boldsymbol{x}) = \frac{-x}{1-x}, \quad t(=\boldsymbol{z}) = \boldsymbol{x}\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{y} = \frac{-y}{1-y},$$

the Schwarz map of the system $\mathcal{E}$ is equivalent to the projectivization of the family of the Abel–Jacobi map of the family $\{C_t\}$ of curves of genus 2, explicitly given as

$$\mathcal{S}_2 : \bigcup_{t\in\mathbb{C}-\{0,1\}} C_t \ni (s,t) \longmapsto \varphi_1(0,t) : t^{-1/3}\varphi_2(0,t) : \varphi_1(s,t) : t^{-1/3}\varphi_2(s,t) \in \mathbb{P}^3.$$

The latter two $\varphi_1(s,t)$ and $t^{-1/3}\varphi_2(s,t)$ are f_1 and f_2 in Section 2. The map by means of the former two

$$\mathcal{S}_1 : \mathbb{P}^1 - \{0,1,\infty\} \ni t \longmapsto \varphi_1(0,t) : \ t^{-1/3}\varphi_2(0,t) \in \mathbb{P}^1$$

is the Schwarz map of the hypergeometric equation $E(\mathbf{2/3}, 4/3)$. Its image is a disc, and the inverse map of $\mathcal{S}_1$ is a single-valued automorphic function on the disc with respect to the triangle group of type $[3,\infty,\infty]$; in other words, the disc is tessellated by Schwarz triangles of type $[3,\infty,\infty]$.
The image under $\mathcal{S}_2$ admits a structure of a fiber bundle over the image of $\mathcal{S}_1$, whose fiber at t is identified with the Jacobian variety of C_t.

A triangle of type $[p,q,r]$ is a hyperbolic triangle with angles $\pi/p, \pi/q$ and π/r; the above triangle has angles $\pi/3, 0$ and 0. The triangle group of type $[p,q,r]$ is the group consisting of the even products of the reflections with the sides of the triangle of type

$[p, q, r]$ as axes. It is known that the triangle group of type $[3, \infty, \infty]$ is conjugate to the congruence subgroup

$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a-1, d-1, c \equiv 0 \mod 3 \right\}.$$

For arithmetic triangle groups, see [**T**].

Other than this family of curves, there are two families of curves of genus 2 branching at four points in $\mathbb{P}^1$; see Appendix 2.

4.4. Monodromy group of $\mathcal{E}$.

From monodromy side, Theorem 4.1 can be understood as follows. Note that the parameters of $\mathcal{E} = E_2\big(4/3, \mathbf{2/3}, \mathbf{4/3}\big)$ satisfy the conditions (3.1) and (3.2). Define M_i and $\widetilde{M_i}$ $(i = 1, \ldots, 5)$ by substituting $\mu_1 = \cdots = \mu_5 = \omega^2 = (-1-\sqrt{-3})/2$ into M_i^{μ} and $\widetilde{M_i}^{-\mu}$ defined in Section 3.5.3, respectively. They are the circuit matrices for $\mathcal{E}$ with respect to

$$ {}^t\Big(\iint_{\square_1} \psi_{\mathcal{E}} dt_1 dt_2, \ldots, \iint_{\square_4} \psi_{\mathcal{E}} dt_1 dt_2\Big)$$

and those for $\mathcal{E}^{\vee} = E_2(-4/3, -2/3, -2/3, -4/3, -4/3)$ with respect to

$$\Big(\iint_{\square_1} \psi_{\mathcal{E}}^{-1} dt_1 dt_2, \ldots, \iint_{\square_4} \psi_{\mathcal{E}}^{-1} dt_1 dt_2\Big),$$

where

$$\psi_{\mathcal{E}} = \frac{1}{\sqrt[3]{t_1(1-t_1)t_2(1-t_2)(1-t_1x-t_2y)^4}}.$$

COROLLARY 4.2. *We have*

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2\omega-1 & -\omega-1 & 0 & 0 \\ -2\omega-1 & 0 & -\omega-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{M_1} = \begin{pmatrix} 1 & 2\omega+1 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & \omega+1 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2\omega-1 & -\omega-1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2\omega-1 & 0 & 0 & -\omega-1 \end{pmatrix}, \quad \widetilde{M_2} = \begin{pmatrix} 1 & 2\omega+1 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & \omega+1 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & -\omega-2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{M_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega-1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} \omega+3 & -2\omega-1 & 0 & 0 \\ -\omega+1 & -\omega-1 & 0 & 0 \\ -\omega+1 & -\omega-2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{M_4} = \begin{pmatrix} -\omega+2 & \omega+2 & 0 & 0 \\ 2\omega+1 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega & 1 & 0 & 1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} \omega+3 & -2\omega-1 & 0 & 0 \\ -\omega+1 & -\omega-1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega+1 & -\omega-2 & 0 & 1 \end{pmatrix}, \quad \widetilde{M_5} = \begin{pmatrix} -\omega+2 & \omega+2 & 0 & 0 \\ 2\omega+1 & \omega & 0 & 0 \\ -\omega & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

They satisfy

$$M_i H \widetilde{M_i} = H, \quad (i = 1, \ldots, 5), \qquad H = \frac{-1}{3}\begin{pmatrix} 1 & \omega & 0 & 0 \\ -\omega-1 & 0 & 0 & 0 \\ -\omega-1 & 0 & \sqrt{-3} & 0 \\ -\omega-1 & 0 & 0 & \sqrt{-3} \end{pmatrix}.$$

By Proposition 3.1 and Remark 3.4, the subspace spanned by solutions $\iint_{\square_1} \psi_{\mathcal{E}} dt_1 dt_2$ and $\iint_{\square_2} \psi_{\mathcal{E}} dt_1 dt_2$ is invariant under the monodromy representation. In fact, the top-left 2×2 block matrices M_i' of M_i $(i = 1, \ldots, 5)$ act on this space. Note that $M_1' = M_2'$, $M_4' = M_5'$. Let G be the group generated by M_1', M_3' and M_5'. The group G is isomorphic to the triangle group $[3, \infty, \infty]$, and is contained in the unitary group

$$\left\{ g \in GL_2(\mathbb{Z}[\omega]) \mid gH'\,{}^t\overline{g} = H' = \begin{pmatrix} -1 & -\omega \\ \omega+1 & 0 \end{pmatrix} \right\}.$$

By a matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -2-\omega \end{pmatrix},$$

the Hermite matrix H' and circuit matrices M_i' $(i = 1, 3, 5)$ are transformed as

$$PH'\,{}^t\overline{P} = \sqrt{-3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$PM_1'P^{-1} = \omega\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \quad PM_3'P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad PM_5'P^{-1} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}.$$

Hence the projectivization of G is isomorphic to the congruence subgroup $\Gamma_1(3)$ of $\mathrm{SL}_2(\mathbb{Z})$ and the ratio $\iint_{\square_1} \psi_{\mathcal{E}} dt_1 dt_2 / \iint_{\square_2} \psi_{\mathcal{E}} dt_1 dt_2$ can be regarded as the map $\mathcal{S}_1$ in Theorem 4.1 and as an element of the upper-half space.

Appendix A. Restriction of the system $E(3,6;a)$ on a 3-dimensional stratum.

Let X_3 denote the stratum defined by $x^2=0$ and let us restrict the system $E(3,6;a)$ to this stratum, which is the space of six lines such that the three lines $\{\ell_1,\ell_3,\ell_6\}$ meet at a point, and nothing further special occurs. This system is denoted by $E(3,6;a)|X_3$ or EX_3. Little is known about this system.

Before stating the proposition in this section, we briefly recall the Appell–Lauricella's system $E_D^{(3)}(a,b_1,b_2,b_3,c;y^1,y^2,y^3)$:

$$\delta_i(\delta+c-1)u - y^i(\delta_i+b_i)(\delta+a)u = 0,$$
$$y^i(\delta_i+b_i)\delta_j u - y^j(\delta_j+b_j)\delta_i u = 0,$$

where (y^1,y^2,y^3) are variables, and $\delta_i = y^i\partial/\partial y^i$ and $\delta=\delta_1+\delta_2+\delta_3$. This is a 3-variable version of the Appell's E_1. It admits solutions given by a power series

$$F_D(a,b_1,b_2,b_3,c;y^1,y^2,y^3) = \sum_{n_1,n_2,n_3}^{\infty} \frac{(a,n_{123})(b_1,n_1)(b_2,n_2)(b_3,n_3)}{(c,n_{123})n_1!n_2!n_3!}(y^1)^{n_1}(y^2)^{n_2}(y^3)^{n_3},$$

where $n_{123}=n_1+n_2+n_3$, and by an integral

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-ty^1)^{-b_1}(1-ty^2)^{-b_2}(1-ty^3)^{-b_3}dt.$$

The collection of solutions is denoted by $S_D^{(3)}(a,b_1,b_2,b_3,c;y^1,y^2,y^3)$.

In this section, we prove the following proposition.

Proposition A.1. *If $a_2+a_4+a_5=1$, then the system $E(3,6;a)|X_3$ is reducible and has a subsystem isomorphic to the Appell–Lauricella's system $E_D^{(3)}=E_D^{(3)}$ in 3 variables with 4 free parameters. More precisely, the collection of the solutions of $E(3,6;a)|X_3$ includes*

$$(1-x^1)^{-a_2}S_D^{(3)}\left(a_3,a_4,1-a_6,a_2,1+a_3-a_5;x^3,x^4,\frac{x^3-x^1}{1-x^1}\right).$$

Here note that $a_1+\cdots+a_6=3$.

If we apply the proposition under the further restriction $x^3=0$, we find a subsystem isomorphic to E_1 in E_2, which is equivalent to Proposition 1.3.

We give three "proof"s: one using power series, one using integral representations, and one manipulating differential equations.

A.1. Power series.

It is known that the system $E(3,6;a)|_{X_3}$ has a solution given by the series

$$F_{X_3}(a_2,a_3,a_4,a_5,a_6;x) := \sum_{n_1,n_3,n_4=0}^{\infty} \frac{(a_5',n_{13})(a_6',n_4)(a_2,n_1)(a_3,n_{34})}{(a_2+a_3+a_4,n_{134})n_1!n_3!n_4!}(x^1)^{n_1}(x^3)^{n_3}(x^4)^{n_4},$$

where $x = (x^1, x^3, x^4), n_{13} = n_1 + n_3, n_{34} = n_3 + n_4, n_{134} = n_1 + n_3 + n_4, a'_5 = 1 - a_5, a'_6 = 1 - a_6$; refer to [**MSY**, p. 47]. A computation shows that the identity

$$F_{X_3}(a_2, a_3, a_4, a_5, a_6; x) = (1 - x^1)^{-a_2} F_D^{(3)}\left(a_3, a_4, 1 - a_6, a_2, 1 + a_3 - a_5; x^3, x^4, \frac{x^3 - x^1}{1 - x^1}\right)$$

holds if and only if $a_2 + a_4 + a_5 = 1$.

A.2. Integral representation.

We manipulate the integral

$$\iint p^{a_1-1} q^{a_2-1} r^{a_3-1} (p + q + r)^{a_4-1} (p + x_1 q + x_3 r)^{a_5-1} (p + x_4 r)^{a_6-1} dp \wedge dq.$$

The three lines

$$\ell_1 : p = 0, \quad \ell_3 : r = 0, \quad \ell_6 : p + x_4 r = 0$$

meet at $0 : 1 : 0$. Introduce a new coordinate Q

$$q = \frac{(p + x_3)(p + 1)Q}{D}, \quad D := x_1(p + 1) - p - x_3 - x_1(p + 1)Q,$$

which send $q = 0, -p - 1, -(p + x_3)/x_1$ to $Q = 0, 1, \infty$. Since

$$q + p + 1 = \frac{(p + 1)N(Q - 1)}{D}, \quad x_1 q + p + x_3 = -\frac{(p + x_3)N}{D},$$
$$dq = -\frac{(p + x_3)(p + 1)N dQ}{D^2} + *dp, \quad N = (1 - x_1)p + x_3 - x_1,$$

we have

$$f = -\iint p^{a_1-1}\{(p + x_3)(p + 1)Q\}^{a_2-1}\{(p + 1)N(Q - 1)\}^{a_4-1}$$
$$\times \{-(p + x_3)N\}^{a_5-1}(p + x_4)^{a_6-1}(p + x_3)(p + 1)N \cdot D^{1-a_{245}} dp \wedge dQ,$$

where $a_{245} = a_2 + a_4 + a_5$. If

$$a_{245} = 1 \qquad (\text{note } : \{2, 4, 5\} = \{1, \ldots, 6\} - \{1, 3, 6\}),$$

then the double integral above becomes the product of the Beta integral

$$\int Q^{a_2-1}(Q - 1)^{a_4-1} dQ$$

and the integral

$$\int p^{a_1-1}(p + 1)^{a_2+a_4-1}(p + x_3)^{a_2+a_5-1}(p + x_4)^{a_6-1}\{(1 - x_1)p + x_3 - x_1\}^{a_4+a_5-1} dp,$$

which can be written as

$$(1-x_1)^{-a_2}\int p^{a_1-1}(p+1)^{-a_5}(p+x_3)^{-a_4}(p+x_4)^{a_6-1}\left(p+\frac{x_3-x_1}{1-x_1}\right)^{-a_2}dp.$$

On the other hand, the integral

$$\int p^{b_1+b_2+b_3-c}(p-1)^{c-a-1}(p-y^1)^{-b_1}(p-y^2)^{-b_2}(p-uy^3)^{-b_3}dp$$

solves $E_D^{(3)}(a,b_1,b_2,b_3,c;y^1,y^2,y^3)$. By solving the system

$$a_1-1=b_1+b_2+b_3-c,\quad -a_5=c-a-1,\quad -a_4=b_1,\quad a_6-1=b_2,\quad -a_2=-b_3,$$

we complete the proof of Proposition A.1

A.3. System of differential equations.

We manipulate the Appell–Lauricella's system $E_D^{(3)}=E_D^{(3)}(a,b_1,b_2,b_3,c;y^1,y^2,y^3)$ and the system EX_3 (given in [**MSY**, p. 24]):

$$\begin{aligned}
&(\theta+a_{234}-1)\theta_1 w-x^1(\theta_1+\theta_3+1-a_5)(\theta_1+a_2)w=0,\\
&(\theta+a_{234}-1)\theta_3 w-x^3(\theta_1+\theta_3+1-a_5)(\theta_3+\theta_4+a_3)w=0,\\
&(\theta+a_{234}-1)\theta_4 w-x^4(\theta_4+1-a_6)(\theta_3+\theta_4+a_3)w=0,\\
&x^3(\theta_1+\theta_3+1-a_5)\theta_4 w-x^4(\theta_4+1-a_6)\theta_3 w=0,\\
&x^1(\theta_1+a_2)\theta_3 w-x^3(\theta_3+\theta_4+a_3)\theta_1 w=0,
\end{aligned}$$

where $\theta_1=x^1\partial/\partial x^1$, $\theta_3=x^3\partial/\partial x^3$, $\theta_4=x^4\partial/\partial x^4$, and $\theta=\theta_1+\theta_3+\theta_4$. We show that the system $E_D^{(3)}$ is a subsystem of EX_3 when $a_2+a_4+a_5=1$.

Change the unknown w of EX_3 into u by

$$w=(1-x^1)^{-p}u,$$

and the variables (x^1,x^3,x^4) into (x,y,z) as

$$x=x^3,\qquad y=x^4,\qquad z=\frac{x^3-x^1}{1-x^1}.$$

Then EX_3 can be written as $L_i u=0,\ 1\le i\le 5$, where

$$\begin{aligned}
L_1:=&\left(\delta_x+\delta_y+\frac{1-z}{1-x}\delta_z+h+a_{234}-1\right)(f\delta_z+h)\\
&-\frac{x-z}{1-z}\left(\delta_x+\frac{1-z}{1-x}\delta_z+h+1-a_5\right)(f\delta_z+h+a_2),\\
L_2:=&\left(\delta_x+\delta_y+\frac{1-z}{1-x}\delta_z+h+a_{234}-1\right)(\delta_x+g\delta_z)\\
&-x\left(\delta_x+\frac{1-z}{1-x}\delta_z+h+1-a_5\right)(\delta_x+\delta_y+g\delta_z+a_3),\\
L_3:=&\left(\delta_x+\delta_y+\frac{1-z}{1-x}\delta_z+h+a_{234}-1\right)\delta_y
\end{aligned}$$

$$
\begin{aligned}
&-y(\delta_y+1-a_6)(\delta_x+\delta_y+g\delta_z+a_3),\\
L_4 &:= x\left(\delta_x+\frac{1-z}{1-x}\delta_z+h+1-a_5\right)\delta_y - y(\delta_y+1-a_6)(\delta_x+g\delta_z),\\
L_5 &:= \frac{x-z}{1-z}(f\delta_z+h+a_2)(\delta_x+g\delta_z) - x(\delta_x+\delta_y+g\delta_z+a_3)(f\delta_z+h),
\end{aligned}
$$

$$
f=\frac{(z-x)(1-z)}{z(1-x)},\qquad g=\frac{x(1-z)}{z(1-x)}.
$$

Write the system $E_D^{(3)}$ as $E_{xy}u=0,\dots,E_{zz}u=0$, where

$$
\begin{aligned}
E_{xy} &:= \delta_{xy}-(b_2y\delta_x-b_1x\delta_y)/(x-y),\\
E_{xz} &:= \delta_{xz}-(b_3z\delta_x-b_1x\delta_z)/(x-z),\\
E_{yz} &:= \delta_{yz}-(b_3z\delta_y-b_2y\delta_z)/(y-z),\\
E_{xx} &:= \delta_{xx}+\delta_{xy}+\delta_{xz}-(((a+b_1)x+1-c)\delta_x+b_1x(\delta_y+\delta_z+a))/(1-x),\\
E_{yy} &:= \delta_{yy}+\delta_{xy}+\delta_{yz}-(((a+b_2)y+1-c)\delta_y+b_2y(\delta_x+\delta_z+a))/(1-y),\\
E_{zz} &:= \delta_{zz}+\delta_{xz}+\delta_{yz}-(((a+b_3)z+1-c)\delta_z+b_3z(\delta_x+\delta_y+a))/(1-z).
\end{aligned}
$$

Eliminating the second derivatives in L_j by using E_{**}, we see that L_j are linear combinations, over $\mathbb{C}(x,y,z)$, of the E_{**}'s if and only if

$$
a_2+a_4+a_5=1 \quad (\Leftrightarrow c=a+b_1+b_3),
$$

and

$$
p=a_2,\quad a=a_3,\quad b_1=a_4,\quad b_2=1-a_6,\quad b_3=a_2,\quad c=1+a_3-a_5.
$$

This completes the proof of Proposition A.1. □

REMARK. Actually we have, under the condition $c=a+b_1+b_3$,

$$
\begin{aligned}
\langle L_1,L_2,L_5\rangle &= \langle E_{xx},E_{xz},E_{zz}\rangle,\\
\langle L_3,L_4\rangle &= \langle E_{yy}-\frac{(z-x)(z-y)}{z(1-x)(1-y)}E_{yz},\ E_{yy}-\frac{(x-z)(x-y)}{x(1-z)(1-y)}E_{yx}\rangle.
\end{aligned}
$$

Appendix B. Families of curves of genus 2.

We encountered a family of curves C_t of genus 2 given as triple covers of $\mathbb{P}^1$. This is the Case 3 in the following Proposition.

PROPOSITION B.1. *A cyclic cover of $\mathbb{P}^1$ branching at four points is of genus 2 only in three cases:*

Case 3 : 3 fold cover with indices 3, 3, 3, 3,
Case 6 : 6 fold cover with indices 2, 2, 3, 3,
Case 4 : 4 fold cover with indices 2, 2, 4, 4.

PROOF. Note that the n-fold cyclic cover C of $\mathbb{P}^1$ branching at four points with indices $k_1, \ldots, k_4$ has the Euler characteristic

$$2n - \sum_{i=1}^{4} \frac{n}{k_i}(k_i - 1).$$

If the genus of C is two (the Euler characteristic of C is -2), we have

$$\sum_{i=1}^{4} \frac{1}{k_i} + \frac{2}{n} = 2, \quad \text{l.c.m.}(k_1, \ldots, k_4) = n, \quad k_i > 1 \ (i = 1, \ldots, 4).$$

We may assume that $k_1 \le k_2 \le k_3 \le k_4$. We show that $k_3 \ge 3$. In fact, if $k_1 = k_2 = k_3 = 2$ then the degree n is an even number $2m$. Thus the cyclic covering C is expressed as

$$S^{2m} = s^m (1 - s)^m (t - s)^m.$$

Since C is irreducible, m is equal to 1 and C is of genus 1. This contradicts to the assumption on the genus.

By using $k_3 \ge 3$, we have

$$2 = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{2}{n} \le \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{2}{n} = \frac{5}{3} + \frac{2}{n},$$

which implies $n \le 6$. We can check that there are only three possibilities for $(k_1, \ldots, k_4)$ given in this proposition. These three cases can be realized by the following families of curves:

$$\begin{aligned}
&\text{Case 3} : C_t^{(3)} : S^3 = s^2(1 - s)(t - s)^2, \qquad t : \text{ parameter,}\\
&\text{Case 6} : C_t^{(6)} : S^6 = s^2(1 - s)^4(t - s)^3, \quad t : \text{ parameter,}\\
&\text{Case 4} : C_t^{(4)} : S^4 = s^2(1 - s)^2(s - t), \qquad t : \text{ parameter.}
\end{aligned}$$

Note that the double cover of the base space of Case 6 branching at the two points of index 2 is equivalent to Case 3. □

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